SPECTRAL THEORY OF THE MULTI-FREQUENCY QUASI-PERIODIC OPERATOR WITH A GEVREY TYPE PERTURBATION

By

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Abstract. In this paper we study the multi-frequency quasi-periodic operator with a Gevrey type perturbation. We first establish the large deviation theorem (LDT) for the multi-dimensional operator with a sub-exponential (or Gevrey) long-range hopping, and then prove the pure point spectrum property. Based on the LDT and the Aubry duality, we show the absence of a point spectrum for the $1D$ exponential long-range operator with a multi-frequency and a Gevrey potential. We also prove the spectrum has positive Lebesgue measure.

1 Introduction and main results

In this paper we study the spectral properties of the multi-frequency long-range quasi-periodic operator with a Gevrey type perturbation. More precisely, we first consider the multi-dimensional quasi-periodic operator with a Gevrey long-range hopping and an analytic potential which satisfies the non-degeneracy condition. We prove such an operator has a pure point spectrum (with sub-exponentially decaying eigenfunctions) in the large coupling regime (see Theorem 1.1 in the following). The Aubry duality of this operator is a $1D$ multi-frequency quasi-periodic operator with an exponential long-range hopping and a Gevrey potential. We show the absence of a point spectrum for the Aubry duality in the small coupling regime (see Theorem 1.2 in the following). We want to mention that in the small coupling regime the non-degeneracy assumption on the potential is not needed. While we can prove the absence of a point spectrum for the Aubry duality, we can not obtain the existence of an absolutely continuous (ac) spectrum via the present method. The Kotani's theory suggests that the existence of an ac spectrum implies the positivity of the Lebesgue measure of the spectrum, which motivates us to study the measure of the spectrum (see Theorem 1.3 in the following) in the small coupling regime.

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We start with the long-range hopping, which is a Toeplitz operator. Let

$$h : \mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d \to \mathbb{R}$$

be a function. We define the Toeplitz operator (with the symbol h) as

$$\mathcal{T}_h(m, n) = \widehat{h}_{m-n}, \quad m, n \in \mathbb{Z}^d,$$

where $\widehat{h}_n = \int_{\mathbb{T}^d} h(\theta)e^{-2\pi i n\cdot\theta} d\theta$. We also define $\delta_{mn} = 1$ if $m = n$, and $\delta_{mn} = 0$ if $m \neq n$.

1.1 Pure point spectrum. We first study the multi-dimensional operator with a Gevrey long-range hopping and an analytic potential.

Assume that υ is Gevrey regular, i.e., $\upsilon(x) \in C^\infty(\mathbb{T}^d, \mathbb{R})$ satisfies for some $\gamma \in (0, 1]$ and $\forall\ n \in \mathbb{Z}^d$

$$|\widehat{\upsilon_n}| \leq e^{-\rho|n|^\gamma}, \tag{1.1}$$

where $\rho > 0$, $|n| = \sup_{1\leq i\leq d} |n_i|$. Notice that υ is analytic if $\gamma = 1$.

We consider the operator

$$\widetilde{H}_{\lambda f,\omega,\theta} = \lambda\mathcal{T}_\upsilon + f(\theta + n\omega)\delta_{nn'}, \quad \theta \in \mathbb{T}^d, \tag{1.2}$$

$$n\omega = (n_1\omega_1, \ldots, n_d\omega_d), \tag{1.3}$$

where $\lambda^{-1} > 0$ is the coupling, θ is the phase, $\omega \in \mathbb{T}^d$ is the frequency and f is a real analytic function satisfying the *non-degeneracy* condition: For all $j = 1, \ldots, d$ and

$$\theta_j^{\neg} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_d) \in \mathbb{T}^{d-1},$$

the map

$$\theta_j \mapsto f(\theta_j, \theta_j^{\neg})$$

is a non-constant function of $\theta_j \in \mathbb{T}$.

We have

Theorem 1.1. *Let $\widetilde{H}_{\lambda f,\omega,\theta}$ be defined by* (1.2)–(1.3) *with υ satisfying* (1.1) *and f satisfying the non-degeneracy condition. Then for any $\varepsilon > 0$, there exists a $\lambda_0 = \lambda_0(d, \gamma, \rho, f, \varepsilon) > 0$ such that the following holds: For $0 \leq \lambda \leq \lambda_0$ and $\theta \in \mathbb{T}^d$, there exists some $\Omega = \Omega(d, \gamma, \rho, \lambda f, \theta) \subset \mathbb{T}^d$ with $\mathrm{mes}(\Omega) \leq \varepsilon$ such that, if $\omega \in \mathbb{T}^d \setminus \Omega$, then $\widetilde{H}_{\lambda f,\omega,\theta}$ has a pure point spectrum with sub-exponentially (exponentially if $\gamma = 1$) decaying eigenfunctions.*

Remark 1.1. This theorem extends the result of Bourgain [Bou07] to the Gevrey long-range hopping case.

The search for the nature of the spectrum and the behaviour of the eigenfunctions for the $1D$ quasi-periodic operator has attracted much attention over the years. Of particular importance is the phenomenon of the Anderson localization (AL), where we say an operator satisfies AL if it has a pure point spectrum with exponentially decaying eigenfunctions. The early results on the AL were perturbative and restricted on "cos" type potentials [Sin87, FSW90]. The first non-perturbative[1] AL was obtained by Jitomirskaya [Jit94] in the almost Matheiu operator (AMO) setting. By developing a new type of KAM arguments, Eliasson [Eli97] proved a pure point spectrum for the $1D$ Schrödinger operator with a large Gevrey potential. Eliasson's result is perturbative and needs the potential to satisfy some transversality condition. Later, the celebrated work of Jitomirskaya [Jit99] indicated that the AL can hold for the AMO with the Diophantine condition if the coupling $\lambda > 1$. Significantly, Bourgain–Goldstein [BG00] established the non-perturbative AL for the $1D$ Schrödinger operator with a single-frequency and an analytic potential. Klein [Kle05, Kle14] proved the AL for the $1D$ Schrödinger operator with a Gevrey potential. In the long-range setting Bourgain–Jitomirskaya [BJ02] proved the non-perturbative AL for the exponential long-range operator with a "cos" potential. In [Bou05] Bourgain extended the result of [BJ02] to an operator with an analytic potential. An improvement of some long-range estimates of [BJ02] has recently been established by Avila–Jitomirskaya [AJ10]. We also mention the work of Jian–Shi–Yuan [JSY19] in which a non-perturbative AL was obtained for some $1D$ long-range block operator. We refer the reader to [AJ09, AYZ17, JL18] for more recent AL results in the $1D$ setting.

In the multi-dimensional case only the perturbative localization can be expected [Bou02b]. The first multi-dimensional localization was obtained by Chulaevsky and Dinaburg [CD93] for a single-phase operator with an exponential long-range hopping. Their perturbative KAM methods seem not applicable in the multi-phase case. Bourgain, Goldstein and Schlag [BGS02] developed a new way to combine the multi-scale analysis developed by Fröhlich–Spencer [FS83] and some of the non-perturbative methods to the case $(n, \theta, \omega) \in \mathbb{Z}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$, and obtained the AL for the large analytic potential. To perform such multi-scale analysis, the sub-linear growth of the number of "bad" small boxes contained in a big box becomes essential. In the single-phase case only the Diophantine condition of the frequency can ensure the sub-linearity property. In the $(n, \theta, \omega) \in \mathbb{Z}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$ case to get the sub-linearity property, additionally arithmetic conditions on the frequency are needed [BGS02]. It was also shown by Bourgain [Bou02a] that the Diophantine

[1] Here, by a non-perturbative argument we mean the argument allows the size of the perturbation to be independent of the frequency.

property of the frequency of the skew shift is sufficient to guarantee the sub-linearity property. For $(n, \theta, \omega) \in \mathbb{Z}^d \times \mathbb{T}^d \times \mathbb{T}^d$ with $d \geq 3$, it is difficult to ensure the sub-linearity property as in the case $d \leq 2$ (or dD with the single-phase). To overcome this problem, Bourgain [Bou07] introduced new methods and allowed the eliminations of the frequency to depend on the potential when proving the LDT. This enables him to extend results of [BGS02] to arbitrary dimension d. The basic techniques of [Bou07] are also semi-algebraic sets arguments and matrix-valued Cartan's estimates, but need more delicate analysis. Recently, methods of Bourgain [Bou07] have been largely extended by Jitomirskaya–Liu–Shi [JLS20] to the long-range operator with $(n, \theta, \omega) \in \mathbb{Z}^d \times \mathbb{T}^b \times \mathbb{T}^b$ for arbitrary b, d. The result of [JLS20] is significantly more general and more technically complex, and can also be viewed as both a clarification and at the same time streamlining of [Bou07]. We also mention the work of Bourgain–Kachkovskiy [BK19] in which the case $(n, \theta, \omega) \in \mathbb{Z}^2 \times \mathbb{T}^2 \times \mathbb{T}$ was studied. For the multi-dimensional long-range operator with a "cos" potential, localization results with the fixed Diophantine frequency have been obtained by Jitomirskaya–Kachkovskiy [JK16] and Ge–You–Zhou [GYZ].

1.2 Absence of a point spectrum. We then study the Aubry duality of (1.2) in the case $f(\theta + n\omega) = g(\theta + n \cdot \omega)$, where $\theta \in \mathbb{T}$, $n \cdot \omega = \sum_{i=1}^d n_i\omega_i$ and g is a non-constant real analytic function. This leads to the $1D$ exponential long-range quasi-periodic operator

$$H_{\lambda\upsilon,\omega,x} = \mathcal{T}_g + \lambda\upsilon(x + \ell\omega)\delta_{\ell\ell'}, \quad x \in \mathbb{T}^d, \ell, \ell' \in \mathbb{Z}, \tag{1.4}$$

where υ is defined by (1.1).

If $g(\theta) = 2\cos 2\pi\theta$, the operator (1.4) becomes the standard multi-frequency quasi-periodic Schrödinger operator. In particular, we call (1.4) the almost Mathieu operator if it is a $1D$ quasi-periodic Schrödinger operator satisfying

$$\upsilon(x) = 2\cos 2\pi x.$$

Denote by $\mathrm{mes}(\cdot)$ the Lebesgue measure. We have

Theorem 1.2. *Let $H_{\lambda\upsilon,\omega,x}$ be defined by* (1.4) *with υ satisfying* (1.1). *Then for any $\varepsilon > 0$, there exists a $\lambda_0 = \lambda_0(g, d, \gamma, \rho, \varepsilon) > 0$ such that the following holds: For $0 \leq \lambda \leq \lambda_0$, there exists some $\Omega = \Omega(g, d, \gamma, \rho, \lambda) \subset \mathbb{T}^d$ with $\mathrm{mes}(\Omega) \leq \varepsilon$ such that, if $\omega \in \mathbb{T}^d \setminus \Omega$, then $H_{\lambda\upsilon,\omega,x}$ has no point spectrum for all $x \in \mathbb{T}^d$.*

Remark 1.2. The non-degeneracy condition on υ is not needed here. In addition, we think the operator should have pure ac spectrum if $0 < \lambda \ll 1$.

In the following we review some results on the ac spectrum. Consider first the one-frequency operator (i.e., $d = 1$) case. As is well-known, the spectrum of the free Laplacian on $\mathbb{Z}$ is pure ac. Thus the question whether the pure ac spectrum property holds for the Schrödinger operator with a small quasi-periodic potential naturally arises. Early results were restricted on the AMO case [BLT83, CD89]. In the continuous setting Eliasson [Eli92] proved a pure ac spectrum for a Schrödinger operator with the Diophantine frequency and a small analytic quasi-periodic potential by using the KAM scheme (see [HA09] for the discrete case). Later, Bourgain–Jitomirskaya [BJ02] developed a non-perturbative argument to handle the $1D$ discrete Schrödinger operator with a small analytic quasi-periodic potential for a.e. phase $x \in \mathbb{T}$. Puig [Pui06] improved partial results of Eliasson [Eli92] to the non-perturbative and discrete setting. The proof of Puig was based on the Aubry duality and a non-perturbative localization result in the exponential long-range Hamiltonian in [BJ02]. Significantly, Avila–Jitomirskaya [AJ10] developed a quantitative version of the duality based on the dual concepts of almost reducibility and almost localization, which ultimately implied a non-perturbative pure ac spectrum result holds for the $1D$ analytic Schrödinger operator with the Diophantine frequency for all phase $x \in \mathbb{T}$. If $0 \leq \lambda < 1$, Avila–Damanik [AD08] proved the pure ac spectrum of the AMO for every irrational frequency and for a.e. $x \in \mathbb{T}$. We also mention the work [AFK11] in which the existence of the ac spectrum is obtained for a $1D$ analytic Schrödinger operator with any irrational frequency. Remarkably, Avila [Avi10, Avi] even established the Almost Reducibility Conjecture and proved the pure ac spectrum for the analytic quasi-periodic Schrödinger operator in the subcritical regime.

Much less is known about the multi-frequency quasi-periodic Schrödinger operator. Based on arguments of [FK09], Bjkerlöv–Krikorian [BK21] showed the existence of the ac spectrum for a smooth multi-frequency quasi-periodic Schrödinger operator without the smallness restriction on the potential. Recently, Hou–Wang–Zhou [HWZ20] proved the existence of the ac spectrum for the analytic multi-frequency quasi-periodic Schrödinger operator with a Liouville frequency. Very recently, Cai [Cai21] obtained the pure ac spectrum for the multi-frequency quasi-periodic Schrödinger operator with a finitely differentiable potential relying on the almost reducibility results of [CCYZ19].

