A $T1$ THEOREM FOR GENERAL CALDERÓN–ZYGMUND OPERATORS WITH COMPARABLE DOUBLING WEIGHTS, AND OPTIMAL CANCELLATION CONDITIONS

By

ERIC T. SAWYER

In memory of Professor Elias M. Stein

Abstract. We begin an investigation into extending the $T1$ theorem of David and Journé, and the corresponding optimal cancellation conditions of Stein, to more general pairs of distinct doubling weights. For example, when $0 < \alpha < n$, and σ and ω are A_∞ weights satisfying the one-tailed Muckenhoupt conditions, and K^α is a smooth fractional CZ kernel, we show there exists a bounded operator $T^\alpha : L^2(\sigma) \to L^2(\omega)$ associated with K^α if and only if there is a positive constant $\mathfrak{A}_{K^\alpha}(\sigma, \omega)$ so that

$$\int_{\|x-x_0\|<N} \left| \int_{\varepsilon<\|x-y\|<N} K^\alpha(x, y) d\sigma(y) \right|^2 d\omega(x) \le \mathfrak{A}_{K^\alpha}(\sigma, \omega) \int_{\|x_0-y\|<N} d\sigma(y),$$
$$\text{for all } 0 < \varepsilon < N \text{ and } x_0 \in \mathbb{R}^n,$$

where $\|y\| \equiv \max_{1\le k\le n} |y_k|$, along with a dual inequality. More generally this holds for measures σ and ω comparable in the sense of Coifman and Fefferman that satisfy a fractional A_∞^α condition.

These results are deduced from the following theorem of $T1$ type, namely that if σ and ω are doubling measures, comparable in the sense of Coifman and Fefferman, and satisfying one-tailed Muckenhoupt conditions, then $T^\alpha : L^2(\sigma) \to L^2(\omega)$ if and only if the dual pair of testing conditions hold, as well as a strong form of the weak boundedness property,

$$\left| \int_F (T^\alpha \mathbf{1}_E) d\omega \right| \le \mathcal{BICT}_{T^\alpha}(\sigma, \omega) \sqrt{|Q_\sigma|_\sigma |Q_\sigma|_\omega}, \quad \text{for all cubes } Q \subset \mathbb{R}^n,$$

where $\mathcal{BICT}_{T^\alpha}(\sigma, \omega)$ is a positive constant called the bilinear cube/indicator testing constant. The comparability of measures and the bilinear cube/indicator testing condition can both be dropped if the stronger indicator/cube testing conditions are assumed.

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 146 (2022)
DOI 10.1007/s11854-022-0198-3

Contents

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 146 (2022)
DOI 10.1007/s11854-022-0198-3

1 Introduction

This paper includes the content of [Saw2], [Saw3] and [Saw4] from the arXiv.

Given a Calderón–Zygmund kernel $K(x, y)$ in Euclidean space $\mathbb{R}^n$, a classical problem for many decades was to identify optimal cancellation conditions on K so that there would exist an associated singular integral operator $Tf(x) \sim \int K(x, y)f(y)dy$ bounded on $L^2(\mathbb{R}^n)$. After a long history, involving contributions by many authors,[1] this effort culminated in the decisive $T1$ theorem of David and Journé [DaJo], in which boundedness of an operator T on $L^2(\mathbb{R}^n)$ associated to K was characterized by

$$T\mathbf{1}, T^*\mathbf{1} \in BMO,$$

together with a weak boundedness property for some $\eta > 0$,

$$\begin{aligned} &\left| \int_Q T\varphi(x)\,\psi(x)dx \right| \\ (1.1)\qquad &\quad \lesssim \sqrt{\|\varphi\|_\infty |Q| + \|\varphi\|_{\mathrm{Lip}\,\eta} |Q|^{1+\frac{\eta}{n}}}\sqrt{\|\psi\|_\infty |Q| + \|\psi\|_{\mathrm{Lip}\,\eta} |Q|^{1+\frac{\eta}{n}}}, \\ &\quad \text{for all } \varphi, \psi \in \mathrm{Lip}\,\eta \text{ with } \operatorname{Supp}\varphi, \operatorname{Supp}\psi \subset Q, \text{ and all cubes } Q \subset \mathbb{R}^n; \end{aligned}$$

equivalently by two testing conditions taken over indicators of cubes,

$$\int_Q |T\mathbf{1}_Q(x)|^2 dx \lesssim |Q| \quad \text{and} \quad \int_Q |T^*\mathbf{1}_Q(x)|^2 dx \lesssim |Q|, \quad \text{all cubes } Q \subset \mathbb{R}^n.$$

The optimal cancellation conditions, which in the words of Stein were ‘a rather direct consequence of’ the $T1$ theorem, were given in [Ste2, Theorem 4, page 306], involving integrals of the kernel over shells—see Theorem 10 below for an extension to certain more general pairs of doubling weights and cubical shells.

[1] See, e.g., [Ste, page 53] for references to the earlier work in this direction.

An obvious next step is to replace Lebsegue measure with a fixed A_2 weight w,

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{w(x)}dx\right) \lesssim 1, \quad \text{all cubes } Q \subset \mathbb{R}^n,$$

and ask when T is bounded on $L^2(w)$, i.e., satisfies the one weight norm inequality. For elliptic Calderón–Zygmund operators T, this question is easily reduced to the David Journé theorem using two results from decades ago, namely the 1956 Stein–Weiss interpolation with change of measures theorem [StWe], and the 1974 Coifman and Fefferman extension [CoFe] of the one weight Hilbert transform inequality of Hunt, Muckenhoupt and Wheeden [HuMuWh], to a large class of general Calderón–Zygmund operators T.[2]

However, for a pair of different measures (σ, ω), the question is wide open in general, and we now turn to a brief discussion of the problem of boundedness of a general Calderón–Zygmund operator T from one general $L^2(\sigma)$ space to another $L^2(\omega)$ space. In the case of the Hilbert transform H in dimension one, the two weight inequality was completely solved in the two part paper [LaSaShUr3]; [Lac], and [Hyt3] where it was shown that H is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the testing and one-tailed Muckenhoupt conditions hold, i.e.,

$$\int_I |H(\mathbf{1}_I\sigma)|^2 d\omega \lesssim \int_I d\sigma \quad \text{and} \quad \int_I |H(\mathbf{1}_I\omega)|^2 d\sigma \lesssim \int_I d\omega, \quad \text{all intervals } I \subset \mathbb{R}^n,$$

$$\left(\int_{\mathbb{R}} \frac{|I|}{|I|^2 + |x-c_I|^2} d\sigma(x)\right)\left(\frac{1}{|I|}\int_I d\omega\right) \lesssim 1 \text{ and its dual}, \quad \text{all intervals } I \subset \mathbb{R}^n,$$

and for fractional Riesz transforms in higher dimensions, it is known that the two weight norm inequality with doubling measures is equivalent to the fractional one-tailed Muckenhoupt and $T1$ cube testing conditions; see [LaWi, Theorem 1.4] and [SaShUr9, Theorem 2.11]. Here a positive measure μ is doubling if

$$\int_{2Q} d\mu \lesssim \int_Q d\mu, \quad \text{all cubes } Q \subset \mathbb{R}^n.$$

However, these results rely on certain 'positivity' properties of the gradient of the kernel (which for the Hilbert transform kernel $\frac{1}{y-x}$ is simply $\frac{d}{dx}\frac{1}{y-x} > 0$ for $x \neq y$), something that is not available for general elliptic, or even strongly elliptic, fractional Calderón–Zygmund operators.

Our point of departure in this paper is the fact, easily proven below, that for doubling weights, certain weak analogues of the pivotal conditions of Nazarov,

[2] Indeed, if T is bounded on $L^2(w)$, then by duality it is also bounded on $L^2(\frac{1}{w})$, and the Stein–Weiss interpolation theorem with change of measure shows that T is bounded on unweighted $L^2(\mathbb{R}^n)$. Conversely, if T is bounded on unweighted $L^2(\mathbb{R}^n)$, the proof in [CoFe] shows that T is bounded on $L^2(w)$ using $w \in A_2$.

Treil and Volberg (often referred to now as NTV) [NTV4] are necessary, and this provides the framework for moving forward.[3] So we will assume that our weight pair (σ, ω) consists of doubling measures, and satisfies at least the classical A_2^α condition of Muckenhoupt, and often the one-tailed versions $\mathcal{A}_2^\alpha$ and $\mathcal{A}_2^{\alpha,*}$ in [SaShUr7]. The former condition is a necessary consequence of boundedness of any elliptic Calderón–Zygmund operator T, and the latter condition is necessary if T is strongly elliptic; see [SaShUr7].

Finally, we will at times also require that the doubling measures σ and ω are comparable in the sense of Coifman and Fefferman [CoFe], which means that the measures are mutually absolutely continuous, uniformly at all scales, i.e., there exist $0 < \beta, \gamma < 1$ such that

$$\frac{|E|_\sigma}{|Q|_\sigma} < \beta \Longrightarrow \frac{|E|_\omega}{|Q|_\omega} < \gamma \quad \text{for all Borel subsets } E \text{ of a cube } Q.$$

This condition is needed to prove the two weight bilinear Carleson Embedding Theorem 20 below, and conversely we show that our bilinear theorem implies it. The point is that if σ and ω are doubling and **comparable**, then a collection of dyadic cubes $\mathcal{F}$ is σ-Carleson if and only if $\mathcal{F}$ is ω-Carleson[4]—see the next section for these definitions.

Remark 1. We do not assume in this paper that the weight pair (σ, ω) satisfies the very strong energy conditions, something that is only necessary for boundedness of the Hilbert transform and its perturbations on the real line (see [SaShUr11] and [Saw]), nor the k-energy dispersed conditions introduced in [SaShUr10], which only hold for perturbations of Riesz transforms in higher dimensions.

The purpose of this paper is to consider measures σ and ω in $\mathbb{R}^n$ that are

- doubling,
- and satisfy the one-tailed Muckenhoupt conditions,

and then:

(1) to characterize the two weight norm inequality, for the class of elliptic α-fractional singular integral operators in $\mathbb{R}^n$, in terms of the $\mathcal{A}_2^\alpha$ conditions and the Indicator/Cube Testing conditions—and if in addition the measures are comparable in the sense of Coifman and Fefferman, then in terms of $\mathcal{A}_2^\alpha$, the Cube Testing conditions, and the Bilinear Indicator/Cube Testing property

[3]We do not know if the usual pivotal conditions hold for doubling measures that satisfy the Muckenhoupt conditions, and we thank Ignacio Uriarte–Tuero for bringing this to our attention by pointing to an error in a previous version of this paper.

[4]We thank Alex Tkachman for pointing to an error in the proof of an earlier incorrect version of Theorem 20, where comparability was needed but not assumed.

(often referred to now as the $\mathcal{BICT}$ property), which plays a role analogous to the weak boundedness property (1.1) with $\gamma = 0$ (see Theorem 6 below),

(2) to eliminate the $\mathcal{BICT}$ property when the measures are both doubling and comparable, with each of them satisfying either the A^{α}_{∞} condition, or the C_q condition of Muckenhoupt for some $q > 2$ (see [Muc] and [Saw1])[5]

   (a) and furthermore, in the case when the measures are A_∞ weights,[6] or, more generally, $C_{2+\varepsilon}$ weights or fractional A^{α}_{∞} measures, to give optimal cancellation conditions on a smooth Calderón–Zygmund kernel in order that there is an associated bounded operator from $L^2(\sigma)$ to $L^2(\omega)$, extending the smooth part of Theorem 4 in [Ste2, Section 3 of Chapter VII] (see Theorem 10 below)

(3) and to give a function theoretic consequence, namely that strong type is equivalent to weak type and dual weak type for elliptic operators; see Corollary 7 below. A one weight version of this result, with optimal A_2 dependence, was obtained by Pérez, Treil and Volberg [PeTrVo, Theorem 2.1].

1.1 Discussion of methodology. Since the weaker pivotal conditions, that can be derived from doubling measures, involve Poisson integrals whose tails have higher powers, we are led naturally to the use of the weighted Alpert wavelets in [RaSaWi], instead of the traditional Haar wavelets, having correspondingly higher order vanishing moments. In order to handle the global form associated with the operator, it suffices to use testing over polynomials times indicators of cubes. However, as pointed out in [RaSaWi], the weighted Alpert wavelets, unlike the weighted Haar wavelets, do not behave well with respect to the famous Paraproduct/Neighbor/Stopping (often referred to now as P/N/S) form decomposition of NTV (because the extension of a nonconstant polynomial from one cube to another is uncontrolled), and so we must divert to an alternate fork in the proof path using the Parallel Corona, in which independent stopping times are used for each function in a bilinear form, in order to handle the local form. In the absense of a P/N/S decomposition, this alternate fork then permits testing over polynomials times indicators of cubes, coupled with testing a bilinear indicator/cube testing property $\mathcal{BICT}$, taken over indicators of subsets of cubes on the left, rather than

[5] In [Ler] Lerner has introduced a strong SC_2 condition on a weight w which characterizes the related inequality $\|Tf\|_{L^{2,\infty}(w)} \lesssim \|Mf\|_{L^2(w)}$.

[6] In the case that both σ and ω are A_∞ weights satisfying $A_2(\sigma, \omega) < \infty$, C. Grigoriadis has shown in arXiv:2009.12091 that the classical pivotal conditions hold, resulting in a $T1$ theorem for nonsmooth kernels.

the cubes themselves,

$$\left|\int_E T^\alpha(\mathbf{1}_F\sigma)d\omega\right| \lesssim \sqrt{|Q|_\sigma |Q|_\omega}, \quad \text{Borel subsets } E, F \text{ of a cube } Q.$$

On the real line, a stronger conjecture is made in [RaSaWi] that the norm inequality holds if testing over these polynomials times indicators of intervals holds, in the presence of energy conditions, and that conjecture remains open at this time.

Moreover, in the proof of our theorem, we will need to bound the L^∞ norm of $L^2(\mu)$-projections onto the space of restrictions to Q of polynomials of degree less than κ (which is trivial when $\kappa = 1$), and for this we use the nondegeneracy conditions

$$\frac{1}{|Q|_\mu}\int_Q \left|P\Big(\frac{x-c_Q}{\ell(Q)}\Big)\right|^2 d\mu(x) \geq c > 0, \tag{1.2}$$

for all cubes Q and normalized polynomials P of degree less than κ, and with μ equal to either measure σ, ω. Such conditions permit control of off-diagonal terms by a Calderón–Zygmund stopping time and corona decomposition. We will see that (1.2) is implied by the doubling property for μ, and provided κ is large enough,[7] doubling is implied by (1.2), providing yet another instance of poor behavior of weighted Alpert wavelets, as compared to that for weighted Haar wavelets. Thus doubling conditions on the weights permit a proof of NTV type as in [NTV4], that both avoids the difficult control of functional energy in [LaSaShUr3] and [SaShUr7], and Lacey's deep breakthrough in controlling the stopping form [Lac], of course at the expense of including bilinear indicator/cube testing.

On the other hand, we are able to replace polynomial testing by the usual Cube Testing in the setting of doubling weights, and if we make one additional assumption, namely that the measures satisfy the fractional A_∞^α condition, then we can do away with the $\mathcal{BICT}$ property as well. Our approach to these results will follow the series of papers [Saw2], [Saw3] and [Saw4]:

(1) First we prove that the two weight norm inequality, for a general elliptic α-fractional Calderón–Zygmund singular integral with comparable doubling weights, is controlled by the classical A_2^α condition of Muckenhoupt, the two dual Polynomial/Cube Testing conditions (referred to now as P/CT), the Bilinear Indicator/Cube Testing property, and a certain weak boundedness property—this latter property is then removed using the doubling properties of the measures together with the $\mathcal{A}_2^\alpha$ conditions.

(2) Second, we replace the P/CT conditions in the previous theorem with the usual Cube Testing condition over indicators of cubes, assuming only that the one-tailed Muckenhoupt constants $\mathcal{A}_2^\alpha$ and $\mathcal{A}_2^{\alpha,*}$ are both finite.

[7]This restriction is removed in Sawyer and Uriarte–Tuero [SaUr].

(3) Third, we eliminate the $\mathcal{BICT}$ from each of the previous two theorems if one of the measures satisfies the fractional A_∞^α condition.
(4) Finally, we use the previous result to extend Stein's cancellation theorem to certain pairs of doubling measures satisfying the one-tailed $\mathcal{A}_2^\alpha$ conditions.

Acknowledgement. We thank the referee for a number of corrections and simplifications of arguments that greatly contribute to the readability of the paper.

2 Two weight $T1$ theorems and cancellation conditions for Calderón–Zygmund operators

Denote by $\mathcal{P}^n$ the collection of cubes in $\mathbb{R}^n$ having sides parallel to the coordinate axes. A positive locally finite Borel measure μ on $\mathbb{R}^n$ is said to satisfy the **doubling condition** if there is a pair of constants $(\beta, \gamma) \in (0,1)^2$, called doubling parameters, such that

$$|\beta Q|_\mu \geq \gamma |Q|_\mu, \quad \text{for all cubes } Q \in \mathcal{P}^n. \tag{2.1}$$

A familiar equivalent reformulation of (2.1) is that there is a positive constant C_{doub}, called the doubling constant, such that $|2Q|_\mu \leq C_{\text{doub}} |Q|_\mu$ for all cubes $Q \in \mathcal{P}^n$.

2.1 Conditions on measures and kernels. We begin with various conditions on measures and measure pairs, with the fractional A_∞^α condition being new, and the others classical. Then we recall ellipticity conditions for Calderón–Zygmund kernels.

2.1.1 The A_∞ and C_p conditions. The absolutely continuous measure

$$d\sigma(x) = s(x)dx$$

is said to be an A_∞ weight if there are constants $0 < \varepsilon, \eta < 1$, called A_∞ parameters, such that

$$\frac{|E|_\sigma}{|Q|_\sigma} < \eta \text{ whenever } E \text{ compact } \subset Q \text{ a cube with } \frac{|E|}{|Q|} < \varepsilon.$$

A useful reformulation given in [CoFe, Theorem III on page 244] is that there is $C > 0$ and an A_∞ exponent $\varepsilon > 0$ such that

$$\frac{|E|_\sigma}{|Q|_\sigma} \leq C \left(\frac{|E|}{|Q|} \right)^\varepsilon \text{ whenever } E \text{ compact } \subset Q \text{ a cube.} \tag{2.2}$$

Recall that there are doubling measures that are mutually singular with respect to Lebesgue measure; for a nice exposition see, e.g., [GaKiSc], and references given there.

Finally an absolutely continuous measure $d\sigma(x) = s(x)dx$ is said to be a C_p weight for $1 < p < \infty$ if

$$\frac{|E|_\sigma}{\int_{\mathbb{R}^n} |M\mathbf{1}_Q|^p d\sigma} \leq C\left(\frac{|E|}{|Q|}\right)^\varepsilon \text{ whenever } E \text{ compact } \subset Q \text{ a cube.}$$

2.1.2 Comparability of measures. A generalization of the A_∞ property to more general pairs (σ, ω) of doubling measures was also given by Coifman and Fefferman in [CoFe]: A pair (σ, ω) of doubling measures is comparable if there are constants $0 < \varepsilon, \eta < 1$, called comparability parameters, such that

$$\frac{|E|_\sigma}{|Q|_\sigma} < \eta \text{ whenever } E \text{ compact } \subset Q \text{ a cube with } \frac{|E|_\omega}{|Q|_\omega} < \varepsilon. \tag{2.3}$$

This condition is easily seen to be symmetric and the reformulation proved in [CoFe] is that there is $C > 0$ and a comparability exponent $\varepsilon > 0$ such that

$$\frac{|E|_\sigma}{|Q|_\sigma} \leq C\left(\frac{|E|_\omega}{|Q|_\omega}\right)^\varepsilon \text{ whenever } E \text{ compact } \subset Q \text{ a cube.} \tag{2.4}$$

A further reformulation is given in [Saw0, Theorem 3 (ii)] in terms of Carleson conditions, namely that a pair (σ, ω) of doubling measures is comparable if and only if

$$\|\mathcal{F}\|_{\mathrm{Car}(\sigma)} \leq C\|\mathcal{F}\|_{\mathrm{Car}(\omega)}, \quad \text{for all grids } \mathcal{F} \subset \mathcal{D}, \tag{2.5}$$

where for a positive locally finite Borel measure μ, we say that a grid $\mathcal{F} \subset \mathcal{D}$ is μ-Carleson if

$$\sum_{Q' \in \mathcal{F}: Q' \subset Q} |Q'|_\mu \leq C|Q|_\mu, \quad \text{for all } Q \in \mathcal{F},$$

and we define the 'Carleson norm' $\|\mathcal{F}\|_{\mathrm{Car}(\mu)}$ of $\mathcal{F}$ to be the infimum of such constants C. We repeat the simple proof here for the sake of completeness. Suppose (2.4) holds. Given $Q \in \mathcal{F}$, let $G_k(Q) \equiv \bigcup_{Q' \in \mathfrak{C}_{\mathcal{F}}^{(k)}} Q'$ be the union of the k^{th}-grandchildren of Q. Since the grid $\mathcal{F}$ satisfies a Carleson condition with respect to ω, there are positive constants $C, \delta > 0$ such that $|G_k(Q)|_\omega \leq C2^{-k\delta}|Q|_\omega$ (this is a well-known consequence of the Carleson condition dating back to a paper of Carleson). Then we have for $Q \in \mathcal{F}$,

$$\begin{aligned}\sum_{Q' \in \mathcal{F}: Q' \subset Q} |Q'|_\sigma &= \sum_{k=0}^\infty |G_k(Q)|_\sigma \leq \sum_{k=0}^\infty |Q|_\sigma C\left(\frac{|G_k(Q)|_\omega}{|Q|_\omega}\right)^\varepsilon \\ &\leq \sum_{k=0}^\infty |Q|_\sigma C(C2^{-k\delta})^\varepsilon \leq C|Q|_\sigma.\end{aligned}$$

Conversely, (2.5) implies (2.3) is easy.

Since comparability of doubling measures is symmetric, (2.5) is equivalent to its dual

$$\|\mathcal{F}\|_{\mathrm{Car}(\omega)} \leq C\|\mathcal{F}\|_{\mathrm{Car}(\sigma)}, \quad \text{for all grids } \mathcal{F} \subset \mathcal{D}. \tag{2.6}$$

This suggests the following extension of comparability to more general pairs of measures.

Definition 2. A pair (σ, ω) of positive locally finite Borel measures is said to be **comparable** if both (2.5) and (2.6) hold.

Note that the equivalence of (2.5) and (2.6) for pairs of doubling measures does not carry over to more general pairs of measures, which explains why we incorporate both conditions (2.5) and (2.6) in the definition of comparability.

2.1.3 The fractional A_∞^α condition.

In order to introduce the larger class of measures satisfying the fractional A_∞^α condition, we define a relative α-capacity $\mathbf{Cap}_\alpha(E;Q)$ of a compact subset E of a cube Q by

$$\begin{aligned}
&\mathbf{Cap}_\alpha(E;Q)\\
&\qquad \equiv \inf\left\{\int h(x)dx : h \geq 0, \operatorname{Supp} h \subset 2Q \text{ and } I_\alpha h \geq (\operatorname{diam} 2Q)^{\alpha-n} \text{ on } E\right\}\\
&\qquad \approx \inf\left\{|2Q|^{\frac{\alpha}{n}-1}\int_{2Q} h(x)dx : h \geq 0, \operatorname{Supp} h \subset 2Q \text{ and } I_\alpha h \geq 1 \text{ on } E\right\}.
\end{aligned}$$

This relative capacity is closely related to the $(\alpha, 1)$-capacity as defined, e.g., in Adams and Hedberg [AdHe], where numerous properties of capacities are developed. We now use this relative capacity to define a fractional A_∞^α condition (different than the A_∞^α condition appearing in [SaWh, page 818]).

Definition 3. A locally finite positive Borel measure ω is said to be an A_∞^α **measure** if

$$\frac{|E|_\omega}{|2Q|_\omega} \leq \eta(\mathbf{Cap}_\alpha(E;Q)), \quad \text{for all compact subsets } E \text{ of a cube } Q,$$
$$\text{for some function } \eta : [0,1] \to [0,1] \text{ with } \lim_{t\searrow 0}\eta(t) = 0.$$

Note that omitting the factor 2 in $|2Q|_\omega$ above makes the condition more restrictive in general, but remains equivalent for doubling measures. We let A_∞^0 be the class of A_∞ weights, so that statements involving both A_∞ and A_∞^α can be given together. Note the inequalities

$$c\left(\frac{|E|}{|Q|}\right)^{1-\frac{\alpha}{n}} \leq \mathbf{Cap}_\alpha(E;Q) \leq \mathbf{Cap}_\beta(E;Q), \quad 0 < \alpha < \beta < n,$$

which hold since

$$\begin{aligned}(\operatorname{diam} 2Q)^{\alpha-n}|E|^{1-\frac{\alpha}{n}} &\leq (\operatorname{diam} 2Q)^{\alpha-n}|\{I_\alpha h \geq (\operatorname{diam} 2Q)^{\alpha-n}\}|^{1-\frac{\alpha}{n}} \\ &\leq \|I_\alpha\|_{L^1\to L^{\frac{n}{n-\alpha},\infty}} \int h\end{aligned}$$

implies

$$\mathbf{Cap}_\alpha(E;Q) \geq \frac{(\operatorname{diam} 2Q)^{\alpha-n}|E|^{1-\frac{\alpha}{n}}}{\|I_\alpha\|_{L^1\to L^{\frac{n}{n-\alpha},\infty}}} = C\Big(\frac{|E|}{|Q|}\Big)^{1-\frac{\alpha}{n}},$$

and since

$$\frac{I_\alpha h(x)}{(\operatorname{diam} 2Q)^{\alpha-n}} = \int_{2Q}\Big(\frac{|x-y|}{\operatorname{diam} 2Q}\Big)^{\alpha-n} dy$$

is decreasing in α. It follows that

$$A_\infty \subset A_\infty^\alpha \subset A_\infty^\beta \quad \text{for } 0<\alpha<\beta<n.$$

2.1.4 The Muckenhoupt conditions.

Definition 4. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$, and denote by $\mathcal{P}^n$ the collection of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. For $0 \leq \alpha < n$, the **classical α-fractional Muckenhoupt condition** for the weight pair (σ, ω) is given by

$$A_2^\alpha(\sigma,\omega) \equiv \sup_{Q\in\mathcal{P}^n} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} < \infty, \tag{2.7}$$

and the corresponding **one-tailed conditions** by

$$\begin{aligned}\mathcal{A}_2^\alpha(\sigma,\omega) &\equiv \sup_{Q\in\mathcal{Q}^n} \mathcal{P}^\alpha(Q,\sigma)\frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} < \infty, \\ \mathcal{A}_2^{\alpha,*}(\sigma,\omega) &\equiv \sup_{Q\in\mathcal{Q}^n} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^\alpha(Q,\omega) < \infty,\end{aligned} \tag{2.8}$$

where the **reproducing Poisson integral** $\mathcal{P}^\alpha$ is given by

$$\mathcal{P}^\alpha(Q,\mu) \equiv \int_{\mathbb{R}^n}\Big(\frac{|Q|^{\frac{1}{n}}}{(|Q|^{\frac{1}{n}}+|x-x_Q|)^2}\Big)^{n-\alpha} d\mu(x).$$

2.1.5 Ellipticity of kernels. Finally, as in [SaShUr7], an α-fractional vector Calderón–Zygmund kernel $K^\alpha = (K_j^\alpha)$ is said to be **elliptic** if there is $c > 0$ such that for each unit vector $\mathbf{u} \in \mathbb{R}^n$ there is j satisfying

$$|K_j^\alpha(x, x+t\mathbf{u})| \geq c t^{\alpha-n}, \quad \text{for all } t > 0;$$

and $K^\alpha = (K_j^\alpha)$ is said to be **strongly elliptic** if for each $m \in \{1, -1\}^n$, there is a sequence of coefficients $\{\lambda_j^m\}_{j=1}^J$ such that

$$\left| \sum_{j=1}^{J} \lambda_j^m K_j^\alpha (x, x + t\mathbf{u}) \right| \geq c t^{\alpha - n}, \quad t \in \mathbb{R} \tag{2.9}$$

holds for all unit vectors $\mathbf{u}$ in the n-ant

$$V_m = \{x \in \mathbb{R}^n : m_i x_i > 0 \text{ for } 1 \leq i \leq n\}, \quad m \in \{1, -1\}^n.$$

For example, the vector Riesz transform kernel is strongly elliptic ([SaShUr7]).

