SHARP PATTERNS FOR SOME SEMILINEAR NONLOCAL DISPERSAL EQUATIONS

By

JIAN-WEN SUN

Abstract. This paper is concerned with the nonlocal dispersal equation

$$\int_{\Omega} J(x-y)u(y)\,dy - u(x) + \lambda u(x) - [c(x)+\delta q(x)]u^p(x) = 0 \text{ in } \bar{\Omega},$$

where J is a nonnegative kernel function, the constants $\lambda > 0$, $\delta > 0$ and $p > 1$, the coefficients $c(x)$, $q(x)$ are nonnegative. We investigate the sharp patterns of positive solutions when $\delta \to 0$. Our study reveals how the existence of sharp profiles is determined by the behavior of $c(x)$ and $q(x)$. We find that sharp profiles are quite different to the results of classical reaction-diffusion equations.

1 Introduction and main results

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous function. It is known that the nonlocal dispersal equation

$$u_t(x,t) = J * u(x,t) - u(x,t) + f(x,u) \tag{1.1}$$

and its various forms arise in the study of different dispersal processes in material science, neurology and genetics (e.g., [3, 21, 22, 28]). As stated in [14], let $u(y,t)$ be the density of population at location y at time t, and $J(x-y)$ be the probability distribution of the population jumping from y to x; then $\int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy$ denotes the rate at which individuals are arriving at location x from all other places and $u(x,t) = \int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to all other places. Thus

$$Du(x,t) = J * u(x,t) - u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t)$$

is the dispersal of population and equation (1.1) describes the change of the population density $u(x,t)$ with the nonlinearity reaction function $f(x,u)$. The operator D is a nonlocal operator since the dispersal of $u(x,t)$ at location x and time t does not

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only depend on $u(x,t)$, but on all the values of u in a fixed spatial neighborhood of x through the term $J * u(x,t)$. There is quite an extensive literature for the study of nonlocal problems recently, among others, the papers [4, 5, 6, 37, 38, 19, 35] and references therein.

In this paper, we consider the nonlocal dispersal equation

$$\int_{\Omega} J(x-y)u(y)\,dy - u(x) + \lambda u(x) - c(x)u^p(x) = 0 \quad \text{in } \bar{\Omega}, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain, $p > 1$ and λ is a real parameter. The coefficient $c \in C(\bar{\Omega})$ is nonnegative and nontrivial. Problem (1.2) has been widely investigated; see, e.g., Bates and Zhao [1], García-Melián and Rossi [17]. On the other hand, the nonlocal dispersal equation (1.2) shares many properties with the following classical reaction-diffusion equation:

$$\begin{cases} \Delta u + \lambda u - c(x)u^p = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which has attracted much attention; see del Pino [9], López-Gómez and Rabinowitz [27] and Ouyang [29]. If the coefficient $c(x)$ has a spatial degeneracy, i.e., it vanishes in some subdomain, both the nonlocal dispersal equation and reaction-diffusion equation shall make a fundamental change [15, 16, 12, 20, 31, 34, 30, 24, 10, 11, 13]. Throughout this paper, we make the following assumptions on $J(x)$, and $c(x)$.

(A1) $J \in C(\mathbb{R}^N)$ is nonnegative, symmetric with unit integral and $J(0) > 0$.

(A2) $c \in C(\bar{\Omega})$ and there exists a non-empty property subdomain $\Omega_0 \subset \Omega$ such that

$$c(x) > 0 \text{ if and only if } x \in \bar{\Omega} \setminus \bar{\Omega}_0.$$

Let $\lambda_P(\Omega)$ be the unique principle eigenvalue of the nonlocal problem

$$\int_{\Omega} J(x-y)\phi(y)\,dy - \phi(x) = -\lambda\phi(x) \quad \text{in } \bar{\Omega}.$$

We then have $0 \leq \lambda_P(\Omega) < 1$ and $\lambda_P(\Omega)$ is monotone with respect to domain Ω; see [18, 21]. The principal eigenvalue $\lambda_P(\Omega)$ is very important in the study of positive solutions of (1.2). By (A2), we know that the coefficient $c(x)$ is degenerate in Ω_0. In this case, the nonlocal equation (1.2) is quite different than the case when $c(x)$ is positive in $\bar{\Omega}$; see [33, 8, 36]. In order to find the sharp influence of a complex environment on the nonlocal dispersal equation (1.2), we consider the asymptotic profile of positive solutions to the nonlocal dispersal problem

$$\int_{\Omega} J(x-y)u(y)\,dy - u(x) + \lambda u(x) - [c(x) + \delta q(x)]u^p(x) = 0 \quad \text{in } \bar{\Omega}, \tag{1.4}$$

where $q \in C(\overline{\Omega})$ is nonnegative and nontrivial and $\delta > 0$ is a parameter. The aim of this paper is to investigate the sharp patterns of positive solutions to (1.4). In fact, we find that the degenerate set $\overline{\Omega}^0 = \{x \in \overline{\Omega} : q(x) = 0\}$ will play a great role on the patterns. Thus we distinguish between the following different situations:

(H1) $\Omega_* = \overline{\Omega}_0 \cap \overline{\Omega}^0 = \emptyset$.
(H2) $\Omega_* = \overline{\Omega}_0 \cap \overline{\Omega}^0 = \overline{\Omega}_0$.
(H3) $\Omega_* = \overline{\Omega}_0 \cap \overline{\Omega}^0$ is a proper subdomain of Ω_0.

We first give the asymptotic profiles of positive solutions when the coefficient $q(x)$ satisfies (H1).