1.3 Lebesgue measure of the spectrum. It is well-known that the spectrum of $H_{\lambda v,\omega,x}$ is independent of $x \in \mathbb{T}^d$ if 1 and ω are rationally independent. In this case we denote by $\Sigma_{\lambda v,\omega}$ the spectrum of $H_{\lambda v,\omega,x}$. We have

Theorem 1.3. *Let v satisfy* (1.1) *and let g be a non-constant analytic function. Then for any $\varepsilon > 0$, there exists a $\lambda_0 = \lambda_0(g, d, \gamma, \rho, \varepsilon) > 0$ such that the following holds: For $0 \le \lambda \le \lambda_0$, there exists some $\Omega = \Omega(g, d, \gamma, \rho, \lambda) \subset \mathbb{T}^d$ with $\mathrm{mes}(\Omega) \le \varepsilon$ such that, if $\omega \in \mathbb{T}^d \setminus \Omega$, then*

$$\mathrm{mes}(\Sigma_{\lambda v,\omega}) \ge c > 0,$$

where $c = c(\lambda_0)$.

Remark 1.3. The study of the Lebesgue measure of the spectrum for the quasi-periodic operator has a long history. The famous Aubry–André conjecture [AA80] states that the measure of the AMO is exactly $|4 - 4\lambda|$ for every frequency $\omega \in \mathbb{R} \setminus \mathbb{Q}$. Before [AK06], only partial results were obtained [HS89, AvMS90, Las94, JK02]. Remarkably, Avila–Krikorian [AK06] settled this conjecture completely. We would also like to mention the recent elegant work [JK19], where a short new proof of zero measure of the spectrum for the critical (i.e., $\lambda = 1$) AMO was given. If one considers the more general Schrödinger operator, there is no explicit representation of the measure of the spectrum. However, based on the LDT and semi-algebraic sets arguments, Bourgain [Bou05] was able to prove that the Lebesgue measure of the spectrum for the $1D$ Schrödinger operator with a single-frequency and an analytic potential is strictly positive. Bourgain's result is non-perturbative. In the present work we extend Bourgain's result to the multi-frequency operator with a Gevrey potential and a long-range hopping (but perturbative).

1.4 Perturbative essentials. As mentioned above, our results and methods are perturbative. Actually, even in the $1D$ Gevrey perturbation case, only perturbative results could be expected. Due to the relatively lower regularity (resp., slower decaying) of the potential (resp., long-range hopping), it seems that only perturbative methods (such as the multi-scale analysis) are applicable. In fact, the appropriate estimates on the Green's functions are key to establish the above spectral results. We can restrict our consideration to the case $(n, \theta, \omega) \in \mathbb{Z} \times \mathbb{T} \times \mathbb{T}$. We denote by $\widetilde{H}_N(\theta)$ the restriction of $\widetilde{H}_{\lambda f,\omega,\theta}$ on $[-N, N] \subset \mathbb{Z}$. Following the non-perturbative techniques (without any inductive arguments) of [BJ02, Bou05], the Green's function

$$G_N(E; \theta) = (\widetilde{H}_N(\theta) - E)^{-1}$$

can be represented via the Cramer's rule as

$$G_N(E; \theta)(m, n) = \frac{\mathcal{M}_{m,n}}{\det(\widetilde{H}_N(\theta) - E)},$$

where $\mathcal{M}_{m,n}$ is the (m,n)-minor of $\widetilde{H}_N(\theta)-E$. As in [BJ02, Bou05], one may show that

$$|\det(\widetilde{H}_N(\theta)-E)|\sim e^{N\int_{\mathbb{T}}\log|f(\theta)-E|\mathrm{d}\theta+o(\lambda)N}$$

for θ being outside a set of measure at most $e^{-N^c}, c\in(0,1)$. Due to the sub-exponentially decaying of $\widehat{v_n}$, the best possible upper bound of $\mathcal{M}_{m,n}$ may be

$$|\mathcal{M}_{m,n}|\leq e^{-\rho|m-n|^{\gamma}+N\int_{\mathbb{T}}\log|f(\theta)-E|\mathrm{d}\theta+o(\lambda)N}.$$

Consequently,

$$|G_N(E;\theta)(m,n)|\leq e^{-\rho|m-n|^{\gamma}+o(\lambda)N}.$$

In the case of $\gamma\in(0,1)$, no off-diagonal decay of $G_N(E;\theta)$ could be expected for $0<\lambda\leq\lambda_0$. This technical difficulty is the main motivation of the present paper to use methods developed by Bourgain [Bou07] and Jitomirskaya–Liu–Shi [JLS20], which depend mainly on the multi-scale analysis. That of course will lead to perturbative results.

1.5 Strategy of the proofs. We outline the proofs. First, we will prove the LDT for Green's functions of $\widetilde{H}_{\lambda f,\omega,\theta}$. This depends on the multi-scale analysis developed in [Bou07, JLS20]. The matrix-valued Cartan's estimates and semi-algebraic geometry arguments play essential roles in this step. In [JLS20] the authors considered the multi-dimensional quasi-periodic operator with the exponentially decaying long-range hopping (which deals with the more complicated b-frequency setting). It turns out that the Gevrey long-range hopping case needs to improve some arguments of [JLS20]:

- In the proof of the resolvent identity (see the Appendix for details) it needs the off-diagonal decaying speeds of the Green's functions to depend on the Gevrey index γ. In the proof of the LDT it also needs to give more delicate estimates on various parameters. The key idea is to remove more θ in some sense when establishing the LDT . This depends sensitively on the Gevrey index γ as well.
- Furthermore, the sub-linear growth property in our setting becomes more precise, which heavily relies on γ.

To prove the pure point spectrum (i.e., Theorem 1.1), it suffices to eliminate the energy in LDT and then apply the Shnol's Theorem. This will be finished by using semi-algebraic sets arguments (including the Yomdin–Gromov triangulation Theorem) as in [Bou07].

To prove the absence of point spectrum (i.e.,Theorem 1.2), we will combine the LDT with a trick originated from Delyon [Del87]. In contrast with [Eli97, Kle05, Kle14], our result holds without any transversality restriction on the Gevrey potential. The proofs of [Kle05, Kle14] dealt with the Schrödinger operator with a Gevrey potential directly. To prove the LDT, Klein performed an inductive scheme as in [BG00, BGS01] and needed the transversality condition of the potential to guarantee the validity of the initial step (or a Łojasiewicz type inequality). Instead, in the present we establish the LDT for the Aubry dual operator of (1.4). It turns out that this operator is actually a multi-dimensional quasi-periodic operator with an analytic potential and a Gevrey long-range hopping.

To prove the spectrum has positive measure (i.e., Theorem 1.3), we will use a renormalization scheme of Bourgain [Bou05] relying on the complexity estimates. In [Bou05] Bourgain directly applied the LDT of [BG00] together with semi-algebraic sets arguments (including theTarski–Seidenberg principle and bounds on the Betti numbers) to construct sufficiently many approximate eigenvalues. However, for the Schrödinger operator with a Gevrey potential, the only known LDTs were proved by Klein [Kle05, Kle14], but require the potential to satisfty the transversality condition. Moreover, Klein's methods seem invalid in the long-range case. To overcome these difficulties, we again use the powerful Aubry duality. Precisely, by the well-known result (see [Pui06, JK16]), we have $\Sigma_{\lambda v,\omega} = \widetilde{\Sigma}$, here $\widetilde{\Sigma}$ denotes the spectrum of the Aubry duality of (1.4). It turns out that this Aubry duality is a multi-dimensional Gevrey long-range operator with an analytic potential. Bourgain [Bou05] claimed that his arguments remain valid for the long-range operator in the $1D$ and single-frequency case once the LDT was established. In this paper we extend Bourgain's method to the multi-dimensional case.

1.6 The structure of this paper. The structure of the paper is as follows. Some preliminaries are introduced in §2. The LDT is established in §3. In §4, §5 and §6, we finish the proof of Theorems 1.1, 1.2 and 1.3, respectively. Some key estimates are included in the Appendix.

2 Preliminaries

2.1 The notations. Let $a > 0, b > 0$. We define $a \lesssim b$ (resp., $a \ll b$) if there is some $\varepsilon > 0$ (resp., small $\varepsilon > 0$) so that $a \leq \varepsilon b$. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. We write $a\pm$ to denote $a \pm \varepsilon$ for some small $\varepsilon > 0$.

For any $x \in \mathbb{R}^d$, let $|x| = \max_{1\le i\le d} |x_i|$. For $\Lambda \subset \mathbb{R}^d$, we introduce

$$\operatorname{diam}(\Lambda) = \sup_{n,n'\in\Lambda} |n - n'|, \quad \operatorname{dist}(m, \Lambda) = \inf_{n\in\Lambda} |m - n|.$$

For $\theta \in \mathbb{R}^d$ and $1 \le j \le d$, let

$$\theta_j^{\urcorner} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_d) \in \mathbb{R}^{d-1}.$$

For $x \in \mathbb{R}^{d_1}$ and $\emptyset \ne X \subset \mathbb{R}^{d_1+d_2}$, define the x-section of X to be

$$X(x) = \{y \in \mathbb{R}^{d_2} : (x, y) \in X\}.$$

For example, $X(\theta_j^{\urcorner}) = \{\theta_j \in \mathbb{T} : (\theta_j, \theta_j^{\urcorner}) \in X\}$ if $\emptyset \ne X \subset \mathbb{T}^d$.

For $x \in \mathbb{R}$, we denote by $[x]$ its integer part.

Throughout this paper, we assume $\rho \in (0, 1)$ for simplicity.

2.2 Some facts on semi-algebraic sets.

Definition 2.1 (Chapter 9, [Bou05]). A set $\mathcal{S} \subset \mathbb{R}^n$ is called a semi-algebraic set if it is a finite union of sets defined by a finite number of polynomial equalities and inequalities. More precisely, let $\{P_1, \ldots, P_s\} \subset \mathbb{R}[x_1, \ldots, x_n]$ be a family of real polynomials whose degrees are bounded by d. A (closed) semi-algebraic set $\mathcal{S}$ is given by an expression

$$\mathcal{S} = \bigcup_j \bigcap_{\ell\in\mathcal{L}_j} \{x \in \mathbb{R}^n : P_\ell(x)\varsigma_{j\ell}0\}, \tag{2.1}$$

where $\mathcal{L}_j \subset \{1, \ldots, s\}$ and $\varsigma_{j\ell} \in \{\ge, \le, =\}$. Then we say that $\mathcal{S}$ has degree at most sd. In fact, the degree of $\mathcal{S}$ which is denoted by $\deg(\mathcal{S})$, is the smallest sd over all representations as in (2.1).

Lemma 2.2 (Tarski–Seidenberg Principle, [Bou05]). *Denote by $(x, y) \in \mathbb{R}^{d_1+d_2}$ the product variable. If $\mathcal{S} \subset \mathbb{R}^{d_1+d_2}$ is semi-algebraic of degree B, then its projections $\mathrm{Proj}_x\mathcal{S} \subset \mathbb{R}^{d_1}$ and $\mathrm{Proj}_y\mathcal{S} \subset \mathbb{R}^{d_2}$ are semi-algebraic of degree at most B^C, where $C = C(d_1, d_2) > 0$.*

Lemma 2.3 ([Bou05]). *Let $\mathcal{S} \subset \mathbb{R}^d$ be a semi-algebraic set of degree B. Then the sum of all Betti numbers of $\mathcal{S}$ is bounded by B^C, where $C = C(d) > 0$.*

Lemma 2.4 ([Bou07]). *Let $\mathcal{S} \subset [0, 1]^{d=d_1+d_2}$ be a semi-algebraic set of degree $\deg(\mathcal{S}) = B$ and $\mathrm{mes}_d(\mathcal{S}) \le \eta$, where*

$$\log B \ll \log \frac{1}{\eta}.$$

Denote by $(x_1, x_2) \in [0, 1]^{d_1} \times [0, 1]^{d_2}$ the product variable. Suppose

$$\eta^{\frac{1}{d}} \le \varepsilon.$$

Then there is a decomposition of $\mathcal{S}$ *as*

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$$

with the following properties. The projection of $\mathcal{S}_1$ *on* $[0,1]^{d_1}$ *has small measure*

$$\mathrm{mes}_{d_1}(\mathrm{Proj}_{x_1}\mathcal{S}_1) \leq B^{C(d)}\varepsilon,$$

and $\mathcal{S}_2$ *has the transversality property*

$$\mathrm{mes}_{d_2}(\mathcal{L} \cap \mathcal{S}_2) \leq B^{C(d)}\varepsilon^{-1}\eta^{\frac{1}{d}},$$

where $\mathcal{L}$ *is any* d_2*-dimensional hyperplane in* $[0,1]^d$ *s.t.*

$$\max_{1\leq j\leq d_1} |\mathrm{Proj}_{\mathcal{L}}(e_j)| < \varepsilon,$$

where we denote by $e_1, \ldots, e_{d_1}$ *the* x_1*-coordinate vectors.*

In [Bou07], Bourgain proved a result for eliminating multi-variables.

Lemma 2.5 (Lemma 1.18, [Bou07])**.** *Let* $\mathcal{S} \subset [0,1]^{d+r}$ *be a semi-algebraic set of degree* B *and such that*

$$\mathrm{mes}_d(\mathcal{S}(y)) < \eta \quad \textit{for}\ \forall\, y \in [0,1]^r.$$

Then the set

$$\left\{(x_1, \ldots, x_{2^r}) \in [0,1]^{d2^r} : \bigcap_{1\leq i\leq 2^r} \mathcal{S}(x_i) \neq \emptyset\right\}$$

is semi-algebraic of degree at most B^C *and measure at most*

$$B^C \eta^{d^{-r}2^{-r(r-1)/2}},$$

where $C = C(d,r) > 0$.

Lemma 2.6 (Lemma 1.20, [Bou07])**.** *Let* $\mathcal{S} \subset [0,1]^{dr}$ *be a semi-algebraic set of degree* B *and* $\mathrm{mes}(\mathcal{S}) < \eta$ *with* $\eta > 0$.