2.2 Standard fractional singular integrals, the norm inequality and testing conditions.

Let $0 \leq \alpha < n$ and $\kappa_1, \kappa_2 \in \mathbb{N}$. We define a standard $(\kappa_1 + \delta, \kappa_2 + \delta)$-smooth α-fractional Calderón–Zygmund kernel $K^\alpha(x, y)$ to be a function

$$K^\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

satisfying the following fractional size and smoothness conditions: For $x \neq y$, and with ∇_1 denoting gradient in the first variable, and ∇_2 denoting gradient in the second variable,

$$\begin{aligned} |\nabla_1^j K^\alpha(x, y)| &\leq C_{CZ} |x - y|^{\alpha - j - n - 1}, \qquad 0 \leq j \leq \kappa_1, \\ |\nabla_1^\kappa K^\alpha(x, y) - \nabla_1^\kappa K^\alpha(x', y)| &\leq C_{CZ} \left(\frac{|x - x'|}{|x - y|} \right)^\delta |x - y|^{\alpha - \kappa_1 - n - 1}, \qquad \frac{|x - x'|}{|x - y|} \leq \frac{1}{2}, \end{aligned} \tag{2.10}$$

and where the same inequalities hold for the adjoint kernel

$$K^{\alpha,*}(x, y) \equiv K^\alpha(y, x),$$

in which x and y are interchanged, and where κ_1 is replaced by κ_2, and ∇_1 by ∇_2.

2.2.1 Defining the norm inequality.

We now turn to a precise definition of the weighted norm inequality

$$\|T_\sigma^\alpha f\|_{L^2(\omega)} \leq \mathfrak{N}_{T^\alpha} \|f\|_{L^2(\sigma)}, \quad f \in L^2(\sigma), \tag{2.11}$$

where of course $L^2(\sigma)$ is the Hilbert space consisting of those functions $f : \mathbb{R}^n \to \mathbb{R}$ for which

$$\|f\|_{L^2(\sigma)} \equiv \sqrt{\int_{\mathbb{R}^n} |f(x)|^2 d\sigma(x)} < \infty,$$

and equipped with the usual inner product. A similar definition holds for $L^2(\omega)$. For a precise definition of (2.11), it is possible to proceed with the notion of associating operators and kernels through the identity (3.6), and more simply by using the notion of restricted boundedness introduced by Liaw and Treil in [LiTr, see Theorem 3.4]. However, we choose to follow the approach in [SaShUr9, see page 314]. So we suppose that K^α is a standard $(\kappa_1+\delta, \kappa_2+\delta)$-smooth α-fractional Calderón–Zygmund kernel, and we introduce a family $\{\eta^\alpha_{\delta,R}\}_{0<\delta<R<\infty}$ of nonnegative functions on $[0,\infty)$ so that the truncated kernels

$$K^\alpha_{\delta,R}(x,y) = \eta^\alpha_{\delta,R}(|x-y|)K^\alpha(x,y)$$

are bounded with compact support for fixed x or y, and uniformly satisfy (2.10). Then the truncated operators

$$T^\alpha_{\sigma,\delta,R}f(x) \equiv \int_{\mathbb{R}^n} K^\alpha_{\delta,R}(x,y)f(y)d\sigma(y), \quad x\in\mathbb{R}^n,$$

are pointwise well-defined, and we will refer to the pair $(K^\alpha, \{\eta^\alpha_{\delta,R}\}_{0<\delta<R<\infty})$ as an α-fractional singular integral operator, which we typically denote by T^α, suppressing the dependence on the truncations. We also consider vector kernels $K^\alpha = (K^\alpha_j)$ where each K^α_j is as above, often without explicit mention. This includes, for example, the vector Riesz transform in higher dimensions.

Definition 5. We say that an α-fractional singular integral operator

$$T^\alpha = (K^\alpha, \{\eta^\alpha_{\delta,R}\}_{0<\delta<R<\infty})$$

satisfies the **norm inequality** (2.11) provided

$$\|T^\alpha_{\sigma,\delta,R}f\|_{L^2(\omega)} \le \mathfrak{N}_{T^\alpha}(\sigma,\omega)\|f\|_{L^2(\sigma)}, \quad f\in L^2(\sigma), 0<\delta<R<\infty.$$

Independence of Truncations: In the presence of the classical Muckenhoupt condition A^α_2, the norm inequality (2.11) is essentially independent of the choice of truncations used, including nonsmooth truncations as well; see [LaSaShUr3]. However, in dealing with the Monotonicity Lemma 27 below, where κ^{th} order Taylor approximations are made on the truncated kernels, it is necessary to use sufficiently smooth truncations. Similar comments apply to the Bilinear Indicator/Cube Testing property (2.15) and the Indicator/Cube Testing conditions (2.14), as well as to the κ-cube testing conditions (6.3) used later in the proof.

The weak type norms $\mathfrak{N}_{\text{weak}\,T^\alpha}(\sigma,\omega)$ and $\mathfrak{N}_{\text{weak}\,T^{\alpha,*}}(\omega,\sigma)$ are the best constants in the inequalities

$$\begin{aligned} &\|T^\alpha_\sigma f\|_{L^{2,\infty}(\omega)} \le \mathfrak{N}_{\text{weak}\,T^\alpha}(\sigma,\omega)\|f\|_{L^2(\sigma)} \text{ and} \\ &\|T^{\alpha,*}_\omega f\|_{L^{2,\infty}(\sigma)} \le \mathfrak{N}_{\text{weak}\,T^{\alpha,*}}(\omega,\sigma)\|f\|_{L^2(\omega)}. \end{aligned} \tag{2.12}$$

2.3 Cube Testing. The **κ-cube testing conditions** associated with an α-fractional singular integral operator T^α introduced in [RaSaWi] are given by

$$(2.13)\quad \begin{aligned} \left(\mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega)\right)^2 &\equiv \sup_{Q\in\mathcal{P}^n}\max_{0\le|\beta|<\kappa}\frac{1}{|Q|_\sigma}\int_Q \left|T_\sigma^\alpha\left(\mathbf{1}_Q m_Q^\beta\right)\right|^2\omega<\infty,\\ \left(\mathfrak{T}_{(T^\alpha)^*}^{(\kappa)}(\omega,\sigma)\right)^2 &\equiv \sup_{Q\in\mathcal{P}^n}\max_{0\le|\beta|<\kappa}\frac{1}{|Q|_\omega}\int_Q \left|\left(T_\sigma^\alpha\right)^*\left(\mathbf{1}_Q m_Q^\beta\right)\right|^2\sigma<\infty, \end{aligned}$$

with $m_Q^\beta(x)\equiv\left(\frac{x-c_Q}{\ell(Q)}\right)^\beta$ for any cube Q and multiindex β, where c_Q is the center of the cube Q, and where we interpret the right-hand sides as holding uniformly over all sufficiently smooth truncations of T^α. Equivalently, in the presence of A_2^α, we can take a single suitable truncation; see Independence of Truncations in Subsubsection 2.2.1 above. We will also use the larger full κ-cube testing conditions in which the integrals over Q are extended to the whole space $\mathbb{R}^n$:

$$\begin{aligned} \left(\mathfrak{F}\mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega)\right)^2 &\equiv \sup_{Q\in\mathcal{P}^n}\max_{0\le|\beta|<\kappa}\frac{1}{|Q|_\sigma}\int_{\mathbb{R}^n} \left|T_\sigma^\alpha\left(\mathbf{1}_Q m_Q^\beta\right)\right|^2\omega<\infty,\\ \left(\mathfrak{F}\mathfrak{T}_{(T^\alpha)^*}^{(\kappa)}(\omega,\sigma)\right)^2 &\equiv \sup_{Q\in\mathcal{P}^n}\max_{0\le|\beta|<\kappa}\frac{1}{|Q|_\omega}\int_{\mathbb{R}^n} \left|\left(T^\alpha\right)_\omega^*\left(\mathbf{1}_Q m_Q^\beta\right)\right|^2\sigma<\infty. \end{aligned}$$

We only use the case $\kappa=1$ in the statements of the four theorems in the next section, and so we will drop the superscript (κ) when $\kappa=1$, e.g., $\mathfrak{T}_{T^\alpha}=\mathfrak{T}_{T^\alpha}^{(1)}$ and $\mathfrak{T}_{(T^\alpha)^*}=\mathfrak{T}_{(T^\alpha)^*}^{(1)}$.

Finally, we define the **Indicator/Cube Testing constants** by

$$(2.14)\quad \begin{aligned} \left(\mathfrak{T}_{T^\alpha}^{IC}(\sigma,\omega)\right)^2 &\equiv \sup_{E\subset Q\in\mathcal{P}^n}\frac{1}{|Q|_\sigma}\int_Q \left|T^\alpha\left(\mathbf{1}_E\sigma\right)\right|^2\omega<\infty,\\ \left(\mathfrak{T}_{(T^\alpha)^*}^{IC}(\omega,\sigma)\right)^2 &\equiv \sup_{E\subset Q\in\mathcal{P}^n}\frac{1}{|Q|_\omega}\int_Q \left|\left(T^\alpha\right)^*\left(\mathbf{1}_E\omega\right)\right|^2\sigma<\infty, \end{aligned}$$

which are larger than the κ-cube testing conditions.

2.4 Bilinear Indicator/Cube Testing. Here we introduce a variant of the weak boundedness property of David and Journe in (1.1), but stronger because we take $\eta=0$ in (1.1)) The Bilinear Indicator/Cube Testing property is

$$(2.15)\quad \mathcal{BICT}_{T^\alpha}(\sigma,\omega)\equiv\sup_{Q\in\mathcal{P}^n}\sup_{E,F\subset Q}\frac{1}{\sqrt{|Q|_\sigma|Q|_\omega}}\left|\int_F T_\sigma^\alpha\left(\mathbf{1}_E\right)\omega\right|<\infty,$$

where the second supremum is taken over all compact sets E and F contained in a cube Q. Note in particular that the bilinear indicator/cube testing property $\mathcal{BICT}_{T^\alpha}(\sigma,\omega)<\infty$ is restricted to considering the same cube Q for each measure σ

and ω—in contrast to the weak boundedness property $\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)} < \infty$ in (6.4) below, that takes the supremum of the inner product over pairs of nearby disjoint cubes Q, Q'. However, in the setting of doubling measures, the latter constant $\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}$ can be controlled by κ^{th}-order testing and the one-tailed Muckenhoupt condition $\mathcal{A}_2^\alpha$ since the cube pairs are disjoint, and hence $\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}$ is removable. On the other hand, the former constant $\mathcal{BICT}_{T^\alpha}$ cannot be controlled in the same way since the cubes coincide, and we are only able to remove $\mathcal{BICT}_{T^\alpha}$ if one of the measures is an A_∞^α or $C_{2+\varepsilon}$ measure.

3 The four main theorems

Here is our general $T1$ theorem for doubling measures, first with indicator/cube testing, then with cube testing and a bilinear indicator/cube testing property when in the addition the measures are comparable. See Lemma 18 below for the definition of the doubling exponent, whose only role here is to determine the degree of smoothness required of the kernel K^α below.

Theorem 6. *Suppose* $0 \leq \alpha < n$, *and* $\kappa_1, \kappa_2 \in \mathbb{N}$ *and* $0 < \delta < 1$. *Let* T^α *be an* α*-fractional Calderón–Zygmund singular integral operator on* $\mathbb{R}^n$ *with a standard* $(\kappa_1 + \delta, \kappa_2 + \delta)$*-smooth* α*-fractional kernel* K^α. *Assume that* σ *and* ω *are doubling measures on* $\mathbb{R}^n$ *with doubling exponents* θ_1 *and* θ_2 *respectively satisfying*

$$\kappa_1 > \theta_1 + \alpha - n \quad \text{and} \quad \kappa_2 > \theta_2 + \alpha - n.$$

Set

$$T_\sigma^\alpha f = T^\alpha(f\sigma)$$

for any smooth truncation of T^α.

Then

$$\mathfrak{N}_{T^\alpha}(\sigma,\omega) \leq C_{\alpha,n}\Big(\sqrt{\mathcal{A}_2^\alpha(\sigma,\omega)+\mathcal{A}_2^\alpha(\omega,\sigma)} + \mathfrak{T}_{T^\alpha}^{IC}(\sigma,\omega) + \mathfrak{T}_{(T^\alpha)^*}^{IC}(\omega,\sigma)\Big), \tag{3.1}$$

and if in addition σ *and* ω *are comparable doubling measures on* $\mathbb{R}^n$ *in the sense of Coifman and Fefferman, then*

$$\begin{aligned}\mathfrak{N}_{T^\alpha}&(\sigma,\omega)\\ &\leq C'_{\alpha,n}\Big(\sqrt{\mathcal{A}_2^\alpha(\sigma,\omega)+\mathcal{A}_2^\alpha(\omega,\sigma)}\\ &\qquad + \mathfrak{T}_{T^\alpha}(\sigma,\omega) + \mathfrak{T}_{(T^\alpha)^*}(\omega,\sigma) + \mathcal{BICT}_{T^\alpha}(\sigma,\omega)\Big),\end{aligned} \tag{3.2}$$

where the constant $C_{\alpha,n}$ *depends also on* C_{CZ} *in* (2.10) *and the doubling exponents* θ_1 *and* θ_2, *and where* $C'_{\alpha,n}$ *also depends on the comparability constants in* (2.3).

Now $\mathcal{A}_2^\alpha$ is necessary for boundedness of a strongly elliptic operator as defined in [SaShUr7]; see also Liaw and Triel [LiTr, Theorem 5.1]. Thus we obtain the following corollary.

Corollary 7. *If, in addition to the hypotheses of Theorem 6, we assume the operator T^α is strongly elliptic, then we can reverse the inequalities in both* (3.1) *and* (3.2)*, i.e.,*

$$\mathfrak{N}_{T^\alpha}(\sigma,\omega) \approx \sqrt{\mathcal{A}_2^\alpha(\sigma,\omega)+\mathcal{A}_2^\alpha(\omega,\sigma)} + \mathfrak{T}_{T^\alpha}^{IC}(\sigma,\omega) + \mathfrak{T}_{(T^\alpha)^*}^{IC}(\omega,\sigma),$$

and if in addition σ and ω are comparable doubling measures on $\mathbb{R}^n$ in the sense of Coifman and Fefferman,

$$\begin{aligned}\mathfrak{N}_{T^\alpha}&(\sigma,\omega)\\ &\approx \sqrt{\mathcal{A}_2^\alpha(\sigma,\omega)+\mathcal{A}_2^\alpha(\omega,\sigma)} + \mathfrak{T}_{T^\alpha}(\sigma,\omega) + \mathfrak{T}_{(T^\alpha)^*}(\omega,\sigma) + \mathfrak{BICT}_{T^\alpha}(\sigma,\omega).\end{aligned} \tag{3.3}$$

In particular, assuming just doubling without comparability of the measures, we have the equivalence of the strong type inequality (2.11) *with both weak type inequalities* (2.12)*, i.e.,*

$$\mathfrak{N}_{T^\alpha}(\sigma,\omega) \approx \mathfrak{N}_{\mathrm{weak}\,T^\alpha}(\sigma,\omega) + \mathfrak{N}_{\mathrm{weak}\,T^{\alpha,*}}(\omega,\sigma).$$

Remark 8. As mentioned earlier, for operators with a partial reversal of energy, it is already known that, for doubling measures, the norm inequalities are characterized by one-tailed Muckenhoupt conditions and the usual $T1$ testing conditions taken over indicators of cubes; see [LaWi] and [SaShUr9]. However, energy reversal fails spectacularly for elliptic operators in general (see [SaShUr4]) and even the weaker energy condition itself fails to be necessary for boundedness of the fractional Riesz transforms with respect to general measures [Saw].

3.1 Elimination of the $\mathfrak{BICT}$.

The following $T1$ theorem provides a Cube Testing extension of the $T1$ theorem of David and Journé [DaJo] to a pair of comparable doubling measures when one of them, hence each of them,[8] satisfies the A_∞ or more generally the A_∞^α condition (and provided the operator is bounded on unweighted $L^2(\mathbb{R}^n)$ when $\alpha=0$).

Theorem 9. *Suppose $0 \le \alpha < n$, and $\kappa_1,\kappa_2 \in \mathbb{N}$ and $0<\delta<1$. Let T^α be an α-fractional Calderón–Zygmund singular integral operator on $\mathbb{R}^n$ with a standard $(\kappa_1+\delta,\kappa_2+\delta)$-smooth α-fractional kernel K^α, and when $\alpha=0$, suppose*

[8] Since if ω is comparable to an A_∞^α measure σ, then $\frac{|E|_\omega}{|Q|_\omega} \le C\left(\frac{|E|_\sigma}{|Q|_\sigma}\right)^\varepsilon \le C\eta(\mathbf{Cap}_\alpha(E;Q))^\varepsilon$ shows that $\omega \in A_\infty^\alpha$.

that T^0 is bounded on unweighted $L^2(\mathbb{R}^n)$. Assume that σ and ω are comparable doubling measures on $\mathbb{R}^n$ that satisfy the one-tailed Muckenhoupt conditions, and with doubling exponents θ_1 and θ_2 respectively satisfying

$$\kappa_1 > \theta_1 + \alpha - n \quad \textit{and} \quad \kappa_2 > \theta_2 + \alpha - n.$$

Furthermore, suppose that the measures are A_∞ weights or more generally,

> *either at least one, and hence each, of σ, ω satisfies the A_∞^α condition,*
> *or at least one of σ, ω is a C_q weight, for some $q > 2$.*

Set

$$T_\sigma^\alpha f = T^\alpha(f\sigma)$$

for any smooth truncation of T^α.

Then

$$\mathfrak{N}_{T^\alpha}(\sigma, \omega) \le C\Big(\sqrt{\mathcal{A}_2^\alpha(\sigma, \omega) + \mathcal{A}_2^\alpha(\omega, \sigma)} + \mathfrak{T}_{T^\alpha}(\sigma, \omega) + \mathfrak{T}_{(T^\alpha)^*}(\omega, \sigma)\Big), \tag{3.4}$$

where the constant C depends on C_{CZ} in (2.10)*, and the appropriate doubling, comparability, A_∞^α, and C_q constants. If T^α is elliptic, and also strongly elliptic if $\frac{n}{2} \le \alpha < n$, the inequality can be reversed.*

3.2 Optimal cancellation conditions for Calderón–Zygmund kernels. In the two weight setting of comparable doubling measures, we give an 'optimal cancellation' analogue of the $T1$ theorem for smooth kernels in the context of singular integrals as defined in [DaJo] or [Ste2, Section 3 of Chapter VII]. We now briefly recall that setup.

For $0 \le \alpha < n$, let T^α be a continuous linear map from rapidly decreasing smooth test functions $\mathcal{S}$ to tempered distributions in $\mathcal{S}'$, to which is associated a kernel $K^\alpha(x, y)$, defined when $x \ne y$, that satisfies the inequalities (more restrictive than those in (2.10) above)

$$|\partial_x^\beta \partial_y^\gamma K^\alpha(x, y)| \le A_{\alpha,\beta,\gamma,n}|x - y|^{\alpha-n-|\beta|-|\gamma|}, \quad \text{for all multiindices } \beta, \gamma; \tag{3.5}$$

such kernels are called **smooth** α-fractional Calderón–Zygmund kernels on $\mathbb{R}^n$. Here we say that an operator T^α is **associated** with a kernel K^α if, whenever $f \in \mathcal{S}$ has compact support, the tempered distribution $T^\alpha f$ can be identified, in the complement of the support, with the function obtained by integration with respect to the kernel, i.e.,

$$T^\alpha f(x) \equiv \int K^\alpha(x, y)f(y)d\sigma(y), \quad \text{for } x \in \mathbb{R}^n \setminus \operatorname{Supp} f. \tag{3.6}$$

The characterization in terms of (3.7) in the next theorem is identical to that in Stein [Ste2, Theorem 4 on page 306], except that the doubling measures σ and ω appear here in place of Legesgue measure in [Ste2], and the Euclidean distance function is replaced by the maximum distance function $\|y\| \equiv \max_{1\leq k\leq n} |y_k|$, whose associated balls are cubes.

Theorem 10. *Let $0 \leq \alpha < n$. Suppose that σ and ω are comparable doubling measures on $\mathbb{R}^n$ that satisfy the one-tailed Muckenhoupt conditions. Suppose also that the measure pair (σ, ω) satisfies the one-tailed conditions in* (2.8)*. Furthermore, suppose that the measures are A_∞ weights or more generally,*

either at least one, and hence each, of σ, ω satisfies the A_∞^α condition,

or at least one of σ, ω is a C_q weight, for some $q > 2$.

Suppose finally that $K^\alpha(x, y)$ is a smooth α-fractional Calderón–Zygmund kernel on $\mathbb{R}^n$. In the case $\alpha = 0$, we also assume there is T^0 associated with the kernel K^0 that is bounded on unweighted $L^2(\mathbb{R}^n)$.

Then there exists a bounded operator $T^\alpha : L^2(\sigma) \to L^2(\omega)$, that is associated with the kernel K^α in the sense that (3.6) *holds, if and only if there is a positive constant $\mathfrak{A}_{K^\alpha}(\sigma, \omega)$ so that*

$$\int_{\|x-x_0\|<N} \left| \int_{\varepsilon<\|x-y\|<N} K^\alpha(x, y) d\sigma(y) \right|^2 d\omega(x) \leq \mathfrak{A}_{K^\alpha}(\sigma, \omega) \int_{\|x_0-y\|<N} d\sigma(y), \tag{3.7}$$
$$\text{for all } 0 < \varepsilon < N \text{ and } x_0 \in \mathbb{R}^n,$$

along with a similar inequality with constant $\mathfrak{A}_{K^{\alpha,*}}(\omega, \sigma)$, in which the measures σ and ω are interchanged and $K^\alpha(x, y)$ is replaced by $K^{\alpha,*}(x, y) = K^\alpha(y, x)$. Moreover, if such T^α has minimal norm, then

$$\|T^\alpha\|_{L^2(\sigma)\to L^2(\omega)} \lesssim \mathfrak{A}_{K^\alpha}(\sigma, \omega) + \mathfrak{A}_{K^{\alpha,*}}(\omega, \sigma) + \sqrt{\mathcal{A}_2^\alpha(\sigma, \omega) + \mathcal{A}_2^\alpha(\omega, \sigma)}, \tag{3.8}$$

with implied constant depending on C_{CZ} in (2.10)*, and the appropriate doubling, A_∞, comparability, and C_q constants. If T^α is strongly elliptic, the inequality can be reversed.*

It should be noted that (3.7) is not simply the testing condition for a truncation of T over subsets of a cube, but instead has the historical form of bounding in some average sense, integrals of the kernel over annuli (of cubes here rather than balls). Nevertheless, this theorem is still a rather direct consequence of Theorem 6, with both doubling and A_2^α playing key roles. The reader can check that a more complicated form of Theorem 10 holds that involves bilinear indicator/cube testing if the A_∞^α conditions on α and ω are dropped.

3.3 The restricted weak type theorem with an A_∞^α measure. Here we eliminate the $\mathcal{BICT}$ from Theorem 6 when one of the measures satisfies either the fractional A_∞^α condition or the $C_{2+\varepsilon}$ condition.[9] Note that we do not assume comparability of measures here, and so conditions imposed on one measure no longer transfer automatically to the other measure. Let $T_\sigma^\alpha f = T^\alpha(f\sigma)$. We say that an α-fractional singular integral operator T^α satisfies the restricted weak type inequality relative to the measure pair (σ, ω) provided $T^\alpha : L^{2,1}(\sigma) \to L^{2,\infty}(\omega)$ where $L^{2,1}(\sigma)$ and $L^{2,\infty}(\omega)$ are the Lorentz spaces as defined, e.g., in [StWe2, page 188]. As shown in [StWe, see Theorem 3.13], this is equivalent to

$$\mathfrak{N}_{T^\alpha}^{\text{restricted weak}}(\sigma,\omega) \equiv \sup_{Q\in\mathcal{P}^n} \sup_{E,F\subset Q} \frac{1}{\sqrt{|E|_\sigma |F|_\omega}} \left| \int_F T_\sigma^\alpha(\mathbf{1}_E)\omega \right| < \infty, \tag{3.9}$$

where the second sup is taken over all compact subsets E, F of the cube Q, and where $0 < \delta < R < \infty$.

Thus we see that the $\mathcal{BICT}$ condition (2.15), having $|Q|_\sigma |Q|_\omega$ in the denominator, is implied by the restricted weak type condition (3.9), having the smaller $|E|_\sigma |F|_\omega$ in the denominator. In the presence of the classical Muckenhoupt condition A_2^α, the restricted weak type inequality in (3.9) is essentially independent of the choice of truncations used—see [LaSaShUr3].

Remark 11. In the special case $\alpha = 0$, we will make the additional assumption that T^0 is bounded on unweighted $L^2(\mathbb{R}^n)$. This is done in order to be able to use the weak type $(1, 1)$ result on Lebesgue measure for maximal truncations of such operators, that follows from standard Calderón–Zygmund theory as in [Ste2, Corollary 2 on page 36].

Theorem 12. *Let $0 \le \alpha < n$. Suppose that σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$ such that*

either at least one of σ, ω satisfies the A_∞^α condition,

or at least one of σ, ω is a C_q weight, for some $q > 2$.

Suppose also that T^α is a standard α-fractional Calderón–Zygmund singular integral in $\mathbb{R}^n$, and that when $\alpha = 0$ the operator T^0 is bounded on unweighted $L^2(\mathbb{R}^n)$. Then the two weight restricted weak type inequality for T^α relative to the measure pair (σ, ω) holds if the classical fractional Muckenhoupt constant A_2^α in (2.7) is finite. Moreover,

$$\mathcal{BICT}_{T^\alpha}(\sigma,\omega) \le \mathfrak{N}_{T^\alpha}^{\text{restricted weak}}(\sigma,\omega) \lesssim \sqrt{A_2^\alpha},$$

[9]Recently, the $\mathcal{BICT}$ has also been eliminated from Theorem 6 when the product measure $\sigma \times \omega$ has an appropriate reverse doubling exponent. See [SaUr].

and provided T^α is elliptic,

$$\mathfrak{BICT}_{T^\alpha}(\sigma,\omega) \approx \mathfrak{N}_{T^\alpha}^{\text{restricted weak}}(\sigma,\omega) \approx \sqrt{A_2^\alpha(\sigma,\omega)},$$

where the implied constants depend on the Calderón–Zygmund norm C_{CZ} in (2.10) *and the A_∞^α or $C_{2+\varepsilon}$ parameters of one of the measures.*

Remark 13. The proof of the theorem shows a bit more, namely that the restricted weak type norms of T^α and its maximal trunction operator $T_\flat^\alpha$ (see below) are equivalent under the hypotheses of the theorem, and including the fractional integral I^α (see below) when $0 < \alpha < n$.

4 Preliminaries

Here we introduce the κ^{th}-order pivotal conditions, recall the weighted Alpert wavelets from [RaSaWi], and establish some connections with doubling weights.

4.1 Necessity of the κ^{th} order Pivotal Condition for doubling weights. The smaller **fractional Poisson integrals** $\mathrm{P}_\kappa^\alpha(Q,\mu)$ used here, in [RaSaWi] and elsewhere, are given by

$$\mathrm{P}_\kappa^\alpha(Q,\mu) = \int_{\mathbb{R}^n} \frac{\ell(Q)^\kappa}{(\ell(Q)+|y-c_Q|)^{n+\kappa-\alpha}} d\mu(y), \quad \kappa \geq 1, \tag{4.1}$$

and the κ^{th}**-order fractional Pivotal Conditions** $\mathcal{V}_2^{\alpha,\kappa}, \mathcal{V}_2^{\alpha,\kappa,*} < \infty$, $\kappa \geq 1$, are given by

$$\begin{aligned} (\mathcal{V}_2^{\alpha,\kappa})^2 &= \sup_{Q\supset\dot{\cup}Q_r} \frac{1}{|Q|_\sigma} \sum_{r=1}^\infty \mathrm{P}_\kappa^\alpha(Q_r,\mathbf{1}_Q\sigma)^2 |Q_r|_\omega, \\ (\mathcal{V}_2^{\alpha,\kappa,*})^2 &= \sup_{Q\supset\dot{\cup}Q_r} \frac{1}{|Q|_\omega} \sum_{r=1}^\infty \mathrm{P}_\kappa^\alpha(Q_r,\mathbf{1}_Q\omega)^2 |Q_r|_\sigma, \end{aligned} \tag{4.2}$$

where the supremum is taken over all subdecompositions of a cube $Q \in \mathcal{P}^n$ into pairwise disjoint subcubes Q_r.