Theorem 1.1. *Assume that (H1) holds. Then there exists a unique positive solution $\theta_\delta \in C(\overline{\Omega})$ to* (1.4) *if and only if $\lambda > \lambda_P(\Omega)$. Meanwhile, the following hold:*

(i) *If $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$, then*

$$\lim_{\delta \to 0+} \theta_\delta(x) = \theta(x) \quad \textit{uniformly in } \overline{\Omega},$$

where $\theta(x)$ is the unique positive solution to (1.2).

(ii) *If $\lambda \geq \lambda_P(\Omega_0)$, then*

$$\lim_{\delta \to 0+} \theta_\delta(x) = \infty \quad \textit{uniformly in } \overline{\Omega}.$$

Remark 1.2. Note that $c(x) = 0$ for $x \in \Omega_0$. We know that (1.2) admits a unique positive solution $\theta \in C(\overline{\Omega})$ if and only if $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$; see [17].

Now we are ready to study the sharp patterns of positive solutions. We find that the existence of sharp profiles is determined by the nonlocal dispersal equation

$$\int_{\Omega_0} J(x-y)u(y)\,dy - u(x) = -\lambda u(x) + q(x)u^p(x) \quad \text{in } \overline{\Omega}_0. \tag{1.5}$$

Letting

$$\vartheta_\delta(x) = \delta^{\frac{1}{p-1}}\theta_\delta(x) \quad \text{and} \quad \eta_\delta(x) = \delta^{\frac{1}{p(p-1)}}\theta_\delta(x),$$

we can establish the sharp profiles of $\theta_\delta(x)$ as follows.

Theorem 1.3. *Assume that* (H1) *holds.*

(i) *If* $\lambda_P(\Omega) < \lambda \le \lambda_P(\Omega_0)$*, then*

$$\lim_{\delta\to 0+} \vartheta_\delta(x) = \lim_{\delta\to 0+} \eta_\delta(x) = 0 \quad \textit{uniformly in } \overline{\Omega}.$$

(ii) *If* $\lambda > \lambda_P(\Omega_0)$*, then*

$$\lim_{\delta\to 0+} \vartheta_\delta(x) = \alpha(x) \quad \textit{uniformly in } \overline{\Omega}_0,$$
$$\lim_{\delta\to 0+} \vartheta_\delta(x) = 0 \quad \textit{uniformly in any compact subset of } \overline{\Omega}\setminus\overline{\Omega}_0,$$

and

$$\lim_{\delta\to 0+} \eta_\delta(x) = \left[\frac{\int_{\Omega_0} J(x-y)\alpha(y)\,dy}{c(x)}\right]^{\frac{1}{p}}$$
$$\textit{uniformly in any compact subset of } \overline{\Omega}\setminus\overline{\Omega}_0,$$

where $\alpha \in C(\overline{\Omega}_0)$ *is the unique positive solution of* (1.5).

The conclusions in Theorems 1.1 and 1.3 give the sharp patterns of positive solutions to (1.4) under the assumptions (H1). If $q(x) > 0$ for $x \in \overline{\Omega}$, we can see that (H1) still holds. If $q(x) = 0$ for $x \in \Omega^0$ and Ω^0 is a proper subset of Ω, then the degeneracy also appears. Thus we know that if $\Omega_* = \emptyset$, the degeneracy of $q(x)$ does not change the sharp profiles of positive solutions to (1.4).

The profiles for positive solutions of the reaction-diffusion equation

$$\begin{cases} \Delta u = -\lambda u + [c(x) + \delta q(x)]u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

have been well investigated; see the works of Du and Li [12], López-Gómez [25], Li, López-Gómez and Sun [23]. Let $\lambda_L(\Omega)$ be the principal eigenvalue of

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we know from [26, 27] that (1.6) admits a unique positive solution $u_\delta(x)$ for $\lambda > \lambda_L(\Omega)$ and the asymptotic profiles of $u_\delta(x)$ with respect to δ are well established. If (H1) holds and $\lambda \in (\lambda_L(\Omega), \lambda_L(\Omega_0))$, then

$$\lim_{\delta\to 0+} u_\delta(x) = u(x) \quad \text{uniformly in } \overline{\Omega},$$

where $u(x)$ is the unique positive solution of (1.3), and

$$\lim_{\delta\to 0+} u_\delta(x) = \infty \quad \text{uniformly in } \overline{\Omega}_0$$

for any $\lambda \geq \lambda_L(\Omega_0)$. In the latter case, we know that $u_\delta(x)$ is still bounded as $\delta \to 0+$ in any compact subset of $\overline{\Omega} \setminus \overline{\Omega}_0$. However, from (ii) of Theorem 1.1 we obtain that the profiles of nonlocal dispersal equation (1.4) are unbounded in $\overline{\Omega}$ as $\delta \to 0+$. It follows from [25, 23] that the asymptotic profiles and sharp patterns of positive solutions for (1.6) are quite different than the cases of (1.4).

Now let us consider the asymptotic profiles when the coefficients $c(x)$ and $q(x)$ are all degenerate in Ω_*. The first case is that $q(x)$ vanishes in Ω_0.

Theorem 1.4. *Assume that* (H2) *holds. Then there exists a unique positive solution* $\theta_\delta \in C(\overline{\Omega})$ *to* (1.4) *if and only if*

$$\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0).$$

Moreover,

$$\lim_{\delta \to 0+} \theta_\delta(x) = \theta(x) \quad \textit{uniformly in } \overline{\Omega};$$

here $\theta(x)$ *is the unique positive solution of* (1.5).

Remark 1.5. Note that

$$c(x) + \delta q(x) = 0 \quad \text{for } x \in \Omega_* \text{ and } \delta > 0;$$

we know that (1.5) admits a unique positive solution $\theta \in C(\overline{\Omega})$ if and only if $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$.

If the degenerate domain of $q(x)$ is a proper subset of Ω_0, we can prove that the sharp patterns of (1.4) change drastically. If (H3) holds, we have the following result.