For $\omega = (\omega_1, \ldots, \omega_r) \in [0,1]^r$ *and* $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r$*, define*

$$n\omega = (n_1\omega_1, \ldots, n_r\omega_r).$$

For any $C > 1$*, define* $\mathcal{N}_1, \ldots, \mathcal{N}_{d-1} \subset \mathbb{Z}^r$ *to be finite sets with the following property:*

$$\min_{1\leq s\leq r} |n_s| > (B \max_{1\leq s\leq r} |m_s|)^C,$$

where $n \in \mathcal{N}_i, m \in \mathcal{N}_{i-1}$ $(2 \leq i \leq d-1)$.

Then there is some $C = C(r,d) > 0$ *such that for* $\max_{n\in\mathcal{N}_{d-1}} |n|^C < \frac{1}{\eta}$*, one has*

$$\mathrm{mes}(\{\omega \in [0,1]^r : \exists\, n^{(i)} \in \mathcal{N}_i \ s.t.,\ (\omega, n^{(1)}\omega, \ldots, n^{(d-1)}\omega) \mod \mathbb{Z}^{dr} \in \mathcal{S}\}) \leq B^C\delta,$$

where

$$\delta^{-1} = \min_{n\in\mathcal{N}_1} \min_{1\leq s\leq r} |n_s|.$$

3 LDT of Green's functions

If $\Lambda \subset \mathbb{Z}^d$, we denote $\widetilde{H}_\Lambda(\theta) = R_\Lambda \widetilde{H}_{\lambda f,\omega,\theta} R_\Lambda$, where R_Λ is the restriction operator. Define the Green's function as

$$G_\Lambda(E;\theta) = (\widetilde{H}_\Lambda(\theta) - E + i0)^{-1}.$$

We denote by Q_N an elementary region of size N centered at 0 (see [JLS20]), which is one of the following regions:

$$Q_N = [-N, N]^d$$

or

$$Q_N = [-N, N]^d \setminus \{n \in \mathbb{Z}^d : \ n_i \varsigma_i 0, \ 1 \le i \le d\},$$

where for $i = 1, 2, \ldots, d$, $\varsigma_i \in \{\{n < 0\}, \{n > 0\}, \emptyset\}$ and at least two ς_i are not $\emptyset$. Denote by $\mathcal{E}_N^0$ the set of all elementary regions of size N centered at 0. Let $\mathcal{E}_N$ be the set of all translates of elementary regions, namely,

$$\mathcal{E}_N := \bigcup_{n \in \mathbb{Z}^d, Q_N \in \mathcal{E}_N^0} \{n + Q_N\}.$$

The main result of this section is

Theorem 3.1 (LDT). *such that for all $N \ge \underline{N}_0$ and $0 < \lambda \le \lambda_0$, the following statements hold:*

- *There is some semi-algebraic set $\Omega_N = \Omega_N(d, \rho, \gamma, \lambda f, c_1) \subset \mathbb{T}^d$ with $\deg(\Omega_N) \le N^{4d}$, and as $\lambda \to \infty$,*

$$\mathrm{mes}(\mathbb{T}^d \setminus \cap_{N \ge \underline{N}_0} \Omega_N) \to 0.$$

- *If $\omega \in \Omega_N$ and $E \in \mathbb{R}$, then there exists some set*

$$X_N = X_N(d, \rho, \gamma, \lambda f, c_1, \omega, E) \subset \mathbb{T}^d$$

such that

$$\sup_{1 \le j \le d, \theta_j^{\neg} \in \mathbb{T}^{d-1}} \mathrm{mes}(X_N(\theta_j^{\neg})) \le e^{-N^{c_1}},$$

and for $\theta \notin X_N$, $Q \in \mathcal{E}_N^0$, one has

$$\|G_Q(E;\theta)\| \le e^{N^{\gamma/2}},$$

$$|G_Q(E;\theta)(n, n')| \le e^{-\frac{(1-5^{-\gamma})\rho}{2}|n-n'|^\gamma} \quad \text{for } |n - n'| \ge N/10.$$

Proof of Theorem 3.1. The proof is based on the multi-scale analysis scheme as in [Bou07, JLS20]. The proof breaks up into three steps.

Step 1: Proof of inductive step.

This will be completed by using semi-algebraic sets arguments and Cartan's estimates as in [Bou07] and [JLS20].

We define for $1 \ll N_1 \in \mathbb{N}$ the scales

$$N_2 \sim N_1^{2/c_1},\ \log N \sim N_1^{c_1}.$$

Then we have

Theorem 3.2. *Let Ω_{N_i} $(i = 1, 2)$ be some semi-algebraic set satisfying $\deg(\Omega_{N_i}) \leq N_i^{4d}$ and let $\overline{\rho_i} \in (0, \rho)$. Assume further the following holds: If $\omega \in \Omega_{N_i}$ and $E \in \mathbb{R}$, then there exists some semi-algebraic set $X_{N_i} \subset \mathbb{T}^d$ satisfying $\deg(X_{N_i}) \leq N_i^{C(d)}$ such that*

$$\sup_{1\leq j\leq d, \theta_j^{\neg}\in\mathbb{T}^{d-1}} \mathrm{mes}(X_{N_i}(\theta_j^{\neg})) \leq e^{-N_i^{c_1}},$$

and for $\theta \notin X_{N_i}$, $Q \in \mathcal{E}^0_{N_i}$, one has

$$\|G_Q(E;\theta)\| \leq e^{N_i^{\gamma/2}}, \tag{3.1}$$

$$|G_Q(E;\theta)(n, n')| \leq e^{-\overline{\rho}_i|n-n'|^{\gamma}} \quad \text{for } |n - n'| \geq N_i/10 \tag{3.2}$$

$$(i = 1, 2).$$

Then there exist positive constants $c_2 < c_3 < c_4 < \gamma/10$ (depending only on d) such that the following holds: There exists some semi-algebraic set $\Omega_N \subset \Omega_{N_1} \cap \Omega_{N_2}$ with

$$\deg(\Omega_N) \leq N^{4d} \quad \text{and} \quad \mathrm{mes}((\Omega_{N_1} \cap \Omega_{N_2})\backslash\Omega_N) \leq N^{-c_2}$$

such that, if $\omega \in \Omega_N$, then for $E \in \mathbb{R}$ and $\theta \in \mathbb{T}^d$, there is $\frac{N^{c_3}}{10} < M < 10N^{c_4}$ such that for all $k \in \Lambda\backslash\overline{\Lambda}$, one has $\theta + k\omega \mod \mathbb{Z}^d \notin X_{N_1}$, where

$$\Lambda = [-M, M]^d, \quad \overline{\Lambda} = [-M^{\frac{\gamma}{10d}}, M^{\frac{\gamma}{10d}}]^d.$$

Proof. The main point of the proof is to eliminate (E, θ) by applying Lemmas 2.5 and 2.6. We refer to [Bou07] for details (see also comments in [JLS20]). We remark that the resolvent identity is actually unnecessary in the proof. □

We then construct the X_N by using Cartan's estimates and the resolvent identity.

Lemma 3.3 (Cartan's estimates, [Bou05]). *Let $T(\theta)$ be a self-adjoint $N \times N$ matrix-valued function of a parameter $\theta \in [-\delta, \delta]$ satisfying the following conditions:*

(i) *$T(\theta)$ is real analytic in $\theta \in [-\delta, \delta]$ and has a holomorphic extension to*

$$\mathcal{D}_\delta = \{\theta \in \mathbb{C} : \ |\Re\theta| \le \delta, |\Im\theta| \le \delta\}$$

satisfying

$$\sup_{\theta \in \mathcal{D}_\delta} \|T(\theta)\| \le K_1, \quad K_1 \ge 1.$$

(ii) *For all $\theta \in [-\delta, \delta]$, there is a subset $V \subset [1, N]$ with*

$$|V| \le M$$

and

$$\|(R_{[1,N]\setminus V} T(\theta) R_{[1,N]\setminus V})^{-1}\| \le K_2, \quad K_2 \ge 1.$$

(iii) $\mathrm{mes}\{\theta \in [-\delta, \delta] : \ \|T^{-1}(\theta)\| \ge K_3\} \le 10^{-3}\delta(1 + K_1)^{-1}(1 + K_2)^{-1}$.

Let

$$0 < \varepsilon \le (1 + K_1 + K_2)^{-10M}.$$

Then

$$\mathrm{mes}\{\theta \in [-\delta/2, \delta/2] : \|T^{-1}(\theta)\| \ge \varepsilon^{-1}\} \le C\delta e^{-\frac{c \log \frac{1}{\varepsilon}}{M \log(K_1 + K_2 + K_3)}}, \tag{3.3}$$

where $C, c > 0$ are some absolute constants.

Applying Cartan's estimates yields the following result.

Theorem 3.4. *Fix $1 \le j \le d$ and $\theta_{\neg j} \in \mathbb{T}^{d-1}$. Write $\theta = (\theta_j, \theta_{\neg j}) \in \mathbb{T}^d$. Assume that the assumptions of Theorem 3.2 are satisfied. Assume further there exist $\widetilde{N} \in [N^{c_3}/4, N^{c_4}]$ and $\overline{\Lambda} \subset \Lambda \in \mathcal{E}_{\widetilde{N}}$ with $\mathrm{diam}(\overline{\Lambda}) \le 4\widetilde{N}^{\frac{\gamma}{10d}}$ such that, for any $k \in \Lambda \setminus \overline{\Lambda}$, there exists some $\mathcal{E}_{N_1} \ni W \subset \Lambda \setminus \overline{\Lambda}$ such that $\mathrm{dist}(k, \Lambda \setminus \overline{\Lambda} \setminus W) \ge N_1/2$, and $\theta + k\omega \bmod \mathbb{Z}^d \notin X_{N_1}$. Let*

$$Y_\theta = \{y \in \mathbb{R} : |y - \theta_j| \le e^{-10\rho N_1^\gamma}, \|G_\Lambda(E; (y, \theta_{\neg j}))\| \ge e^{\widetilde{N}^{\gamma/2}}\}.$$

Then for $\omega \in \mathbf{\Omega}_{N_1} \cap \mathbf{\Omega}_{N_2}$, one has

$$\mathrm{mes}(Y_\theta) \le e^{-\widetilde{N}^{\gamma/3}}.$$

Proof. The proof is similar to that in [JLS20]. Let $\mathcal{D}$ be the $e^{-10\rho N_1^\gamma}$-neighbourhood of θ_j in the complex plane, i.e.,

$$\mathcal{D} = \{y \in \mathbb{C} : |\Im y| \le e^{-10\rho N_1^\gamma}, |\Re y - \theta_j| \le e^{-10\rho N_1^\gamma}\}.$$

Applying Theorem 3.2 yields for all $k \in \Lambda\backslash\bar{\Lambda}$ and $Q \in \mathcal{E}^0_{N_1}$,

$$\|G_Q(E;\theta + k\omega)\| \le e^{{N_1}^{\gamma/2}}, \tag{3.4}$$

$$|G_Q(E;\theta + k\omega)(n, n')| \le e^{-\bar{\rho}_1|n-n'|^\gamma} \quad \text{for } |n - n'| \ge N_1/10. \tag{3.5}$$

Note that for all $n, n' \in [-N_1, N_1]^d$, one has

$$e^{-10\rho N_1^\gamma} < e^{-3\bar{\rho}_1 N_1^\gamma - \bar{\rho}_1|n-n'|^\gamma}.$$

Then by Lemma A.1, (3.4) and (3.5), we have for any $y \in \mathcal{D}$, $Q \in \mathcal{E}^0_{N_1}$ and $k \in \Lambda\backslash\bar{\Lambda}$,

$$\|G_Q(E;(\theta_j + y, \theta_j^{\neg}) + k\omega)\| \le 2e^{{N_1}^{\gamma/2}}, \tag{3.6}$$

$$|G_Q(E;(\theta_j + y, \theta_j^{\neg}) + k\omega)(n, n')| \le 2e^{-\bar{\rho}_1|n-n'|^\gamma} \quad \text{for } |n - n'| \ge N_1/10. \tag{3.7}$$

Applying Lemma A.2 with $M_1 = M_0 = N_1$ implies for any $y \in \mathcal{D}$,

$$\|G_{\Lambda\backslash\bar{\Lambda}}(E;(\theta_j + y, \theta_j^{\neg}))\| \le 4(2N_1 + 1)^d e^{{N_1}^{\gamma/2}} \le e^{2{N_1}^{\gamma/2}}. \tag{3.8}$$

We want to use Lemma 3.3 to finish the proof. For this purpose, let

$$T(y) = \widetilde{H}_\Lambda((\theta_j + y, \theta_j^{\neg})) - E, \delta = \delta_1 = 2e^{-10\rho N_1^\gamma}. \tag{3.9}$$

It suffices to verify the assumptions of Lemma 3.3. Obviously, $K_1 = O(1)$. By (3.8), one has

$$M = |\bar{\Lambda}| \le 100^d \widetilde{N}^{\gamma/10}, \quad K_2 = e^{2{N_1}^{\gamma/2}}. \tag{3.10}$$

Since $\omega \in \Omega_{N_2}$, (3.1) and (3.2) hold at scale N_2 for y being outside a set of measure at most $e^{-N_2^{c_1}}$. Applying Lemma A.2 with $M_0 = M_1 = N_2$ yields

$$\|T^{-1}(y)\| \le 4(2N_2 + 1)^d e^{{N_2}^{\gamma/2}} \le e^{2{N_2}^{\gamma/2}} = K_3$$

for y being outside a set of measure at most

$$(2\widetilde{N} + 1)^d e^{-N_2^{c_1}} \le e^{-N_2^{c_1}/2}.$$

It follows from $100N_1^\gamma < N_2^{c_1}$ that

$$10^{-3}\delta_1(1 + K_1)^{-1}(1 + K_2)^{-1} \ge e^{-N_2^{c_1}/2}.$$

This verifies (iii) of Lemma 3.3. For $\varepsilon = e^{-\widetilde{N}^{\gamma/2}}$ one has, by (3.9) and (3.10),

$$\varepsilon < (1 + K_1 + K_2)^{-10M}.$$

By (3.3) of Lemma 3.3, we obtain

$$\operatorname{mes}(Y_\theta) \le e^{-\frac{c\widetilde{N}^{\gamma/2}}{N_2\widetilde{N}^{\gamma/10}\log\widetilde{N}}} \le e^{-\widetilde{N}^{\gamma/3}}.$$

□

Combining Theorems 3.2 and 3.4 yields

Theorem 3.5. *Let $\omega \in \Omega_N$ and fix $N_\star \in [N, N^2]$. If $E \in \mathbb{R}$ and $c_1 < \gamma c_3/10$, then there exists some set $X_{N_\star} = X_{N_\star}(E, \omega) \subset \mathbb{T}^d$ such that*

$$\sup_{1\le j\le d,\theta_j^\neg\in\mathbb{T}^{d-1}} \operatorname{mes}(X_{N_\star}(\theta_j^\neg)) \le e^{-N_\star^{c_1}},$$

and for $\theta \notin X_{N_\star}$, $Q \in \mathcal{E}^0_{N_\star}$, one has

$$|G_Q(E;\theta)(n,n')| \le e^{-(\bar{\rho}_1 - \frac{C}{N_1^{\gamma/2}})|n-n'|^\gamma} \quad \text{for } |n-n'| \ge N_\star/10,$$

where $C = C(d, \gamma, \rho) > 0$.