We begin with the elementary derivation of κ^{th}-order pivotal conditions from doubling assumptions. From Lemma 19 below, a doubling measure ω with doubling parameters $0 < \beta, \gamma < 1$ as in (2.1) has a 'doubling exponent' $\theta > 0$ and a positive constant c depending on β, γ that satisfy the condition

$$|2^{-j}Q|_\omega \geq c2^{-j\theta}|Q|_\omega, \quad \text{for all } j \in \mathbb{N}.$$

We can then exploit the doubling exponents $\theta = \theta(\beta, \gamma)$ of the doubling measures σ and ω in order to derive certain κ^{th}-order pivotal conditions $\mathcal{V}_2^{\alpha,\kappa}, \mathcal{V}_2^{\alpha,\kappa,*} < \infty$. Indeed, if ω has doubling exponent θ and $\kappa > \theta + \alpha - n$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus I} \frac{\ell(I)^\kappa}{(\ell(I) + |x - c_I|)^{n+\kappa-\alpha}} d\omega(x) \\
&\quad = \sum_{j=1}^{\infty} \ell(I)^{\alpha-n} \int_{2^j I \setminus 2^{j-1} I} \frac{1}{(1 + \frac{|x-c_I|}{\ell(I)})^{n+\kappa-\alpha}} d\omega(x) \\
&\quad \lesssim |I|^{\frac{\alpha}{n}-1} \sum_{j=1}^{\infty} 2^{-j(n+\kappa-\alpha)} |2^j I|_\omega \qquad (4.3) \\
&\quad \lesssim |I|^{\frac{\alpha}{n}-1} \sum_{j=1}^{\infty} 2^{-j(n+\kappa-\alpha)} \frac{1}{c 2^{-j\theta}} |I|_\omega \\
&\quad \le C_{n,\kappa,\alpha,(\beta,\gamma)} |I|^{\frac{\alpha}{n}-1} |I|_\omega,
\end{aligned}$$

provided $n + \kappa - \alpha - \theta > 0$, i.e., $\kappa > \theta + \alpha - n$. It follows that if $I \dot{\bigcup}_{r=1}^{\infty} I_r$ is a subdecomposition of I into pairwise disjoint cubes I_r, and $\kappa > \theta + \alpha - n$, then

$$\begin{aligned}
\sum_{r=1}^{\infty} \mathrm{P}_\kappa^\alpha(I_r, \omega)^2 |I_r|_\sigma &\lesssim \sum_{r=1}^{\infty} (|I_r|^{\frac{\alpha}{n}-1} |I_r|_\omega)^2 |I_r|_\sigma = \sum_{r=1}^{\infty} \frac{|I_r|_\omega |I_r|_\sigma}{|I|^{2(1-\frac{\alpha}{n})}} |I_r|_\omega \\
&\lesssim A_2^\alpha \sum_{r=1}^{\infty} |I_r|_\omega = A_2^\alpha |I|_\omega,
\end{aligned}$$

which gives

$$\mathcal{V}_2^{\alpha,\kappa,*} \le C_{\kappa,(\beta,\gamma)} A_2^\alpha, \qquad \kappa > \theta + \alpha - n, \tag{4.4}$$

where the constant $C_{\kappa,(\beta,\gamma)}$ depends on the doubling parameters (β, γ) and on κ. Thus the dual κ^{th}-order pivotal condition is controlled by A_2^α provided $\kappa + n - \alpha$ exceeds the doubling exponent of the measure ω. A similar result holds for $\mathcal{V}_2^{\alpha,\kappa}$ if $\kappa + n - \alpha$ exceeds the doubling exponent of σ.

Remark 14. The integers κ may have to be taken quite large depending on the doubling exponent of the doubling measures. In fact, the proof of Lemma 19 shows that we may take $\theta = \frac{\log_2 \frac{1}{\gamma}}{\log_2 \frac{1}{\beta}}$, and so we need $\kappa > \frac{\log_2 \frac{1}{\gamma}}{\log_2 \frac{1}{\beta}} + \alpha - n$, where β and γ are the doubling parameters for the measure. Since $C_{\text{doub}} = \frac{1}{\gamma}$ when $\beta = \frac{1}{2}$, we can equivalently write $\kappa > \log_2 C_{\text{doub}} + \alpha - n$, where $\log_2 C_{\text{doub}}$ can be thought of as the 'upper dimension' of the doubling measure. Indeed, in the case $\alpha = 0$ and $d\sigma(x) = d\omega(x) = dx$ on $\mathbb{R}^n$, we have $|\beta Q| = \beta^n |Q|$ implies

$$\theta = \frac{n \log_2 \frac{1}{\beta}}{\log_2 \frac{1}{\beta}} = n.$$

4.2 Weighted Alpert bases for $L^2(\mu)$ and L^∞ control of projections. The proof of Theorem 6 will require weighted wavelets with higher vanishing moments in order to accommodate the Poisson integrals with smaller tails. We now briefly recall the construction of weighted Alpert wavelets in [RaSaWi]. Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$, and fix $\kappa \in \mathbb{N}$. For $Q \in \mathcal{P}^n$, the collection of cubes with sides parallel to the coordinate axes, denote by $L^2_{Q;k}(\mu)$ the finite-dimensional subspace of $L^2(\mu)$ that consists of linear combinations of the indicators of the children $\mathfrak{C}(Q)$ of Q multiplied by polynomials of degree less than κ, and such that the linear combinations have vanishing μ-moments on the cube Q up to order $\kappa - 1$:

$$L^2_{Q;\kappa}(\mu) \equiv \left\{ f = \sum_{Q' \in \mathfrak{C}(Q)} \mathbf{1}_{Q'} p_{Q';\kappa}(x) : \int_Q f(x) x^\beta d\mu(x) = 0, \text{ for } 0 \le |\beta| < \kappa \right\},$$

where

$$p_{Q';\kappa}(x) = \sum_{\beta \in \mathbb{Z}^n_+ : |\beta| \le \kappa - 1} a_{Q';\alpha} x^\beta$$

is a polynomial in $\mathbb{R}^n$ of degree $|\beta| = \beta_1 + \cdots + \beta_n$ less than κ. Here $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$. Let $d_{Q;\kappa} \equiv \dim L^2_{Q;\kappa}(\mu)$ be the dimension of the finite-dimensional linear space $L^2_{Q;\kappa}(\mu)$. Now define

$$\mathcal{F}^\kappa_\infty(\mu) \equiv \{ \beta \in \mathbb{Z}^n_+ : |\beta| \le \kappa - 1 : x^\beta \in L^2(\mu) \}, \quad \text{and}$$
$$\mathcal{P}^n_{\mathbb{R}^n;\kappa}(\mu) \equiv \operatorname{Span}\{ x^\beta \}_{\beta \in \mathcal{F}^\kappa_\infty}.$$

Let $\triangle^\mu_{Q;\kappa}$ denote orthogonal projection onto the finite-dimensional subspace $L^2_{Q;\kappa}(\mu)$, let $\mathbb{E}^\mu_{Q;\kappa}$ denote orthogonal projection onto the finite-dimensional subspace

$$\mathcal{P}^n_{Q;\kappa}(\sigma) \equiv \operatorname{Span}\{ \mathbf{1}_Q x^\beta : 0 \le |\beta| < \kappa \},$$

and let $\triangle^\mu_{\mathbb{R}^n;\kappa}$ denote orthogonal projection onto $\mathcal{P}^n_{\mathbb{R}^n;\kappa}(\mu)$.

The following theorem was proved in [RaSaWi], which establishes the existence of Alpert wavelets, for $L^2(\mu)$ in all dimensions, having the three important properties of orthogonality, telescoping and moment vanishing.

Theorem 15 (Weighted Alpert Bases)**.** *Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$, fix $\kappa \in \mathbb{N}$, and fix a dyadic grid $\mathcal{D}$ in $\mathbb{R}^n$.*

(1) *Then $\{ \triangle^\mu_{\mathbb{R}^n;\kappa} \} \cup \{ \triangle^\mu_{Q;\kappa} \}_{Q \in \mathcal{D}}$ is a complete set of orthogonal projections in $L^2_{\mathbb{R}^n}(\mu)$ and*

$$f = \triangle^\mu_{\mathbb{R}^n;\kappa} f + \sum_{Q \in \mathcal{D}} \triangle^\mu_{Q;\kappa} f, \qquad f \in L^2_{\mathbb{R}^n}(\mu),$$

$$\langle \triangle^\mu_{\mathbb{R}^n;\kappa} f, \triangle^\mu_{Q;\kappa} f \rangle = \langle \triangle^\mu_{P;\kappa} f, \triangle^\mu_{Q;\kappa} f \rangle = 0 \qquad \text{for } P \ne Q,$$

where convergence in the first line holds both in $L^2_{\mathbb{R}^n}(\mu)$ norm and pointwise μ-almost everywhere.

(2) *Moreover, we have the telescoping identities*

$$\mathbf{1}_Q \sum_{Q\subsetneqq I\subset P} \triangle^{\mu}_{I;\kappa} = \mathbb{E}^{\mu}_{Q;\kappa} - \mathbb{E}^{\mu}_{P;\kappa} \quad \textit{for } P, Q \in \mathcal{D} \textit{ with } Q \subsetneqq P, \tag{4.5}$$

(3) *and the moment vanishing conditions*

$$\int_{\mathbb{R}^n} \triangle^{\mu}_{Q;\kappa} f(x) x^{\beta} d\mu(x) = 0, \quad \textit{for } Q \in \mathcal{D},\ \beta \in \mathbb{Z}^n_+,\ 0 \le |\beta| < \kappa. \tag{4.6}$$

We can fix an orthonormal basis $\{h^{\mu,a}_{Q;\kappa}\}_{a\in\Gamma_{Q,n,\kappa}}$ of $L^2_{Q;\kappa}(\mu)$ where $\Gamma_{Q,n,\kappa}$ is a convenient finite index set. Then

$$\{h^{\mu,a}_{Q;\kappa}\}_{a\in\Gamma_{Q,n,\kappa} \text{ and } Q\in\mathcal{D}}$$

is an orthonormal basis for $L^2(\mu)$, with the understanding that we add an orthonormal basis of $\mathcal{P}^{\kappa}_{\mathbb{R}^n}(\mu)$ if it is nontrivial. In particular, we have from the theorem above that (at least when $\mathcal{P}^{\kappa}_{\mathbb{R}^n}(\mu) = \{0\}$)

$$\|f\|^2_{L^2(\mu)} = \sum_{Q\in\mathcal{D}} \| \triangle^{\mu}_Q f\|^2_{L^2(\mu)} = \sum_{Q\in\mathcal{D}} |\widehat{f}(Q)|^2,$$
$$|\widehat{f}(Q)|^2 \equiv \sum_{a\in\Gamma_{Q,n,\kappa}} |\langle f, h^{\mu,a}_Q\rangle_{\mu}|^2.$$

In the case $\kappa = 1$, this construction reduces to the familiar Haar wavelets, where with $\mathbb{E}^{\mu}_I = \mathbb{E}^{\mu,1}_I$ we have the following useful bound:

$$\|\mathbb{E}^{\mu}_I f\|_{L^{\infty}_I(\mu)} = \left\| \left\langle f, \frac{1}{\sqrt{|I|_{\mu}}}\mathbf{1}_I \right\rangle \frac{1}{\sqrt{|I|_{\mu}}}\mathbf{1}_I \right\|_{L^{\infty}_I(\mu)} = |E^{\mu}_I f| \le E^{\mu}_I |f|.$$

We will consider below an analogous bound for the Alpert projections $\mathbb{E}^{\mu}_{I;\kappa}$ when $\kappa > 1$, that is of the form

$$\|\mathbb{E}^{\mu,\kappa}_I f\|_{L^{\infty}_I(\mu)} \lesssim E^{\mu}_I |f|, \quad \text{for all } f \in L^1_{\text{loc}}(\mu). \tag{4.7}$$

This will require certain energy nondegeneracy conditions to be imposed on μ, which turn out to be essentially equivalent to doubling conditions (thus limiting our application of Alpert wavelets to doubling measures in this paper).

4.2.1 Doubling and energy nondegeneracy conditions.

We will need the following relation between energy nondegeneracy and doubling conditions. We say that a polynomial $P(y) = \sum_{0\le|\beta|<\kappa} c_{\beta} y^{\beta}$ of degree less than κ is **normalized** if

$$\sup_{y\in Q_0} |P(y)| = 1, \quad \text{where } Q_0 \equiv \prod_{i=1}^{n} \left[-\frac{1}{2}, \frac{1}{2}\right).$$

Remark 16. Since all norms on a finite-dimensional vector space are equivalent, we have

$$\|P\|_{L^\infty(Q_0)} \approx |P(0)| + \|\nabla P\|_{L^\infty(Q_0)}, \quad \deg P < \kappa, \tag{4.8}$$

with implicit constants depending only on n and κ, and so a compactness argument shows there is $\varepsilon_\kappa > 0$ such that for every normalized polynomial P of degree less than κ, there is a ball $B(y, \varepsilon_\kappa) \subset Q_0$ on which P is nonvanishing.

Definition 17. Denote by c_Q the center of the cube Q, and by $\ell(Q)$ its side length, and for any polynomial P set

$$P^Q(y) \equiv P(c_Q + \ell(Q)y).$$

We say that $P(x)$ is **Q-normalized** if P^Q is normalized. Denote by $(\mathcal{P}_\kappa^Q)_{\text{norm}}$ the set of Q-normalized polynomials of degree less than κ.

Thus a Q-normalized polynomial has its supremum norm on Q equal to 1. Recall from (2.1) that a locally finite positive Borel measure μ on $\mathbb{R}^n$ is **doubling** if there exist constants $0 < \beta, \gamma < 1$ such that

$$|\beta Q|_\mu \geq \gamma |Q|_\mu, \quad \text{for all cubes } Q \text{ in } \mathbb{R}^n. \tag{4.9}$$

Note that $\sup_{y \in Q_0} |P(y)| = \|\mathbf{1}_{Q_0} P\|_{L^\infty(\mu)}$ for any cube Q_0, polynomial P, and doubling measure μ. The following lemma on doubling measures is well known.

Lemma 18. *Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$. Then μ is doubling if and only if there exists a positive constant θ, called the doubling exponent, such that*

$$|2^{-k} Q|_\mu \geq 2^{-\theta k} |Q|_\mu, \quad \textit{for all cubes } Q \textit{ in } \mathbb{R}^n \textit{ and } k \in \mathbb{N}.$$

Proof. Suppose there are $0 < \beta, \gamma < 1$ such that $|\beta Q|_\mu \geq \gamma |Q|_\mu$ for all cubes Q in $\mathbb{R}^n$. Iteration of this inequality leads to $|\beta^j Q|_\mu \geq \gamma^j |Q|_\mu$. Now choose $t > 0$ so that $\beta \leq 2^{-t} < 2\beta$, which then gives

$$\begin{aligned}
|2^{-k} Q|_\sigma &= |(2^{-t})^{\frac{k}{t}} Q|_\sigma \geq |\beta^{\frac{k}{t}} Q|_\sigma \geq |\beta^{[\frac{k}{t}]} Q|_\sigma \\
&\geq \gamma^{[\frac{k}{t}]} |Q|_\sigma = 2^{-[\frac{k}{t}] \log_2 \frac{1}{\gamma}} |Q|_\sigma \geq 2^{-\frac{k}{t} \log_2 \frac{1}{\gamma}} |Q|_\sigma = 2^{-\theta k} |Q|_\sigma
\end{aligned}$$

with

$$\theta = \frac{\log_2 \frac{1}{\gamma}}{t} \geq \frac{\log_2 \frac{1}{\gamma}}{\log_2 \frac{1}{\beta}} > 0.$$

The converse statement is trivial with $\beta = \frac{1}{2}$ and $\gamma = 2^{-\theta} = \frac{1}{C_{\text{doub}}}$. □

The doubling exponent $\theta = \log_2 C_{\mathrm{doub}}$ can be thought of as the upper dimension of μ. Here now is the connection between doubling measures and energy degeneracy. We thank Ignacio Uriarte-Tuero for pointing to a gap in the proof of part (2) in the first version of this paper.

Lemma 19. *Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$.*

(1) *If μ is doubling on $\mathbb{R}^n$, then for every $\kappa \in \mathbb{N}$ there exists a positive constant C_κ such that*

$$|Q|_\mu \leq C_\kappa \int_Q |P(x)|^2 d\mu(x), \quad \textit{for all cubes } Q \textit{ in } \mathbb{R}^n, \tag{4.10}$$
$$\textit{and for all } Q\textit{-normalized polynomials } P \textit{ of degree less than } \kappa.$$

(2) *Conversely, if* (4.10) *holds for some positive integer $\kappa > 2n$, then μ is doubling.*

Proof. Fix a cube Q and a positive integer $\kappa \in \mathbb{N}$. By Remark 16, there is a positive integer $L = L(\kappa) \in \mathbb{N}$ with the property that for every Q-normalized polynomial P of degree less than κ on $\mathbb{R}^n$, at least one of the dyadic children $K \in \mathfrak{C}^{(L)}(Q)$ at level L beneath Q satisfies $3K \subset Q \setminus Z_P$, where Z_P is the zero set of the polynomial P. Furthermore, if P is a Q-normalized polynomial of degree less than κ, then $P^Q(y) \equiv P(c_Q + \ell(Q)y)$ is normalized and $P(x) = P^Q(\frac{x-c_Q}{\ell(Q)})$, and so we have from (4.8) the inequality

$$|P(x)| = \left|P^Q\Big(\frac{x-c_Q}{\ell(Q)}\Big)\right| \geq c\Big(\operatorname{dist}\Big(\frac{x-c_Q}{\ell(Q)}, Z_{P^Q}\Big)\Big)^\kappa = c\Big(\frac{\operatorname{dist}(x, Z_P)}{\ell(Q)}\Big)^\kappa, \quad x \in Q.$$

Moreover, $Q \subset 2^{L+1}K$, and hence we have the lower bound

$$\begin{aligned}\int_Q |P(x)|^2 d\sigma(x) &\geq c^2 \int_K \Big(\frac{\operatorname{dist}(x, Z_P)}{\ell(Q)}\Big)^{2\kappa} d\sigma(x) \geq c^2 \int_K \Big(\frac{\ell(K)}{\ell(Q)}\Big)^{2\kappa} d\sigma(x)\\ &= c^2 2^{-2\kappa L}|K|_\sigma \geq c^2 2^{-2\kappa L} 2^{-(L+1)\theta}|2^{L+1}K|_\sigma \geq c_\kappa |Q|_\sigma,\end{aligned}$$

where $c_\kappa = c^2 2^{-2\kappa L} 2^{-(L+1)\theta}$. Thus (4.10) holds with $C_\kappa = \frac{1}{c_\kappa}$.

Conversely, assume that (4.10) holds for some $\kappa > 2n$. Momentarily fix a cube Q. Then the polynomial

$$P(x) \equiv \prod_{i=1}^n \Big[1 - \Big(\frac{x_i - (c_Q)_i}{\ell(Q)}\Big)^2\Big]$$

is Q-normalized of degree less than κ, vanishes on the boundary of Q, and is 1 at the center c_Q of Q. Now using that $2n < \kappa$ in (4.10), there is $\beta < 1$, sufficiently close to 1, and independent of the cube Q, so that

$$|Q|_\mu \leq C_\kappa \int_Q |P|^2 d\mu = C_\kappa \left\{ \int_{Q\setminus\beta Q} |P|^2 d\mu + \int_{\beta Q} |P|^2 d\mu \right\}$$
$$\leq \frac{1}{2}|Q\setminus\beta Q|_\mu + C_\kappa|\beta Q|_\mu \leq \frac{1}{2}|Q|_\mu + C_\kappa|\beta Q|_\mu .$$

Thus we have

$$|Q|_\mu \leq 2C_\kappa|\beta Q|_\mu ,$$

which is (4.9) with $\gamma = \frac{1}{2C_\kappa}$. □

4.2.2 Control of Alpert projections. For $n, \kappa \in \mathbb{N}$, let $\mathcal{P}^n_\kappa$ denote the finite-dimensional vector space of real polynomials $P(x)$ on $\mathbb{R}^n$ with degree less than κ, i.e., $P(x) = \sum_{0\leq|\beta|<\kappa} c_\beta x^\beta$ where $\beta = (\beta_i)^n_{i=} \in \mathbb{Z}^n_+$ and $|\beta| = \sum^n_{i=1} \beta_i$. Then denote by $\mathcal{P}^n_{I;\kappa}$ the space of restrictions of polynomials in $\mathcal{P}^n_\kappa$ to the interval I, also denoted $\mathcal{P}^n_{I;\kappa}(\mu)$ when we wish to emphasize the underlying measure. Now let $\{b^j_{I;\kappa}\}^N_{j=1}$ be an orthonormal basis for $\mathcal{P}^n_{I;\kappa}$ with the inner product of $L^2(\mu)$. If we assume that μ is doubling, and define the polynomial P_j by

$$P_j(x) = \frac{1}{\|b^j_{I;\kappa}\|_{L^\infty_I(\mu)}} b^j_{I;\kappa}(x),$$

then $P_j \in (\mathcal{P}^I_\kappa)_{\text{norm}}$ is I-normalized, and so part (1) of Lemma 19 shows that

$$\frac{1}{\|b^j_{I;\kappa}\|^2_{L^\infty_I(\mu)}} = \int_I \left| \frac{1}{\|b^j_{I;\kappa}\|_{L^\infty_I(\mu)}} b^j_{I;\kappa}(x) \right|^2 d\mu(x) = \int_I |P_j(x)|^2 d\mu(x) \approx |I|_\mu .$$

This then gives (4.7):

$$\|\mathbb{E}^{\mu,\kappa}_I f\|_{L^\infty_I(\mu)} = \left\| \sum_{j=1}^N \langle f, b^j_{I;\kappa}\rangle b^j_{I;\kappa} \right\|_{L^\infty_I(\mu)} \leq \sum_{j=1}^N |\langle f, P_j\rangle| \|b^j_{I;\kappa}\|_{L^\infty_I(\mu)} \|b^j_{I;\kappa}\|_{L^\infty_I(\mu)}$$
$$\leq \sum_{j=1}^N \left(\int_I |f| d\mu \right) \|b^j_{I;\kappa}\|^2_{L^\infty_I(\mu)} \lesssim \sum_{j=1}^N \frac{1}{|I|_\mu} \int_I |f| d\mu = N\, E^\mu_I |f|.$$

We also record the following additional consequence of (4.10):

$$\|\mathbb{E}^{\mu,\kappa}_I f\|^2_{L^\infty_I(\mu)} |I|_\mu \lesssim \|\mathbb{E}^{\mu,\kappa}_I f\|^2_{L^2_I(\mu)}, \tag{4.11}$$

which follows from

$$\|\mathbb{E}^{\mu,\kappa}_I f\|^2_{L^\infty_I(\mu)} |I|_\mu \lesssim \left(\sum_{j=1}^N |\langle f, b^j_{I;\kappa}\rangle| \right)^2 \left(\max_{1\leq j\leq N} \|b^j_{I;\kappa}\|_{L^\infty_I(\mu)} \right)^2 |I|_\mu$$
$$\lesssim N \sum_{j=1}^N |\langle f, b^j_{I;\kappa}\rangle|^2 = N \|\mathbb{E}^{\mu,\kappa}_I f\|^2_{L^2_I(\mu)}.$$

4.3 A two weight bilinear Carleson Embedding Theorem. The classical Carleson Embedding Theorem [NTV4] states that for any dyadic grid $\mathcal{D}$, and any sequence $\{c_I\}_{I\in\mathcal{D}}$ of nonnegative numbers indexed by $\mathcal{D}$,

$$\sum_{I\in\mathcal{D}} c_I \left(\frac{1}{|I|_\sigma}\int_I f d\sigma\right)^2 \leq C\|f\|^2_{L^2(\sigma)} \tag{4.12}$$

for all nonnegative $f \in L^2(\sigma)$, if and only if the sequence $\{c_I\}_{I\in\mathcal{D}}$ satisfies a Carleson condition

$$\sum_{I\in\mathcal{D}:\ I\subset J} c_I \leq C'|J|_\sigma, \quad \text{for all } J \in \mathcal{D}. \tag{4.13}$$

Moreover, $C' \leq C \leq 4C'$. The two weight bilinear analogue of (4.12) is the inequality

$$\sum_{I\in\mathcal{D}} a_I \left(\frac{1}{|I|_\sigma}\int_I f d\sigma\right)\left(\frac{1}{|I|_\omega}\int_I g d\omega\right) \leq C\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}, \tag{4.14}$$

which is equivalent to the pair of Carleson-type conditions,

$$\sum_{I',I\in\mathcal{D}:\ I'\subset I\subset K} \frac{a_{I'}a_I}{|I|_\sigma} \leq C'|K|_\omega \text{ and } \sum_{I',I\in\mathcal{D}:\ I'\subset I\subset K} \frac{a_{I'}a_I}{|I|_\omega} \leq C'|K|_\sigma, \quad \text{for all cubes } K \in \mathcal{D}. \tag{4.15}$$

Indeed, (4.14) is equivalent to

$$\int\left|\sum_{I\in\mathcal{D}} \frac{a_I}{|I|_\sigma}\left(\frac{1}{|I|_\omega}\int_I g d\omega\right)\mathbf{1}_I(y)\right|^2 d\sigma(y) \leq C^2\|g\|^2_{L^2(\omega)},$$

which by [NTV] and [LaSaUr2] is equivalent to the pair of testing conditions

$$\begin{aligned} &\int_K\left|\sum_{I\in\mathcal{D}:\ I\subset K} \frac{a_I}{|I|_\sigma}\mathbf{1}_I(y)\right|^2 d\sigma(y) \leq C^2|K|_\omega\,, \quad \text{for all cubes } K \in \mathcal{D}, \\ &\int_K\left|\sum_{I\in\mathcal{D}:\ I\subset K} \frac{a_I}{|I|_\omega}\mathbf{1}_I(y)\right|^2 d\omega(y) \leq C^2|K|_\sigma, \quad \text{for all cubes } K \in \mathcal{D}. \end{aligned} \tag{4.16}$$

However, the Carleson-type conditions in (4.15) are too strong for our purposes in this paper, and instead, we prove a bilinear extension of the Carleson Embedding Theorem (related to the Bilinear Imbedding Theorem of Nazarov, Treil and Volberg in [NTV, page 915]) which uses the more familiar bilinear Carleson condition in (4.18) below—at the expense of assuming comparability of the measure pair as in Definition 2 above.

Given any subset $\mathcal{A}$ of the dyadic grid $\mathcal{D}$, we view $\mathcal{A}$ as a subtree of $\mathcal{D}$, and denote by $\mathfrak{C}_{\mathcal{A}}(A)$ the set of $\mathcal{A}$-children of A in the tree $\mathcal{A}$, and by $\mathcal{C}_{\mathcal{A}}(A)$ the $\mathcal{A}$-corona of A in the tree $\mathcal{A}$, so that

$$\mathcal{C}_{\mathcal{A}}(A) = \bigcup_{A' \in \mathfrak{C}_{\mathcal{A}}(A)} \{I \in \mathcal{D} : A' \subsetneqq I \subset A\}.$$

Theorem 20 (Two weight bilinear Carleson Embedding Theorem, cf. [NTV]). *Suppose σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$, and that $\mathcal{D}$ is a dyadic grid.*

(1) *Suppose further that $\{a_I\}_{I\in\mathcal{D}}$ is a sequence of nonnegative real numbers indexed by $\mathcal{D}$. If in addition σ and ω are comparable in the sense of Definition 2, then*

$$\sum_{I\in\mathcal{D}} a_I \left(\sup_{K\in\mathcal{D}:\ K\supset I} \frac{1}{|K|_\sigma} \int_K f d\sigma \right) \left(\sup_{L\in\mathcal{D}:\ L\supset I} \frac{1}{|L|_\omega} \int_L g d\omega \right) \leq C \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} \tag{4.17}$$

for all nonnegative $f \in L^2(\sigma)$ and nonnegative $g \in L^2(\omega)$, if and only if the sequence $\{a_I\}_{I\in\mathcal{D}}$ satisfies the bilinear Carleson condition,

$$\sum_{I\in\mathcal{D}:\ I\subset J} a_I \leq C' \sqrt{|J|_\sigma |J|_\omega}, \quad \text{for all } J \in \mathcal{D}, \tag{4.18}$$

where $C' \leq C \lesssim C'$.