Theorem 1.6. *Assume that* (H3) *holds. Then there exists a unique solution* $\theta_\delta \in C(\overline{\Omega})$ *to* (1.4) *if and only if* $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_*)$.

(i) *If* $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$, *then*

$$\lim_{\delta \to 0+} \theta_\delta(x) = \theta(x) \quad \textit{uniformly in } \overline{\Omega},$$

where $\theta(x)$ *is the unique positive solution of* (1.2).

(ii) *If* $\lambda_P(\Omega_0) \leq \lambda < \lambda_P(\Omega_*)$, *then*

$$\lim_{\delta \to 0+} \theta_\delta(x) = \infty \quad \textit{uniformly in } \overline{\Omega}.$$

Letting $\vartheta_\delta(x) = \delta^{\frac{1}{p-1}} \theta_\delta(x)$ and $\eta_\delta(x) = \delta^{\frac{1}{p(p-1)}} \theta_\delta(x)$, we can obtain the sharp profiles of (1.4) as follows.

Theorem 1.7. *Assume that* (H3) *holds.*

(i) *If* $\lambda_P(\Omega) < \lambda \le \lambda_P(\Omega_0)$*, then*

$$\lim_{\delta\to 0+} \vartheta_\delta(x) = \lim_{\delta\to 0+} \eta_\delta(x) = 0 \quad \textit{uniformly in } \overline{\Omega}.$$

(ii) *If* $\lambda_P(\Omega_0) < \lambda < \lambda_P(\Omega_*)$*, then*

$$\lim_{\delta\to 0+} \vartheta_\delta(x) = \alpha_*(x) \qquad \textit{uniformly in } \overline{\Omega}_0,$$

$$\lim_{\delta\to 0+} \vartheta_\delta(x) = 0 \qquad \textit{uniformly in any compact subset of } \overline{\Omega}\setminus\overline{\Omega}_0,$$

and

$$\lim_{\delta\to 0+} \eta_\delta(x) = \left[\frac{\int_{\Omega_0} J(x-y)\alpha_*(y)\,dy}{c(x)}\right]^{\frac{1}{p}}$$

$$\textit{uniformly in any compact subset of } \overline{\Omega}\setminus\overline{\Omega}_0,$$

where $\alpha_* \in C(\overline{\Omega}_0)$ *is the unique positive solution to the nonlocal dispersal equation* (1.5).

The sharp asymptotic profile of positive solutions to (1.4) is given by Theorems 1.4–1.7 if the degenerate domains of $c(x)$ and $q(x)$ are mixed. We find that if (H2) holds, i.e., both $c(x)$ and $q(x)$ are degenerate in the same domain Ω_0, the positive solution will converge to the unique positive solution of the initial equation without perturbation. In this case, no sharp pattern appears. However, if (H3) holds, we can see that asymptotic behavior changes a lot and there exist sharp profiles for the positive solutions of (1.4). Thus we know that the degenerate domain Ω^0 of $q(x)$ plays a great role in the sharp patterns of positive solutions to nonlocal dispersal equation (1.4).

In the present paper, we establish that the asymptotic profiles in the degeneracy domain are different than the domain without degeneracy for nonlocal dispersal equation (1.4). We prove that the nonlocal effect and the degenerate heterogenous nonlinearities $c(x)$, $q(x)$ make the positive solutions of (1.4) blow up, but the blow-up speeds are determined by $q(x)$. The similar problem for the general semilinear elliptic equation is studied in the author's recent joint paper with Li and López-Gómez [23]. The profiles for two kinds of equations are quite different. In fact, the sharp patterns only appear in the degenerate domain for the elliptic problem, where the singular boundary problem is very important. However, we know that sharp patterns appear in the whole domain Ω with two different speeds for the nonlocal dispersal problem.

The rest of this paper is organized as follows. In Section 2, we investigate the asymptotic profiles if (H1) holds. The behavior of the principal eigenfunction with respect to the parameter is also obtained. Section 3 is devoted to the proof of sharp profiles if (H2) or (H3) holds.

2 Asymptotic profiles with different degenerate domains

In this section, we investigate the asymptotic profiles of positive solutions to (1.4) when the coefficients $c(x)$ and $q(x)$ are degenerate in different domains. To do this, we first give the existence and uniqueness of positive solutions. Throughout this section, we assume that $q(x)$ satisfies (H1).

Lemma 2.1. *Assume that $\delta > 0$. Then there exists a unique positive solution $\theta_\delta \in C(\overline{\Omega})$ to* (1.4) *for every $\lambda > \lambda_P(\Omega)$ and no bounded positive solution for $\lambda \leq \lambda_P(\Omega)$.*

Proof. The proof follows a similar argument as in [18]; we omit the details here. □

2.1 The nonlocal eigenvalue problem. In order to investigate the limiting behavior of positive solutions to (1.4), we consider the nonlocal eigenvalue problem

$$\int_\Omega J(x-y)u(y)\,dy - u(x) - \mu c(x)u(x) = -\sigma u(x) \quad \text{in } \overline{\Omega}, \tag{2.1}$$

where $\mu > 0$. It follows form [33, 7] that (2.1) admits a unique principal eigenvalue $\sigma_P(\mu,\Omega)$ for $\mu \geq 0$ if $c(x)$ has a spatial degeneracy. We first recall a basic result of the eigenvalue problem (2.3) [33].

Lemma 2.2. *Problem* (2.1) *admits a unique principal eigenvalue $\sigma_P(\mu,\Omega)$ for every $\mu \geq 0$. Moreover, $\sigma_P(\mu,\Omega)$ is increasing with respect to μ and*

$$\lim_{\mu\to\infty} \sigma_P(\mu,\Omega) = \lambda_P(\Omega_0).$$

We shall investigate the asymptotic behavior of positive eigenfunctions to (2.1).