Proof. Fix $1 \le j \le d$, $\theta_j^\neg \in \mathbb{T}^{d-1}$ and $\theta = (\theta_j, \theta_j^\neg) \in \mathbb{T}^d$. As done in [JLS20] by using Theorem 3.2, for such θ and any $n \in Q \in \mathcal{E}^0_{N_\star}$, there exist $\frac{1}{4}N^{c_3} \le \widetilde{N}_{n,\theta} \le N^{c_4}$, $\Lambda_{n,\theta} \in \mathcal{E}_{\widetilde{N}}$ and $\overline{\Lambda}_{n,\theta}$, such that

$$n \in \overline{\Lambda}_{n,\theta} \subset \Lambda_{n,\theta} \subset Q, \quad \operatorname{dist}(n, Q\backslash\Lambda_{n,\theta}) \ge \widetilde{N}/2, \quad \operatorname{diam}(\overline{\Lambda}_{n,\theta}) \le 4\widetilde{N}_{n,\theta}^{\frac{\gamma}{10d}}.$$

Moreover, for any $k \in \Lambda_{n,\theta}\backslash\overline{\Lambda}_{n,\theta}$, we have $\theta + k\omega \mod \mathbb{Z}^d \notin X_{N_1}$, and there exists some $\mathcal{E}_{N_1} \ni W \subset \Lambda_{n,\theta}\backslash\overline{\Lambda}_{n,\theta}$ such that

$$k \in W, \quad \operatorname{dist}(k, \Lambda_{n,\theta}\backslash\overline{\Lambda}_{n,\theta}\backslash W) \ge N_1/2.$$

We now fix the above $\widetilde{N}_{n,\theta}$, $\overline{\Lambda}_{n,\theta}$, $\Lambda_{n,\theta}$ throughout the set

$$\{(y, \theta_j^\neg) \in \mathbb{R}^d : |y - \theta_j| \le e^{-10\rho N_1^\gamma}\}.$$

Recalling Lemma A.1 and the above constructions, the assumptions of Theorem 3.4 are essentially satisfied. Applying Theorem 3.4 implies that there exists a set $Y_{n,\theta} \subset \{y \in \mathbb{R} : |y - \theta_j| \le e^{-10\rho N_1^\gamma}\}$ such that

$$\operatorname{mes}(Y_{n,\theta}) \le e^{-\widetilde{N}_{n,\theta}^{\gamma/3}}, \tag{3.11}$$

and for $\theta_j \notin Y_{n,\theta}$, one has

$$\|G_{\Lambda_{n,\theta}}(E;\theta)\| \le e^{\widetilde{N}_{n,\theta}^{\gamma/2}}.$$

Applying Lemma A.3 with $M_0 = N_1$, $\Lambda = \Lambda_{n,\theta}$ and $\Lambda_1 = \overline{\Lambda}_{n,\theta}$ yields

$$|G_{\Lambda_{n,\theta}}(E;\theta)(n,n')| \le e^{-(\bar{\rho} - \frac{C}{N_1^{\gamma/2}})|n-n'|^\gamma} \quad \text{for } |n-n'| \ge \widetilde{N}_{n,\theta}/10.$$

Cover $[0, 1]$ by pairwise disjoint $e^{-10\rho N_1^\gamma}$-size intervals and let

$$X_{N_\star}(\theta_j^\neg) = \bigcup_{Q\in\mathcal{E}^0_{N_\star}, n\in Q, \theta=(\theta_j,\theta_j^\neg)} Y_{n,\theta}. \tag{3.12}$$

We remark that while $\theta = (\theta_j, \theta_j^{\neg})$ varies on a line for a fixed $\theta_j^{\neg}$, the total number of $Y_{n,\theta}$ is bounded by $e^{10\rho N_1^{\gamma}}$. Thus by (3.11), (3.12) and $c_1 < \gamma c_3/10$, one has

$$\mathrm{mes}(X_{N_\star}(\theta_j^{\neg})) \le C(2N+1)^d e^{10\rho N_1^{\gamma}} e^{-\widetilde{N}_{n,\theta}^{\gamma/3}} \le e^{-N_\star^{c_3\gamma/7}} \le e^{-N_\star^{c_1}}.$$

Suppose now $\theta \notin X_{N_\star}$. Applying Lemma A.2 with $\Lambda = Q \in \mathcal{E}^0_{N_\star}$, $M_0 = \frac{1}{4}N^{c_3}$ and $M_1 = \widetilde{N}_{n,\theta} \le N^{c_4}$, one has

$$\|G_Q(E;\theta)\| \le 4(2N^{c_4}+1)^d e^{N^{c_4\gamma/2}} \le e^{N_\star^{\gamma/2}}.$$

Applying Lemma A.3 with $\Lambda = Q$, $M_0 = \frac{1}{4}N^{c_3}$, $M_1 = \widetilde{N}_{n,\theta} \le N^{c_4}$ and $\Lambda_1 = \emptyset$, we have

$$|G_Q(E;\theta)(n,n')| \le e^{-(\bar{\rho}_1 - \frac{C}{N_1^{\gamma/2}})|n-n'|^{\gamma}} \quad \text{for } |n-n'| \ge N_\star/10.$$

This proves the theorem. □

Step 2: Proof of initial step.

Lemma 3.6. *Let*

$$X_N = \bigcup_{|n|\le N} \{\theta : |f(\theta + n\omega) - E| < \delta\}.$$

Then we have, for any $1 \le j \le d$,

$$\sup_{\theta_j^{\neg}\in\mathbb{T}^{d-1}} \mathrm{mes}(X_N(\theta_j^{\neg})) \le C(2N+1)^d \delta^c,$$

where $C = C(f) > 0$, $c = c(f) > 0$. *Moreover, if* $\lambda^{-1} \ge 2\delta^{-1}(2N+1)^d$, *then for any* $\theta \notin X_N$, $\omega \in \mathbb{T}^d$ *and* $\Lambda \subset [-N,N]^d$, *we have*

$$\|G_\Lambda(E;\theta)\| \le 2\delta^{-1},$$
$$|G_\Lambda(E;\theta)(n,n')| \le 2\delta^{-1} e^{-\rho|n-n'|^{\gamma}}.$$

Proof. The measure bound follows from a Łojasiewicz type inequality (see Lemma 5.2 of [JLS20]) and the non-degeneracy condition of f immediately.

The Green's function estimates follow from the Neumann series argument. For details, we refer to [JLS20] (or the proof of Lemma A.1, which deals with some more complicated setting). □

Step 3: Completion of the proof.

This will follow from Theorem 3.5, Lemma 3.6 and a multi-scale induction. For details, we refer to [JLS20]. □

4 Proof of Theorem 1.1

The key point of the proof is to eliminate the energy E in the LDT and this needs to remove further ω by semi-algebraic geometry arguments (i.e., Lemma 2.4).

Proof of Theorem 1.1. The proof is rather standard and based on Theorems 3.1, 3.2 and Lemma 2.4. We refer to [Bou07] for details. □

5 Proof of Theorem 1.2

In this section we will prove Theorem 1.2 by using the LDT and Delyon's trick [Del87].

Fix

$$\overline{\rho} = (1 - 5^{-\gamma})\rho.$$

We have Poisson's identity: For $\widetilde{H}(\theta)\xi = E\xi$ and $n \in \Lambda \subset \mathbb{Z}^d$,

$$\xi_n = -\lambda \sum_{n' \in \Lambda, n'' \notin \Lambda} G_\Lambda(E;\theta)(n, n')\widehat{v}_{n'-n''}\xi_{n''}. \tag{5.1}$$

Proof of Theorem 1.2. Let $\omega \in \bigcap_{N \geq \underline{N}_0} \Omega_N$ and $0 < \lambda \leq \lambda_0$ be as in Theorem 3.1. Suppose $H_{\lambda v,\omega,x}$ has some eigenvalue E. Then there must be some $0 \neq \psi = \{\psi_\ell\}_{\ell \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ so that

$$\sum_{\ell' \in \mathbb{Z}} \widehat{g}_{\ell - \ell'}\psi_{\ell'} + (\lambda v(x + \ell\omega) - E)\psi_\ell = 0.$$

Define

$$F(\theta) = \sum_{\ell \in \mathbb{Z}} \psi_\ell e^{2\pi i \ell \theta}$$

and

$$\xi_n(\theta) = e^{2\pi i n \cdot x}F(\theta + n \cdot \omega).$$

We have

$$\|F\|_{L^2(\mathbb{T})} = \|\psi\|_{\ell^2(\mathbb{Z})} > 0 \tag{5.2}$$

and by direct computation

$$(g(\theta) - E)F(\theta) + \lambda \sum_{k \in \mathbb{Z}^d} \widehat{v}_k \xi_k(\theta) = 0. \tag{5.3}$$

Then

$$\begin{aligned}\int_{\mathbb{T}} \sum_{n \in \mathbb{Z}^d} \frac{|\xi_n(\theta)|^2}{1 + |n|^{2d}} d\theta &= \sum_{n \in \mathbb{Z}^d} \frac{\|F\|^2_{L^2(\mathbb{T})}}{1 + |n|^{2d}} \\ &\leq C\|F\|^2_{L^2(\mathbb{T})} < \infty.\end{aligned}$$

This implies that for a.e. θ, we have $\sum_{n\in\mathbb{Z}^d} \frac{|\xi_n(\theta)|^2}{1+|n|^{2d}} < \infty$ and

$$|\xi_n(\theta)| \le C(\theta, d)|n|^d, \ C(\theta, d) > 0.$$

We let $\theta = \theta + n \cdot \omega$ in (5.3). Then

$$(g(\theta + n \cdot \omega) - E)F(\theta + n \cdot \omega) + \lambda \sum_{k\in\mathbb{Z}^d} \widehat{v}_k e^{2\pi i k\cdot x} F(\theta + (n + k) \cdot \omega) = 0.$$

Multiplying by $e^{2\pi i n\cdot x}$ in the above equality implies

$$(g(\theta + n \cdot \omega) - E)\xi_n(\theta) + \lambda \sum_{k\in\mathbb{Z}^d} \widehat{v}_{n-k}\xi_k(\theta) = 0. \tag{5.4}$$

Now let $X_N = X_N(\omega, E)$ be as in Theorem 3.1. We define

$$\Theta = \bigcup_{M\ge \underline{N}_0} \bigcap_{N\ge M} X_N.$$

Then by $\mathrm{mes}(X_N) \le e^{-N^{c_1}}$, one has $\mathrm{mes}(\Theta) = 0$. Fix $\theta \in \mathbb{T} \setminus \Theta$. Then there exists $M \ge \underline{N}_0$ such that

$$\theta \notin X_N \quad \text{for } N \ge M.$$

Recalling (5.1), (5.4) and Theorem 3.1, one has, for $N \ge M \gg 1$,

$$\begin{aligned}
|F(\theta)| = |\xi_0(\theta)| &= \left| \sum_{|n|\le N, |n'|>N} G_{[-N,N]^d}(E; \theta)(0, n)\widehat{v}_{n-n'}\xi_{n'}(\theta) \right| \\
&\le C(\theta, d) \sum_{|n|\le N, |n'|>N} e^{-\frac{\bar{\rho}}{2}|n|^\gamma + \frac{\bar{\rho}}{2}(N/10)^\gamma + N^{\gamma/2}} e^{-\rho|n-n'|^\gamma} |n'|^d \\
&\le C(\theta, d)N^d \sum_{|n'|>N} e^{-\frac{\bar{\rho}}{2}|n'|^\gamma + \frac{\bar{\rho}}{2}(N/10)^\gamma + N^{\gamma/2}} |n'|^d \\
&= o(N).
\end{aligned}$$

Letting $N \to \infty$, we have $F(\theta) = 0$ for a.e. $\theta \in \mathbb{T} \setminus \Theta$. Thus $\|F\|_{L^2(\mathbb{T})} = 0$, which contradicts (5.2).