(2) *The inequality*

$$\sum_{I\in\mathcal{D}} a_I \left(\frac{1}{|I|_\sigma} \int_I f d\sigma \right) \left(\frac{1}{|I|_\omega} \int_I g d\omega \right) \leq C \left\{ \frac{1}{\sqrt{|J|_\sigma |J|_\omega}} \sum_{I\in\mathcal{D}:\ I\subset J} a_I \right\} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} \tag{4.19}$$

holds if and only if σ and ω are comparable in the sense of Definition 2.

Proof. Part (1): The necessity of the bilinear Carleson condition follows upon setting $f = g = \mathbf{1}_J$ in the bilinear inequality, since then for $I \subset J$ we have

$$\sup_{K\in\mathcal{D}:\ K\supset I} \frac{1}{|K|_\sigma} \int_K f d\sigma \geq \frac{1}{|I|_\sigma} \int_I \mathbf{1}_J d\sigma = 1 \text{ and similarly } \sup_{L\in\mathcal{D}:\ L\supset I} \frac{1}{|L|_\omega} \int_L g d\omega \geq 1,$$

which gives

$$\sum_{I\in\mathcal{D}:\ I\subset J} a_I \leq C \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} = C \sqrt{|J|_\sigma |J|_\omega}.$$

For the converse assertion, fix $\Gamma \geq 4$, and let $\mathcal{A}$ be a collection of Γ-Calderón–Zygmund stopping cubes for $f \in L^2(\sigma)$, and let $\mathcal{B}$ be a collection of Γ-Calderón–Zygmund stopping cubes for $g \in L^2(\omega)$. Then we have

$$\begin{aligned} \frac{1}{|A'|_\sigma}\int_{A'} fd\sigma &> \Gamma \frac{1}{|A|_\sigma}\int_A fd\sigma, \quad A' \in \mathfrak{C}_{\mathcal{A}}(A), \\ \frac{1}{|I|_\sigma}\int_I fd\sigma &\leq \Gamma \frac{1}{|A|_\sigma}\int_A fd\sigma, \quad I \in \mathcal{C}_{\mathcal{A}}(A), \\ \sum_{A' \in \mathcal{A}:\ A' \subset A} |A'|_\sigma &\leq C_\Gamma |A|_\sigma, \end{aligned} \tag{4.20}$$

and similarly

$$\begin{aligned} \frac{1}{|B'|_\omega}\int_{B'} gd\omega &> \Gamma \frac{1}{|B|_\omega}\int_B gd\omega, \quad B' \in \mathfrak{C}_{\mathcal{B}}(B), \\ \frac{1}{|J|_\omega}\int_J gd\omega &\leq \Gamma \frac{1}{|B|_\omega}\int_B gd\omega, \quad J \in \mathcal{C}_{\mathcal{B}}(B), \\ \sum_{B' \in \mathcal{B}:\ B' \subset B} |B'|_\omega &\leq C_\Gamma |B|_\omega. \end{aligned}$$

Now we estimate the left-hand side of (4.17),

$$\begin{aligned} &\sum_{I \in \mathcal{D}} a_I \left(\sup_{K \in \mathcal{D}:\ K \supset I} \frac{1}{|K|_\sigma}\int_K fd\sigma \right)\left(\sup_{L \in \mathcal{D}:\ L \supset I} \frac{1}{|L|_\omega}\int_L gd\omega \right) \\ &= \sum_{A \in \mathcal{A}} \sum_{B \in \mathcal{B}} \sum_{I \in \mathcal{D}:\ I \in \mathcal{C}_{\mathcal{A}}(A) \cap \mathcal{C}_{\mathcal{B}}(B)} a_I \left(\sup_{K \in \mathcal{D}:\ K \supset I} \frac{1}{|K|_\sigma}\int_K fd\sigma \right)\left(\sup_{L \in \mathcal{D}:\ L \supset I} \frac{1}{|L|_\omega}\int_L gd\omega \right) \\ &\leq \Gamma^2 \sum_{A \in \mathcal{A}} \sum_{B \in \mathcal{B}} \left\{ \sum_{I \in \mathcal{D}:\ I \in \mathcal{C}_{\mathcal{A}}(A) \cap \mathcal{C}_{\mathcal{B}}(B)} a_I \right\} \left(\frac{1}{|A|_\sigma}\int_A fd\sigma \right)\left(\frac{1}{|B|_\omega}\int_B gd\omega \right). \end{aligned}$$

Since (4.18) implies

$$\sum_{I \in \mathcal{D}:\ I \in \mathcal{C}_{\mathcal{A}}(A) \cap \mathcal{C}_{\mathcal{B}}(B)} a_I \leq \begin{cases} C' \min\{\sqrt{|A|_\sigma |A|_\omega}, \sqrt{|B|_\sigma |B|_\omega}\} & \text{if } A \cap B \neq \emptyset, \\ 0 & \text{if } A \cap B = \emptyset, \end{cases}$$

we conclude that the left-hand side of (4.17) is at most

$$\begin{aligned} &C'\Gamma^2 \sum_{A \in \mathcal{A}} \sum_{B \in \mathcal{B}:\ B \in \mathcal{C}_{\mathcal{A}}(A)} \sqrt{|B|_\sigma |B|_\omega} \left(\frac{1}{|A|_\sigma}\int_A fd\sigma \right)\left(\frac{1}{|B|_\omega}\int_B gd\omega \right) \\ &+ C'\Gamma^2 \sum_{B \in \mathcal{B}} \sum_{A \in \mathcal{A}:\ A \in \mathcal{C}_{\mathcal{B}}(B)} \sqrt{|A|_\sigma |A|_\omega} \left(\frac{1}{|A|_\sigma}\int_A fd\sigma \right)\left(\frac{1}{|B|_\omega}\int_B gd\omega \right) \\ &\equiv S_1 + S_2. \end{aligned}$$

By symmetry it suffices to bound the first sum S_1. By Cauchy–Schwarz, we have

$$\sum_{B\in\mathcal{B}:\ B\in\mathcal{C}_{\mathcal{A}}(A)} \sqrt{|B|_\sigma |B|_\omega}\left(\frac{1}{|B|_\omega}\int_B g d\omega\right) \leq \sqrt{\sum_{B\in\mathcal{B}:\ B\in\mathcal{C}_{\mathcal{A}}(A)} |B|_\sigma}\sqrt{\sum_{B\in\mathcal{B}:\ B\in\mathcal{C}_{\mathcal{A}}(A)} |B|_\omega\left(\frac{1}{|B|_\omega}\int_B g d\omega\right)^2}.$$

We now invoke the comparability assumption on the measures σ and ω, which implies that the grid $\mathcal{B}$ is also σ-Carleson, hence $\sum_{B\in\mathcal{B}:\ B\in\mathcal{C}_{\mathcal{A}}(A)} |B|_\sigma \leq C|A|_\sigma$. Thus we conclude that

$$\begin{aligned}
S_1 &\leq C'\Gamma^2 \sum_{A\in\mathcal{A}}\left(\frac{1}{|A|_\sigma}\int_A f d\sigma\right)\sqrt{|A|_\sigma}\sqrt{\sum_{B\in\mathcal{B}:\ B\in\mathcal{C}_{\mathcal{A}}(A)} |B|_\omega\left(\frac{1}{|B|_\omega}\int_B g d\omega\right)^2} \\
&\leq C'\Gamma^2\sqrt{\sum_{A\in\mathcal{A}} |A|_\sigma\left(\frac{1}{|A|_\sigma}\int_A f d\sigma\right)^2}\sqrt{\sum_{A\in\mathcal{A}}\sum_{B\in\mathcal{B}:\ B\in\mathcal{C}_{\mathcal{A}}(A)} |B|_\omega\left(\frac{1}{|B|_\omega}\int_B g d\omega\right)^2} \\
&\leq C\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)},
\end{aligned}$$

with C depending on C' and Γ, upon applying the usual Carleson Embedding Theorem to both stopping collections $\mathcal{A}$ and $\mathcal{B}$. Indeed, we take

$$c_I \equiv \begin{cases} |I|_\sigma & \text{if } I \in \mathcal{A} \\ 0 & \text{if } I \notin \mathcal{A} \end{cases}$$

in (4.12), and note that $\{c_I\}_{I\in\mathcal{D}}$ satisfies the Carleson condition (4.13) with $C' = C_\Gamma$ by the third line in (4.20), it then follows from (4.12) that

$$\sum_{A\in\mathcal{A}} |A|_\sigma\left(\frac{1}{|A|_\sigma}\int_A f d\sigma\right)^2 \leq C_\Gamma\|f\|^2_{L^2(\sigma)}.$$

Similarly, we obtain

$$\sum_{A\in\mathcal{A}}\sum_{B\in\mathcal{B}:\ B\in\mathcal{C}_{\mathcal{A}}(A)} |B|_\omega\left(\frac{1}{|B|_\omega}\int_B g d\omega\right)^2 \leq C\|g\|^2_{L^2(\omega)}.$$

Part (2): It remains to show that if (4.19) holds, then σ and ω are comparable in the sense of Definition 2. So let the dyadic grid $\mathcal{F}$ be ω-Carleson, and let $C, \delta > 0$ be such that $|G_k(Q)|_\omega \leq C2^{-k\delta}|Q|_\omega$ for all $Q \in \mathcal{F}$. Define

$$a_I \equiv \begin{cases} \sqrt{|I|_\sigma |I|_\omega} & \text{if } I \in \mathcal{F} \\ 0 & \text{if } I \notin \mathcal{F} \end{cases}$$

and note that if $\{M_i\}_{i=1}^{\infty}$ are the maximal cubes in $\mathcal{F}$ that are contained in J, then

$$\begin{aligned}
\sum_{I\in\mathcal{D}:\ I\subset J} a_I &= \sum_{i=1}^{\infty}\sum_{I\in\mathcal{F}:\ I\subset M_i}\sqrt{|I|_\sigma|I|_\omega} = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\sum_{I\in\mathfrak{C}_{\mathcal{F}}^{(k)}(M_i)}\sqrt{|I|_\sigma|I|_\omega} \\
&\le \sum_{i=1}^{\infty}\sqrt{\sum_{k=0}^{\infty}2^{-k\varepsilon}\sum_{I\in\mathfrak{C}_{\mathcal{F}}^{(k)}(M_i)}|I|_\sigma}\sqrt{\sum_{k=0}^{\infty}2^{k\varepsilon}\sum_{I\in\mathfrak{C}_{\mathcal{F}}^{(k)}(M_i)}|I|_\omega} \\
&\le \sum_{i=1}^{\infty}\sqrt{\sum_{k=0}^{\infty}2^{-k\varepsilon}|M_i|_\sigma}\sqrt{\sum_{k=0}^{\infty}2^{k\varepsilon}C2^{-k\delta}|M_i|_\omega} \\
&\le C\sum_{i=1}^{\infty}\sqrt{|M_i|_\sigma|M_i|_\omega} \le C\sqrt{\sum_{i=1}^{\infty}|M_i|_\sigma}\sqrt{\sum_{i=1}^{\infty}|M_i|_\omega} \le C\sqrt{|J|_\sigma}\sqrt{|J|_\omega}.
\end{aligned}$$

Thus from (4.19) we obtain

$$\sum_{I\in\mathcal{F}}\sqrt{|I|_\sigma|I|_\omega}\left(\frac{1}{|I|_\sigma}\int_I fd\sigma\right)\left(\frac{1}{|I|_\omega}\int_I gd\omega\right) \le C\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)},$$

and then from (4.16) we conclude

$$\begin{aligned}
\sum_{I\in\mathcal{F}:\ I\subset K}|I|_\sigma &= \sum_{I\in\mathcal{F}:\ I\subset K}\frac{|I|_\omega|I|_\sigma}{|I|_\omega^2}|I|_\omega = \int_K\sum_{I\in\mathcal{F}:\ I\subset K}\left|\frac{\sqrt{|I|_\omega|I|_\sigma}}{|I|_\omega}\mathbf{1}_I(y)\right|^2 d\omega(y) \\
&\le \int_K\left|\sum_{I\in\mathcal{F}:\ I\subset K}\frac{\sqrt{|I|_\omega|I|_\sigma}}{|I|_\omega}\mathbf{1}_I(y)\right|^2 d\omega(y) \le C^2|K|_\sigma,
\end{aligned}$$

for all cubes $K\in\mathcal{F}$,

which is $\|\mathcal{F}\|_{\mathrm{Car}(\sigma)} \le C\|\mathcal{F}\|_{\mathrm{Car}(\omega)}$. A dual argument gives $\|\mathcal{F}\|_{\mathrm{Car}(\omega)} \le C\|\mathcal{F}\|_{\mathrm{Car}(\sigma)}$, and so σ and ω are comparable in the sense of Definition 2. □

5 Controlling polynomial testing conditions by $T1$ and $\mathcal{A}_2$

Here we show that the familiar $T1$ testing conditions over indicators of cubes imply the Tp testing conditions over polynomials times indicators of cubes. To highlight the main idea, we begin with the simpler case of dimension $n = 1$. We start with the elementary formula for recovering a linear function, restricted to an interval, from indicators of intervals:

$$\text{(5.1)}\qquad \mathbf{1}_{[a,b)}(y)\left(\frac{y-a}{b-a}\right) = \int_a^b \mathbf{1}_{[r,b)}(y)\frac{dr}{b-a}, \quad \text{for all } y\in\mathbb{R}.$$

We conclude that for any locally finite positive Borel measure σ, and any operator T bounded from $L^2(\sigma)$ to $L^2(\omega)$,

$$T_\sigma\Big(\mathbf{1}_{[a,b)}(y)\Big(\frac{y-a}{b-a}\Big)\Big)(x) = T_\sigma\bigg(\int_a^b \mathbf{1}_{[r,b)}(y)\frac{dr}{b-a}\bigg)(x) = \int_a^b (T_\sigma\mathbf{1}_{[r,b)})(x)\frac{dr}{b-a},$$

where T_σ has moved inside the integral since truncations of fractional Calderón–Zygmund operators have bounded compactly supported kernels. We then use the testing estimate

$$\|T_\sigma\mathbf{1}_{[r,b)}\|^2_{L^2(\omega)} \le (\mathfrak{F}\mathfrak{T}_T)^2|[r,b)|_\sigma,$$

together with Minkowski's inequality $\|\int f\| \le \int \|f\|$, to obtain

$$\begin{aligned}
&\Big\|T_\sigma\Big[\mathbf{1}_{[a,b)}(y)\Big(\frac{y-a}{b-a}\Big)\Big]\Big\|_{L^2(\omega)} = \bigg\|T_\sigma\bigg[\int_a^b \mathbf{1}_{[r,b)}(y)\frac{dr}{b-a}\bigg]\bigg\|_{L^2(\omega)}\\
&\le \int_a^b \|T_\sigma[\mathbf{1}_{[r,b)}(y)]\|_{L^2(\omega)}\frac{dr}{b-a} \le \int_a^b \mathfrak{F}\mathfrak{T}_T\sqrt{|[r,b)|_\sigma}\frac{dr}{b-a}\\
&\le \mathfrak{F}\mathfrak{T}_T\sqrt{\int_a^b |[r,b)|_\sigma\frac{dr}{b-a}} = \mathfrak{F}\mathfrak{T}_T\sqrt{\int_a^b\bigg(\int_{[r,b)} d\sigma(y)\bigg)\frac{dr}{b-a}}\\
&= \mathfrak{F}\mathfrak{T}_T\sqrt{\int_{[a,b)}\bigg(\int_a^y \frac{dr}{b-a}\bigg)d\sigma(y)} = \mathfrak{F}\mathfrak{T}_T\sqrt{\int_{[a,b)}\frac{y-a}{b-a}d\sigma(y)}\\
&\le \mathfrak{F}\mathfrak{T}_T\sqrt{|[a,b)|_\sigma},
\end{aligned}$$

and hence $\mathfrak{F}\mathfrak{T}_T^{(1)} \le \mathfrak{F}\mathfrak{T}_T^{(0)} \equiv \mathfrak{F}\mathfrak{T}_T$. Similarly, the identity

$$\mathbf{1}_{[a,b)}(y)\Big(\frac{y-a}{b-a}\Big)^2 = \int_a^b 1_{[r,b)}(y)2\Big(\frac{y-r}{b-a}\Big)\frac{dr}{b-a}, \quad \text{for all } y \in \mathbb{R},$$

shows that

$$\begin{aligned}
&\Big\|T\Big[\mathbf{1}_{[a,b)}(y)\Big(\frac{y-a}{b-a}\Big)^2\Big]\Big\|_{L^2(\omega)} = \bigg\|T\bigg[\int_a^b \mathbf{1}_{[r,b)}(y)2\Big(\frac{y-r}{b-a}\Big)dr\bigg]\bigg\|_{L^2(\omega)}\\
&\le 2\int_a^b \Big\|T\Big[\mathbf{1}_{[r,b)}(y)\Big(\frac{y-r}{b-a}\Big)\Big]\Big\|_{L^2(\omega)}\frac{dr}{b-a} \le 2\mathfrak{F}\mathfrak{T}_T^{(1)}\sqrt{|[a,b)|_\sigma},
\end{aligned}$$

and hence $\mathfrak{F}\mathfrak{T}_T^{(2)} \le 2\mathfrak{F}\mathfrak{T}_T^{(1)}$. Continuing in this manner we obtain

$$\mathfrak{F}\mathfrak{T}_T^{(\kappa)} \le \kappa\mathfrak{F}\mathfrak{T}_T^{(\kappa-1)}, \quad \text{for all } \kappa \ge 1,$$

which when iterated gives

$$\mathfrak{F}\mathfrak{T}_T^{(\kappa)}(\sigma,\omega) \le \kappa!\mathfrak{F}\mathfrak{T}_T(\sigma,\omega).$$

By a result of Hytönen [Hyt3] (see also [SaShUr12] for the straightforward extension to fractional singular integrals), the full testing constant $\mathfrak{F}\mathfrak{T}_T(\sigma,\omega)$ in dimension $n = 1$ is controlled by the usual testing constant $\mathfrak{T}_T(\sigma,\omega)$ and the one-tailed Muckenhoupt condition $\mathcal{A}_2^\alpha$. Thus we have proved the following lemma for the case when $T = T^\alpha$ is a fractional Calderón–Zygmund operator in dimension $n = 1$:

Lemma 21. *Suppose that σ and ω are locally finite positive Borel measures on $\mathbb{R}$ and $\kappa \in \mathbb{N}$. If T^α is a bounded α-fractional Calderón–Zygmund operator from $L^2(\sigma)$ to $L^2(\omega)$, then we have*

$$\mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega) \le \kappa!\mathfrak{T}_{T^\alpha}(\sigma,\omega) + C_\kappa \mathcal{A}_2^\alpha(\sigma,\omega), \quad \kappa \ge 1,$$

where the constant C_κ depends on the kernel constant C_{CZ} in (2.10)*, but is independent of the operator norm $\mathfrak{N}_{T^\alpha}(\sigma,\omega)$.*

5.1 The higher dimensional case. The higher dimensional version of this lemma will include a small multiple of the operator norm $\mathfrak{N}_T(\sigma,\omega)$ in place of the one-tailed Muckenhoupt constant $\mathcal{A}_2^\alpha(\sigma,\omega)$ on the right-hand side, since we no longer have available an analogue of Hytönen's result; see [GrPa]. Nevertheless, we show below that for doubling measures, the two testing conditions are equivalent in the presence of one-tailed Muckenhoupt conditions (2.8) in all dimensions, and so we will be able to prove a $T1$ theorem in higher dimensions in certain cases.

Theorem 22. *Suppose that σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$, and let $\kappa \in \mathbb{N}$. If T is a bounded operator from $L^2(\sigma)$ to $L^2(\omega)$, then for every* $0 < \varepsilon < 1$*, there is a positive constant $C(\kappa,\varepsilon)$ such that*

$$\mathfrak{F}\mathfrak{T}_T^{(\kappa)}(\sigma,\omega) \le C(\kappa,\varepsilon)\mathfrak{F}\mathfrak{T}_T(\sigma,\omega) + \varepsilon\mathfrak{N}_T(\sigma,\omega), \quad \kappa \ge 1,$$

and where the constants $C(\kappa,\varepsilon)$ depend only on κ and ε, and not on the operator norm $\mathfrak{N}_T(\sigma,\omega)$.

Proof. We begin with the following geometric observation, similar to a construction used in the recursive control of the nearby form in [SaShUr12]. Let $R = [0,1)^{n-1} \times [0,t)$ be a rectangle in $\mathbb{R}^n$ with $0 < t < 1$. Then given $0 < \varepsilon < 1$, there is a positive integer $m \in \mathbb{N}$ and a dyadic number $t^* \equiv \frac{b}{2^m}$ with $0 \le b < 2^m$, so that

$$\begin{aligned} & R = E \dot\cup \left\{ \dot{\bigcup_{i=1}^{B}} K_i \right\}; \\ (5.2) \qquad & E = [0,1)^{n-1} \times [t^*, t) \text{ with } |t - t^*| < \varepsilon, \\ & B \le 2^{nm-n-m+2}, \end{aligned}$$

and where the K_i are pairwise disjoint cubes inside R. To see (5.2) we choose $m \in \mathbb{N}$ so that $\frac{1}{2^m} < \varepsilon$ and then let $b \in \mathbb{N}$ satisfy $2^m t - 1 \leq b < 2^m t$. Then with $t^* = \frac{b}{2^m}$ we have $|t - t^*| < \frac{1}{2^m} < \varepsilon$. Now expand t^* in binary form,

$$t^* = b_1 \frac{1}{2} + b_2 \frac{1}{4} + \cdots + b_{m-1} \frac{1}{2^{m-1}}, \quad b_k \in \{0, 1\}.$$

Then for each k with $b_k = 1$ we decompose the rectangle

$$R_k \equiv [0, 1)^{n-1} \times \left[b_1 \frac{1}{2} + b_2 \frac{1}{4} + \cdots + b_{k-1} \frac{1}{2^{k-1}}, b_1 \frac{1}{2} + b_2 \frac{1}{4} + \cdots + b_{k-1} \frac{1}{2^{k-1}} + \frac{1}{2^k} \right)$$

into $2^{(n-1)k}$ pairwise disjoint dyadic cubes of side length $\frac{1}{2^k}$. Then we take the collection of all such cubes, noting that the number B of such cubes is at most

$$\sum_{k=1}^{m-1} 2^{(n-1)k} \leq 2 \cdot 2^{(n-1)(m-1)} = 2^{nm-n-m+2},$$

and label them as $\{K_i\}_{i=1}^B$ with $B \leq 2^{nm-n-m+2}$. Finally we note that

$$\dot{\bigcup_{i=1}^{B}} K_i = \dot{\bigcup_{k:\ b_k=1}} R_k = [0, 1)^{n-1} \times [0, t^*).$$

This completes the proof of (5.2). Note that we may arrange to have $m \approx \ln \frac{1}{\varepsilon}$.

We also have the same result for the complementary rectangle

$$R = [0, 1)^{n-1} \times [r, 1)$$

by simply reflecting about the plane $y_n = \frac{1}{2}$ and taking $r = 1 - t$. It is in this complementary form that we will use (5.2).

Again we start by considering the full testing condition $\mathfrak{F}\mathfrak{T}_T^1$ over linear functions, and we begin by estimating

$$\|T_\sigma[\mathbf{1}_Q(y) y_j]\|_{L^2(\omega)}^2, \quad Q \in \mathcal{P}^n, 1 \leq j \leq n.$$

In order to reduce notational clutter in appealing to the complementary form of the geometric observation above, we will suppose, without loss of generality, that $Q = [0, 1)^n$ is the unit cube in $\mathbb{R}^n$, and that $j = n$. Then we have

$$\mathbf{1}_{[0,1)^n}(y) y_n = \int_0^1 \mathbf{1}_{[0,1)^{n-1} \times [r,1)}(y) dr, \quad \text{for all } y \in \mathbb{R}^n,$$

and

$$T_\sigma(\mathbf{1}_{[0,1)^n}(y) y_n)(x) = T_\sigma \left(\int_0^1 \mathbf{1}_{[0,1)^{n-1} \times [r,1)}(y) dr \right)(x) = \int_0^1 (T_\sigma \mathbf{1}_{[0,1)^{n-1} \times [r,1)})(x) dr.$$

The norm estimate is complicated by the lack of Hytönen's result in higher dimensions, and we compensate by using the complementary form of the geometric observation (5.2), together with a simple probability argument. Let $[r, 1) = [r, r^*)\dot\cup[r^*, 1)$ and write

$$\begin{aligned}
\|T_\sigma \mathbf{1}_{[0,1)^{n-1}\times[r,1)}\|^2_{L^2(\omega)} &= \int \left|T_\sigma\{\mathbf{1}_{[0,1)^{n-1}\times[r,r^*)} + \mathbf{1}_{[0,1)^{n-1}\times[r^*,1)}\}(x)\right|^2 d\omega(x) \\
&= \int \left|T_\sigma\left\{\mathbf{1}_{[0,1)^{n-1}\times[r,r^*)} + \sum_{i=1}^{B}\mathbf{1}_{K_i}\right\}(x)\right|^2 d\omega(x) \\
&\lesssim \int |T_\sigma \mathbf{1}_{[0,1)^{n-1}\times[r,r^*)}(x)|^2 d\omega(x) + \sum_{i=1}^{B}\int |T_\sigma \mathbf{1}_{K_i}(x)|^2 d\omega(x) \\
&\le \int |T_\sigma \mathbf{1}_{[0,1)^{n-1}\times[r,r^*)}(x)|^2 d\omega(x) + (\mathfrak{F}\mathfrak{T}_T)^2 \sum_{i=1}^{B} |K_i|_\sigma .
\end{aligned}$$

First, we apply a simple probability argument to the integral over r of the last integral above by pigeonholing the values taken by $r^* \in \{\frac{b}{2^m}\}_{0\le b<2^m}$:

$$\begin{aligned}
&\int_0^1 \int |T_\sigma \mathbf{1}_{[0,1)^{n-1}\times[r,r^*)}(x)|^2 d\omega(x) dr \\
&\quad\le \mathfrak{N}_T(\sigma,\omega)^2 \int_0^1 \left\{\int_{[0,1)^{n-1}\times[r,r^*)} d\sigma\right\} dr \\
&\quad= \mathfrak{N}_T(\sigma,\omega)^2 \sum_{0<b\le 2^m} \int_{\frac{b-1}{2^m}}^{\frac{b}{2^m}} \left\{\int_{[0,1)^{n-1}\times[r,\frac{b}{2^m})} d\sigma\right\} dr \\
&\quad\le \mathfrak{N}_T(\sigma,\omega)^2 \int_{[0,1)^n} \left\{\int_{y_n-\varepsilon}^{y_n} dr\right\} d\sigma(y_1,\ldots,y_n) \\
&\quad\le \mathfrak{N}_T(\sigma,\omega)^2 \int_{[0,1)^n} \varepsilon d\sigma(y_1,\ldots,y_n) \\
&\quad= \varepsilon \mathfrak{N}_T(\sigma,\omega)^2 |[0,1)^n|_\sigma ,
\end{aligned}$$

since $\frac{b-1}{2^m} \le r \le y_n < \frac{b}{2^m}$ implies $y_n - \varepsilon < y_n - \frac{1}{2^m} \le r \le y_n$.