Lemma 2.3. *Let $\phi_\mu(x)$ and $\psi(x)$ be the positive eigenfunctions associated with $\sigma_P(\mu,\Omega)$ for $\mu \geq 0$ and $\lambda_P(\Omega_0)$ such that*

$$\max_{\overline{\Omega}} \phi_\mu(x) = \max_{\overline{\Omega}_0} \psi(x) = 1,$$

respectively. Then we have

$$\lim_{\mu\to\infty} \phi_\mu(x) = \psi(x) \quad \textit{uniformly in } \overline{\Omega}_0,$$

and

$$\lim_{\mu\to\infty} \phi_\mu(x) = 0 \quad \textit{uniformly in any compact subset of } \overline{\Omega} \setminus \overline{\Omega}_0.$$

Proof. Take $x_1, x_2 \in \overline{\Omega}_0$. We know from (2.1) that

$$[1 - \sigma_P(\mu, \Omega)][\phi_\mu(x_1) - \phi_\mu(x_2)] = \int_\Omega [J(x_1 - y) - J(x_2 - y)]\phi_\mu(y)\, dy.$$

Since $\lambda_P(\Omega) \le \sigma_P(\mu, \Omega) < \lambda_P(\Omega_0) < 1$ for $\mu \ge 0$ and $\phi_\mu(x) \le 1$ for $x \in \overline{\Omega}$, we get

$$|\phi_\mu(x_1) - \phi_\mu(x_2)| \le \frac{\int_\Omega |J(x_1 - y) - J(x_2 - y)|\, dy}{1 - \lambda_P(\Omega_0)}.$$

Then, subject to a subsequence, we know that there exists $\hat{\phi} \in C(\overline{\Omega}_0)$ such that $0 \le \hat{\phi}(x) \le 1$ in $\overline{\Omega}_0$ and

$$\lim_{\mu\to\infty} \phi_\mu(x) = \hat{\phi}(x) \quad \text{uniformly in } \overline{\Omega}_0. \tag{2.2}$$

Using (2.1) we get

$$\phi_\mu(x) \le \frac{\int_\Omega J(x - y)dy}{1 - \lambda_P(\Omega_0) + \mu c(x)}$$

for $x \in \overline{\Omega}$. Thus we know that

$$\lim_{\mu\to\infty} \phi_\mu(x) = 0 \tag{2.3}$$

for any $x \in \overline{\Omega} \setminus \overline{\Omega}_0$ and

$$\lim_{\mu\to\infty} \phi_\mu(x) = 0 \quad \text{uniformly in any compact subset of } \overline{\Omega} \setminus \overline{\Omega}_0.$$

Now we show that $\hat{\phi}(x) = \psi(x)$ for $x \in \overline{\Omega}_0$. In view of (2.2) and (2.3), we get

$$\int_{\Omega_0} J(x - y)\hat{\phi}(y)dy - \hat{\phi}(x) + \lambda_P(\Omega_0)\hat{\phi}(x) = 0 \quad \text{in } \overline{\Omega}_0 \tag{2.4}$$

by the dominated convergence theorem. Note that $\max_{\overline{\Omega}} \phi_\mu(x) = 1$. We can find $x_\mu \in \overline{\Omega}$ such that $\phi_\mu(x_\mu) = 1$ for $\mu > 0$. Then we have

$$\int_\Omega J(x_\mu - y)\phi_\mu(y)dy - 1 - \mu c(x_\mu) = -\sigma_P(\mu, \Omega). \tag{2.5}$$

Since Ω is bounded, without loss of generality, we may assume that $x_\mu \to x_0$ as $\mu \to \infty$ for some $x_0 \in \overline{\Omega}$. We know from (2.5) that

$$c(x_0) = \lim_{\mu\to\infty} c(x_\mu) = \lim_{\mu\to\infty} \frac{\int_\Omega J(x_\mu - y)\phi_\mu(y)dy - 1 + \sigma_P(\mu, \Omega)}{\mu} = 0;$$

this also gives that $x_0 \in \overline{\Omega}_0$.

It follows from (2.1) that

$$\int_{\Omega} J(x_0 - y)\phi_\mu(y)dy - \phi_\mu(x_0) = -\sigma_P(\mu, \Omega)\phi_\mu(x_0),$$

and by (2.2)–(2.3) we obtain

$$\int_{\Omega_0} J(x_0 - y)\hat{\phi}(y)dy = [1 - \lambda_P(\Omega_0)]\hat{\phi}(x_0).$$

Then we get from (2.5) that

$$\begin{aligned}\lim_{\mu\to\infty} \mu c(x_\mu) &= \int_{\Omega_0} J(x_0 - y)\hat{\phi}(y)dy - 1 + \lambda_N(\Omega_0)\\ &= [1 - \lambda_N(\Omega_0)][\hat{\phi}(x_0) - 1] \le 0.\end{aligned}$$

Since $\mu c(x_\mu) \ge 0$ for $\mu > 0$, we must have $\hat{\phi}(x_0) = 1.$ In view of (2.4), we know

$$\hat{\phi}(x) > 0 \quad \text{in } \overline{\Omega}_0$$

from the maximum principle. Since $\psi(x)$ is a positive eigenfunction of (2.4), we obtain

$$\hat{\phi}(x) = c\psi(x) \quad \text{in } \overline{\Omega}_0$$

for some constant $c > 0$. However, we get from $\max_{\overline{\Omega}} \psi(x) = 1$ that $c = 1$; this also shows that (2.2) holds for the entire sequences. □

2.2 Asymptotic profile for positive solutions. In this subsection, we investigate the asymptotic behavior of positive solutions to (1.4). To this end, we show that the positive solution is monotone with respect to δ.