This proves Theorem 1.2. □

6 Proof of Theorem 1.3

In this section we will prove Theorem 1.3 by applying the LDT. The main idea of the proof is from Bourgain [Bou05], where the $1D$ analytic Schrödinger operator with the single-frequency was investigated. For $f(\theta + n\omega) = g(\theta + n \cdot \omega)$, we denote by $\widetilde{\Sigma}$ the spectrum of $\widetilde{H}_{\lambda f,\omega,\theta}$, which is also independent of θ. Thus to prove Theorem 1.3, it suffices to show $\widetilde{\Sigma}$ has positive Lebesgue measure.

For simplicity, we write $\widetilde{H}(\theta) = \widetilde{H}_{\lambda f,\omega,\theta}$ and

$$\widetilde{H}_N(\theta) = R_\Lambda \widetilde{H}(\theta) R_\Lambda \quad \text{for } \Lambda \in \mathcal{E}_N^0.$$

We denote by $\{e_k : k \in \mathbb{Z}^d\}$ (resp., $\langle \cdot, \cdot \rangle$) the standard orthogonal basis (resp., inner product) on $\ell^2(\mathbb{Z}^d)$.

Lemma 6.1. *Let* $\omega \in \bigcap_{N \geq \underline{N}_0} \Omega_N$ *and* $N_0 \gg \underline{N}_0$. *Then there exists a positive constant* $\lambda_0 = \lambda_0(N_0) \ll 1$ *such that the following holds: If* $0 \leq \lambda \leq \lambda_0$, *then there exist an interval* $I_0 \subset [0, 1]$ *and a continuous function* $E_{I_0}(\cdot)$ *on* I_0 *satisfying*

$$|I_0| \geq N_0^{-C_1}$$

and for $\theta \in I_0$,

$$\min_{\xi \in \mathrm{Span}\{e_k : k \in \mathbb{Z}^d, |k| \leq N_0\},\ \|\xi\|=1} \|(\widetilde{H}(\theta) - E_{I_0}(\theta))\xi\| \leq e^{-c_5 (\log N_0)^{\gamma / c_1}},$$

where $0 < c_5 = c_5(\gamma, \rho) \ll 1$ *and* $C_1 = C_1(d) > 1$.

Proof. Fix any θ. Denote by $\lambda_s(\theta),\ 1 \leq s \leq (2N_0 + 1)^d$ (resp., $\phi_s,\ \|\phi_s\| = 1$) the eigenvalues (resp., corresponding eigenvectors) of $\widetilde{H}_{N_0}(\theta)$, where $N_0 \gg 1$ will be specified later. Then one has

$$e_0 = \sum_{1 \leq s \leq (2N_0+1)^d} \langle e_0, \phi_s \rangle \phi_s. \tag{6.1}$$

Obviously, we have

$$\|(\widetilde{H}(\theta) - f(\theta)) e_0\| \leq \sum_{m \in \mathbb{Z}^d} \lambda e^{-\rho |m|^\gamma} \leq C(\rho, \gamma, d)\lambda.$$

Thus

$$\begin{aligned} (\widetilde{H}_{N_0}(\theta) - f(\theta)) e_0 &= (\widetilde{H}(\theta) - f(\theta)) e_0 - R_{\mathbb{Z}^d \setminus [-N_0, N_0]^d} \widetilde{H}(\theta) e_0 \\ &= O(\lambda). \end{aligned} \tag{6.2}$$

On the other hand, we have

$$\begin{aligned} \widetilde{H}_{N_0}(\theta) e_0 &= \sum_{1 \leq s \leq (2N_0+1)^d} \langle e_0, \phi_s \rangle \widetilde{H}_{N_0}(\theta) \phi_s \\ &= \sum_{1 \leq s \leq (2N_0+1)^d} \langle e_0, \phi_s \rangle \lambda_s(\theta) \phi_s. \end{aligned} \tag{6.3}$$

Thus by combining (6.1), (6.2) and (6.3), we obtain

$$\begin{aligned} &\left(\sum_{1 \leq s \leq (2N_0+1)^d} |\langle e_0, \phi_s \rangle|^2 |\lambda_s(\theta) - f(\theta)|^2 \right)^{1/2} \\ &\quad = \left\| \sum_{1 \leq s \leq (2N_0+1)^d} \langle e_0, \phi_s \rangle (\lambda_s(\theta) - f(\theta)) \phi_s \right\| \\ &\quad \leq C\lambda. \end{aligned} \tag{6.4}$$

Since $1 = \|e_0\|^2 = \sum_{1\le s\le (2N_0+1)^d} |\langle e_0, \phi_s\rangle|^2$, there exists some $s_\star \in [1, (2N_0+1)^d]$ so that

$$|\langle e_0, \phi_{s_\star}\rangle| \ge (2N_0+1)^{-d/2}, \tag{6.5}$$

which together with (6.4) implies

$$|\lambda_{s_\star}(\theta) - f(\theta)| \le C(2N_0+1)^{d/2}\lambda. \tag{6.6}$$

Recall that $\omega \in \Omega_{N_0}$, $N_0 \gg \underline{N}_0$. We have by Theorem 3.2, there exist

$$M_0 \sim (\log N_0)^{1/c_1} \ge \underline{N}_0 \quad \text{and} \quad N_0^{c_3}/10 \le M_1 \le 10N_0^{c_4}$$

so that $\theta + n\omega \mod \mathbb{Z}^d \notin X_{M_0}$ for all n satisfying

$$N_0^{c_3}/10 \le M_1^{\gamma/(10d)} \le |n| \le M_1 \le 10N_0^{c_4}.$$

Fix $M_1^{\gamma/(10d)} \le |n| \le M_1$. Then we can find $Q(n) \in \mathcal{E}_{M_0}$ so that

$$\begin{aligned} \operatorname{dist}(n, \Lambda \setminus \Lambda_1 \setminus Q(n)) &\ge M_0/2, \\ \|G_{Q(n)}(E;\theta)\| &\le e^{M_0^{\gamma/2}}, \\ |G_{Q(n)}(E;\theta)(k,k')| &\le e^{-\frac{\bar\rho}{2}|k-k'|^\gamma} \quad \text{for } |k-k'| \ge M_0/10. \end{aligned}$$

Thus by the Poisson's identity (5.1), we have, for $M_0 \ge M_0(\gamma, \bar\rho) \gg 1$ and $\|\phi_{s_\star}\| = 1$,

$$\begin{aligned} |\phi_{s_\star}(n)| &= \left| \sum_{n_1\in Q(n), n_2\in\Lambda\setminus\Lambda_1\setminus Q(n)} \lambda G_{Q(n)}(E;\theta)(n,n_1)\widehat{v}_{n_1-n_2}\phi_{s_\star}(n_2)\right| \\ &\le \sum_{n_1\in Q(n), n_2\in\Lambda\setminus\Lambda_1\setminus Q(n)} e^{M_0^{\gamma/2}+\frac{\bar\rho}{2}(M_0/10)^\gamma - \frac{\bar\rho}{2}|n-n_1|^\gamma - \rho|n_1-n_2|^\gamma} \\ &\le \sum_{n_1\in Q(n), n_2\in\Lambda\setminus\Lambda_1\setminus Q(n)} e^{M_0^{\gamma/2}+\frac{\bar\rho}{2}(M_0/10)^\gamma - \frac{\bar\rho}{2}|n-n_2|^\gamma} \\ &\le \sum_{n_1\in Q(n), n_2\in\Lambda\setminus\Lambda_1\setminus Q(n)} e^{M_0^{\gamma/2}+\frac{\bar\rho}{2}(M_0/10)^\gamma - \frac{\bar\rho}{2}(M_0/2)^\gamma} \\ &\le e^{-c(\log N_0)^{\gamma/c_1}}. \end{aligned} \tag{6.7}$$

We define

$$J = [M_1 + M_1^{\gamma/(10d)}/2], \quad \Lambda = [-J, J]^d \subset [-N_0, N_0]^d.$$

Then by (6.5),

$$\|R_\Lambda \phi_{s_\star}\| \ge (2N_0+1)^{-d/2}.$$

Define now

$$\psi = \frac{R_\Lambda \phi_{s_\star}}{\|R_\Lambda \phi_{s_\star}\|}.$$

Since $(\widetilde{H}_{N_0}(\theta) - \lambda_{s_\star}(\theta))\phi_{s_\star} = 0$, we have

$$R_\Lambda(\widetilde{H}(\theta) - \lambda_{s_\star}(\theta))\psi = -\|R_\Lambda \phi_{s_\star}\|^{-1} R_\Lambda \widetilde{H}(\theta) R_{[-N_0,N_0]^d \setminus \Lambda} \phi_{s_\star}. \tag{6.8}$$

Thus by direct computations, we obtain

$$\begin{aligned}(\widetilde{H}(\theta) - \lambda_{s_\star}(\theta))\psi &= R_{\mathbb{Z}^d \setminus [-N_0,N_0]^d} \widetilde{H}(\theta)\psi + R_{[-N_0,N_0]^d \setminus \Lambda} \widetilde{H}(\theta)\psi \\ &\quad + R_\Lambda(\widetilde{H}(\theta) - \lambda_{s_\star}(\theta))\psi \\ &= \|R_\Lambda \phi_{s_\star}\|^{-1} R_{\mathbb{Z}^d \setminus [-N_0,N_0]^d} \widetilde{H}(\theta) R_\Lambda \phi_{s_\star} \\ &\quad + \|R_\Lambda \phi_{s_\star}\|^{-1} R_{[-N_0,N_0]^d \setminus \Lambda} \widetilde{H}(\theta) R_\Lambda \phi_{s_\star} \\ &\quad + (-\|R_\Lambda \phi_{s_\star}\|^{-1} R_\Lambda \widetilde{H}(\theta) R_{[-N_0,N_0]^d \setminus \Lambda} \phi_{s_\star}) \text{ (by (6.8))} \\ &= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).\end{aligned}$$

For (I), we have

$$\begin{aligned}\|(\mathrm{I})\|^2 &\le \lambda^2 (2N_0+1)^d \sum_{|m|>N_0} \left(\sum_{|n|\le J} e^{-\rho|m-n|^\gamma} \right)^2 \\ &\le \lambda^2 (2N_0+1)^{2d} e^{2\rho J^\gamma} \left(\sum_{|m|>N_0} e^{-\rho|m|^\gamma} \right)^2 \\ &\le e^{-\rho N_0^\gamma} \text{ (since } J \le 10N_0^{c_4}).\end{aligned} \tag{6.9}$$

For (II), we have, with (6.7),

$$\begin{aligned}\|(\mathrm{II})\|^2 &\le \lambda^2 (2N_0+1)^d \sum_{J<|m|\le N_0} \left| \sum_{|n|\le J} e^{-\rho|m-n|^\gamma} \phi_{s_\star}(n) \right|^2 \\ &\le \lambda^2 (2N_0+1)^d \sum_{J<|m|\le N_0} \left(\sum_{|n|\le M_1^{\gamma/(10d)}} e^{-\rho|m-n|^\gamma} \right)^2 \\ &\quad + \lambda^2 (2N_0+1)^d \sum_{J<|m|\le N_0} \left(\sum_{M_1^{\gamma/(10d)} \le |n| \le J} e^{-c(\log N_0)^{\gamma/c_1}} \right)^2 \\ &\le \lambda^2 (10N_0)^{3d} e^{-cJ^\gamma} + \lambda^2 (10N_0)^{3d} e^{-c(\log N_0)^{\gamma/c_1}} \\ &\le e^{-3c_5(\log N_0)^{\gamma/c_1}}.\end{aligned} \tag{6.10}$$

Similarly, for (III), we have

$$
\begin{aligned}
\|(III)\|^2 &\le \lambda^2(2N_0+1)^d \sum_{|m|\le J}\Big|\sum_{J\le|n|\le N_0} e^{-\rho|m-n|^\gamma}\phi_{s_\star}(n)\Big|^2\\
&\le \lambda^2(2N_0+1)^d \sum_{|m|\le J}\Big(\sum_{J\le|n|\le M_1} e^{-c(\log N_0)^{\gamma/c_1}}\Big)^2\\
&\quad + \lambda^2(2N_0+1)^d \sum_{|m|\le J}\Big(\sum_{M_1\le|n|\le N_0} e^{-\rho|m-n|^\gamma}\Big)^2\\
&\le e^{-3c_5(\log N_0)^{\gamma/c_1}}.
\end{aligned}
\tag{6.11}
$$

Thus combining (6.9), (6.10) and (6.11), we obtain

$$\min_{\xi\in \mathrm{Span}\{e_k:k\in\mathbb{Z}^d,|k|\le J\},\ \|\xi\|=1} \|(\widetilde{H}(\theta)-\lambda_{s_\star}(\theta))\xi\| \le e^{-c_5(\log N_0)^{\gamma/c_1}},$$

or equivalently

$$\|(R_\Lambda(\widetilde{H}(\theta)-\lambda_{s_\star}(\theta))^*(\widetilde{H}(\theta)-\lambda_{s_\star}(\theta))R_\Lambda)^{-1}\| \ge e^{2c_5(\log N_0)^{\gamma/c_1}}. \tag{6.12}$$

Define for $1\le s_\star\le(2N_0+1)^d$ and $J\in[N_0^{c_3}/10, N_0^{c_4}]$ the set $\Gamma_{s_\star,J}\subset[0,1]$ of θ for which (6.6) and (6.12) hold. It is well-known that $\lambda_{s_\star}(\theta)$ is Lipschitz continuous in f (see [Tao12] for details). By a standard truncation argument, we can replace $f(\theta)$ by a polynomial in θ of degree CN_0^2. Notice that $\lambda_{s_\star}(\theta)$ satisfies the equation

$$\zeta^D + \sum_{r<D} c_r(\theta)\zeta^D = 0,$$

where $D=(2N_0+1)^d$ and $c_r(\theta)$ are polynomials of degree at most N_0^C. Expressing (6.12) by Cramer's rule, a polynomial condition

$$P(\theta,\zeta) > 0$$

is obtained in $(\theta, \zeta=\lambda_{s_\star}(\theta))$. Recalling Lemmas 2.2 and 2.3, $\Gamma_{s_\star,J}$ can be decomposed into N_0^C many intervals $I'\subset\Gamma_{s_\star,J}$. For each such I', we set $E_{I'}(\theta)=\lambda_{s_\star}(\theta)$, $\theta\in I'$. Let $\mathcal{F}_0$ be the collection of all such intervals I' (counting all possible $s_\star, J$). Then $\#\mathcal{F}_0\le N_0^{C_1}$. In particular, for $\theta\in I'\subset\Gamma_{s_\star,J}$, we have

$$\min_{\xi\in \mathrm{Span}\{e_k:k\in\mathbb{Z}^d,|k|\le N_0\},\ \|\xi\|=1} \|(\widetilde{H}(\theta)-E_{I'}(\theta))\xi\| \le e^{-c_5(\log N_0)^{\gamma/c_1}}.$$