Combining estimates, and setting $R_r \equiv [0,1)^{n-1} \times [r,1)$ for convenience, we obtain

$$\begin{aligned}
\|T_\sigma[\mathbf{1}_{R_r}(y) y_n]\|_{L^2(\omega)} &= \left\| T_\sigma\left[\int_0^1 \mathbf{1}_{R_r}(y) dr\right]\right\|_{L^2(\omega)} \\
&\le \int_0^1 \|T_\sigma[\mathbf{1}_{R_r}(y)]\|_{L^2(\omega)} dr \\
&\le \mathfrak{F}\mathfrak{T}_T(\sigma,\omega)\int_0^1 \sqrt{|R_r|_\sigma} dr + \varepsilon \mathfrak{N}_T(\sigma,\omega) |[0,1)^n|_\sigma ,
\end{aligned}$$

where

$$\int_0^1 \sqrt{|R_r|_\sigma} dr \leq \sqrt{\int_0^1 |[r,b)|_\sigma \frac{dr}{b-a}} = \sqrt{\int_0^1 \int_{[0,1)^{n-1}\times[r,1)} d\sigma(y)dr}$$
$$= \sqrt{\int_{[0,1)^n} \int_{[0,y_n)} drd\sigma(y)} = \sqrt{\int_{[0,1)^n} y_n d\sigma(y)}.$$

Noting that

$$\sqrt{\int_{[0,1)^n} y_n d\sigma(y)} \leq |[0,1)^n|_\sigma,$$

that the same estimates hold for y_j in place of y_n, and finally that there are appropriate analogues of these estimates for all cubes $Q \in \mathcal{P}^n$ in place of $[0,1)^n$, we see that

$$\mathfrak{F}\mathfrak{T}_T^{(1)}(\sigma,\omega) \leq C_{m,0}\mathfrak{F}\mathfrak{T}(\sigma,\omega) + \varepsilon\mathfrak{N}_T(\sigma,\omega).$$

Similarly, for each $i < n$ we can consider the monomial $y_i y_n$, and obtain from the above argument with y_i included in the integrand that

$$\begin{aligned} &\|T_\sigma[\mathbf{1}_{R_r}(y)y_i y_n]\|_{L^2(\omega)} \\ &\lesssim \sqrt{\int \left|T_\sigma(\mathbf{1}_{[0,1)^{n-1}\times[r,r^*)}(y)y_i)(x)\right|^2 d\omega(x)} + \mathfrak{F}\mathfrak{T}_T^{(1)}|[0,1)^n|_\sigma. \end{aligned} \tag{5.3}$$

For the monomial y_n^2 we use the identity

$$\mathbf{1}_{[0,1)^n}(y)y_n^2 = \int_0^1 \mathbf{1}_{[0,1)^{n-1}\times[r,1)}(y)2(y_n - r)dr, \quad \text{for all } y \in \mathbb{R}^n,$$

to obtain

$$\begin{aligned} &\|T_\sigma[\mathbf{1}_{R_r}(y)y_n^2]\|_{L^2(\omega)} \\ &\lesssim \sqrt{\int \left|T_\sigma(\mathbf{1}_{[0,1)^{n-1}\times[r,r^*)}(y)(y_n - r))(x)\right|^2 d\omega(x)} + \mathfrak{F}\mathfrak{T}_T^{(1)}|[0,1)^n|_\sigma. \end{aligned}$$

Then in either case, integrating in r, using the simple probability argument above, and finally using the appropriate analogues of these estimates for all cubes $Q \in \mathcal{P}^n$ in place of $[0,1)^n$, we obtain

$$\mathfrak{F}\mathfrak{T}_T^{(2)}(\sigma,\omega) \leq C_{m,1}\mathfrak{F}\mathfrak{T}_T^{(1)}(\sigma,\omega) + \varepsilon\mathfrak{N}_T(\sigma,\omega).$$

Continuing in this way, using the identity

$$\mathbf{1}_{[0,1)^n}(y)y^\beta = \int_0^1 \mathbf{1}_{[0,1)^{n-1}\times[r,1)}(y)(y_1^{\beta_1}\cdots y_{n-1}^{\beta_{n-1}})(2\beta_n(y_n - r)^{\beta_n - 1})dr, \quad \text{for all } y \in \mathbb{R}^n,$$

yields the inequality

$$\mathfrak{FT}_T^{(\kappa)}(\sigma,\omega) \leq C_{m,\kappa-1}\mathfrak{FT}_T^{(\kappa-1)}(\sigma,\omega) + \varepsilon\mathfrak{N}_T(\sigma,\omega), \qquad \kappa \in \mathbb{N}.$$

Iteration then gives

$$\begin{aligned}
&\mathfrak{FT}_T^{(\kappa)}(\sigma,\omega) \\
&\quad \leq \varepsilon\mathfrak{N}_T(\sigma,\omega) + C_{m,\kappa-1}\mathfrak{FT}_T^{(\kappa-1)}(\sigma,\omega) \\
&\quad \leq \varepsilon\mathfrak{N}_T(\sigma,\omega) + C_{m,\kappa-1}\{\varepsilon\mathfrak{N}_T(\sigma,\omega) + C_{m,\kappa-2}\mathfrak{FT}_T^{(\kappa-2)}(\sigma,\omega)\} \\
&\quad \ \vdots \\
&\quad \leq \varepsilon\{1 + C_{m,\kappa-1} + C_{m,\kappa-1}C_{m,\kappa-2} + \cdots + C_{m,\kappa-1}C_{m,\kappa-2}\cdots C_{m,0}\}\mathfrak{N}_T(\sigma,\omega) \\
&\qquad + \{C_{m,\kappa-1}C_{m,\kappa-2} + \cdots + C_{m,\kappa-1}C_{m,\kappa-2}\cdots C_{0m,}\}\mathfrak{FT}_T(\sigma,\omega) \\
&\quad = \varepsilon A(\kappa,\varepsilon)\mathfrak{N}_T(\sigma,\omega) + B(\kappa,\varepsilon)\mathfrak{FT}_T(\sigma,\omega),
\end{aligned}$$

where the constants $A(\kappa,\varepsilon)$ and $B(\kappa,\varepsilon)$ are independent of the operator norm $\mathfrak{N}_T(\sigma,\omega)$. Here we have taken $m \approx \log_2 \frac{1}{\varepsilon}$. This completes the proof of Theorem 22. □

We have already pointed out in dimension $n = 1$ the equivalence of full testing with the usual 1-testing in the presence of one-tailed Muckenhoupt conditions. In higher dimensions the same is true for at least doubling measures. For this we use a quantitative expression of the fact that doubling measures don't charge the boundaries of cubes [Ste2, see, e.g., 8.6 (b) on page 40].

Lemma 23. *Suppose σ is a doubling measure on $\mathbb{R}^n$ and that $Q \in \mathcal{P}^n$. Then for $0 < \delta < 1$ we have*

$$|Q \setminus (1-\delta)Q|_\sigma \leq \frac{C}{\ln\frac{1}{\delta}}|Q|_\sigma.$$

Proof. Let $\delta = 2^{-m}$. Denote by $\mathfrak{C}^{(m)}(Q)$ the set of m^{th} generation dyadic children of Q, so that each $I \in \mathfrak{C}^{(m)}(Q)$ has side length $\ell(I) = 2^{-m}\ell(Q)$, and define the collections

$$\begin{aligned}
\mathfrak{G}^{(m)}(Q) &\equiv \{I \in \mathfrak{C}^{(m)}(Q) : I \subset Q \text{ and } \partial I \cap \partial Q \neq \emptyset\}, \\
\mathfrak{H}^{(m)}(Q) &\equiv \{I \in \mathfrak{C}^{(m)}(Q) : 3I \subset Q \text{ and } \partial(3I) \cap \partial Q \neq \emptyset\}.
\end{aligned}$$

Then

$$Q \setminus (1-\delta)Q = \mathfrak{G}^{(m)}(Q) \quad \text{and} \quad (1-\delta)Q = \dot{\bigcup_{k=2}^{m}}\mathfrak{H}^{(k)}(Q).$$

From the doubling condition we have $|3I|_\sigma \leq D|I|_\sigma$ for all cubes I, and so

$$\begin{aligned}|\mathfrak{H}^{(k)}(Q)|_\sigma &= \sum_{I\in\mathfrak{H}^{(k)}(Q)} |I|_\sigma \geq \sum_{I\in\mathfrak{H}^{(k)}(Q)} \frac{1}{D}|3I|_\sigma = \frac{1}{D}\int\Big(\sum_{I\in\mathfrak{H}^{(k)}(Q)} \mathbf{1}_{3I}\Big)d\sigma \\ &\geq \frac{1}{D}\int\Big(\sum_{I\in\mathfrak{G}^{(k)}(Q)} \mathbf{1}_{I}\Big)d\sigma = \frac{1}{D}|\mathfrak{G}^{(k)}(Q)|_\sigma \geq \frac{1}{D}|\mathfrak{G}^{(m)}(Q)|_\sigma \\ &= \frac{1}{D}|Q\setminus(1-\delta)Q|_\sigma.\end{aligned}$$

Thus we have

$$|Q|_\sigma \geq \sum_{k=2}^{m} |\mathfrak{H}^{(k)}(Q)|_\sigma \geq \frac{m-1}{D}|Q\setminus(1-\delta)Q|_\sigma,$$

which proves the lemma. □

Proposition 24. Suppose that σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$, and that σ is doubling. Then for $0<\varepsilon<1$ there is a positive constant $C(\varepsilon)$ such that

$$\mathfrak{F}\mathfrak{T}_T(\sigma,\omega) \leq \mathfrak{T}_T(\sigma,\omega) + C(\varepsilon)\mathcal{A}_2^\alpha(\sigma,\omega) + \varepsilon\mathfrak{N}_T(\sigma,\omega).$$

Proof. Let $\delta>0$ be defined by the equation $\varepsilon = \frac{C}{\ln\frac{1}{\delta}}$, i.e., $\delta = e^{-\frac{C}{\varepsilon}}$. Then we write

$$\begin{aligned}&\int_{\mathbb{R}^n} |T_\sigma \mathbf{1}_Q|^2 d\omega \\ &\quad= \int_Q |T_\sigma\mathbf{1}_Q|^2 d\omega + \int_{\mathbb{R}^n\setminus Q} \left|T_\sigma\mathbf{1}_{(1-\delta)Q} + T_\sigma\mathbf{1}_{Q\setminus(1-\delta)Q}\right|^2 d\omega \\ &\quad\leq \mathfrak{T}_T(\sigma,\omega)^2|Q|_\sigma + 2\int_{\mathbb{R}^n\setminus Q}\left|T_\sigma\mathbf{1}_{(1-\delta)Q}\right|^2 d\omega + 2\int_{\mathbb{R}^n\setminus Q}\left|T_\sigma\mathbf{1}_{Q\setminus(1-\delta)Q}\right|^2 d\omega \\ &\quad\leq \mathfrak{T}_T(\sigma,\omega)^2|Q|_\sigma + C\frac{1}{\delta}\mathcal{A}_2^\alpha(\sigma,\omega)|Q|_\sigma + 2\mathfrak{N}_T^2(\sigma,\omega)|Q\setminus(1-\delta)Q|_\sigma.\end{aligned}$$

Now invoke Lemma 23 to obtain

$$\int_{\mathbb{R}^n} |T_\sigma\mathbf{1}_Q|^2 d\omega \leq \mathfrak{T}_T(\sigma,\omega)^2|Q|_\sigma + C\frac{1}{\delta}\mathcal{A}_2^\alpha(\sigma,\omega)|Q|_\sigma + \varepsilon\mathfrak{N}_T^2(\sigma,\omega)|Q|_\sigma,$$

with $\varepsilon = \frac{2C}{\ln\frac{1}{\delta}}$. □

In the sequel we will want to combine Theorem 22 and Proposition 24 into the following single estimate.

Corollary 25. *Suppose that σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$, and that σ is doubling. Then for $\kappa\in\mathbb{N}$ and $0<\varepsilon<1$, there is a positive constant $C_{\kappa,\varepsilon}$ such that*

$$\mathfrak{F}\mathfrak{T}_T^{(\kappa)}(\sigma,\omega) \leq C_{\kappa,\varepsilon}[\mathfrak{T}_T(\sigma,\omega) + \mathcal{A}_2^\alpha(\sigma,\omega)] + \varepsilon\mathfrak{N}_T(\sigma,\omega).$$

6 Proof of the Tp theorem with $\mathcal{BICT}$ and doubling weights

We will prove Theorem 6 by adapting the beautiful pivotal argument of Nazarov, Treil and Volberg in [NTV4], that uses weighted Haar wavelets and random grids, to a weaker κ^{th}-order pivotal condition with Alpert wavelets and the Parallel Corona decomposition, the latter being used to circumvent difficulties in establishing the paraproduct decomposition using weighted Alpert wavelets. More precisely, we will work in the one-grid world, where the Alpert wavelet expansions for f and g in $L^2(\sigma)$ and $L^2(\omega)$ respectively are taken with respect to a common grid $\mathcal{D}$, and follow the standard NTV argument for $T1$-type theorems already in the literature (see, e.g., [NTV4], the two-part paper [LaSaShUr3], [Lac], [Hyt3] and [SaShUr7]), i.e., using NTV random grids $\mathcal{D}$ and goodness, but using pivotal conditions when possible to avoid functional energy, and using the Parallel Corona and κ-Cube Testing and Bilinear Indicator/Cube Testing to avoid paraproduct terms, which as observed earlier behave poorly with respect to weighted Alpert wavelets of order greater than 1. But first we extend the scope of the Indicator/Cube Testing condition and the Bilinear Indicator/Cube Testing property.

6.1 Extending indicators to bounded functions. It was observed in [LaSaUr1] that the supremum over $\mathbf{1}_E$ in the Indicator/Cube testing condition (2.14) can be replaced with the logically larger supremum over an arbitrary function h with $|h| \leq 1$. Here we extend the analogue of this observation to hold for the Bilinear Indicator/Cube Testing constant $\mathcal{BICT}_{T^\alpha}(\sigma, \omega)$.

Lemma 26. *Let σ and ω be positive locally finite Borel measures on $\mathbb{R}^n$, and let T^α be a standard α-fractional singular integral operator on $\mathbb{R}^n$. Then*

$$\mathcal{BICT}_{T^\alpha}(\sigma, \omega) \leq \sup_{Q \in \mathcal{P}^n} \sup_{\substack{\|f\|_{L^\infty(\sigma)} \leq 1 \\ \|g\|_{L^\infty(\omega)} \leq 1}} \frac{1}{\sqrt{|Q|_\sigma |Q|_\omega}} \left| \int_Q T^\alpha_\sigma(\mathbf{1}_Q f) g \omega \right| \leq 4\mathcal{BICT}_{T^\alpha}(\sigma, \omega).$$

Proof. Given a cube Q and a bounded function $f \in L^\infty(\sigma)$, define

$$h_Q[f](x) \equiv \begin{cases} \frac{|T^\alpha_\sigma(\mathbf{1}_Q f)(x)|}{T^\alpha_\sigma(\mathbf{1}_Q f)(x)} & \text{if } T^\alpha_\sigma(\mathbf{1}_Q f)(x) \neq 0 \\ 0 & \text{if } T^\alpha_\sigma(\mathbf{1}_Q f)(x) = 0 \end{cases}$$
$$= \mathbf{1}_{F_+[f]}(x) - \mathbf{1}_{F_-[f]}(x),$$

where the sets

$$F_+[f] \equiv \{x \in Q : T^\alpha_\sigma(\mathbf{1}_Q f)(x) > 0\},$$
$$F_-[f] \equiv \{x \in Q : T^\alpha_\sigma(\mathbf{1}_Q f)(x) < 0\},$$

both depend on f. Then we have

$$\begin{aligned}
&\sup_{Q\in\mathcal{P}^n}\sup_{\substack{\|f\|_{L^\infty(\sigma)}\le 1\\ \|g\|_{L^\infty(\omega)}\le 1}}\frac{1}{\sqrt{|Q|_\sigma|Q|_\omega}}\left|\int_Q T_\sigma^\alpha(\mathbf{1}_Q f)g\,d\omega\right|\\
&\qquad=\sup_{Q\in\mathcal{P}^n}\sup_{\|f\|_{L^\infty(\sigma)}\le 1}\frac{1}{\sqrt{|Q|_\sigma|Q|_\omega}}\int_Q |T_\sigma^\alpha(\mathbf{1}_Q f)|\,d\omega\\
&\qquad=\sup_{Q\in\mathcal{P}^n}\sup_{\|f\|_{L^\infty(\sigma)}\le 1}\frac{1}{\sqrt{|Q|_\sigma|Q|_\omega}}\int_Q T_\sigma^\alpha(\mathbf{1}_Q f)h_Q[f]\,d\omega\\
&\qquad=\sup_{Q\in\mathcal{P}^n}\sup_{\|f\|_{L^\infty(\sigma)}\le 1}\frac{1}{\sqrt{|Q|_\sigma|Q|_\omega}}\int_Q f(T_\omega^{\alpha,*}(\mathbf{1}_{F_+[f]}-\mathbf{1}_{F_-[f]}))\,d\sigma\\
&\qquad\le\sup_{Q\in\mathcal{P}^n}\frac{1}{\sqrt{|Q|_\sigma|Q|_\omega}}\sup_{\|f\|_{L^\infty(\sigma)}\le 1}\int_Q |T_\omega^{\alpha,*}(\mathbf{1}_{F_+[f]}-\mathbf{1}_{F_-[f]})|\,d\sigma.
\end{aligned}$$

But now

$$\sup_{\|f\|_{L^\infty(\sigma)}\le 1}\int_Q |T_\omega^{\alpha,*}(\mathbf{1}_{F_+[f]}-\mathbf{1}_{F_-[f]})|\,d\sigma=\int_Q T_\omega^{\alpha,*}(\mathbf{1}_{F_+[f]}-\mathbf{1}_{F_-[f]})k_Q[f]\,d\sigma$$

where

$$\begin{aligned}
k_Q[f](y)&\equiv\begin{cases}\frac{|T_\omega^{\alpha,*}(\mathbf{1}_{F_+[f]}-\mathbf{1}_{F_-[f]})(y)|}{T_\omega^{\alpha,*}(\mathbf{1}_{F_+[f]}-\mathbf{1}_{F_-[f]})(y)} & \text{if } T_\omega^{\alpha,*}(\mathbf{1}_{F_+[f]}-\mathbf{1}_{F_-[f]})(y)\neq 0\\ 0 & \text{if } T_\omega^{\alpha,*}(\mathbf{1}_{F_+[f]}-\mathbf{1}_{F_-[f]})(y)=0\end{cases}\\
&=\mathbf{1}_{E_+[f]}(y)-\mathbf{1}_{E_-[f]}(y),
\end{aligned}$$

where the sets

$$\begin{aligned}
E_+[f]&\equiv\{y\in Q: T_\omega^{\alpha,*}(\mathbf{1}_{F_+[f]}-\mathbf{1}_{F_-[f]})(y)>0\},\\
E_-[f]&\equiv\{y\in Q: T_\omega^{\alpha,*}(\mathbf{1}_{F_+[f]}-\mathbf{1}_{F_-[f]})(y)<0\},
\end{aligned}$$

also both depend on f. Thus we have shown that

$$\begin{aligned}
&\sup_{Q\in\mathcal{P}^n}\sup_{\substack{\|f\|_{L^\infty(\sigma)}\le 1\\ \|g\|_{L^\infty(\omega)}\le 1}}\frac{1}{\sqrt{|Q|_\sigma|Q|_\omega}}\left|\int_Q T_\sigma^\alpha(\mathbf{1}_Q f)g\,d\omega\right|\\
&\le\sup_{Q\in\mathcal{P}^n}\sup_{\|f\|_{L^\infty(\sigma)}\le 1}\frac{1}{\sqrt{|Q|_\sigma|Q|_\omega}}\int_Q(\mathbf{1}_{E_+[f]}-\mathbf{1}_{E_-[f]})T_\omega^{\alpha,*}(\mathbf{1}_{F_+[f]}-\mathbf{1}_{F_-[f]})\,d\sigma\\
&\le 4\sup_{Q\in\mathcal{P}^n}\sup_{E,F\subset Q}\frac{1}{\sqrt{|Q|_\sigma|Q|_\omega}}\left|\int_E T_\omega^{\alpha,*}(\mathbf{1}_F)\,d\sigma\right|=4\mathcal{BICT}_{T^\alpha}(\sigma,\omega).
\end{aligned}$$

The converse inequality

$$\mathcal{BICT}_{T^\alpha}(\sigma,\omega)\le\sup_{Q\in\mathcal{P}^n}\sup_{\substack{\|f\|_{L^\infty(\sigma)}\le 1\\ \|g\|_{L^\infty(\omega)}\le 1}}\frac{1}{\sqrt{|Q|_\sigma|Q|_\omega}}\left|\int_Q T_\sigma^\alpha(\mathbf{1}_Q f)g\,d\omega\right|$$

is trivial. □

6.2 Initial steps. The first step in the proof of Theorem 6 is to expand an inner product $\langle T_\sigma^\alpha f, g\rangle_{L^2(\omega)}$ in weighted Alpert projections $\triangle_{I;\kappa_1}^\sigma f$ and $\triangle_{J;\kappa_2}^\omega g$ associated with a fixed dyadic grid $\mathcal{D}$:

$$\langle T_\sigma^\alpha f, g\rangle_{L^2(\omega)} = \sum_{I,J\in\mathcal{D}} \langle T_\sigma^\alpha \triangle_{I;\kappa_1}^\sigma f, \triangle_{J;\kappa_2}^\omega g\rangle_{L^2(\omega)}. \tag{6.1}$$

We next wish to reduce the above sum to $(I,J)\in\mathcal{D}\times\mathcal{D}$ such that $I\subset I_0$ and $J\subset J_0$ where I_0 and J_0 are large cubes in $\mathcal{D}$, and for this we will use, in a standard way, the testing conditions over polynomials of degree less than κ. This reduced sum is then decomposed into many separate sums according to the relative sizes of Calderón–Zygmund stopping cubes, i.e., first into the Parallel Corona decomposition, then further into Near, Disjoint and Far forms, and then finally according to the locations and goodness of the intervals I and J. Each of the resulting forms is then controlled using widely different techniques.

A crucial tool from [RaSaWi] is the estimate for $L^2(\omega)$ norms of Alpert projections $\|\triangle_{J;\kappa}^\omega T^\alpha\mu\|_{L^2(\omega)}^2$, called the Monotonicity Lemma below (see [LaWi] and also [SaShUr7]), and which is improved by the extra vanishing moments of Alpert wavelets to the following NTV type estimate,

$$\|\triangle_{J;\kappa}^\omega T^\alpha\mu\|_{L^2(\omega)}^2 \lesssim \left(\frac{\mathrm{P}_\kappa^\alpha(J,\mu)}{\ell(J)^\kappa}\right)^2 \sum_{|\beta|=\kappa-1} \|(x-m_J^\kappa)^\beta\|_{L^2(\mathbf{1}_J\omega)}^2,$$

which in turn can then be controlled by a κ^{th}-order pivotal condition, **weaker** than the usual pivotal condition with $\kappa=1$. The telescoping identities (4.5) reduce sums of consecutive Alpert projections $\triangle_{I;\kappa}^\mu$ to differences of projections $\mathbb{E}_{Q;\kappa}^\mu$ onto spaces of polynomials of degree at most $\kappa-1$. Since by (4.7), the sup norms of these latter projections are controlled by Calderón–Zygmund averages, we are able to obtain an analogue of the Intertwining Proposition in [SaShUr7], which controls the Far forms. The Near forms are controlled by the κ-Cube Testing conditions and Bilinear Indicator/Cube Testing property.

Underlying all of this analysis, however, is the powerful tool of Nazarov, Treil and Volberg introduced in [NTV1], that restricts wavelet expansions to good cubes, thus permitting the geometric decay necessary to control off-diagonal terms in the presence of some appropriate side condition—such as a pivotal or energy condition, which can be thought of as a proof catalyst.

Before proceeding with the Parallel Corona decomposition and the subsequent elements of the proof of Theorem 6 in Subsection 6.5 below, we give detailed analogues of the Monotonicity Lemma and Intertwining Proposition in the setting of Alpert wavelets.

6.3 The Monotonicity Lemma. For $0 \leq \alpha < n$ and $m \in \mathbb{R}_+$, we recall from (4.1) the m^{th}-order fractional Poisson integral

$$\mathrm{P}_m^\alpha(J,\mu) \equiv \int_{\mathbb{R}^n} \frac{|J|^m}{(|J| + |y - c_J|)^{m+n-\alpha}} d\mu(y),$$

where $\mathrm{P}_1^\alpha(J,\mu) = \mathrm{P}^\alpha(J,\mu)$ is the standard Poisson integral. The following extension of the Lacey–Wick formulation [LaWi] of the Monotonicity Lemma to weighted Alpert wavelets is due to Rahm, Sawyer and Wick [RaSaWi]. Since the proof in [RaSaWi] is given only for dimension $n = 1$, we include the straightforward extension to the higher dimensional operators considered here.

Lemma 27 (Monotonicity [RaSaWi])**.** *Let $0 \leq \alpha < n$, and $\kappa_1, \kappa_2 \in \mathbb{N}$ and $0 < \delta < 1$. Suppose that I and J are cubes in $\mathbb{R}^n$ such that $J \subset 2J \subset I$, and that μ is a signed measure on $\mathbb{R}^n$ supported outside I. Finally suppose that T^α is a standard $(\kappa_1 + \delta, \kappa_2 + \delta)$-smooth fractional singular integral on $\mathbb{R}^n$ with kernel $K^\alpha(x,y) = K_y^\alpha(x)$. Then*

$$\| \triangle_{J;\kappa}^\omega T^\alpha \mu \|_{L^2(\omega)}^2 \lesssim \Phi_\kappa^\alpha(J,\mu)^2 + \Psi_\kappa^\alpha(J,|\mu|)^2, \tag{6.2}$$

where for a measure ν,

$$\Phi_\kappa^\alpha(J,\nu)^2 \equiv \sum_{|\beta|=\kappa} \left| \int (K_y^\alpha)^{(\kappa)}(m_J^\kappa) d\nu(y) \right|^2 \| \triangle_{J;\kappa}^\omega x^\beta \|_{L^2(\omega)}^2,$$

$$\Psi_\kappa^\alpha(J,|\nu|)^2 \equiv \left(\frac{\mathrm{P}_{\kappa+\delta}^\alpha(J,|\nu|)}{|J|^{\frac{\kappa}{n}}} \right)^2 \| |x - m_J^\kappa|^\kappa \|_{L^2(\mathbf{1}_J\omega)}^2,$$

where $m_J^\kappa \in J$ satisfies $\| |x - m_J^\kappa|^\kappa \|_{L^2(\mathbf{1}_J\omega)}^2 = \inf_{m \in J} \| |x - m|^\kappa \|_{L^2(\mathbf{1}_J\omega)}^2$.

Proof of Lemma 27. The proof is an easy adaptation of the one-dimensional proof in [RaSaWi], which was in turn adapted from the proofs in [LaWi] and [SaShUr7], but using a κ^{th}-order Taylor expansion instead of a first-order expansion on the kernel $(K_y^\alpha)(x) = K^\alpha(x,y)$. Due to the importance of this lemma, as explained above, we repeat the short argument.