Lemma 2.4. *Assume that $\lambda > \lambda_P(\Omega)$ and $\delta > 0$. Let $\theta_\delta(x)$ be the unique positive solution to* (1.4). *Then we know that*

$$\theta_{\delta_2}(x) < \theta_{\delta_1}(x) \tag{2.6}$$

for $x \in \overline{\Omega}$, provided $\delta_2 > \delta_1 > 0$. Meanwhile, if $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$, then

$$\lim_{\delta\to 0+} \theta_\delta(x) = \theta(x) \quad \textit{uniformly in } \overline{\Omega}; \tag{2.7}$$

here $\theta(x)$ is the unique positive solution of (1.2).

Proof. Since $\delta_2 > \delta_1 > 0$, we can see that $\theta_{\delta_2}(x)$ is a lower-solution to (1.4) with $\delta = \delta_1$. But we know that $\theta_{\delta_1}(x)$ is the unique solution of (1.4) with $\delta = \delta_1$, then (2.6) is followed by an upper-lower solutions argument.

If $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$, since $\theta_\delta(x) < \theta(x)$ for $x \in \bar{\Omega}$, we know that there exists a positive function $\theta_0(x)$ such that

$$\lim_{\delta\to 0+} \theta_\delta(x) = \theta_0(x)$$

for $x \in \bar{\Omega}$. Applying the dominated convergence theorem we have

$$\int_\Omega J(x-y)\theta_0 dy - \theta_0(x) = -\lambda\theta_0(x) + c(x)\theta_0^p(x). \tag{2.8}$$

Since (2.8) admits a unique positive solution for $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$, we must have $\theta_0(x) = \theta(x)$ for $x \in \bar{\Omega}$. It follows from Dini's theorem that (2.7) holds. □

Lemma 2.5. *Assume that* $\delta > 0$ *and* $\lambda \geq \lambda_P(\Omega_0)$. *Let* $\theta_\delta \in C(\bar{\Omega})$ *be the unique positive solution to* (1.4). *Then*

$$\lim_{\delta\to 0+} \theta_\delta(x) = \infty \quad \textit{uniformly in } \bar{\Omega}.$$

Proof. Let $\sigma_P(\mu, \Omega)$ be the unique principal eigenvalue to (2.1) for $\mu > 0$ associated with a positive eigenfunction $\phi_\mu(x)$ such that $\max_{\bar{\Omega}} \phi_\mu(x) = 1$. Since $\sigma_P(\mu, \Omega) < \lambda_P(\Omega_0)$ for $\mu > 0$ and $\lambda \geq \lambda_P(\Omega_0)$, we can take δ small such that

$$\lambda - \sigma_P(\mu, \Omega) \geq \mu\delta q(x)$$

for $x \in \bar{\Omega}$. Then we can check that $\mu^{\frac{1}{p-1}}\phi_\mu(x)$ is a lower-solution to (1.4). Note that $\theta_\delta(x)$ is monotone with respect to δ. By the uniqueness of positive solutions, we get

$$\mu^{\frac{1}{p-1}}\phi_\mu(x) \leq \lim_{\delta\to 0+} \theta_\delta(x)$$

for $x \in \bar{\Omega}$. Letting $\mu \to \infty$, we get from Lemma 2.3 that

$$\lim_{\delta\to 0+} \theta_\delta(x) = \infty \quad \text{uniformly in } \bar{\Omega}_0.$$

Meanwhile, we can see that

$$\int_\Omega J(x-y)\theta_\delta(y)dy = [1 - \lambda + (c(x) + \delta q(x))\theta_\delta^{p-1}(x)]\theta_\delta(x),$$

and

$$\int_\Omega J(x-y)\theta_\delta(y)dy \geq \int_{\Omega_0} J(x-y)\theta_\delta(y)dy,$$

so we have that

$$\lim_{\delta\to 0+} \theta_\delta(x) = \infty \quad \text{uniformly in } \bar{\Omega}.$$ □

The proof of Theorem 1.1 is followed by Lemmas 2.4–2.5.

2.3 Sharp profile for positive solutions. In this subsection, we establish the sharp profile for positive solutions of (1.4). Set $\vartheta_\delta(x) = \delta^{\frac{1}{p-1}}\theta_\delta(x)$. Then $\vartheta_\delta(x)$ satisfies

$$\int_\Omega J(x-y)\vartheta_\delta(y)dy - \vartheta_\delta(x) = -\lambda\vartheta_\delta(x) + \left[\frac{c(x)}{\delta} + q(x)\right]\vartheta_\delta^p(x) \quad \text{in } \overline{\Omega}. \tag{2.9}$$

In order to give some estimate to $\vartheta_\delta(x)$, we first consider the nonlocal dispersal equation

$$\int_\Omega J(x-y)u(y)dy - u(x) = -\lambda u(x) + q(x)u^p(x) \quad \text{in } \overline{\Omega}. \tag{2.10}$$

If $q(x) > 0$ for $x \in \overline{\Omega}$, then no spatial degeneracy appears in (2.10) and it admits a unique continuous positive solution $\hat{v}(x)$ for $\lambda > \lambda_P(\Omega)$. Inspired by the works of Du and Li [12], López-Gómez [25], we then have the following estimate.

Lemma 2.6. *Assume that* $\lambda > \lambda_P(\Omega)$ *and* $\delta > 0$. *Assume further that* $q(x) > 0$ *for* $x \in \overline{\Omega}$. *Let* $\hat{u}(x)$ *be the positive solution of* (2.10). *Then we have*

$$\vartheta_\delta(x) \le \hat{v}(x) \tag{2.11}$$

for $x \in \overline{\Omega}$.