We observe that

$$\begin{aligned}[f_{\min}, f_{\max}] &= \bigcup_{s_\star, J} f(\Gamma_{s_\star, J}) \\ &\subset \bigcup_{s_\star, J} \bigcup_{I' \subset \Gamma_{s_\star, J}} (\lambda_{s_\star}(I') + [-CN_0^{d/2}\lambda, CN_0^{d/2}\lambda]) \quad \text{(by (6.6))} \\ &= \bigcup_{I' \in \mathcal{F}_0} (E_{I'}(I') + [-CN_0^{d/2}\lambda, CN_0^{d/2}\lambda]).\end{aligned}$$

Thus for $N_0 \gg \underline{N}_0$ and $\lambda \leq \lambda_0(N_0) \ll 1$, we get

$$0 < f_{\max} - f_{\min} \leq \operatorname{mes}\Big(\bigcup_{I' \in \mathcal{F}_0} E_{I'}(I')\Big) + N_0^{C_1}\lambda \leq \operatorname{mes}\Big(\bigcup_{I' \in \mathcal{F}_0} E_{I'}(I')\Big) + \sqrt{\lambda}.$$

Define I_0 to be the interval in $\mathcal{F}_0$ with the maximal length. Then by

$$[0, 1] \subset \bigcup_{I' \in \mathcal{F}_0} I' \quad \text{and} \quad \#\mathcal{F}_0 \leq N_0^{C_1},$$

we obtain $|I_0| \geq N_0^{-C_1}$. If $\theta \in I_0$, we have

$$\min_{\xi \in \operatorname{Span}\{e_k : k \in \mathbb{Z}^d, |k| \leq N_0\},\ \|\xi\|=1} \|(\widetilde{H}(\theta) - E_{I_0}(\theta))\xi\| \leq e^{-c_5(\log N)^{\gamma/c_1}}.$$

This proves the lemma. □

The following lemma is an inductive extension of Lemma 6.1.

Lemma 6.2. *Let $\gamma/c_1 > 100$. Let $I \subset [0, 1]$ be an interval and $E(\theta) \in \sigma(\widetilde{H}_N(\theta))$ be a continuous function on I. Assume again that*

$$\min_{\xi \in \operatorname{Span}\{e_k : k \in \mathbb{Z}^d, |k| \leq N\},\ \|\xi\|=1} \|(\widetilde{H}(\theta) - E(\theta))\xi\| \leq e^{-c_5(\log N)^{\gamma/c_1}}, \tag{6.13}$$

where $c_5 > 0$ is given by Lemma 6.1.

Let

$$\mathbb{N} \ni N_1 \sim e^{(\log N)^{10}}. \tag{6.14}$$

Then there exists a system $(I', E_{I'}(\cdot))_{I' \in \mathcal{F}_1}$ such that the following holds: $\mathcal{F}_1$ is a collection of at most $N_1^{C_1}$ intervals $I' \subset I$ so that $E_{I'}(\theta) \in \sigma(\widetilde{H}_{N_1}(\theta))$ is a continuous function on I', and for $\theta \in I'$,

$$\min_{\xi \in \operatorname{Span}\{e_k : k \in \mathbb{Z}^d, |k| \leq N_1\},\ \|\xi\|=1} \|(\widetilde{H}(\theta) - E_{I'}(\theta))\xi\| \leq e^{-c_5(\log N_1)^{\gamma/c_1}}. \tag{6.15}$$

Moreover,

$$\operatorname{mes}\Big(\bigcup_{I' \in \mathcal{F}_1} E_{I'}(I')\Big) \geq \operatorname{mes}(E(I)) - \frac{1}{N_1}. \tag{6.16}$$

Proof. The proof is similar to that of Lemma 6.1. Fix a $\theta \in I$. Choose a ξ with $\|\xi\| = 1$ and $\xi \in \text{Span}\{e_k : k \in \mathbb{Z}^d, |k| \leq N\}$ so that (6.13) holds. Denote by $\lambda_s(\theta)$, $1 \leq s \leq (2N_1+1)^d$ (resp., ϕ_s, $\|\phi_s\| = 1$) the eigenvalues (resp., corresponding eigenvectors) of $\widetilde{H}_{N_1}(\theta)$. Then one has

$$\xi = \sum_{1 \leq s \leq (2N_1+1)^d} \langle \xi, \phi_s \rangle \phi_s.$$

Obviously, we have

$$\|(\widetilde{H}(\theta) - E(\theta))\xi\| \leq e^{-c_5 (\log N)^{\gamma/c_1}}.$$

Thus

$$\|(\widetilde{H}_{N_1}(\theta) - E(\theta))\xi\| = \|(\widetilde{H}(\theta) - E(\theta))\xi - R_{\mathbb{Z}^d \setminus [-N_1, N_1]^d} \widetilde{H}(\theta)\xi\| \leq 2e^{-c_5(\log N)^{\gamma/c_1}}.$$

On the other hand, one has

$$\widetilde{H}_{N_1}(\theta) = \sum_{1 \leq s \leq (2N_1+1)^d} \langle \xi, \phi_s \rangle \lambda_s(\theta) \phi_s.$$

Thus

$$\left(\sum_{|s| \leq N_1} |\langle \xi, \phi_s \rangle|^2 |\lambda_s(\theta) - E(\theta)|^2 \right)^{1/2} \leq 2e^{-c_5(\log N)^{\gamma/c_1}}.$$

Since $\|\xi\| = 1$, there exists an $s_\star \in [1, (2N_1+1)^d]$ so that

$$|\langle \xi, \phi_{s_\star} \rangle| \geq (2N_1+1)^{-d/2}$$

and

$$|\lambda_{s_\star}(\theta) - E(\theta)| \leq 2(2N_1+1)^{d/2} e^{-c_5(\log N)^{\gamma/c_1}}. \tag{6.17}$$

As in the proof of Lemma 6.1, we have, for some $M_1 \in [N_1^{c_3}/10, 10N_1^{c_4}]$,

$$|\phi_{s_\star}(n)| \leq e^{-c(\log N_1)^{\gamma/c_1}} \quad \text{for } M_1^{\gamma/(10d)} \leq |n| \leq M_1.$$

Note that for $J = [(M_1^{\gamma/(10d)} + M_1)/2]$ and $\Lambda = [-J, J]^d$, one has

$$\|R_\Lambda \phi_{s_\star}\| \geq (2N_1+1)^{-d/2}.$$

Define

$$\psi = \frac{R_\Lambda \phi_{s_\star}}{\|R_\Lambda \phi_{s_\star}\|}.$$

Similar to the proof of Lemma 6.1, we have

$$\min_{\xi \in \text{Span}\{e_k : k \in \mathbb{Z}^d, |k| \leq J\},\ \|\xi\|=1} \|(\widetilde{H}(\theta) - \lambda_{s_\star}(\theta))\xi\| \leq e^{-c_5(\log N_1)^{\gamma/c_1}},$$

or equivalently

$$\|(R_\Lambda(\widetilde{H}(\theta)-\lambda_{s_\star}(\theta))^*(\widetilde{H}(\theta)-\lambda_{s_\star}(\theta))R_\Lambda)^{-1}\| \geq e^{2c_5(\log N_1)^{\gamma/c_1}}. \tag{6.18}$$

Similarly, we define for $1 \leq s_\star \leq (2N_1+1)^d$ and $N_1^{c_3}/10 \leq J \leq 10N_1^{c_4}$ the set $\Gamma_{s_\star,J} \subset I$ of θ for which (6.17) and (6.18) hold. Using semi-algebraic sets arguments as previously, $\Gamma_{s_\star,J}$ can be decomposed into N_1^C many intervals $I' \subset \Gamma_{s_\star,J}$. For each such I', we set $E_{I'}(\theta) = \lambda_{s_\star}(\theta)$, $\theta \in I'$. Let $\mathcal{F}_1$ be the collection of all such intervals I' (counting all possible $s_\star, J$). Then $\#\mathcal{F}_1 \leq N_1^{C_1}$. In particular, for $\theta \in I' \subset \Gamma_{s_\star,J}$, we have

$$\min_{\xi\in \mathrm{Span}\{e_k:k\in\mathbb{Z}^d,|k|\leq N_1\},\ \|\xi\|=1} \|(\widetilde{H}(\theta)-E_{I'}(\theta))\xi\| \leq e^{-c_5(\log N_1)^{\gamma/c_1}}.$$

This proves (6.15). Observe that again since (6.17),

$$\begin{aligned}
E(I) &= \bigcup_{s_\star,J} E(\Gamma_{s_\star,J})\\
&\subset \bigcup_{s_\star,J}\bigcup_{I'\subset\Gamma_{s_\star,J}} (\lambda_{s_\star}(I') + [-2(2N_1+1)^{d/2}e^{-c_5(\log N)^{\gamma/c_1}}, 2(2N_1+1)^{d/2}e^{-c_5(\log N)^{\gamma/c_1}}])\\
&= \bigcup_{I'\in\mathcal{F}_1} (E_{I'}(I') + [-2(2N_1+1)^{d/2}e^{-c_5(\log N)^{\gamma/c_1}}, 2(2N_1+1)^{d/2}e^{-c_5(\log N)^{\gamma/c_1}}]).
\end{aligned}$$

Thus by (6.14), we obtain

$$\begin{aligned}
\mathrm{mes}(E(I)) &\leq \mathrm{mes}\left(\bigcup_{I'\in\mathcal{F}_1} E_{I'}(I')\right) + N_1^C e^{-c_5(\log N)^{\gamma/c_1}}\\
&\leq \mathrm{mes}\left(\bigcup_{I'\in\mathcal{F}_1} E_{I'}(I')\right) + \frac{1}{N_1}.
\end{aligned}$$

This proves (6.16). □

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. Choose $N_s \sim e^{(\log N_{s-1})^{10}}$ $(s \geq 1)$, where N_0 is given by Lemma 6.1. Then applying Lemmas 6.1 and 6.2 yields a system $(I, E_I(\cdot))_{I\in\mathcal{F}_s}$ satisfying, for $\theta \in I \in \mathcal{F}_s$,

$$\mathrm{dist}(E_I(\theta), \widetilde{\Sigma}) \leq e^{-c_5(\log N_s)^{\gamma/c_1}}. \tag{6.19}$$

Moreover, for any $s \geq 1$, one has

$$\begin{aligned}
\mathrm{mes}\left(\bigcup_{I\in\mathcal{F}_s} E_I(I)\right) &\geq \mathrm{mes}\left(\bigcup_{I\in\mathcal{F}_{s-1}} E_I(I)\right) - \frac{1}{N_s} \geq \mathrm{mes}(E_{I_0}(I_0)) - \sum_{s\geq 1}\frac{1}{N_s}\\
&\geq \frac{\mathrm{mes}(E_{I_0}(I_0))}{2},
\end{aligned}$$

where E_{I_0}, I_0 are given by Lemma 6.1. Define

$$\Omega = \bigcap_{s\geq 0} \bigcup_{I\in\mathcal{F}_s} E_I(I).$$

Since (6.19), we have

$$\Omega \subset \widetilde{\Sigma}$$

and

$$\mathrm{mes}(\Omega) \geq \frac{\mathrm{mes}(E_{I_0}(I_0))}{2}.$$

Thus it suffices to establish some lower bound on $\mathrm{mes}(E_{I_0}(I_0))$.