Let $\{h_{J;\kappa}^{\mu,a}\}_{a \in \Gamma_{J,n,\kappa}}$ be an orthonormal basis of $L_{J;\kappa}^2(\mu)$ consisting of Alpert functions as above. Now we use the $(\kappa + \delta)$-smooth Calderón–Zygmund smoothness estimate (2.10), together with Taylor's formula

$$K_y^\alpha(x) = \mathrm{Tay}(K_y^\alpha)(x,c) + \frac{1}{\kappa!} \sum_{|\beta|=\kappa} (K_y^\alpha)^{(\beta)}(\theta(x,c))(x-c)^\beta;$$

$$\mathrm{Tay}(K_y^\alpha)(x,c) \equiv K_y^\alpha(c) + [(x-c)\cdot\nabla] K_y^\alpha(c) + \cdots + \frac{1}{(\kappa-1)!} [(x-c)\cdot\nabla]^{\kappa-1} K_y^\alpha(c),$$

and the vanishing means of the vector of Alpert functions $h^{\omega}_{J;\kappa} = \{h^{\omega,a}_{J;\kappa}\}_{a\in\Gamma_{J,n,\kappa}}$, to obtain

$$\begin{aligned}
&\langle T^{\alpha}\mu, h^{\omega}_{J;\kappa}\rangle_{L^2(\omega)} \\
&\quad = \int\left\{\int K^{\alpha}(x,y)h^{\omega}_{J;\kappa}(x)d\omega(x)\right\}d\mu(y) = \int\langle K^{\alpha}_y, h^{\omega}_{J;\kappa}\rangle_{L^2(\omega)}d\mu(y) \\
&\quad = \int\langle K^{\alpha}_y(x) - \mathrm{Tay}(K^{\alpha}_y)(x, m^{\kappa}_J), h^{\omega}_{J;\kappa}(x)\rangle_{L^2(\omega)}d\mu(y) \\
&\quad = \int\left\langle\frac{1}{\kappa!}\sum_{|\beta|=\kappa}(K^{\alpha}_y)^{(\beta)}(\theta(x,m^{\kappa}_J))(x-m^{\kappa}_J)^{\beta}, h^{\omega}_{J;\kappa}(x)\right\rangle_{L^2(\omega)}d\mu(y) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{some } \theta(x,m^{\kappa}_J)\in J) \\
&\quad = \sum_{|\beta|=\kappa}\left\langle\left[\int\frac{1}{\kappa!}\sum_{|\beta|=\kappa}(K^{\alpha}_y)^{(\beta)}(m^{\kappa}_J)d\mu(y)\right](x-m^{\kappa}_J)^{\beta}, h^{\omega}_{J;\kappa}\right\rangle_{L^2(\omega)} \\
&\qquad + \sum_{|\beta|=\kappa}\left\langle\left[\int\frac{1}{\kappa!}\left[\sum_{|\beta|=\kappa}(K^{\alpha}_y)^{(\beta)}(\theta(x,m^{\kappa}_J)) - \sum_{|\beta|=\kappa}(K^{\alpha}_y)^{(\beta)}(m^{\kappa}_J)\right]d\mu(y)\right]\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\times (x-m^{\kappa}_J)^{\beta}, h^{\omega}_{J;\kappa}\right\rangle_{L^2(\omega)}.
\end{aligned}$$

Then using that $\int(K^{\alpha}_y)^{(\beta)}(m^{\kappa}_J)d\mu(y)$ is independent of $x\in J$, and that

$$\langle(x-m^{\kappa}_J)^{\beta}, h^{\omega}_{J;\kappa}\rangle_{L^2(\omega)} = \langle x^{\beta}, h^{\omega}_{J;\kappa}\rangle_{L^2(\omega)}$$

by moment vanishing of the Alpert wavelets, we can continue with

$$\begin{aligned}
&\langle T^{\alpha}\mu, h^{\omega}_{J;\kappa}\rangle_{L^2(\omega)} \\
&\ = \left[\int\frac{1}{\kappa!}\sum_{|\beta|=\kappa}(K^{\alpha}_y)^{(\beta)}(m^{\kappa}_J)d\mu(y)\right]\cdot\langle x^{\beta}, h^{\omega}_{J;\kappa}\rangle_{L^2(\omega)} \\
&\qquad + \frac{1}{\kappa!}\sum_{|\beta|=\kappa}\left\langle\left[\int\left[\sum_{|\beta|=\kappa}(K^{\alpha}_y)^{(\beta)}(\theta(x,m^{\kappa}_J)) - \sum_{|\beta|=\kappa}(K^{\alpha}_y)^{(\beta)}(m^{\kappa}_J)\right]d\mu(y)\right]\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\times (x-m^{\kappa}_J)^{\beta}, h^{\omega}_{J;\kappa}\right\rangle_{L^2(\omega)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left|\langle T^{\alpha}\mu, h^{\omega}_{J;\kappa}\rangle_{L^2(\omega)} - \left[\int\frac{1}{\kappa!}\sum_{|\beta|=\kappa}(K^{\alpha}_y)^{(\beta)}(m^{\kappa}_J)d\mu(y)\right]\cdot\langle x^{\beta}, h^{\omega}_{J;\kappa}\rangle_{L^2(\omega)}\right| \\
&\quad \le \frac{1}{\kappa!}\sum_{|\beta|=\kappa}\left|\left\langle\left[\int\sup_{\theta\in J}|(K^{\alpha}_y)^{(\beta)}(\theta) - (K^{\alpha}_y)^{(\beta)}(m^{\kappa}_J)|d|\mu|(y)\right]|x-m^{\kappa}_J|^{\kappa}, |h^{\omega}_{J;\kappa}|\right\rangle_{L^2(\omega)}\right| \\
&\quad \lesssim C_{CZ}\frac{\mathrm{P}^{\alpha}_{\kappa+\delta}(J,|\mu|)}{|J|^{\kappa}}\left\||x-m^{\kappa}_J|^{\kappa}\right\|_{L^2(\mathbf{1}_J\omega)}
\end{aligned}$$

where in the last line we have used

$$\begin{aligned}\int \sup_{\theta\in J}\left|(K_y^\alpha)^{(\beta)}(\theta)-(K_y^\alpha)^{(\beta)}(m_J^\kappa)\right|d|\mu|(y)&\\ &\lesssim C_{CZ}\int\left(\frac{|J|}{|y-c_J|}\right)^\delta\frac{d|\mu|(y)}{|y-c_J|^{\kappa+1-\alpha}}\\ &=C_{CZ}\frac{\mathrm{P}^\alpha_{\kappa+\delta}(J,|\mu|)}{|J|^\kappa}.\end{aligned}$$

Thus with

$$\mathbf{v}_J^\beta=\frac{1}{\kappa!}\int(K_y^\alpha)^{(\beta)}(m_J^\kappa)d\mu(y),$$

and noting that the functions $\{\mathbf{v}_J^\beta\cdot h^{\omega,a}_{J;\kappa}\}_{a\in\Gamma_{J,n,\kappa}}$ are orthonormal in $a\in\Gamma_{J,n,\kappa}$ for each β and J, we have

$$\begin{aligned}\left|\mathbf{v}_J^\beta\cdot\langle x^\beta,h^\omega_{J;\kappa}\rangle_{L^2(\omega)}\right|^2&=\sum_{a\in\Gamma_{J,n,\kappa}}\left|\langle x^\beta,\mathbf{v}_J^\beta\cdot h^{\omega,a}_{J;\kappa}\rangle_{L^2(\omega)}\right|^2\\ &=\|\triangle^\omega_{J;\kappa}\mathbf{v}_J^\beta x^\beta\|^2_{L^2(\omega)}\\ &=|\mathbf{v}_J^\beta|^2\|\triangle^\omega_{J;\kappa}x^\beta\|^2_{L^2(\omega)},\end{aligned}$$

and hence

$$\begin{aligned}&\|\triangle^\omega_{J;\kappa}T^\alpha\mu\|^2_{L^2(\omega)}\\ &\quad=\left|\langle T^\alpha\mu,h^\omega_{J;\kappa}\rangle_{L^2(\omega)}\right|^2\\ &\quad=\sum_{|\beta|=\kappa}|\mathbf{v}_J^\beta|^2\|\triangle^\omega_{J;\kappa}x^\kappa\|^2_{L^2(\omega)}+O\left(\frac{\mathrm{P}^\alpha_{\kappa+\delta}(J,|\mu|)}{|J|^{\frac{\kappa}{n}}}\right)^2\||x-m_J^\kappa|^\kappa\|_{L^2(\mathbf{1}_J\omega)}.\end{aligned}$$

Thus we conclude that

$$\begin{aligned}\|\triangle^\omega_{J;\kappa}T^\alpha\mu\|^2_{L^2(\omega)}\leq C_1\sum_{|\beta|=\kappa}\left|\left|\frac{1}{\kappa!}\int(K_y^\alpha)^{(\beta)}(m_J)d\mu(y)\right|^2\right|^2\|\triangle^\omega_{J;\kappa}x^\kappa\|^2_{L^2(\omega)}&\\ +C_2\left(\frac{\mathrm{P}^\alpha_{\kappa+\delta}(J,|\mu|)}{|J|^{\frac{\kappa}{n}}}\right)^2\||x-m_J^\kappa|^\kappa\|^2_{L^2(\mathbf{1}_J\omega)},&\end{aligned}$$

where

$$\sum_{|\beta|=\kappa}\left|\frac{1}{\kappa!}\int(K_y^\alpha)^{(\beta)}(m_J)d\mu(y)\right|^2\lesssim\left(\frac{\mathrm{P}^\alpha_\kappa(J,|\mu|)}{|J|^{\frac{\kappa}{n}}}\right)^2.\qquad\square$$

The following Energy Lemma follows from the above Monotonicity Lemma in a standard way; see, e.g., [SaShUr7]. Given a subset $\mathcal{J}\subset\mathcal{D}$, define the projection $\mathsf{P}^\omega_{\mathcal{J}}\equiv\sum_{J'\in\mathcal{J}}\triangle^\omega_{J';\kappa}$, and given a cube $J\in\mathcal{D}$, define the projection

$$\mathsf{P}^\omega_J\equiv\sum_{J'\in\mathcal{D}:\ J'\subset J}\triangle^\omega_{J';\kappa}.$$

Lemma 28 (Energy Lemma)**.** *Fix $\kappa \geq 1$. Let J be a cube in $\mathcal{D}$. Let Ψ_J be an $L^2(\omega)$ function supported in J with vanishing ω-means up to order less than κ, and let $\mathcal{J} \subset \mathcal{D}$ be such that $J' \subset J$ for every $J' \in \mathcal{J}$. Let ν be a positive measure supported in $\mathbb{R}^n \setminus \gamma J$ with $\gamma > 1$, and for each $J' \in \mathcal{J}$, let $d\nu_{J'} = \varphi_{J'} d\nu$ with $|\varphi_{J'}| \leq 1$. Let T^α be a standard α-fractional singular integral operator with $0 \leq \alpha < n$. Then we have*

$$\begin{aligned}
&\left|\sum_{J'\in\mathcal{J}} \langle T^\alpha(\nu_{J'}), \triangle^\omega_{J';\kappa}\Psi_J\rangle_\omega\right| \\
&\qquad \lesssim C_\gamma \sum_{J'\in\mathcal{J}} \Phi^\alpha_\kappa(J',\nu) \| \triangle^\omega_{J';\kappa} \Psi_J\|_{L^2(\mu)} \\
&\qquad \lesssim C_\gamma \sqrt{\sum_{J'\in\mathcal{J}} \Phi^\alpha_\kappa(J',\nu)^2} \sqrt{\sum_{J'\in\mathcal{J}} \| \triangle^\omega_{J';\kappa} \Psi_J\|^2_{L^2(\mu)}} \\
&\qquad \leq C_\gamma \Big(\frac{\mathrm{P}^\alpha_\kappa(J,\nu)}{|J|^{\frac{\kappa}{n}}} \|\mathsf{P}^\omega_{\mathcal{J}} x\|_{L^2(\omega)} + \frac{\mathrm{P}^\alpha_{\kappa+\delta}(J,\nu)}{|J|^{\frac{\kappa}{n}}} \| |x - m^\kappa_J|^\kappa \|_{L^2(\mathbf{1}_J\omega)} \Big) \|\mathsf{P}^\omega_{\mathcal{J}} \Psi_J\|_{L^2(\mu)},
\end{aligned}$$

and in particular the 'energy' estimate

$$\begin{aligned}
&|\langle T^\alpha \varphi\nu, \Psi_J\rangle_\omega| \\
&\quad \lesssim C_\gamma \Big(\frac{\mathrm{P}^\alpha_\kappa(J,\nu)}{|J|^{\frac{\kappa}{n}}} \|\mathsf{P}^\omega_J x\|_{L^2(\omega)} + \frac{\mathrm{P}^\alpha_{\kappa+\delta}(J,\nu)}{|J|^{\frac{\kappa}{n}}} \| |x - m^\kappa_J|^\kappa \|_{L^2(\mathbf{1}_J\omega)} \Big) \left\| \sum_{J'\subset J} \triangle^\omega_{J';\kappa} \Psi_J \right\|_{L^2(\omega)},
\end{aligned}$$

where

$$\left\| \sum_{J'\subset J} \triangle^\omega_{J';\kappa} \Psi_J \right\|_{L^2(\omega)} \lesssim \|\Psi_J\|_{L^2(\omega)},$$

and the 'pivotal' bound

$$|\langle T^\alpha(\varphi\nu), \Psi_J\rangle_{L^2(\omega)}| \lesssim C_\gamma \mathrm{P}^\alpha_k(J,\nu) \sqrt{|J|_\omega} \|\Psi_J\|_{L^2(\omega)},$$

for any function φ with $|\varphi| \leq 1$.

6.3.1 Comparison of the k^{th}-order pivotal constant and the usual pivotal constant. As in [RaSaWi], where the corresponding estimate for k^{th}-order energy constants was obtained, we clearly have the inequality

$$\begin{aligned}
\mathrm{P}^\alpha_k(J, \mathbf{1}_I\sigma) &= \int_{\mathbb{R}^n} \frac{|J|^k}{(\ell(J) + |y - c_J|)^{k+n-\alpha}} d\sigma(y) \\
&= \int_{\mathbb{R}} \Big(\frac{|J|}{\ell(J) + |y - c_J|} \Big)^{k-\ell} \frac{|J|^\ell}{(\ell(J) + |y - c_J|)^{\ell+n-\alpha}} d\sigma(y) \\
&\leq \int_{\mathbb{R}} \frac{|J|^\ell}{(\ell(J) + |y - c_J|)^{\ell+n-\alpha}} d\sigma(y) = \mathrm{P}^\alpha_\ell(J, \mathbf{1}_I\sigma),
\end{aligned}$$

for $1 \leq \ell \leq k$, and as a consequence, we obtain the decrease of the pivotal constants $\mathcal{V}_2^{\alpha,k}$ in k:

$$\mathcal{V}_2^{\alpha,k} \leq \mathcal{V}_2^{\alpha,\ell}, \quad \text{for } 1 \leq \ell \leq k.$$

6.4 The Intertwining Proposition. Here we prove the Intertwining Proposition of [SaShUr7, Proposition 9.4 on page 123] by appealing to the κ^{th}-order pivotal condition rather than functional energy, and by using instead of the Indicator/Cube Testing conditions (2.14), the weaker κ-Cube Testing conditions (2.13) similar to those introduced in [RaSaWi]:

$$\begin{aligned}
\left(\mathfrak{T}_{T^\alpha}^{(\kappa_1)}(\sigma,\omega)\right)^2 &\equiv \sup_{Q\in\mathcal{P}^n} \max_{0\leq|\beta|<\kappa_1} \frac{1}{|Q|_\sigma} \int_Q \left|T_\sigma^\alpha(\mathbf{1}_Q m_Q^\beta)\right|^2 \omega < \infty, \\
\left(\mathfrak{T}_{(T^\alpha)^*}^{(\kappa_2)}(\omega,\sigma)\right)^2 &\equiv \sup_{Q\in\mathcal{P}^n} \max_{0\leq|\beta|<\kappa_2} \frac{1}{|Q|_\omega} \int_Q \left|(T_\sigma^\alpha)^*(\mathbf{1}_Q m_Q^\beta)\right|^2 \sigma < \infty,
\end{aligned} \tag{6.3}$$

with

$$m_Q^\beta(x) \equiv \left(\frac{x-c_Q}{\frac{\sqrt{n}}{2}\ell(Q)}\right)^\beta$$

for any cube Q and multiindex β, where c_Q is the center of the cube Q. (The factor $\frac{\sqrt{n}}{2}$in the denominator ensures that $m_Q^\beta \in (\mathcal{P}_\kappa^Q)_{\text{norm}}$ has supremum norm 1 on Q.) In this way we will avoid using the one-tailed Muckenhoupt conditions, relying instead on only the simpler classical condition A_2^α, while requiring the κ-Cube Testing condition and a certain weak boundedness property. Later on we will use the one-tailed Muckenhoupt conditions to both eliminate the weak boundedness property and reduce κ-Cube testing to the usual testing over indicators.

6.4.1 Three NTV estimates. But first, we recall three estimates of Nazarov, Treil and Volberg [NTV4], in a form taken from [SaShUr7, Lemmas 7.1 and 7.2 on page 101], where the 'one-tailed' Muckenhoupt constants are not needed, only the classical Muckenhoupt constant A_2^α. The weak boundedness constant $\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega)$ appearing in estimate (6.5) below is

$$\begin{aligned}
&\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega) \\
&= \sup_{\mathcal{D}\in\Omega} \sup_{\substack{Q,Q'\in\mathcal{D} \\ Q\subset 3Q'\setminus Q' \text{ or } Q'\subset 3Q\setminus Q}} \frac{1}{\sqrt{|Q|_\sigma |Q'|_\omega}} \sup_{\substack{f\in(\mathcal{P}_Q^{\kappa_1})_{\text{norm}} \\ g\in(\mathcal{P}_{Q'}^{\kappa_2})_{\text{norm}}}} \left|\int_{Q'} T_\sigma^\alpha(\mathbf{1}_Q f) g\, d\omega\right| < \infty,
\end{aligned} \tag{6.4}$$

where the space $(\mathcal{P}_\kappa^Q)_{\text{norm}}$ of Q-normalized polynomials of degree less than κ is defined in Definition 17 above. Note that this notion of weak boundedness, which unlike the Bilinear Indicator/Cube Testing property, involves only pairs of disjoint cubes. Finally, we need the concept of $(\mathbf{r},\varepsilon)$-goodness introduced first in [NTV1], and used later in [NTV3] and [NTV4], and then in virtually every paper on the subject thereafter.

Definition 29. Let $\mathcal{D}$ be a dyadic grid. Given $\mathbf{r}\in\mathbb{N}$ and $0<\varepsilon<1$, called **goodness parameters**, a cube $Q\in\mathcal{D}$ is said to be $(\mathbf{r},\varepsilon)$-bad if there is a supercube $I\supset Q$ with

$$\ell(I)\geq 2^{\mathbf{r}}\ell(Q)$$

that satisfies

$$\operatorname{dist}(Q,\partial I)<2\sqrt{n}|Q|^{\varepsilon}|I|^{1-\varepsilon}.$$

Otherwise Q is said to be $(\mathbf{r},\varepsilon)$-good. The collection of $(\mathbf{r},\varepsilon)$-good cubes in $\mathcal{D}$ is denoted $\mathcal{D}^{\mathrm{good}}$. Finally, a function $f\in L^2(\mu)$ is said to be good if

$$f=\sum_{I\in\mathcal{D}^{\mathrm{good}}}\triangle^{\mu}_{I;\kappa}f.$$

It is shown in [NTV1], [NTV3] and [NTV4] for the two-grid world, and in [HyPeTrVo, Section 4] for the one-grid world, that in order to prove a two weight testing theorem, it suffices to obtain estimates for good functions, uniformly over all dyadic grids, provided $\mathbf{r}\in\mathbb{N}$ is chosen large enough depending on the choice of ε satisfying $0<\varepsilon<1$. We assume this reduction is in force for an appropriate $\varepsilon>0$ from now on.

Lemma 30. *Suppose T^{α} is a standard fractional singular integral with $0\leq\alpha<n$, and that all of the cubes $I,J\in\mathcal{D}$ below are* $(\mathbf{r},\varepsilon)$-good *with goodness parameters ε and* $\mathbf{r}$. *Fix $\kappa_1,\kappa_2\geq 1$ and a positive integer $\rho>\mathbf{r}$. For $f\in L^2(\sigma)$ and $g\in L^2(\omega)$ we have*

$$\begin{aligned}\sum_{\substack{I,J\in\mathcal{D}\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq\ell(I)}}&\left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I;\kappa_1}f\right),\triangle^{\omega}_{J;\kappa_2}g\right\rangle_{\omega}\right|\\ &\lesssim\left(\mathfrak{T}^{(\kappa_1)}_{T^{\alpha}}(\sigma,\omega)+\mathfrak{T}^{(\kappa_2)}_{(T^{\alpha})^*}(\omega,\sigma)+\mathcal{WBP}^{(\kappa_1,\kappa_2)}_{T^{\alpha}}(\sigma,\omega)+\sqrt{A^{\alpha}_2(\sigma,\omega)}\right)\\ &\quad\times\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}\end{aligned}\tag{6.5}$$

and

$$\begin{aligned}\sum_{\substack{(I,J)\in\mathcal{D}^{\sigma}\times\mathcal{D}^{\omega}\\ I\cap J=\emptyset\text{ and }\frac{\ell(J)}{\ell(I)}\notin[2^{-\rho},2^{\rho}]}}&\left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I;\kappa_1}f\right),\triangle^{\omega}_{J;\kappa_2}g\right\rangle_{\omega}\right|\\ &\lesssim\sqrt{A^{\alpha}_2(\sigma,\omega)}\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.\end{aligned}\tag{6.6}$$

Justification: Using weak boundedness together with the L^∞ control

$$\|\mathbb{E}_I^{\sigma,\kappa} f\|_{L_I^\infty(\sigma)} \lesssim E_I^\sigma |f|$$

of Alpert expectations given by (4.7), the proof in [SaShUr7] adapts readily to obtain (6.5), as we sketch below. The proof of (6.6) is virtually identical to the corresponding proofs in [SaShUr7], and we will not repeat those details here.

The inequality (6.5) is the only place in the proof where the weak boundedness constant $\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}$ is used, and this constant $\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}$ will be eliminated by exploiting the doubling properties of the measures in the final subsubsection of the proof. This avoids the more difficult surgery argument that was used to eliminate a weak boundedness property by Lacey and Wick in [LaWi]. Moreover, surgery requires the use of two independent families of grids, something we do not have in this proof.

Sketch of Proof of (6.5)**.** First, following [SaShUr7], which in turn followed [NTV4], we reduce matters to the case when $J \subset I$. Then we break up the Alpert projections $\triangle_{I;\kappa_1}^\sigma f$ and $\triangle_{J;\kappa_2}^\omega g$ according to expectations over their respective children:

$$\triangle_{I;\kappa_1}^\sigma f = \sum_{I' \in \mathfrak{C}(I)} (\triangle_{I;\kappa_1}^\sigma f)\mathbf{1}_{I'} = \sum_{I' \in \mathfrak{C}(I)} \|(\triangle_{I;\kappa_1}^\sigma f)\mathbf{1}_{I'}\|_\infty P_{I';\kappa_1}^\sigma f,$$
$$\triangle_{J;\kappa_2}^\omega g = \sum_{J' \in \mathfrak{C}(J)} (\triangle_{J;\kappa_2}^\omega g)\mathbf{1}_{J'} = \sum_{J' \in \mathfrak{C}(J)} \|(\triangle_{J;\kappa_2}^\omega g)\mathbf{1}_{J'}\|_\infty Q_{J';\kappa_2}^\omega g,$$

where

$$P_{I';\kappa_1}^\sigma f = \frac{(\triangle_{I;\kappa_1}^\sigma f)\mathbf{1}_{I'}}{\|(\triangle_{I;\kappa_1}^\sigma f)\mathbf{1}_{I'}\|_\infty}$$

and

$$Q_{J';\kappa_2}^\omega g = \frac{(\triangle_{J;\kappa_2}^\omega g)\mathbf{1}_{J'}}{\|(\triangle_{J;\kappa_2}^\omega g)\mathbf{1}_{J'}\|_\infty},$$

to further reduce matters to proving that

$$\sum_{\substack{I,J \in \mathcal{D}:\ J \subset I \\ 2^{-\rho}\ell(I) \le \ell(J) \le \ell(I)}} \ \sum_{I' \in \mathfrak{C}(I), J' \in \mathfrak{C}(J)} \left\|(\triangle_{I;\kappa_1}^\sigma f)\mathbf{1}_{I'}\right\|_\infty \left\|(\triangle_{J;\kappa_2}^\omega g)\mathbf{1}_{J'}\right\|_\infty \left|\langle T_\sigma^\alpha(P_{I';\kappa_1}^\sigma f), Q_{J';\kappa_2}^\omega g\rangle_\omega\right|$$

is dominated by the right-hand side of (6.5). Note that $P_{I';\kappa_1}^\sigma f \in (\mathcal{P}_{\kappa_1}^{I'})_{\mathrm{norm}}$ and $Q_{J';\kappa_2}^\omega g \in (\mathcal{P}_{\kappa_2}^{J'})_{\mathrm{norm}}$ are L^∞ normalized. Then with $\mathcal{NTV}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega)$ denoting the constant in parentheses on the right-hand side of (6.5), we continue with

$$\begin{aligned}
&\sum_{\substack{I,J\in\mathcal{D}:\ J\subset I\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq\ell(I)}} |\langle T^\alpha_\sigma(\triangle^\sigma_{I;\kappa_1}f), \triangle^\omega_{J;\kappa_2}g\rangle_\omega| \\
&\quad\lesssim \sum_{\substack{I,J\in\mathcal{D}:\ J\subset I\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq\ell(I)}} \sum_{I'\in\mathfrak{C}(I),J'\in\mathfrak{C}(J)} \|(\triangle^\sigma_{I;\kappa_1}f)\mathbf{1}_{I'}\|_\infty \|(\triangle^\omega_{J;\kappa_2}g)\mathbf{1}_{J'}\|_\infty \\
&\qquad\qquad \times |\langle T^\alpha_\sigma(P^\sigma_{I';\kappa_1}f), Q^\omega_{J';\kappa_2}g\rangle_\omega| \\
&\quad\lesssim \sum_{\substack{I,J\in\mathcal{D}:\ J\subset I\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq\ell(I)}} \sum_{I'\in\mathfrak{C}(I),J'\in\mathfrak{C}(J)} \|(\triangle^\sigma_{I;\kappa_1}f)\mathbf{1}_{I'}\|_\infty \|(\triangle^\omega_{J;\kappa_2}g)\mathbf{1}_{J'}\|_\infty \\
&\qquad\qquad \times \mathcal{NTV}^{(\kappa_1,\kappa_2)}_{T^\alpha}(\sigma,\omega)\sqrt{|I'|_\sigma |J'|_\omega} \\
&\quad\lesssim \mathcal{NTV}^{(\kappa_1,\kappa_2)}_{T^\alpha}(\sigma,\omega)(\sigma,\omega)\sqrt{\sum_{I\in\mathcal{D}}\|\triangle^\sigma_{I;\kappa_1}f\|^2_{L^2(\sigma)}}\sqrt{\sum_{J\in\mathcal{D}}\|\triangle^\omega_{J;\kappa_2}g\|^2_{L^2(\omega)}},
\end{aligned}$$

since (4.11) yields both

$$\sum_{I'\in\mathfrak{C}(I)} \|(\triangle^\sigma_{I;\kappa_1}f)\mathbf{1}_{I'}\|^2_\infty |I'|_\sigma \lesssim \|\triangle^\sigma_{I;\kappa_1}f\|^2_{L^2(\sigma)}$$

and

$$\sum_{J'\in\mathfrak{C}(J)} \|(\triangle^\omega_{J;\kappa_2}g)\mathbf{1}_{J'}\|^2_\infty |J'|_\omega \lesssim \|\triangle^\omega_{J;\kappa_2}g\|^2_{L^2(\omega)},$$

and since the restriction $2^{-\rho}\ell(I)\leq\ell(J)\leq\ell(I)$ gives bounded overlap in the sum over $I,J\in\mathcal{D}$ with $J\subset I$. Now we finish by applying the orthonormality of Alpert projections, namely

$$\|f\|^2_{L^2(\sigma)} = \sum_{I\in\mathcal{D}}\|\triangle^\sigma_{I;\kappa_1}f\|^2_{L^2(\sigma)} \quad\text{and}\quad \|g\|^2_{L^2(\omega)} = \sum_{J\in\mathcal{D}}\|\triangle^\omega_{J;\kappa_2}g\|^2_{L^2(\omega)} \qquad \square$$

.

Lemma 31. *Suppose T^α is a standard fractional singular integral with $0\leq\alpha<n$, that all of the cubes $I,J\in\mathcal{D}$ below are $(\mathbf{r},\varepsilon)$-*good *with goodness parameters ε and $\mathbf{r}$, that $\rho>\mathbf{r}$, that $f\in L^2(\sigma)$ and $g\in L^2(\omega)$, that $\mathcal{F}\subset\mathcal{D}^\sigma$ is σ-Carleson, i.e.,*

$$\sum_{F'\in\mathcal{F}:\ F'\subset F} |F'|_\sigma \lesssim |F|_\sigma, \quad F\in\mathcal{F},$$

that there is a numerical sequence $\{\alpha_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$ such that

$$\sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)^2|F|_\sigma \leq \|f\|^2_{L^2(\sigma)}, \tag{6.7}$$

and finally that for each pair of cubes $(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega$, there is a bounded function $\beta_{I,J}$ supported in $I\setminus 2J$ satisfying

$$\|\beta_{I,J}\|_\infty \leq 1.$$

Then with $\kappa \geq 1$ *we have*

$$\sum_{\substack{(F,J)\in\mathcal{F}\times\mathcal{D}^{\omega}\\ F\cap J=\emptyset \text{ and } \ell(J)\leq 2^{-\rho}\ell(F)}} \left|\left\langle T_{\sigma}^{\alpha}\left(\beta_{F,J}\mathbf{1}_{F}\alpha_{\mathcal{F}}(F)\right), \triangle_{J;\kappa}^{\omega}g\right\rangle_{\omega}\right| \lesssim \sqrt{A_2^{\alpha}}\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}. \tag{6.8}$$

The proof of (6.8) is again virtually identical to the corresponding proof in [SaShUr7], and we will not repeat the details here.