Proof. Since

$$\int_\Omega J(x-y)\vartheta_\delta(y)dy - \vartheta_\delta(x) + \lambda\vartheta_\delta(x) - q(x)\vartheta_\delta^p(x) = \frac{c(x)}{\delta}\vartheta_\delta^p(x) \ge 0$$

for $x \in \overline{\Omega}$, we get that $\vartheta_\delta(x)$ is a lower-solution of (2.10). By the uniqueness of positive solutions and an upper-lower solutions argument we obtain (2.11). □

If Ω^0 is a proper subset, we shall consider the nonlocal dispersal equation

$$\int_\Omega J(x-y)u(y)dy - u(x) = -\lambda u(x) + [c(x) + q(x)]u^p(x) \quad \text{in } \overline{\Omega}. \tag{2.12}$$

We can see that $c(x) + q(x) > 0$ for $x \in \overline{\Omega}$ and there exists a unique continuous positive solution $\hat{u}(x)$ to (2.12) for $\lambda > \lambda_P(\Omega)$. Similar to Lemma 2.6 we have the following result.

Lemma 2.7. *Assume that* $\lambda > \lambda_P(\Omega)$ *and* $0 < \delta \le 1$. *Assume further that* Ω^0 *is a proper subset of* Ω. *Let* $\hat{u}(x)$ *be the positive solution of* (2.10). *Then we have*

$$\vartheta_\delta(x) \le \hat{u}(x) \tag{2.13}$$

for $x \in \overline{\Omega}$.

Remark 2.8. In fact we can see that (2.13) still holds if $q(x) > 0$ for $x \in \bar{\Omega}$. However, if Ω^0 is a proper subset of Ω, we can see that (2.11) is not valid since there is no positive solution to (2.10) for $\lambda \geq \lambda(\Omega^0)$.

Now let us consider the nonlocal dispersal equation

$$\int_{\Omega_0} J(x-y)u(y)dy - u(x) = -\lambda u(x) + q(x)u^p(x) \quad \text{in } \bar{\Omega}_0. \tag{2.14}$$

We know that (2.14) admits a unique continuous positive solution $\bar{u}(x)$ for $\lambda > \lambda_P(\Omega_0)$. Then by the method of upper-lower solutions, we obtain the lower bound for $\vartheta_\delta(x)$.

Lemma 2.9. *Assume that $\lambda > \lambda_P(\Omega)$ and $\delta > 0$. Let $\bar{u}(x)$ be the positive solution of* (2.14). *Then we have*

$$\vartheta_\delta(x) \geq \bar{u}(x)$$

for $x \in \bar{\Omega}_0$.

In order to establish the sharp profiles, we need the following technical lemma.

Lemma 2.10. *Assume that $\lambda \geq \lambda_P(\Omega_0)$ and $\delta > 0$. Then there exists $\tau > 0$ which is independent of δ such that*

$$1 - \lambda + q(x)\vartheta_\delta^{p-1}(x) \geq \tau$$

for $x \in \bar{\Omega}_0$.

Proof. We only prove the case $\lambda > \lambda_P(\Omega_0)$, since $0 < \lambda_P(\Omega) < \lambda_P(\Omega_0) < 1$. It follows from (2.9) that

$$\int_{\Omega_0} J(x-y)\vartheta_\delta(y)dy \leq \int_{\Omega} J(x-y)\vartheta_\delta(y)dy = [1 - \lambda + q(x)\vartheta_\delta^{p-1}(x)]\vartheta_\delta(x)$$

and so

$$1 - \lambda + q(x)\vartheta_\delta^{p-1}(x) \geq 0 \quad \text{in } \bar{\Omega}_0$$

for $x \in \bar{\Omega}_0$. But we know from Lemmas 2.7–2.9 that

$$\int_{\Omega_0} J(x,y)\bar{u}(y)dy \leq [1 - \lambda + q(x)\vartheta_\delta^{p-1}(x)]\hat{u}(x)$$

for $x \in \bar{\Omega}_0$. Since $\hat{u}(x) > 0$ and $\bar{u}(x) > 0$ for $x \in \bar{\Omega}_0$, we complete the proof. □

At the end of this section, we prove the main theorem.

Proof of Theorem 1.3. We first prove the second claim. Set

$$l(x) = 1 - \lambda + q(x)\vartheta_\delta^{p-1}(x).$$

We know from Lemma 2.7 that $l(x) \geq \tau$ for $x \in \overline{\Omega}_0$ and $\delta > 0$.

For any $x_1, x_2 \in \overline{\Omega}_0$, without loss of generality, we assume that $\vartheta_\delta(x_1) \geq \vartheta_\delta(x_2)$. A direct computation from (2.9) gives

$$\begin{aligned}
&\int_\Omega [J(x_1 - y) - J(x_2 - y)]\vartheta_\delta(y)dy \\
&\quad = [1 - \lambda + pq(x_2)\hat{\upsilon}^{\,p-1}][\vartheta_\delta(x_1) - \vartheta_\delta(x_2)] + [q(x_1) - q(x_2)]\vartheta_\delta^p(x_1) \\
&\quad \geq [1 - \lambda + q(x_2)\vartheta_\delta^{p-1}(x_2)][\vartheta_\delta(x_1) - \vartheta_\delta(x_2)] + [q(x_1) - q(x_2)]\vartheta_\delta^p(x_1) \\
&\quad \geq \tau[\vartheta_\delta(x_1) - \vartheta_\delta(x_2)] + [q(x_1) - q(x_2)]\vartheta_\delta^p(x_1),
\end{aligned}$$

where $\hat{\upsilon}$ is between $\vartheta_\delta(x_1)$ and $\vartheta_\delta(x_2)$. Thus by Lemma 2.10,

$$|\vartheta_\delta(x_1) - \vartheta_\delta(x_2)| \leq \frac{1}{\tau}\left[\int_\Omega |J(x_1, y) - J(x_2, y)|\hat{u}(y)dy + [q(x_1) - q(x_2)]\hat{u}^p(x_1)\right]$$

for any $x_1, x_2 \in \overline{\Omega}_0$. Subject to a subsequence, a simple compactness argument gives that there exists $\omega \in C(\overline{\Omega})$ such that $\omega(x) > 0$ for $x \in \overline{\Omega}_0$ and

$$\lim_{\delta \to 0+} \vartheta_\delta(x) = \omega(x) \quad \text{uniformly in } \overline{\Omega}_0. \tag{2.15}$$