Recall that $E_{I_0}(\cdot)$ is continuous on I_0 and $|I_0| \geq N_0^{-C_1}$. We can write $E_{I_0}(I_0) = [E_0+\varepsilon, E_0-\varepsilon]$ for some $E_0 \in E_{I_0}(I_0)$ and $\varepsilon \geq 0$. We need to establish some concrete lower bound on ε. Choose $\underline{N}_0 \leq M \ll N_0$ and apply the LDT (i.e., Theorem 3.1) at scale M, where M will be specified later. We have

$$\|G_M(E_0;\theta)\| \leq e^{M^{\gamma/2}},$$
$$|G_M(E_0;\theta)(n,n')| \leq e^{-\frac{(1-5^{-\gamma})\rho}{2}|n-n'|^\gamma} \quad \text{for } |n-n'| \geq M/10$$

provided θ is outside a set $\Theta \subset [0,1]$ with $\mathrm{mes}(\Theta) \leq e^{-M^{c_1}}$. Paving $[-N_0,N_0]^d$ with $Q \in \mathcal{E}_M$, then we have, by Lemma A.2,

$$\|G_{N_0}(E_0;\theta)\| \leq (10M)^d e^{M^{\gamma/2}} \leq e^{2M^{\gamma/2}} \tag{6.20}$$

provided θ is outside a set $\Theta_1 \subset [0,1]$ with $\mathrm{mes}(\Theta_1) \leq (10N_0)^d e^{-M^{c_1}}$. Fix

$$M \sim (\log N_0)^{3/(2c_1)}.$$

Then

$$(10N_0)^d e^{-M^{c_1}} < \frac{N_0^{-C_1}}{2}$$

and thus $([0,1]\setminus\Theta_1)\cap I_0 \neq \emptyset$. We pick a $\theta_0 \in ([0,1]\setminus\Theta_1)\cap I_0$ and a ξ with $\|\xi\|=1$ so that

$$|(\widetilde{H}(\theta_0) - E_{I_0}(\theta_0))\xi\| \leq e^{-c_5(\log N_0)^{\gamma/c_1}}.$$

Note that

$$\begin{aligned}\|(\widetilde{H}_{N_0}(\theta_0)-E_0)\xi\| &= \|(\widetilde{H}(\theta_0)-E_{I_0}(\theta_0))\xi - (E_0 - E_{I_0}(\theta_0))\xi\| \\ &= \|(\widetilde{H}(\theta_0)-E_{I_0}(\theta_0))\xi - R_{\mathbb{Z}^d\setminus[-N_0,N_0]^d}\widetilde{H}(\theta_0)\xi \\ &\quad - (E_0 - E_{I_0}(\theta_0))\xi\| \\ &\leq 2e^{-c_5(\log N_0)^{\gamma/c_1}} + \varepsilon.\end{aligned} \tag{6.21}$$

Recalling (6.20), we have

$$\|G_{N_0}(E_0;\theta_0)\| \le e^{2(\log N_0)^{3\gamma/(4c_1)}}. \tag{6.22}$$

Combining (6.21) and (6.22) yields

$$e^{-2(\log N_0)^{3\gamma/(4c_1)}} \le 2e^{-c_5(\log N_0)^{\gamma/c_1}} + \varepsilon$$

and

$$\varepsilon \ge \frac{1}{2}e^{-2(\log N_0)^{3\gamma/(4c_1)}}.$$

In conclusion, we have shown

$$\operatorname{mes}(\widetilde{\Sigma}) \ge e^{-10(\log N_0)^{3\gamma/(4c_1)}} > 0.$$

This proves Theorem 1.3. □

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Appendix A

We write $G_{(\cdot)} = G_{(\cdot)}(E;\theta)$ for simplicity. Let $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$. Let $\Lambda = \Lambda_1 \cup \Lambda_2$. If $m \in \Lambda_1$ and $n \in \Lambda$, we have

$$G_\Lambda(m,n) = G_{\Lambda_1}(m,n)\chi_{\Lambda_1}(n) - \lambda \sum_{n' \in \Lambda_1, n'' \in \Lambda_2} G_{\Lambda_1}(m,n')\mathcal{T}_v(n',n'')G_\Lambda(n'',n). \tag{A.1}$$

We first prove a useful perturbation argument (see Lemma A.1 of [Shi19] for a more general form with $\gamma = 1$).

Lemma A.1. *Fix $\overline{\rho} > 0$. Let $\Lambda \subset \mathbb{Z}^d$ satisfy $\Lambda \in \mathcal{E}_N$ and let A, B be two linear operators on $\mathbb{C}^\Lambda$. We assume*

$$\|A^{-1}\| \le e^{N^{\gamma/2}},$$
$$|A^{-1}(n,n')| \le e^{-\overline{\rho}|n-n'|^\gamma} \quad \textit{for } |n-n'| \ge N/10.$$

Suppose that for all $n, n' \in \Lambda$,

$$|(B-A)(n,n')| \le e^{-3\overline{\rho}N^\gamma - \overline{\rho}|n-n'|^\gamma}.$$

Then

$$\|B^{-1}\| \le 2\|A^{-1}\|,$$
$$|B^{-1}(n,n')| \le |A^{-1}(n,n')| + e^{-\bar\rho|n-n'|^\gamma}.$$

Proof. Obviously, $B = A(I + A^{-1}(B-A))$. We write $P = A^{-1}(B-A)$. Then by the assumptions, $\|P\| \le 1/2$, which together with the Neumann series argument implies

$$\|B^{-1}\| \le \sum_{s\ge 0} 2^{-s}\|A^{-1}\| = 2\|A^{-1}\|.$$

Observing that for any $m, n \in \Lambda$,

$$|A^{-1}(m,n)| \le e^{N^{\gamma/2}+\bar\rho(N/10)^\gamma - \bar\rho|m-n|^\gamma},$$

then for $m^0 = m$, $m^s = n$ and $s \ge 1$, we have

$$P^s(m,n) = \sum_{m^1,\dots,m^{s-1},n^1,\dots,n^s\in\Lambda} \prod_{t=1}^{s} A^{-1}(m^{t-1},n^t)(B-A)(n^t,m^t).$$

Thus for $s \ge 1$ and $N \gg 1$, one has

$$|P^s(m,n)| \le (CN)^{2sd} e^{s(N^{\gamma/2}-2\bar\rho N^\gamma)-\bar\rho|m-n|^\gamma} \le e^{-3\bar\rho s N^\gamma/2 - \bar\rho|m-n|^\gamma}.$$

As a result, we obtain

$$\begin{aligned}
|B^{-1}(n,n')| &\le |A^{-1}(n,n')| + \sum_{m\in\Lambda}\sum_{s\ge1} |P^s(n,m)|\cdot|A^{-1}(m,n')| \\
&\le |A^{-1}(n,n')| + \sum_{m\in\Lambda}\sum_{s\ge1} e^{-3\bar\rho s N^\gamma/2-\bar\rho|m-n|^\gamma}\cdot|A^{-1}(m,n')| \\
&\le |A^{-1}(n,n')| + \sum_{m\in\Lambda,|m-n'|\le N/10}\ \sum_{s\ge1} e^{-3\bar\rho s N^\gamma/2-\bar\rho|m-n|^\gamma+N^{\gamma/2}} \\
&\quad + \sum_{m\in\Lambda,|m-n'|\ge N/10}\ \sum_{s\ge1} e^{-3\bar\rho s N^\gamma/2-\bar\rho|m-n|^\gamma} e^{-\bar\rho|m-n'|^\gamma} \\
&\le |A^{-1}(n,n')| + \sum_{m\in\Lambda,|m-n'|\le N/10} e^{-\bar\rho N^\gamma/4+N^{\gamma/2}-\bar\rho|n-n'|^\gamma} \\
&\quad + \sum_{m\in\Lambda,\ |m-n'|>N/10} e^{-\bar\rho N^\gamma/4-\bar\rho|n-n'|^\gamma} \\
&\le |A^{-1}(n,n')| + e^{-\bar\rho|n-n'|^\gamma}.
\end{aligned}$$

□

Lemma A.2. *Let* $\bar{\rho} \in (\varepsilon, \rho]$, $M_1 \leq N$ *and* $\operatorname{diam}(\Lambda) \leq 2N+1$. *Suppose that for any* $n \in \Lambda$, *there exists some* $W = W(n) \in \mathcal{E}_M$ *with* $M_0 \leq M \leq M_1$ *such that* $n \in W \subset \Lambda$, $\operatorname{dist}(n, \Lambda \backslash W) \geq M/2$ *and*

$$\|G_W\| \leq 2e^{M^{\gamma/2}}, \tag{A.2}$$

$$|G_W(n,n')| \leq 2e^{-\bar{\rho}|n-n'|^{\gamma}} \quad \text{for } |n-n'| \geq M/10. \tag{A.3}$$

We assume that $M_0 \geq M_0(\varepsilon, \gamma, d) \gg 1$. *Then*

$$\|G_\Lambda\| \leq 4(2M_1+1)^d e^{M_1{}^{\gamma/2}}.$$

Proof. We fix $n, n' \in \Lambda$ and $W = W(n)$ as in the assumptions. Then

$$|W| \leq (2M+1)^d.$$

By (A.2) and (A.3), one has, for all $k, k' \in W$,

$$|G_W(k,k')| \leq 2e^{M^{\gamma/2}+\bar{\rho}(M/10)^{\gamma}} e^{-\bar{\rho}|k-k'|^{\gamma}}.$$

Applying (A.1) with $\Lambda_1 = W = W(n)$, one has

$$\begin{aligned}
&|G_\Lambda(n,n')| \\
&\leq |G_W(n,n')|\chi_W(n') \\
&\quad + 2\lambda \sum_{\substack{n_1 \in W \\ n_2 \in \Lambda\backslash W}} e^{M^{\gamma/2}+\bar{\rho}(M/10)^{\gamma}} e^{-\bar{\rho}|n-n_1|^{\gamma}-\rho|n_1-n_2|^{\gamma}} |G_\Lambda(n_2,n')| \\
&\leq |G_W(n,n')|\chi_W(n') \\
&\quad + 2\lambda \sum_{\substack{n_1 \in W \\ n_2 \in \Lambda\backslash W}} e^{M^{\gamma/2}+\bar{\rho}(M/10)^{\gamma}} e^{-\bar{\rho}|n-n_2|^{\gamma}} |G_\Lambda(n_2,n')| \\
&\leq |G_W(n,n')|\chi_W(n') \\
&\quad + 2\lambda(2M+1)^d e^{M^{\gamma/2}+\bar{\rho}(M/10)^{\gamma}} \sum_{\substack{n_2 \in \Lambda \\ |n_2-n| \geq M/2}} e^{-\bar{\rho}|n-n_2|^{\gamma}} |G_\Lambda(n_2,n')| \\
&\leq |G_W(n,n')|\chi_W(n') + 2\lambda(2M+1)^d e^{M^{\gamma/2}-\varepsilon(M/10)^{\gamma}} \sup_{n_2 \in \Lambda} |G_\Lambda(n_2,n')|,
\end{aligned} \tag{A.4}$$

where the third inequality holds since $\operatorname{dist}(n, \Lambda \backslash W) \geq M/2$. Summing over $n' \in \Lambda$ in (A.4) and using $M_0 \geq M_0(\varepsilon, \gamma, d) \gg 1$ yield (since $0 < \lambda < 1$)

$$\sup_{n \in \Lambda} \sum_{n' \in \Lambda} |G_\Lambda(n,n')| \leq 2(2M_1+1)^d e^{M_1{}^{\gamma/2}} + \frac{1}{2} \sup_{n_2 \in \Lambda} \sum_{n' \in \Lambda} |G_\Lambda(n_2,n')|.$$

This lemma then follows from Schur's test and the self-adjointness of G_Λ. □

Lemma A.3. *Let* $\Lambda_1 \subset \Lambda \subset \mathbb{Z}^d$ *satisfy* $\mathrm{diam}(\Lambda) \le 2N+1$ *and* $\mathrm{diam}(\Lambda_1) \le N^{\frac{\gamma}{3d}}$. *Let* $M_0 \ge (\log N)^{2/\gamma}$ *and* $\overline{\rho} \in [(1-5^{-\gamma})/10, (1-5^{-\gamma})\rho]$. *Suppose that for any* $n \in \Lambda\backslash\Lambda_1$, *there exists some* $W = W(n) \in \mathcal{E}_M$ *with* $M_0 \le M \le N^{\gamma/3}$ *such that* $n \in W \subset \Lambda\backslash\Lambda_1$, $\mathrm{dist}(n, \Lambda\backslash\Lambda_1\backslash W) \ge M/2$ *and*

$$\|G_W\| \le e^{M^{\gamma/2}},$$
$$|G_W(n,n')| \le e^{-\overline{\rho}|n-n'|^\gamma} \quad \text{for } |n-n'| \ge M/10.$$

Suppose that

$$\|G_\Lambda\| \le e^{N^{\gamma/2}}. \tag{A.5}$$

Then

$$|G_\Lambda(n,n')| \le e^{-(\overline{\rho}-\frac{C}{M_0^{\gamma/2}})|n-n'|^\gamma} \quad \text{for } |n-n'| \ge N/10,$$

where $C = C(d,\rho,\gamma) > 0$.

Proof. We first assume $n \in \Lambda \setminus \Lambda_1, n' \in \Lambda_1$ and $|n-n'| \ge N^{\gamma/2}$. We let $W = W(n) \subset \Lambda\setminus\Lambda_1$ satisfy the assumptions as above. Note that for $|n-n_2| \ge M/2$ and $0 < \rho < (1-5^{-\gamma})\rho$, one has

$$e^{-\rho|n-n_2|^\gamma+\rho(M/10)^\gamma} \le e^{-\overline{\rho}|n-n_2|^\gamma}. \tag{A.6}$$