We will also need the following Poisson estimate, that is a straightforward extension of the case $m = 1$ due to NTV in [NTV4].

Lemma 32. *Fix* $m \geq 1$. *Suppose that* $J \subset I \subset K$ *and that*

$$\operatorname{dist}(J, \partial I) > 2\sqrt{n}\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}.$$

Then

$$\mathrm{P}_m^{\alpha}(J, \sigma\mathbf{1}_{K\setminus I}) \lesssim \left(\frac{\ell(J)}{\ell(I)}\right)^{m-\varepsilon(n+m-\alpha)}\mathrm{P}_m^{\alpha}(I, \sigma\mathbf{1}_{K\setminus I}). \tag{6.9}$$

Proof. We have

$$\mathrm{P}_m^{\alpha}(J, \sigma\chi_{K\setminus I}) \approx \sum_{k=0}^{\infty} 2^{-km}\frac{1}{|2^kJ|^{1-\frac{\alpha}{n}}}\int_{(2^kJ)\cap(K\setminus I)} d\sigma,$$

and $(2^kJ)\cap(K\setminus I) \neq \emptyset$ requires

$$\operatorname{dist}(J, K\setminus I) \leq c2^k\ell(J),$$

for some dimensional constant $c > 0$. Let k_0 be the smallest such k. By our distance assumption we must then have

$$2\sqrt{n}\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon} \leq \operatorname{dist}(J, \partial I) \leq c2^{k_0}\ell(J),$$

or

$$2^{-k_0-1} \leq c\left(\frac{\ell(J)}{\ell(I)}\right)^{1-\varepsilon}.$$

Now let k_1 be defined by $2^{k_1} \equiv \frac{\ell(I)}{\ell(J)}$. Then assuming $k_1 > k_0$ (the case $k_1 \leq k_0$ is similar) we have

$$\begin{aligned}
&\mathrm{P}_m^{\alpha}(J, \sigma\chi_{K\setminus I})\\
&\quad\approx \left\{\sum_{k=k_0}^{k_1} + \sum_{k=k_1}^{\infty}\right\}2^{-km}\frac{1}{|2^kJ|^{1-\frac{\alpha}{n}}}\int_{(2^kJ)\cap(K\setminus I)} d\sigma\\
&\quad\lesssim 2^{-k_0m}\frac{|I|^{1-\frac{\alpha}{n}}}{|2^{k_0}J|^{1-\frac{\alpha}{n}}}\left(\frac{1}{|I|^{1-\frac{\alpha}{n}}}\int_{(2^{k_1}J)\cap(K\setminus I)} d\sigma\right) + 2^{-k_1m}\mathrm{P}_m^{\alpha}(I, \sigma\chi_{\setminus I})\\
&\quad\lesssim \left(\frac{\ell(J)}{\ell(I)}\right)^{(1-\varepsilon)(n+m-\alpha)}\left(\frac{\ell(I)}{\ell(J)}\right)^{n-\alpha}\mathrm{P}_m^{\alpha}(I, \sigma\chi_{K\setminus I}) + \left(\frac{\ell(J)}{\ell(I)}\right)^{m}\mathrm{P}_m^{\alpha}(I, \sigma\chi_{K\setminus I}),
\end{aligned}$$

which is the inequality (6.9). □

6.4.2 Stopping data. Next we review the notion of stopping data from [LaSaShUr3].

Definition 33. Suppose we are given a positive constant $C_0 \geq 4$, a subset $\mathcal{F}$ of the dyadic quasigrid $\mathcal{D}$ (called the stopping times), and a corresponding sequence $\alpha_{\mathcal{F}} \equiv \{\alpha_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$ of nonnegative numbers $\alpha_{\mathcal{F}}(F) \geq 0$ (called the stopping data). Let $(\mathcal{F}, \prec, \pi_{\mathcal{F}})$ be the tree structure on $\mathcal{F}$ inherited from $\mathcal{D}$, and for each $F \in \mathcal{F}$ denote by $\mathcal{C}_{\mathcal{F}}(F) = \{I \in \mathcal{D} : \pi_{\mathcal{F}} I = F\}$ the corona associated with F:

$$\mathcal{C}_{\mathcal{F}}(F) = \{I \in \mathcal{D} : I \subset F \text{ and } I \not\subset F' \text{ for any } F' \prec F\}.$$

We say the triple $(C_0, \mathcal{F}, \alpha_{\mathcal{F}})$ constitutes **stopping data** for a function $f \in L^1_{loc}(\mu)$ if

(1) $\mathbb{E}^{\mu}_I |f| \leq \alpha_{\mathcal{F}}(F)$ for all $I \in \mathcal{C}_F$ and $F \in \mathcal{F}$,
(2) $\sum_{F' \preceq F} |F'|_\mu \leq C_0 |F|_\mu$ for all $F \in \mathcal{F}$,
(3) $\sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)^2 |F|_\mu \leq C_0^2 \|f\|^2_{L^2(\mu)}$,
(4) $\alpha_{\mathcal{F}}(F) \leq \alpha_{\mathcal{F}}(F')$ whenever $F', F \in \mathcal{F}$ with $F' \subset F$,
(5) $\| \sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F) \mathbf{1}_F \|^2_{L^2(\mu)} \leq C_0' \|f\|^2_{L^2(\mu)}$.

Definition 34. If $(C_0, \mathcal{F}, \alpha_{\mathcal{F}})$ constitutes stopping data for a function $f \in L^1_{loc}(\mu)$, we refer to the othogonal weighted Alpert decomposition

$$f = \sum_{F\in\mathcal{F}} \mathsf{P}^{\mu}_{\mathcal{C}_{\mathcal{F}}(F)} f; \quad \mathsf{P}^{\mu}_{\mathcal{C}_{\mathcal{F}}(F)} f \equiv \sum_{I\in\mathcal{C}_{\mathcal{F}}(F)} \triangle^{\mu}_{I;\kappa} f,$$

as the **corona decomposition** of f associated with the stopping times $\mathcal{F}$.

It is often convenient to extend the definition of $\alpha_{\mathcal{F}}$ from $\mathcal{F}$ to the entire grid $\mathcal{D}$ by setting

$$\alpha_{\mathcal{F}}(I) \equiv \sup_{F\in\mathcal{F}: F\supset I} \alpha_{\mathcal{F}}(F).$$

When we wish to emphasize the dependence of $\alpha_{\mathcal{F}}$ on f we will write $\alpha_{\mathcal{F};f}$.

Comments on stopping data: Property (1) says that $\alpha_{\mathcal{F}}(F)$ bounds the averages of f in the corona $\mathcal{C}_F$, and property (2) says that the cubes at the tops of the coronas satisfy a Carleson condition relative to the weight μ. Note that a standard 'maximal cube' argument extends the Carleson condition in property (2) to the inequality

$$\sum_{F'\in\mathcal{F}:\ F'\subset A} |F'|_\mu \leq C_0 |A|_\mu \quad \text{for all open sets } A \subset \mathbb{R}^n.$$

Property (3) is the quasiorthogonality condition that says the sequence of functions $\{\alpha_{\mathcal{F}}(F)\mathbf{1}_F\}_{F\in\mathcal{F}}$ is in the vector-valued space $L^2(\ell^2; \mu)$, and property (4) says that the control on stopping data is nondecreasing on the stopping tree $\mathcal{F}$. (For the Calderón–Zgumund stopping times above, we have the stronger property that $\alpha_{\mathcal{F}}(F') > C_0 \alpha_{\mathcal{F}}(F)$ when F' is an $\mathcal{F}$-child of F, and this stronger property implies both (2) and (3).) Finally, property (5) is a consequence of (2) and (3) that says the sequence $\{\alpha_{\mathcal{F}}(F)\mathbf{1}_F\}_{F\in\mathcal{F}}$ has a quasiorthogonal property relative to f with a constant C_0' depending only on C_0. Indeed, the Carleson condition (2) implies a geometric decay in levels of the tree $\mathcal{F}$, namely that there are positive constants C_1 and ε, depending on C_0, such that if $\mathfrak{C}_{\mathcal{F}}^{(m)}(F)$ denotes the set of m^{th} generation children of F in $\mathcal{F}$,

$$\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(m)}(F):} |F'|_\mu \leq (C_1 2^{-\varepsilon m})^2 |F|_\mu, \quad \text{for all } m \geq 0 \text{ and } F \in \mathcal{F},$$

and the proof of Property (5) follows from this in a standard way; see, e.g., [SaShUr7].

Define Alpert corona projections

$$\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{F}}(F)} \equiv \sum_{I\in\mathcal{C}_{\mathcal{F}}(F)} \triangle^\sigma_{I;\kappa_1} \quad \text{and} \quad \mathsf{P}^\omega_{\mathcal{C}_{\mathcal{F}}^{\tau-\text{shift}}(F)} \equiv \sum_{J\in\mathcal{C}_{\mathcal{F}}^{\tau-\text{shift}}(F)} \triangle^\omega_{J;\kappa_2},$$

where

$$\mathcal{C}_{\mathcal{F}}^{\tau-\text{shift}}(F) \equiv [\mathcal{C}_{\mathcal{F}}(F) \setminus \mathcal{N}^\tau_{\mathcal{D}}(F)] \cup \bigcup_{F'\in\mathcal{F}} \mathcal{N}^\tau_{\mathcal{D}}(F');$$

$$\text{here } \mathcal{N}^\tau_{\mathcal{D}}(E) \equiv \{J \in \mathcal{D} : J \subset E \text{ and } \ell(J) \geq 2^\tau \ell(E)\}.$$

Thus the shifted corona $\mathcal{C}_{\mathcal{F}}^{\tau-\text{shift}}(F)$ has the top τ levels from $\mathcal{C}_{\mathcal{F}}(F)$ removed, and includes the first τ levels from each of its $\mathcal{F}$-children, even if some of them were initially removed. Keep in mind that we are restricting the Alpert supports of f and g to good functions so that

$$\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{F}}(F)} f = \sum_{I\in\mathcal{C}_{\mathcal{F}}^{\text{good}}(F)} \triangle^\sigma_{I;\kappa_1} \text{ and } \mathsf{P}^\omega_{\mathcal{C}_{\mathcal{F}}^{\tau-\text{shift}}(F)} g = \sum_{J\in\mathcal{C}_{\mathcal{F}}^{\text{good},\tau-\text{shift}}(F)} \triangle^\omega_{J;\kappa_2},$$

where

$$\mathcal{C}_{\mathcal{F}}^{\text{good}}(F) \equiv \mathcal{C}_{\mathcal{F}}(F) \cap \mathcal{D}^{\text{good}} \quad \text{and} \quad \mathcal{C}_{\mathcal{F}}^{\text{good},\tau-\text{shift}}(F) \equiv \mathcal{C}_{\mathcal{F}}^{\tau-\text{shift}}(F) \cap \mathcal{D}^{\text{good}}.$$

Note also that we suppress the integers κ_1 and κ_2 from the notation for the corona projections $\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{F}}(F)}$ and $\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{F}}^{\tau-\text{shift}}(F)}$. Finally note that we do not assume that σ is doubling for the next proposition, although the assumptions come close to forcing this.

6.4.3 The main Intertwining Proposition. Here now is the Intertwining Proposition with a proof obtained by adapting the argument in Nazarov, Treil and Volberg [NTV4] to the argument in [SaShUr7], and using weaker pivotal conditions with Alpert wavelets. Recall that $0 < \varepsilon < 1$ and $\mathbf{r}$ is chosen sufficiently large depending on ε. Later, in using the Intertwining Proposition to control the Far form in Subsubsection 6.6.2 below, we will need to resolve the difference between the shifted coronas used here and the parallel coronas used there.

Proposition 35 (The Intertwining Proposition)**.** Suppose that $\mathcal{F}$ is σ-Carleson, that $(C_0, \mathcal{F}, \alpha_{\mathcal{F};f})$ constitutes stopping data for f for all $f \in L^2(\sigma)$, and that

$$\| \triangle^{\sigma}_{I;\kappa_1} f \|_{L^\infty(\sigma)} \leq C\alpha_{\mathcal{F};f}(I), \quad f \in L^2(\sigma), I \in \mathcal{D}.$$

Then for good functions $f \in L^2(\sigma)$ and $g \in L^2(\omega)$, and with $\kappa_1, \kappa_2 \geq 1$, we have

$$\left| \sum_{F\in\mathcal{F}} \sum_{I:\, I \supsetneqq F} \langle T^{\alpha}_{\sigma} \triangle^{\sigma}_{I;\kappa_1} f, \mathsf{P}^{\omega}_{\mathcal{C}^{\tau-\mathrm{shift}}_{\mathcal{F}}(F)} g \rangle_{\omega} \right| \lesssim \left(\mathcal{V}^{\alpha,\kappa_1}_2 + \sqrt{A^{\alpha}_2} + \mathfrak{T}^{(\kappa_1)}_{T^{\alpha}} \right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$

Proof. We write the left-hand side of the display above as

$$\begin{aligned} \sum_{F\in\mathcal{F}} \sum_{I:\, I \supsetneqq F} \langle T^{\alpha}_{\sigma} \triangle^{\sigma}_{I;\kappa_1} f, g_F \rangle_{\omega} &= \sum_{F\in\mathcal{F}} \left\langle T^{\alpha}_{\sigma} \left(\sum_{I:\, I \supsetneqq F} \triangle^{\sigma}_{I;\kappa_1} f \right), g_F \right\rangle_{\omega} \\ &\equiv \sum_{F\in\mathcal{F}} \langle T^{\alpha}_{\sigma} f_F, g_F \rangle_{\omega}, \end{aligned}$$

where

$$g_F = \mathsf{P}^{\omega}_{\mathcal{C}^{\tau-\mathrm{shift}}_{\mathcal{F}}(F)} g \quad \text{and} \quad f_F \equiv \sum_{I:\, I \supsetneqq F} \triangle^{\sigma}_{I;\kappa_1} f.$$

Note that g_F is supported in F, and that f_F is the restriction of a polynomial of degree less than κ to F. We next observe that the cubes I occurring in this sum are linearly and consecutively ordered by inclusion, along with the cubes $F' \in \mathcal{F}$ that contain F. More precisely, we can write

$$F \equiv F_0 \subsetneqq F_1 \subsetneqq F_2 \subsetneqq \cdots \subsetneqq F_n \subsetneqq F_{n+1} \subsetneqq \cdots \subsetneqq F_N$$

where $F_m = \pi^m_{\mathcal{F}} F$ is the m^{th} ancestor of F in the tree $\mathcal{F}$ for all $m \geq 1$. We can also write

$$F = F_0 \subsetneqq I_1 \subsetneqq I_2 \subsetneqq \cdots \subsetneqq I_k \subsetneqq I_{k+1} \subsetneqq \cdots \subsetneqq I_K = F_N$$

where $I_k = \pi^k_{\mathcal{D}} F$ is the k^{th} ancestor of F in the tree $\mathcal{D}$ for all $k \geq 1$. There is a (unique) subsequence $\{k_m\}^N_{m=1}$ such that

$$F_m = I_{k_m}, \quad 1 \leq m \leq N.$$

Then we have

$$f_F(x) = \sum_{\ell=1}^{\infty} \triangle^{\sigma}_{I_\ell;\kappa_1} f(x).$$

Assume now that $k_m \leq k < k_{m+1}$. We denote the $2^n - 1$ siblings of I by $\theta(I)$, $\theta \in \Theta$, i.e., $\{\theta(I)\}_{\theta\in\Theta} = \mathfrak{C}_{\mathcal{D}}(\pi_{\mathcal{D}} I) \setminus \{I\}$. There are two cases to consider here:

$$\theta(I_k) \notin \mathcal{F} \quad \text{and} \quad \theta(I_k) \in \mathcal{F}.$$

Suppose first that $\theta(I_k) \notin \mathcal{F}$. Then $\theta(I_k) \in \mathcal{C}^{\sigma}_{F_{m+1}}$ and using a telescoping sum, we compute that for

$$x \in \theta(I_k) \subset I_{k+1} \setminus I_k \subset F_{m+1} \setminus F_m$$

we have

$$|f_F(x)| = \left| \sum_{\ell=k}^{\infty} \triangle^{\sigma}_{I_\ell;\kappa_1} f(x) \right| = |\mathbb{E}^{\sigma}_{\theta(I_k)} f(x) - \mathbb{E}^{\sigma}_{I_K} f(x)| \lesssim E^{\sigma}_{F_{m+1}} |f|,$$

by (4.7).

On the other hand, if $\theta(I_k) \in \mathcal{F}$, then $I_{k+1} \in \mathcal{C}^{\sigma}_{F_{m+1}}$ and we have for $x \in \theta(I_k)$ that

$$|f_F(x) - \triangle^{\sigma}_{\theta(I_k);\kappa_1} f(x)| = \left| \sum_{\ell=k+1}^{\infty} \triangle^{\sigma}_{I_\ell;\kappa_1} f(x) \right| = |\mathbb{E}^{\sigma}_{I_{k+1};\kappa_1} f(x) - \mathbb{E}^{\sigma}_{I_K;\kappa_1} f(x)| \lesssim E^{\sigma}_{F_{m+1}} |f|,$$

by (4.7) again. Now we write

$$\begin{aligned} &f_F = \varphi_F + \psi_F, \\ &\text{where } \varphi_F \equiv \sum_{1\leq k<\infty, \theta:\ \theta(I_k)\in\mathcal{F}} \mathbf{1}_{\theta(I_k)} \triangle^{\sigma}_{I_k;\kappa_1} f \text{ and } \psi_F = f_F - \varphi_F; \\ &\sum_{F\in\mathcal{F}} \langle T^{\alpha}_{\sigma} f_F, g_F \rangle_\omega = \sum_{F\in\mathcal{F}} \langle T^{\alpha}_{\sigma} \varphi_F, g_F \rangle_\omega + \sum_{F\in\mathcal{F}} \langle T^{\alpha}_{\sigma} \psi_F, g_F \rangle_\omega. \end{aligned}$$

We can apply (6.8) using $\theta(I_k) \in \mathcal{F}$ to the first sum here to obtain

$$\begin{aligned} \left| \sum_{F\in\mathcal{F}} \langle T^{\alpha}_{\sigma} \varphi_F, g_F \rangle_\omega \right| &\lesssim \sqrt{A^{\alpha}_2} \left\| \sum_{F\in\mathcal{F}} \varphi_F \right\|_{L^2(\sigma)} \left\| \sum_{F\in\mathcal{F}} g_F \right\|^2_{L^2(\omega)} \\ &\lesssim \sqrt{A^{\alpha}_2} \|f\|_{L^2(\sigma)} \left[\sum_{F\in\mathcal{F}} \|g_F\|^2_{L^2(\omega)} \right]^{\frac{1}{2}}. \end{aligned}$$

Turning to the second sum we note that

$$\begin{aligned} \psi_F(x) = f_F(x) - \varphi_F(x) &= \sum_{\ell=1}^{\infty} [1 - \mathbf{1}_{\mathcal{F}}(\theta(I_\ell)) \mathbf{1}_{\theta(I_\ell)}(x)] \triangle^{\sigma}_{I_\ell;\kappa_1} f(x) \\ &= \sum_{m=1}^{\infty} \sum_{\ell=k_{m-1}}^{k_m} [1 - \mathbf{1}_{\mathcal{F}}(\theta(I_\ell)) \mathbf{1}_{\theta(I_\ell)}(x)] \triangle^{\sigma}_{I_\ell;\kappa_1} f(x) \equiv \sum_{m=1}^{\infty} \psi^{(m)}_F(x), \end{aligned}$$

where

$$\psi_F^m(x) = \sum_{\ell=k_{m-1}}^{k_m} [1 - \mathbf{1}_{\mathcal{F}}(\theta(I_\ell))\mathbf{1}_{\theta(I_\ell)}(x)] \,\triangle_{I_\ell;\kappa}^{\sigma} f(x)$$
$$= \begin{cases} \mathbb{E}_{I_{\ell+1};\kappa_1} f - \mathbb{E}_{\pi_{\mathcal{F}}^{(m)} F;\kappa_1} f & \text{if } x \in \theta(I_\ell) \text{ and } \theta(I_\ell) \in \mathcal{F},\ k_m \le \ell \le k_{m+1}, \\ \mathbb{E}_{\theta(I_\ell);\kappa} f - \mathbb{E}_{\pi_{\mathcal{F}}^{(m)} F;\kappa_1} f & \text{if } x \notin \theta(I_\ell) \text{ or } \theta(I_\ell) \notin \mathcal{F},\ k_m \le \ell \le k_{m+1}. \end{cases}$$

Now we write

$$\begin{aligned} \sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha \psi_F, g_F\rangle_\omega &= \sum_{m=1}^{\infty} \sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha \psi_F^m, g_F\rangle_\omega \\ &= \sum_{m=1}^{\infty} \sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha \mathbf{1}_{\pi_{\mathcal{F}}^{m+1} F \setminus \pi_{\mathcal{F}}^{m} F} \psi_F^m, g_F\rangle_\omega \\ &\equiv \sum_{m=1}^{\infty} \sum_{F\in\mathcal{F}} \mathcal{I}_m(F), \end{aligned}$$

where

$$\mathcal{I}_m(F) = \langle T_\sigma^\alpha (\mathbf{1}_{\pi_{\mathcal{F}}^{m+1} F \setminus \pi_{\mathcal{F}}^{m} F} \psi_F^m), g_F\rangle_\omega. \tag{6.10}$$

We then note that (4.7) once more gives

$$|\psi_F^m| \lesssim E_{F_{m+1}}^{\sigma} |f| \lesssim \alpha_{\mathcal{F}}(\pi_{\mathcal{F}}^{m+1} F) \mathbf{1}_{\pi_{\mathcal{F}}^{m+1} F \setminus \pi_{\mathcal{F}}^{m} F},$$

and so

$$\begin{aligned} |\psi_F| &\le \sum_{m=0}^{N} (E_{F_{m+1}}^{\sigma} |f|) \mathbf{1}_{F_{m+1}\setminus F_m} \\ &= (E_F^\sigma |f|)\mathbf{1}_F + \sum_{m=0}^{N} (E_{\pi_{\mathcal{F}}^{m+1} F}^{\sigma} |f|) \mathbf{1}_{\pi_{\mathcal{F}}^{m+1} F \setminus \pi_{\mathcal{F}}^{m} F} \\ &= (E_F^\sigma |f|)\mathbf{1}_F + \sum_{F'\in\mathcal{F}:\ F\subset F'} (E_{\pi_{\mathcal{F}} F'}^{\sigma} |f|) \mathbf{1}_{\pi_{\mathcal{F}} F' \setminus F'} \\ &\le \alpha_{\mathcal{F}}(F)\mathbf{1}_F + \sum_{F'\in\mathcal{F}:\ F\subset F'} \alpha_{\mathcal{F}}(\pi_{\mathcal{F}} F') \mathbf{1}_{\pi_{\mathcal{F}} F' \setminus F'} \\ &\le \alpha_{\mathcal{F}}(F)\mathbf{1}_F + \sum_{F'\in\mathcal{F}:\ F\subset F'} \alpha_{\mathcal{F}}(\pi_{\mathcal{F}} F') \mathbf{1}_{\pi_{\mathcal{F}} F'} \mathbf{1}_{F^c}, \quad \text{for all } F \in \mathcal{F}. \end{aligned}$$

Now we write

$$\sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha \psi_F, g_F\rangle_\omega = I + II;$$
$$\text{where } I \equiv \sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha (\mathbf{1}_F \psi_F), g_F\rangle_\omega \quad \text{and} \quad II \equiv \sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha (\mathbf{1}_{F^c} \psi_F), g_F\rangle_\omega.$$

Then by κ-Cube Testing (6.3), and the fact that $\psi_F \mathbf{1}_F$ is a polynomial on F bounded in modulus by $\alpha_{\mathcal{F}}(F)$, we have

$$\left|\langle T_\sigma^\alpha(\psi_F), g_F\rangle_\omega\right| \leq \|T_\sigma^\alpha(\psi_F \mathbf{1}_F)\|_{L^2(\omega)} \|g_F\|_{L^2(\omega)} \leq \mathfrak{T}_{T^\alpha}^{(\kappa)} \alpha_{\mathcal{F}}(F) \sqrt{|F|_\sigma} \|g_F\|_{L^2(\omega)},$$

and then quasi-orthogonality yields

$$\begin{aligned} |I| &\leq \sum_{F\in\mathcal{F}} \left|\langle T_\sigma^\alpha \mathbf{1}_F \psi_F, g_F\rangle_\omega\right| \lesssim \mathfrak{T}_{T^\alpha}^{(\kappa)} \sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F) \sqrt{|F|_\sigma} \|g_F\|_{L^2(\omega)} \\ &\lesssim \mathfrak{T}_{T^\alpha}^{(\kappa)} \|f\|_{L^2(\sigma)} \left[\sum_{F\in\mathcal{F}} \|g_F\|_{L^2(\omega)}^2\right]^{\frac{1}{2}}. \end{aligned}$$

On the other hand, $\mathbf{1}_{F^c} \psi_F$ is supported outside F, and each J in the Alpert support of g_F is $(\mathbf{r}, \varepsilon)$-deeply embedded in F, which we write as $J \Subset_{\mathbf{r},\varepsilon} F$. So if we denote by

$$\mathcal{M}_{(\mathbf{r},\varepsilon)\text{- deep}}^{\text{good}}(F) \equiv \{\text{maximal good } J \Subset_{\mathbf{r},\varepsilon} F\}$$

the set of maximal intervals that are both good and $(\mathbf{r}, \varepsilon)$-deeply embedded in F, then

$$F = \dot{\bigcup_{K\in\mathcal{M}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)}} K = \dot{\bigcup_{G\in\mathcal{M}_{(\mathbf{r},\varepsilon)\text{- deep}}^{\text{good}}(F)}} G,$$

where each $G \in \mathcal{M}_{(\mathbf{r},\varepsilon)\text{- deep}}^{\text{good}}(F)$ is contained in some $K \in \mathcal{M}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)$.