On the other hand, since

$$\int_\Omega J(x - y)\vartheta_\delta(y)dy - \vartheta_\delta(x) + \lambda\vartheta_\delta(x) = \left[\frac{c(x)}{\delta} + q(x)\right]\vartheta_\delta^p(x) \quad \text{in } \overline{\Omega},$$

by (2.11) we obtain

$$\begin{aligned}
\left[\frac{c(x)}{\delta} + q(x)\right]\vartheta_\delta^p(x) &\leq \int_\Omega J(x - y)\hat{u}(y)dy + \hat{u}(x) + \lambda\hat{u}(x) \\
&\leq (2 + \lambda)\max_{\overline{\Omega}} \hat{u}(x)
\end{aligned}$$

and

$$\vartheta_\delta(x) \leq \left[\frac{(2 + \lambda)\max_{\overline{\Omega}} \hat{u}(x)}{\frac{c(x)}{\delta} + q(x)}\right]^{\frac{1}{p}}$$

for $\overline{\Omega} \setminus \overline{\Omega}_0$. Hence

$$\lim_{\delta \to 0+} \vartheta_\delta(x) = 0 \quad \text{uniformly in any compact subset of } \overline{\Omega} \setminus \overline{\Omega}_0. \tag{2.16}$$

In view of (2.15) and (2.16), by the dominated convergence theorem, we know that

$$\int_{\Omega_0} J(x,y)\omega(y)dy - \omega(x) = -\lambda\omega(x) + q(x)\omega^p(x) \quad \text{in } \overline{\Omega}_0. \tag{2.17}$$

As (2.17) admits a unique continuous positive solution for $\lambda > \lambda_P(\Omega_0)$, we must have $\omega(x) = \alpha(x)$ for $x \in \overline{\Omega}_0$; here $\alpha(x)$ is the unique positive solution of (1.5). This also implies that (2.15) holds for the entire original sequences.

At last, we can see that $\eta_\delta(x) = \delta^{\frac{1}{p(p-1)}}\theta_\delta(x)$ satisfies

$$\int_\Omega J(x-y)\eta_\delta(y)dy - \eta_\delta(x) = -\lambda\eta_\delta(x) + [c(x) + \delta q(x)]\frac{\eta_\delta^p(x)}{\delta^{1/p}} \quad \text{in } \overline{\Omega}$$

and so

$$\int_\Omega J(x-y)\vartheta_\delta(y)dy - \vartheta_\delta(x) = -\lambda\vartheta_\delta(x) + [c(x) + \delta q(x)]\eta_\delta^p(x) \quad \text{in } \overline{\Omega}.$$

A simple computation yields

$$\eta_\delta(x) = \left[\frac{\int_\Omega J(x-y)\vartheta_\delta(y)dy - \vartheta_\delta(x) + \lambda\vartheta_\delta(x)}{c(x) + \delta q(x)}\right]^{\frac{1}{p}}.$$

Using (2.15)–(2.16), we get

$$\lim_{\delta\to 0+} \eta_\delta(x) = \left[\frac{\int_{\Omega_0} J(x-y)\alpha(y)dy}{c(x)}\right]^{\frac{1}{p}} \quad \text{uniformly in any compact subset of } \overline{\Omega}\setminus\overline{\Omega}_0.$$

Now we prove the first claim. If $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$, the conclusion is followed by Lemma 2.4. If $\lambda = \lambda_P(\Omega_0)$, since the only nonnegative solution to (2.17) is the trivial solution, a similar argument as above gives that

$$\lim_{\delta\to 0+} \vartheta_\delta(x) = 0 \quad \text{uniformly in } \overline{\Omega}.$$

□

3 Asymptotic profiles with mixed degenerate domain

In this section, we investigate the profiles of positive solutions to (1.4) when the coefficients $c(x)$ and $q(x)$ are degenerate in a common domain. To do this, we first give the existence and uniqueness of positive solutions; one can see [18, 33] for a similar proof.

Lemma 3.1. *Assume that* (H2) *holds. Then there exists a unique solution* $\theta_\delta \in C(\overline{\Omega})$ *to* (1.4) *for* $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$. *Meanwhile, there exists no bounded positive solution for* $\lambda \le \lambda_P(\Omega)$ *and* $\lambda \ge \lambda_P(\Omega_0)$.

Lemma 3.2. *Assume that (H2)Potext holds. Let $\theta_\delta(x)$ be the unique positive solution of* (1.4) *for $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$ and $\delta \geq 0$. Then we know that*

$$\theta_{\delta_2}(x) < \theta_{\delta_1}(x)$$

for $x \in \bar{\Omega}$ and $\delta_2 > \delta_1 \geq 0$. Meanwhile, we have

$$\lim_{\delta\to 0+} \theta_\delta(x) = \theta(x) \quad \textit{uniformly in } \bar{\Omega},$$

here $\theta(x) = \theta_0(x)$ is the unique positive solution of (1.5).

Proof. Since $\delta_2 > \delta_1 \geq 0$ and $q(x)$ is nonnegative, we know that $\theta_{\delta_2}(x)$ is a lower-solution of (1.4) with $\delta = \delta_1$. Using the fact that the positive solution is unique, we obtain $\theta_{\delta_2}(x) < \theta_{\delta_1}(x)$ for $x \in \bar{\Omega}$.