Recall that $0 < \lambda < 1$ and $|n-n'| \ge N^{\gamma/2} > 10N^{\gamma/3} > \mathrm{diam}(W)$. Applying (A.1) with $\Lambda_1 = W = W(n)$ yields

$$\begin{aligned}
|G_\Lambda(n,n')| &\le \sum_{\substack{n_1\in W, |n_1-n|\le \frac{M}{10}\\ n_2\in\Lambda\backslash W}} e^{M^{\gamma/2}} e^{-\rho|n_1-n_2|^\gamma}|G_\Lambda(n_2,n')| \\
&\quad + \sum_{\substack{n_1\in W, |n_1-n|\ge \frac{M}{10}\\ n_2\in\Lambda\backslash W}} e^{-\overline{\rho}|n-n_1|^\gamma} e^{-\rho|n_1-n_2|^\gamma}|G_\Lambda(n_2,n')| \\
&\le \sum_{\substack{n_1\in W, |n_1-n|\le \frac{M}{10}\\ n_2\in\Lambda\backslash W}} e^{M^{\gamma/2}} e^{-\rho|n-n_2|^\gamma+\rho(M/10)^\gamma}|G_\Lambda(n_2,n')| \\
&\quad + \sum_{\substack{n_1\in W, |n_1-n|\ge \frac{M}{10}\\ n_2\in\Lambda\backslash W}} e^{-\overline{\rho}|n-n_2|^\gamma}|G_\Lambda(n_2,n')| \\
&\le \sum_{\substack{n_1\in W, |n_1-n|\le \frac{M}{10}\\ n_2\in\Lambda\backslash W}} e^{M^{\gamma/2}} e^{-\overline{\rho}|n-n_2|^\gamma}|G_\Lambda(n_2,n')| \qquad \text{(by (A.6))} \\
&\quad + \sum_{\substack{n_1\in W, |n_1-n|\ge \frac{M}{10}\\ n_2\in\Lambda\backslash W}} e^{-\overline{\rho}|n-n_2|^\gamma}|G_\Lambda(n_2,n')| \\
&\le 2(2N+1)^{2d} \sup_{n_2\in\Lambda\backslash W} e^{-(\overline{\rho}-\frac{C}{M_0^{\gamma/2}})|n-n_2|^\gamma}|G_\Lambda(n_2,n')|,
\end{aligned} \tag{A.7}$$

where the last inequality holds because $|n-n_2| \geq M/2$ and $M \geq M_0$. Iterating (A.7) until $|n_2-n'| \leq N^{\gamma/2}$ (but stopping at most $\frac{C|n-n'|^\gamma}{M_0^\gamma}$ steps), we have, since $|n-n'| \geq N^{\gamma/2}$ and $M_0 \geq (\log N)^{2/\gamma}$,

$$\begin{aligned}|G_\Lambda(n,n')| &\leq (10N)^{\frac{C|n-n'|^\gamma}{M_0^\gamma}} e^{-(\bar{\rho}-\frac{C}{M_0^{\gamma/2}})(|n-n'|^\gamma - N^{\gamma^2/2})} e^{N^{\gamma/2}} \\ &\leq e^{-(\bar{\rho}-\frac{C}{M_0^{\gamma/2}}-\frac{C\log N}{M_0^\gamma})|n-n'|^\gamma+2N^{\gamma/2}} \quad \text{(since } 0<\rho<1) \\ &\leq e^{-(\bar{\rho}-\frac{C}{M_0^{\gamma/2}})|n-n'|^\gamma+2N^{\gamma/2}}.\end{aligned}$$

Recalling (A.5) again, we obtain, for all $n \in \Lambda \setminus \Lambda_1$, $n' \in \Lambda_1$,

$$|G_\Lambda(n,n')| \leq e^{-(\bar{\rho}-\frac{C}{M_0^{\gamma/2}})|n-n'|^\gamma+3N^{\gamma/2}}.$$

Then by the self-adjointness of G_Λ, one has, for $n \in \Lambda_1$, $n' \in \Lambda \setminus \Lambda_1$,

$$|G_\Lambda(n,n')| \leq e^{-(\bar{\rho}-\frac{C}{M_0^{\gamma/2}})|n-n'|^\gamma+3N^{\gamma/2}}. \tag{A.8}$$

We now assume $n, n' \in \Lambda$ satisfy $|n-n'| \geq N^{\gamma/2}$. By $\operatorname{diam}(\Lambda_1) \leq N^{\frac{\gamma}{3d}}$, at least one of n, n' must be in $\Lambda \setminus \Lambda_1$. From the above discussions, It suffices to assume $n, n' \in \Lambda \setminus \Lambda_1$. Similar to the proof of (A.7), we have

$$|G_\Lambda(n,n')| \leq 2(2N+1)^{2d} \sup_{n_2 \in \Lambda\setminus W} e^{-(\bar{\rho}-\frac{C}{M_0^{\gamma/2}})|n-n_2|^\gamma} |G_\Lambda(n_2,n')|, \tag{A.9}$$

where $|n-n_2| \geq M/2$. Hence iterating (A.9) until $n_2 \in \Lambda_1$ (but stopping at most $\frac{C|n-n'|^\gamma}{M_0^\gamma}$ steps), we have, for $|n-n'| \geq N^{\gamma/2}$ (and some $n_2 \in \Lambda_1$),

$$\begin{aligned}|G_\Lambda(n,n')| &\leq (10N)^{\frac{C|n-n'|^\gamma}{M_0^\gamma}} e^{-(\bar{\rho}-\frac{C}{M_0^{\gamma/2}})|n-n_2|^\gamma} |G_\Lambda(n_2,n')| \\ &\leq (10N)^{\frac{C|n-n'|^\gamma}{M_0^\gamma}} e^{-(\bar{\rho}-\frac{C}{M_0^{\gamma/2}})|n-n_2|^\gamma} e^{-(\bar{\rho}-\frac{C}{M_0^{\gamma/2}})|n_2-n'|^\gamma+3N^{\gamma/2}} \quad \text{(by (A.8))} \\ &\leq e^{-(\bar{\rho}-\frac{C}{M_0^{\gamma/2}})|n-n'|^\gamma+3N^{\gamma/2}}.\end{aligned}$$

Finally, since $|n-n'| \geq N/10$, we have $\frac{N^{\gamma/2}}{|n-n'|^\gamma} \ll M_0^{-\gamma/2}$.

This finishes the proof. □

References

[AA80] S. Aubry and G. André, *Analyticity breaking and Anderson localization in incommensurate lattices*, in *Group Theoretical Methods in Physics (Proc. Eighth Internat. Colloq., Kiryat Anavim, 1979)*, Hilger, Bristol, 1980, pp. 133–164.

[AD08] A. Avila and D. Damanik, *Absolute continuity of the integrated density of states for the almost Mathieu operator with non-critical coupling*, Invent. Math. **172** (2008), 439–453.

[AFK11] A. Avila, B. Fayad and R. Krikorian, *A KAM scheme for* $\mathrm{SL}(2,\mathbb{R})$ *cocycles with Liouvillean frequencies*, Geom. Funct. Anal. **21** (2011), 1001–1019.

[AJ09] A. Avila and S. Jitomirskaya, *The Ten Martini Problem*, Ann. of Math. (2) **170** (2009), 303–342.

[AJ10] A. Avila and S. Jitomirskaya, *Almost localization and almost reducibility*, J. Eur. Math. Soc. (JEMS) **12** (2010), 93–131.

[AK06] A. Avila and R. Krikorian, *Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles*, Ann. of Math. (2) **164** (2006), 911–940.

[Avi] A. Avila, *KAM, Lyapunov exponents and the spectral dichotomy for one-frequency Schrödinger operators*, in preparation.

[Avi10] A. Avila, *Almost reducibility and absolute continuity I*, arXiv:1006.0704 [math.DS].

[AvMS90] J. Avron, P. H. M. van Mouche and B. Simon, *On the measure of the spectrum for the almost Mathieu operator*. Comm. Math. Phys. **132** (1990), 103–118.

[AYZ17] A. Avila, J. You and Q. Zhou, *Sharp phase transitions for the almost Mathieu operator*, Duke Math. J. **166** (2017), 2697–2718.

[BG00] J. Bourgain and M. Goldstein, *On nonperturbative localization with quasi-periodic potential*, Ann. of Math. (2) **152** (2000), 835–879.

[BGS01] J. Bourgain, M. Goldstein and W. Schlag, *Anderson localization for Schrödinger operators on* $\mathbb{Z}$ *with potentials given by the skew-shift*, Comm. Math. Phys. **220** (2001), 583–621.

[BGS02] J. Bourgain, M. Goldstein and W. Schlag, *Anderson localization for Schrödinger operators on* $\mathbf{Z}^2$ *with quasi-periodic potential*, Acta Math. **188** (2002), 41–86.

[BJ02] J. Bourgain and S. Jitomirskaya, *Absolutely continuous spectrum for 1D quasiperiodic operators*, Invent. Math. **148** (2002), 453–463.

[BK19] J. Bourgain and I. Kachkovskiy, *Anderson localization for two interacting quasiperiodic particles*, Geom. Funct. Anal. **29** (2019), 3–43.

[BK21] K. Bjerklöv and R. Krikorian, *Coexistence of absolutely continuous and pure point spectrum for kicked quasiperiodic potentials*, J. Spectr. Theory **11** (2021), 1215–1254.

[BLT83] J. Bellissard, R. Lima and D. Testard, *A metal-insulator transition for the almost Mathieu model*, Comm. Math. Phys. **88** (1983), 207–234.

[Bou02a] J. Bourgain, *Estimates on Green's functions, localization and the quantum kicked rotor model*, Ann. of Math. (2) **156** (2002), 249–294.

[Bou02b] J. Bourgain, *On the spectrum of lattice Schrödinger operators with deterministic potential. II*, J. Anal. Math. **88** (2002), 221–254.

[Bou05] J. Bourgain, *Green's Function Estimates for Lattice Schrödinger Operators and Applications*, Princeton University Press, Princeton, NJ, 2005.

[Bou07] J. Bourgain, *Anderson localization for quasi-periodic lattice Schrödinger operators on* $\mathbb{Z}^d$*, d arbitrary*, Geom. Funct. Anal. **17** (2007), 682–706.

[Cai21] A. Cai, *The absolutely continuous spectrum of finitely differentiable quasi-periodic Schrödinger operators*, arXiv:2103.15525 [math.DS].

[CCYZ19] A. Cai, C. Chavaudret, J. You and Q. Zhou. *Sharp Hölder continuity of the Lyapunov exponent of finitely differentiable quasi-periodic cocycles*, Math. Z. **291** (2019), 931–958.

[CD89] V. Chulaevsky and F. Delyon, *Purely absolutely continuous spectrum for almost Mathieu operators*, J. Statist. Phys. **55** (1989), 1279–1284.

[CD93] V. A. Chulaevsky and E. I. Dinaburg, *Methods of KAM-theory for long-range quasi-periodic operators on* $\mathbf{Z}^\nu$. *Pure point spectrum*, Comm. Math. Phys. **153** (1993), 559–577, 1993.

[Del87] F. Delyon, *Absence of localisation in the almost Mathieu equation*, J. Phys. A **20** (1987), L21–L23.

[Eli92] L. H. Eliasson, *Floquet solutions for the* 1*-dimensional quasi-periodic Schrödinger equation*, Comm. Math. Phys. **146** (1992), 447–482.

[Eli97] L. H. Eliasson, *Discrete one-dimensional quasi-periodic Schrödinger operators with pure point spectrum*, Acta Math. **179** (1997), 153–196.

[FK09] B. Fayad and R. Krikorian, *Rigidity results for quasiperiodic* $\mathrm{SL}(2,\mathbb{R})$*-cocycles*, J. Mod. Dyn. **3** (2009), 497–510.

[FS83] J. Fröhlich and T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Comm. Math. Phys. **88** (1983), 151–184.

[FSW90] J. Fröhlich, T. Spencer and P. Wittwer, *Localization for a class of one-dimensional quasi-periodic Schrödinger operators*, Comm. Math. Phys. **132** (1990), 5–25.

[GYZ] L. Ge, J. You and Q. Zhou, *Exponential dynamical localization: Criterion and Applications*, Ann. Sci. Éc. Norm. Supér, to appear.

[HA09] S. Hadj Amor, *Hölder continuity of the rotation number for quasi-periodic co-cycles in* $\mathrm{SL}(2,\mathbb{R})$, Comm. Math. Phys. *287* (2009), 565–588.

[HS89] B. Helffer and J. Sjöstrand, *Semiclassical analysis for Harper's equation. III. Cantor structure of the spectrum*, Mém. Soc. Math. France (N.S.) **39** (1989), 1–124.

[HWZ20] X. Hou, J. Wang and Q. Zhou, *Absolutely continuous spectrum of multifrequency quasiperiodic Schrödinger operator*, J. Funct. Anal. **279** (2020), Article no. 108632.

[Jit94] S. Jitomirskaya, *Anderson localization for the almost Mathieu equation: a nonperturbative proof*, Comm. Math. Phys. **165** (1994), 49–57.

[Jit99] S. Jitomirskaya, *Metal-insulator transition for the almost Mathieu operator*, Ann. of Math. (2) **150** (1999), 1159–1175.

[JK02] S. Jitomirskaya and I. Krasovsky, *Continuity of the measure of the spectrum for discrete quasiperiodic operators*, Math. Res. Lett. **9** (2002), 413–421.

[JK16] S. Jitomirskaya and I. Kachkovskiy, L^2*-reducibility and localization for quasiperiodic operators*, Math. Res. Lett. **23** (2016), 431–444.

[JK19] S. Jitomirskaya and I. Krasovsky, *Critical almost Mathieu operator: hidden singularity, gap continuity, and the Hausdorff dimension of the spectrum*, arXiv:1909.04429 [math.SP]

[JL18] S. Jitomirskaya and W. Liu, *Universal hierarchical structure of quasiperiodic eigenfunctions*, Ann. of Math. (2) **187** (2018), 721–776.

[JLS20] S. Jitomirskaya, W. Liu and Y. Shi, *Anderson localization for multi-frequency quasi-periodic operators on* $\mathbb{Z}^d$, Geom. Funct. Anal. **30** (2020), 457–481.

[JSY19] W. Jian, Y. Shi and X. Yuan, *Anderson localization for one-frequency quasi-periodic block operators with long-range interactions*, J. Math. Phys. **60** (2019), Article no. 063504.

[Kle05] S. Klein, *Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function*, J. Funct. Anal. **218** (2005), 255–292.

[Kle14] S. Klein, *Localization for quasiperiodic Schrödinger operators with multivariable Gevrey potential functions*, J. Spectr. Theory **4** (2014), 431–484.

[Las94] Y. Last, *Zero measure spectrum for the almost Mathieu operator*, Comm. Math. Phys. *164* (1994), 421–432.

[Pui06] J. Puig, *A nonperturbative Eliasson's reducibility theorem*, Nonlinearity **19** (2006), 355–376.

[Shi19] Y. Shi, *Analytic solutions of nonlinear elliptic equations on rectangular tori*, J. Differential Equations **267** (2019), 5576–5600.

[Sin87] Ya. G. Sinai, *Anderson localization for one-dimensional difference Schrödinger operator with quasiperiodic potential*, J. Statist. Phys. **46** (1987), 861–909.

[Tao12] T. Tao, *Topics in Random Matrix Theory*, American Mathematical Society, Providence, RI, 2012.

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