On the other hand, we know that

$$\theta_\delta(x) < \theta(x)$$

for $x \in \bar{\Omega}$ and there exists a positive function $\theta_0(x)$ such that

$$\lim_{\delta\to 0+} \theta_\delta(x) = \theta_0(x)$$

and $\theta_0(x)$ satisfies

$$\int_\Omega J(x-y)\theta_0(y)dy - \theta_0(x) = -\lambda\theta_0(x) + c(x)\theta_0^p(x).$$

This also shows that $\theta_0(x) = \theta(x)$ for $x \in \bar{\Omega}$ and $\theta_\delta(x) \to \theta(x)$ uniformly in $\bar{\Omega}$ as $\delta \to 0+$. □

At the end of this section, we consider the case that $q(x)$ satisfies (H3). Then we know that $c(x) + \delta q(x) = 0$ for $x \in \bar{\Omega}_*$ and $\delta > 0$. We have the following result.

Lemma 3.3. *Assume that* (H3) *holds. Then there exists a unique solution $\theta_\delta \in C(\bar{\Omega})$ to* (1.4) *for $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_*)$. Meanwhile, there exists no bounded positive solution for $\lambda \leq \lambda_P(\Omega)$ and $\lambda \geq \lambda_P(\Omega_*)$.*

Lemma 3.4. *Assume that* (H3) *holds. Let $\theta_\delta(x)$ be the unique positive solution of* (1.4) *for $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_*)$ and $\delta > 0$. Then we know that*

$$\theta_{\delta_2}(x) < \theta_{\delta_1}(x)$$

for $x \in \bar{\Omega}$ and $\delta_2 > \delta_1 > 0$. Meanwhile, we have

$$\lim_{\delta\to 0+} \theta_\delta(x) = \theta(x) \quad \textit{uniformly in } \bar{\Omega}, \tag{3.1}$$

for $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$; here $\theta(x) = \theta_0(x)$ is the unique positive solution of (1.5).

Proof. Note that (H3) holds. We know that $\lambda_P(\Omega_0) < \lambda_P(\Omega_*)$; see [18]. Then we can establish that (3.1) holds by an upper-lower solutions argument, since (1.5) admits a unique positive solution for $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$. □

If $\lambda_P(\Omega_0) \le \lambda < \lambda_P(\Omega_*)$, a similar argument as in the proof of Lemma 2.5 gives that the unique positive solution blows up in Ω as $\delta \to 0+$; we omit the details.

Lemma 3.5. *Let $\theta_\delta \in C(\overline{\Omega})$ be the unique positive solution to* (1.4) *for $\lambda_P(\Omega_0) \le \lambda < \lambda_P(\Omega_*)$. Then*

$$\lim_{\delta \to 0+} \theta_\delta(x) = \infty \quad \textit{uniformly in } \overline{\Omega}.$$

In order to analyze the sharp profiles, we consider the nonlocal dispersal equations

$$\int_\Omega J(x-y)u(y)dy - u(x) = -\lambda u(x) + [c(x) + q(x)]u^p(x) \quad \text{in } \overline{\Omega} \tag{3.2}$$

and

$$\int_{\Omega_0} J(x-y)u(y)dy - u(x) = -\lambda u(x) + q(x)u^p(x) \quad \text{in } \overline{\Omega}_0. \tag{3.3}$$

If (H3) holds, then $c(x) + q(x) = 0$ for $x \in \Omega_*$ and there exists a unique continuous positive solution $\hat{u}(x)$ to (3.2) for $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_*)$. Meanwhile, we know that (3.3) admits a unique continuous positive solution $\overline{u}(x)$ for $\lambda_P(\Omega_0) < \lambda < \lambda_P(\Omega_*)$. Since $\vartheta_\delta(x) = \delta^{\frac{1}{p-1}} \theta_\delta(x)$ satisfies

$$\int_\Omega J(x-y)\vartheta_\delta(y)dy - \vartheta_\delta(x) = -\lambda \vartheta_\delta(x) + \left[\frac{c(x)}{\delta} + q(x)\right]\vartheta_\delta^p(x) \quad \text{in } \overline{\Omega},$$

by the method of upper-lower solutions, we obtain the following lemmas.

Lemma 3.6. *Assume that* (H3) *holds. Assume further that $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_*)$ and $\delta > 0$. Let $\hat{u}(x)$ be the positive solution of* (3.2)*. Then we have*

$$\vartheta_\delta(x) \le \hat{u}(x)$$

for $x \in \overline{\Omega}$.

Lemma 3.7. *Assume that* (H3) *holds. Assume further that $\lambda_P(\Omega_0) < \lambda < \lambda_P(\Omega_*)$ and $\delta > 0$. Let $\overline{u}(x)$ be the positive solution of* (3.3)*. Then we have*

$$\vartheta_\delta(x) \ge \overline{u}(x)$$

for $x \in \overline{\Omega}_0$.

In order to establish the sharp profiles, we need the following technical lemma.

Corollary 3.8. *Assume that* (H3) *holds. Assume further that* $\lambda_P(\Omega_0) < \lambda < \lambda_P(\Omega_*)$ *and* $\delta > 0$. *Then there exists* $\tau > 0$, *which is independent of* δ, *such that*

$$1 - \lambda + q(x)\vartheta_\delta^{p-1}(x) \geq \tau$$

for $x \in \overline{\Omega}_0$.

The conclusion of Theorem 1.7 can be proved by a similar argument as in the proof of Theorem 1.3 with the help of Lemmas 3.6–3.7 and Corollary 3.8.

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Jian-Wen Sun
SCHOOL OF MATHEMATICS AND STATISTICS,
GANSU KEY LABORATORY OF APPLIED MATHEMATICS AND COMPLEX SYSTEMS,
LANZHOU UNIVERSITY
LANZHOU, 730000, P. R. CHINA
email: jianwensun@lzu.edu.cn

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