Thus we can apply the Energy Lemma 28 to obtain from (6.10) that

$$\begin{aligned} |II| &= \left|\sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha(\mathbf{1}_{F^c}\psi_F), g_F\rangle_\omega\right| = \left|\sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}} \mathcal{I}_m(F)\right| \leq \sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}} |\mathcal{I}_m(F)| \\ &\lesssim \sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)\text{- deep}}^{\text{good}}(F)} \frac{\mathrm{P}_{\kappa_1}^\alpha\left(J, \alpha_{\mathcal{F}}(\pi_{\mathcal{F}}^{m+1}F)\mathbf{1}_{\pi_{\mathcal{F}}^{m+1}F\setminus\pi_{\mathcal{F}}^m F}\sigma\right)}{|J|^{\frac{\kappa}{n}}} \\ &\qquad\qquad \times \sqrt{\sum_{|\beta|=\kappa} \|\mathsf{Q}_{\mathcal{C}_F^{\tau\text{- shift}};J}^\omega x^\beta\|_{L^2(\omega)}^2} \|\mathsf{P}_J^\omega g_F\|_{L^2(\omega)} \\ &\quad + \sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)\text{- deep}}^{\text{good}}(F)} \frac{\mathrm{P}_{\kappa_1+\delta'}^\alpha\left(J, \alpha_{\mathcal{F}}(\pi_{\mathcal{F}}^{m+1}F)\mathbf{1}_{\pi_{\mathcal{F}}^{m+1}F\setminus\pi_{\mathcal{F}}^m F}\sigma\right)}{|J|^{\frac{\kappa}{n}}} \\ &\qquad\qquad \times \left\||x - m_J^\kappa|^\kappa\right\|_{L^2(\omega)} \|\mathsf{P}_J^\omega g_F\|_{L^2(\omega)} \\ &\equiv II_G + II_B. \end{aligned}$$

Then we have that $|II_G|$ is bounded by

$$\sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(\pi_{\mathcal{F}}^{m+1}F)$$
$$\times\left\{\sum_{J\in\mathcal{M}^{\mathrm{good}}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)}\mathrm{P}^{\alpha}_{\kappa_1}(J,\mathbf{1}_{\pi_{\mathcal{F}}^{m+1}F\setminus\pi_{\mathcal{F}}^{m}F}\sigma)\sqrt{\sum_{|\beta|=\kappa}\|\mathsf{Q}^{\omega}_{\mathcal{C}_F^{\tau\text{- shift}};J}x^{\beta}\|^2_{L^2(\omega)}}\|\mathsf{P}^{\omega}_J g_F\|_{L^2(\omega)}\right\}$$

$$\lesssim\sum_{m=1}^{\infty}\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(\pi_{\mathcal{F}}^{m+1}F)\tag{6.11}$$
$$\times\left\{\sum_{J\in\mathcal{M}^{\mathrm{good}}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)}\mathrm{P}^{\alpha}_{\kappa_1}(J,\mathbf{1}_{\pi_{\mathcal{F}}^{m+1}F\setminus\pi_{\mathcal{F}}^{m}F}\sigma)^2\sum_{|\beta|=\kappa}\|\mathsf{Q}^{\omega}_{\mathcal{C}_F^{\tau\text{- shift}};J}x^{\beta}\|^2_{L^2(\omega)}\right\}^{\frac{1}{2}}$$
$$\times\left\{\sum_{J\in\mathcal{M}^{\mathrm{good}}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)}\|\mathsf{P}^{\omega}_J g_F\|^2_{L^2(\omega)}\right\}^{\frac{1}{2}}.$$

We now reindex the last sum in (6.11) above using $F^*=\pi_{\mathcal{F}}^{m+1}F$ to rewrite it as

$$\sum_{m=1}^{\infty}\sum_{F^*\in\mathcal{F}}\alpha_{\mathcal{F}}(F^*)\sum_{F'\in\mathfrak{C}_{\mathcal{F}}(F^*)}\sum_{F\in\mathfrak{C}^{(m)}_{\mathcal{F}}(F')}\left\{\sum_{J\in\mathcal{M}^{\mathrm{good}}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)}\mathrm{P}^{\alpha}_{\kappa}(J,\mathbf{1}_{F^*\setminus F'}\sigma)^2\right.$$
$$\left.\times\sum_{|\beta|=\kappa}\left\|\mathsf{Q}^{\omega}_{\mathcal{C}_F^{\mathrm{good},\tau\text{- shift}};J}\left(\frac{x}{\ell(J)}\right)^{\beta}\right\|^2_{L^2(\omega)}\right\}^{\frac{1}{2}}\tag{6.12}$$
$$\times\left\{\sum_{J\in\mathcal{M}^{\mathrm{good}}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)}\|\mathsf{P}^{\omega}_J g_F\|^2_{L^2(\omega)}\right\}^{\frac{1}{2}}.$$

Using (6.9) with $m=\kappa$ and $\mu=\sigma$, we obtain that for $J\in\mathcal{M}^{\mathrm{good}}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)$ and $I=\pi_{\mathcal{F}}^{m-1}F$, we have

$$\frac{\ell(J)}{\ell(\pi_{\mathcal{F}}^{m-1}F)}=2^{-k}$$

for some $k\geq m-1$, and hence

$$\begin{aligned}\mathrm{P}^{\alpha}_{\kappa_1}(J,\mathbf{1}_{F^*\setminus F'}\sigma)^2&\leq\left(\frac{\ell(J)}{\ell(\pi_{\mathcal{F}}^{m-1}F)}\right)^{2\kappa-2\varepsilon(n+\kappa-\alpha)}\mathrm{P}^{\alpha}_{\kappa_1}(\pi_{\mathcal{F}}^{m-1}F,\mathbf{1}_{\pi_{\mathcal{F}}^{m+1}F\setminus\pi_{\mathcal{F}}^{m}F}\sigma)^2\\&=(2^{-k})^{2\kappa-2\varepsilon(n+\kappa-\alpha)}\mathrm{P}^{\alpha}_{\kappa_1}(\pi_{\mathcal{F}}^{m-1}F,\mathbf{1}_{\pi_{\mathcal{F}}^{m+1}F\setminus\pi_{\mathcal{F}}^{m}F}\sigma)^2.\end{aligned}$$

Now we pigeonhole the intervals J by side length in the sum over $J\in\mathcal{M}^{\mathrm{good}}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)$ in the first factor in braces in (6.12) to obtain that it satisfies, under the assumptions $F' \in \mathfrak{C}_{\mathcal{F}}(F^*)$ and $F \in \mathfrak{C}_{\mathcal{F}}^{(m)}(F')$,

$$\begin{aligned}
&\sum_{J\in\mathcal{M}^{\mathrm{good}}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)} \mathrm{P}^{\alpha}_{\kappa_1}(J,\mathbf{1}_{F^*\setminus F'}\sigma)^2 \sum_{|\beta|=\kappa}\left\|\mathsf{Q}^{\omega}_{\mathcal{C}_F^{\tau-\mathrm{shift}};J}\left(\frac{x}{\ell(J)}\right)^{\beta}\right\|^2_{L^2(\omega)}\\
&\lesssim \sum_{k=m-1}^{\infty}(2^{-k})^{2\kappa-2\varepsilon(n+\kappa-\alpha)}\mathrm{P}^{\alpha}_{\kappa_1}(\pi^{m-1}_{\mathcal{F}}F,\mathbf{1}_{\pi^{m+1}_{\mathcal{F}}F\setminus\pi^{m}_{\mathcal{F}}F}\sigma)^2\\
&\quad\times \sum_{\substack{J\in\mathcal{M}^{\mathrm{good}}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)\\ \ell(J)=2^{-k}\ell(\pi^{m-1}_{\mathcal{F}}F)}}\ \sum_{|\beta|=\kappa}\left\|\mathsf{Q}^{\omega}_{\mathcal{C}_F^{\tau\text{- shift}};J}\left(\frac{x}{\ell(J)}\right)^{\beta}\right\|^2_{L^2(\omega)}\\
&\lesssim \sum_{k=m-1}^{\infty}(2^{-k})^{2\kappa-2\varepsilon(n+\kappa-\alpha)}\mathrm{P}^{\alpha}_{\kappa}(\pi^{m-1}_{\mathcal{F}}F,\mathbf{1}_{\pi^{m+1}_{\mathcal{F}}F\setminus\pi^{m}_{\mathcal{F}}F}\sigma)^2 \sum_{\substack{J\in\mathcal{M}^{\mathrm{good}}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)\\ \ell(J)=2^{-k}\ell(\pi^{m-1}_{\mathcal{F}}F)}} |J|_{\omega}\\
&\lesssim (2^{-m})^{2\kappa-2\varepsilon(n+\kappa-\alpha)}\mathrm{P}^{\alpha}_{\kappa_1}(\pi^{m-1}_{\mathcal{F}}F,\mathbf{1}_{\pi^{m+1}_{\mathcal{F}}F\setminus\pi^{m}_{\mathcal{F}}F}\sigma)^2 \sum_{J\in\mathcal{M}^{\mathrm{good}}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)} |J|_{\omega}
\end{aligned}$$

so that altogether we have

$$\begin{aligned}
|II_G| &\lesssim \sum_{F\in\mathcal{F}}\left\{\sum_{m=1}^{\infty}\alpha_{\mathcal{F}}(\pi^{m+1}_{\mathcal{F}}F)^2(2^{-m})^{\kappa-\varepsilon(n+\kappa-\alpha)}\mathrm{P}^{\alpha}_{\kappa_1}(\pi^{m-1}_{\mathcal{F}}F,\mathbf{1}_{\pi^{m+1}_{\mathcal{F}}F\setminus\pi^{m}_{\mathcal{F}}F}\sigma)^2|F|_{\omega}\right\}^{\frac{1}{2}}\\
&\qquad\times C_{\varepsilon,\alpha}\|g_F\|_{L^2(\omega)}\\
&\lesssim\left\{\sum_{m=1}^{\infty}(2^{-m})^{\kappa-\varepsilon(n+\kappa-\alpha)}\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(\pi^{m+1}_{\mathcal{F}}F)^2\mathrm{P}^{\alpha}_{\kappa_1}(\pi^{m-1}_{\mathcal{F}}F,\mathbf{1}_{\pi^{m+1}_{\mathcal{F}}F\setminus\pi^{m}_{\mathcal{F}}F}\sigma)^2|F|_{\omega}\right\}^{\frac{1}{2}}\\
&\quad\times\sqrt{\sum_{F\in\mathcal{F}}\|g_F\|^2_{L^2(\omega)}}\\
&\lesssim(\mathcal{V}^{\alpha,\kappa_1}_2)^2\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)},
\end{aligned}$$

since $\kappa-\varepsilon(n+\kappa-\alpha)>0$ implies

$$C_{\varepsilon,\alpha}=\sqrt{\sum_{m=1}^{\infty}(2^{-m})^{\kappa-\varepsilon(n+\kappa-\alpha)}}<\infty,$$

and since for each fixed $m \geq 1$ we have

$$\begin{aligned}
&\sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(\pi_{\mathcal{F}}^{m+1}F)^2 \mathrm{P}_{\kappa_1}^{\alpha}(\pi_{\mathcal{F}}^{m-1}F, \mathbf{1}_{\pi_{\mathcal{F}}^{m+1}F\setminus\pi_{\mathcal{F}}^{m}F}\sigma)^2 |F|_\omega \\
&\qquad = \sum_{F'\in\mathcal{F}} \alpha_{\mathcal{F}}(\pi_{\mathcal{F}}F')^2 \sum_{F''\in\mathfrak{C}_{\mathcal{F}}(F')} \mathrm{P}_{\kappa_1}^{\alpha}(F'', \mathbf{1}_{\pi_{\mathcal{F}}F'\setminus F'}\sigma)^2 \sum_{F\in\mathfrak{C}_{\mathcal{F}}^{m-1}(F'')} |F|_\omega \\
&\qquad \leq \sum_{F'\in\mathcal{F}} \alpha_{\mathcal{F}}(\pi_{\mathcal{F}}F')^2 \sum_{F''\in\mathfrak{C}_{\mathcal{F}}(F')} \mathrm{P}_{\kappa_1}^{\alpha}(F'', \mathbf{1}_{\pi_{\mathcal{F}}F'\setminus F'}\sigma)^2 |F''|_\omega \\
&\qquad = \sum_{F^*\in\mathcal{F}} \alpha_{\mathcal{F}}(F^*)^2 \sum_{F''\in\mathfrak{C}_{\mathcal{F}}^{(2)}(F^*)} \mathrm{P}_{\kappa_1}^{\alpha}(F'', \mathbf{1}_{F^*\setminus\pi_{\mathcal{F}}F''}\sigma)^2 |F''|_\omega \\
&\qquad \leq \sum_{F'\in\mathcal{F}} \alpha_{\mathcal{F}}(\pi_{\mathcal{F}}F')^2 (\mathcal{V}_2^{\alpha,\kappa_1})^2 |\pi_{\mathcal{F}}F'|_\sigma \leq (\mathcal{V}_2^{\alpha,\kappa_1})^2 \|f\|_{L^2(\sigma)}^2 .
\end{aligned}$$

In term II_B the expressions $\| |x - m_J^{\kappa_1}|^{\kappa_1} \|_{L^2(\omega)}^2$ are no longer 'almost orthogonal' in J, and we must instead exploit the extra decay in the Poisson integral $\mathrm{P}_{\kappa+\delta'}^{\alpha}$ due to the addition of $\delta' > 0$, along with goodness of the cubes J. This idea was already used by M. Lacey and B. Wick in [LaWi] in a similar situation, and subsequently exploited in [SaShUr7]. As a consequence of this decay we will be able to bound II_B directly by the κ^{th}-order pivotal condition, without having to invoke the more difficult functional energy condition of [LaSaShUr3] and [SaShUr7]. For the decay, we follow [SaShUr7] and use the 'large' function

$$\Phi \equiv \sum_{F''\in\mathcal{F}} \alpha_{\mathcal{F}}(F'') \mathbf{1}_{F''}$$

that dominates $|\psi_F|$ for all $F \in \mathcal{F}$, and compute that

$$\begin{aligned}
\frac{\mathrm{P}_{\kappa_1+\delta'}^{\alpha}(J, \Phi\sigma)}{|J|^{\frac{\kappa}{n}}} &= \int_{F^c} \frac{|J|^{\frac{\delta'}{n}}}{|y - c_J|^{n+\kappa+\delta-\alpha}} \Phi(y) d\sigma(y) \\
&\leq \sum_{t=0}^{\infty} \int_{\pi_{\mathcal{F}}^{t+1}F\setminus\pi_{\mathcal{F}}^{t}F} \left(\frac{|J|^{\frac{1}{n}}}{\operatorname{dist}(c_J, (\pi_{\mathcal{F}}^t F)^c)} \right)^{\delta'} \frac{1}{|y - c_J|^{n+\kappa_1-\alpha}} \Phi(y) d\sigma(y) \\
&\leq \sum_{t=0}^{\infty} \left(\frac{|J|^{\frac{1}{n}}}{\operatorname{dist}(c_J, (\pi_{\mathcal{F}}^t F)^c)} \right)^{\delta'} \frac{\mathrm{P}_{\kappa_1}^{\alpha}(J, \mathbf{1}_{\pi_{\mathcal{F}}^{t+1}F\setminus\pi_{\mathcal{F}}^{t}F}\Phi\sigma)}{|J|^{\frac{\kappa}{n}}},
\end{aligned}$$

and then use the goodness inequality

$$\begin{aligned}
\operatorname{dist}(c_J, (\pi_{\mathcal{F}}^t F)^c) &\geq \frac{1}{2} \ell(\pi_{\mathcal{F}}^t F)^{1-\varepsilon} \ell(J)^{\varepsilon} \geq \frac{1}{2} 2^{t(1-\varepsilon)} \ell(F)^{1-\varepsilon} \ell(J)^{\varepsilon} \\
&\geq 2^{t(1-\varepsilon)-1} \ell(J),
\end{aligned}$$

to conclude that

$$\left(\frac{\mathrm{P}^{\alpha}_{\kappa_1+\delta'}(J,\mathbf{1}_{F^c}\Phi\sigma)}{|J|^{\frac{\kappa}{n}}}\right)^2 \lesssim \left(\sum_{t=0}^{\infty}2^{-t\delta'(1-\varepsilon)}\frac{\mathrm{P}^{\alpha}_{\kappa_1}(J,\mathbf{1}_{\pi^{t+1}_{\mathcal{F}}F\setminus\pi^t_{\mathcal{F}}F}\Phi\sigma)}{|J|^{\frac{\kappa}{n}}}\right)^2 \tag{6.13}$$
$$\lesssim \sum_{t=0}^{\infty}2^{-t\delta'(1-\varepsilon)}\left(\frac{\mathrm{P}^{\alpha}_{\kappa_1}(J,\mathbf{1}_{\pi^{t+1}_{\mathcal{F}}F\setminus\pi^t_{\mathcal{F}}F}\Phi\sigma)}{|J|^{\frac{\kappa}{n}}}\right)^2 .$$

Now we apply Cauchy–Schwarz to obtain

$$\begin{aligned}
II_B &= \sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}\frac{\mathrm{P}^{\alpha}_{\kappa_1+\delta'}(J,\mathbf{1}_{F^c}\Phi\sigma)}{|J|^{\frac{\kappa}{n}}}\left\| |x-m^{\kappa_1}_J|^{\kappa_1}\right\|_{L^2(\omega)}\left\|\mathsf{P}^{\omega}_J g_F\right\|_{L^2_{\mathcal{H}_2}(\omega)}\\
&\le \left(\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)}\left(\frac{\mathrm{P}^{\alpha}_{\kappa_1+\delta'}(J,\mathbf{1}_{F^c}\Phi\sigma)}{|J|^{\frac{\kappa}{n}}}\right)^2\left\| |x-m^{\kappa_1}_J|^{\kappa_1}\right\|^2_{L^2(\omega)}\right)^{\frac{1}{2}}\\
&\quad\times\left[\sum_F\|g_F\|^2_{L^2_{\mathcal{H}_2}(\omega)}\right]^{\frac{1}{2}}\\
&\equiv \sqrt{II_{\mathrm{energy}}}\left[\sum_F\|g_F\|^2_{L^2_{\mathcal{H}_2}(\omega)}\right]^{\frac{1}{2}},
\end{aligned}$$

and it remains to estimate II_{energy}. From (6.13) and the κ^{th} order pivotal condition we have

$$\begin{aligned}
II_{\mathrm{energy}} &\le \sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)}\sum_{t=0}^{\infty}2^{-t\delta'(1-\varepsilon)}\left(\frac{\mathrm{P}^{\alpha}_{\kappa_1}(J,\mathbf{1}_{\pi^{t+1}_{\mathcal{F}}F\setminus\pi^t_{\mathcal{F}}F}\Phi\sigma)}{|J|^{\frac{\kappa}{n}}}\right)^2\left\| |x-m^{\kappa_1}_J|^{\kappa_1}\right\|^2_{L^2(\omega)}\\
&= \sum_{t=0}^{\infty}2^{-t\delta'(1-\varepsilon)}\sum_{G\in\mathcal{F}}\sum_{F\in\mathfrak{C}^{(t+1)}_{\mathcal{F}}(G)}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)}\left(\frac{\mathrm{P}^{\alpha}_{\kappa_1}(J,\mathbf{1}_{G\setminus\pi^t_{\mathcal{F}}F}\Phi\sigma)}{|J|^{\frac{\kappa}{n}}}\right)^2|J|^{\frac{k}{n}}|J|_{\omega}\\
&\lesssim \sum_{t=0}^{\infty}2^{-t\delta'(1-\varepsilon)}\sum_{G\in\mathcal{F}}\alpha_{\mathcal{F}}(G)^2\sum_{F\in\mathfrak{C}^{(t+1)}_{\mathcal{F}}(G)}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)\text{- deep}}(F)}\mathrm{P}^{\alpha}_{\kappa_1}(J,\mathbf{1}_{G\setminus\pi^t_{\mathcal{F}}F}\sigma)^2|J|_{\omega}\\
&\lesssim \sum_{t=0}^{\infty}2^{-t\delta'(1-\varepsilon)}\sum_{G\in\mathcal{F}}\alpha_{\mathcal{F}}(G)^2((\mathcal{V}^{\alpha,\kappa_1}_{\alpha})^2+A^{\alpha}_2)|G|_{\sigma}\lesssim((\mathcal{V}^{\alpha,\kappa_1}_{\alpha})^2+A^{\alpha}_2)\|f\|^2_{L^2(\sigma)}.
\end{aligned}$$

This completes the proof of the Intertwining Proposition 35. □

6.4.4 An alternate Intertwining Corollary.

We will also need an alternate version of the Intertwining Proposition 35 in which J and I are at least τ levels apart, but where the proximity of J and I to F is reversed, namely the cubes J are close to F but the cubes I are not. We exploit the doubling property of σ to obtain this alternate version as a relatively simple corollary of the Intertwining Proposition 35.

Corollary 36 (The alternate Intertwining Corollary)**.** *Suppose that σ is a doubling measure, that $\mathcal{F}$ is σ-Carleson, that $(C_0, \mathcal{F}, \alpha_{\mathcal{F};f})$ constitutes stopping data for f for all $f \in L^2(\sigma)$, and that*

$$\| \triangle^{\sigma}_{I;\kappa_1} f \|_{L^\infty(\sigma)} \leq C\alpha_{\mathcal{F};f}(I), \quad f \in L^2(\sigma), I \in \mathcal{D}.$$

Let $\mathcal{W}_F$ be any subset of $\mathcal{C}_{\mathcal{F}}(F)$. Then for good *functions $f \in L^2(\sigma)$ and $g \in L^2(\omega)$, and with $\kappa_1, \kappa_2 \geq 1$, we have*

$$\left| \sum_{F\in\mathcal{F}} \sum_{I:\ I \supsetneqq \pi^{(\tau)}_{\mathcal{D}} F} \langle T^{\alpha}_{\sigma} \triangle^{\sigma}_{I;\kappa_1} f, \mathsf{P}^{\omega}_{\mathcal{W}_F} g \rangle_{\omega} \right| \lesssim (\mathcal{V}_2^{\alpha,\kappa_1} + \sqrt{A_2^{\alpha}} + \mathfrak{T}^{(\kappa_1)}_{T^{\alpha}}) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$

Note that the cubes J in $\mathcal{W}_F$ can be close to F, but that the cubes I with $I \supsetneqq \pi^{(\tau)}_{\mathcal{D}} F$ are far from F.

Proof. We will apply the Intertwining Proposition 35 to stopping data $(C_0, \mathcal{H}, \alpha_{\mathcal{H};f})$ derived from the τ-grandparents of cubes in $\mathcal{F}$, where

$$\mathcal{H} \equiv \{ \pi^{(\tau)}_{\mathcal{D}} A : A \in \mathcal{A} \}.$$

Since σ is doubling we conclude that the collection of τ-grandparents $\mathcal{H}$ also satisfies a σ-Carleson condition. In fact, if $H = \pi^{(\tau)}_{\mathcal{D}} A \subset \pi^{(\tau)}_{\mathcal{D}} B = K$, then $A \subset K$, and so if $\mathcal{M}^{(\tau)}_K$ is the collection of maximal cubes $A \in \mathcal{A}$ for which $\pi^{(\tau)}_{\mathcal{D}} A \subset K$, we have

$$\begin{aligned}
\sum_{H\in\mathcal{H}:\ H\subset K} |H|_{\sigma} &= \sum_{A\in\mathcal{A}:\ \pi^{(\tau)}_{\mathcal{D}}A\subset K} |\pi^{(\tau)}_{\mathcal{D}} A|_{\sigma} = \sum_{M\in\mathcal{M}^{(\tau)}_K} \sum_{A\in\mathcal{A}:\ A\subset M} |\pi^{(\tau)}_{\mathcal{D}} A|_{\sigma} \\
&\leq C_{\tau} \sum_{M\in\mathcal{M}^{(\tau)}_K} \sum_{A\in\mathcal{A}:\ A\subset M} |A|_{\sigma} \leq C_{\tau} C_{\mathrm{Carleson}} \sum_{M\in\mathcal{M}^{(\tau)}_K} |M|_{\sigma} \\
&\leq C_{\tau} C_{\mathrm{Carleson}} |M|_{\sigma}.
\end{aligned}$$

Moreover, from this σ-Carleson condition, and the generalized Carleson Embedding Theorem, we obtain the following quasi-orthogonality inequality

$$\sum_{H\in\mathcal{H}} |H|_{\sigma} \left(\sup_{H'\in\mathcal{D}:\ H'\supset H} \frac{1}{|H'|_{\sigma}} \int_{H'} |f| d\sigma \right)^2 \lesssim \|f\|^2_{L^2(\sigma)}. \tag{6.14}$$

Indeed, this follows from interpolating the trivial estimate

$$A : L^{\infty}(\sigma) \to \ell^{\infty}(\mathcal{H})$$

for the sublinear operator

$$Af(H) \equiv \sup_{H'\in\mathcal{D}:\ H'\supset H} E^{\sigma}_{H'} |f|$$

with the weak type estimate

$$A : L^1(\sigma) \to \ell^{1,\infty}(\mathcal{H}),$$

which in turn follows by applying the Carleson condition to the maximal cubes M for which $Af(M) > \lambda$, $\lambda > 0$. Finally, set

$$\alpha_{\mathcal{H};f}(H) \equiv \sup_{H' \in \mathcal{D}:\ H' \supset H} \frac{1}{|H'|_\sigma} \int_{H'} |f| d\sigma, \quad H \in \mathcal{H},$$

so that the triple $(C_0, \mathcal{H}, \alpha_{\mathcal{H};f})$ constitutes stopping data for the function $f \in L^2(\sigma)$ in the sense of Definition 33. Now define an Alpert projection $\widehat{g}$ so that

$$\sum_{F \in \mathcal{F}} \mathsf{P}^{\omega}_{\mathcal{W}_F} g = \sum_{F \in \mathcal{F}} \mathsf{P}^{\omega}_{\mathcal{C}_{\mathcal{F}}(F)} \widehat{g}.$$

Then $\|\widehat{g}\|_{L^2(\omega)} \leq \|g\|_{L^2(\omega)}$ and the Intertwining Proposition 35 yields

$$\left| \sum_{H \in \mathcal{H}} \sum_{I:\ I \supsetneqq H} \langle T^\alpha_\sigma \triangle^\sigma_{I;\kappa_1} f, \mathsf{P}^{\omega}_{\mathcal{C}^{\tau-\mathrm{shift}}_{\mathcal{H}}(H)} \widehat{g} \rangle_\omega \right| \lesssim (\mathcal{V}^{\alpha,\kappa_1}_2 + \sqrt{A^\alpha_2} + \mathfrak{T}^{(\kappa_1)}_{T^\alpha}) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$

Unravelling the definitions shows that this inequality is precisely the conclusion of the Alternate Intertwining Corollary 36. □

6.5 The Parallel Corona. Armed with the Monotonicity Lemma and the Intertwining Proposition from the previous two subsections, we can now give the proof of Theorem 6, for which it suffices to show that

$$\begin{aligned} &|\langle T^\alpha_\sigma f, g \rangle_{L^2(\omega)}| \\ &\lesssim \left(\mathfrak{T}_{T^\alpha} + \mathfrak{T}^*_{T^\alpha} + \mathcal{BICT}_{T^\alpha} + \mathcal{V}^{\alpha,\kappa_1}_2 + \mathcal{V}^{\alpha,\kappa_2,*}_2 + \sqrt{\mathcal{A}^\alpha_2} + \sqrt{\mathcal{A}^{\alpha,*}_2} \right) \\ &\quad \times \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}, \end{aligned} \tag{6.15}$$

since by (4.4),

$$\begin{aligned} \mathcal{V}^{\alpha,\kappa_1}_2 + \mathcal{V}^{\alpha,\kappa_2,*}_2 &\leq C_{n,\alpha,\kappa_1,(\beta_1,\gamma_1),\kappa_2,(\beta_2,\gamma_2)} \sqrt{A^\alpha_2}, \\ &\text{for } \kappa_1 > \theta_1 + \alpha - n \text{ and } \kappa_2 > \theta_2 + \alpha - n. \end{aligned}$$

Note that as above we are abbreviating $\mathfrak{T}^{(\kappa)}_{(T^\alpha)^*}(\omega, \sigma)$ with $\mathfrak{T}^{(\kappa),*}_{T^\alpha}$.

As a first step, we will prove the weaker inequality

$$\begin{aligned} &|\langle T^\alpha_\sigma f, g \rangle_{L^2(\omega)}| \\ &\lesssim (\mathfrak{T}^{(\kappa)}_{T^\alpha} + \mathfrak{T}^{(\kappa),*}_{T^\alpha} + \mathcal{BICT}_{T^\alpha} + \mathcal{V}^{\alpha,\kappa_1}_2 + \mathcal{V}^{\alpha,\kappa_2,*}_2 + \sqrt{A^\alpha_2} + \mathcal{WBP}^{(\kappa_1,\kappa_2)}_{T^\alpha}) \\ &\quad \times \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}, \end{aligned} \tag{6.16}$$

in which we only need the classical Muckenhoupt constant A_2^α, and then replace κ-testing with 1-testing, and remove the κ^{th}-order weak boundedness constant $\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega)$ all at the price of using the one-tailed constants $\mathcal{A}_2^\alpha, \mathcal{A}_2^{\alpha,*}$ instead of A_2^α.

A crucial result of Nazarov, Treil and Volberg in [NTV1], [NTV3] and [NTV4] shows that all of the cubes I and J in the sum

$$\langle T_\sigma^\alpha f, g\rangle_{L^2(\omega)} = \sum_{I,J\in\mathcal{D}} \langle T_\sigma^\alpha(\triangle_{I;\kappa_1}^\sigma f), \triangle_{J;\kappa_2}^\omega g\rangle_{L^2(\omega)}$$

may be assumed $(\mathbf{r},\varepsilon)$-good.

6.5.1 The Calderón–Zygmund corona construction. Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$. Let $\mathcal{F}$ be a collection of Calderón–Zygmund stopping cubes for f, and let

$$\mathcal{D} = \bigcup_{F\in\mathcal{F}} \mathcal{C}_{\mathcal{F}}(F)$$

be the associated corona decomposition of the dyadic grid $\mathcal{D}$. Then we have

$$\begin{aligned} E_{F'}^\mu|f| &> C_0 E_F^\mu|f| \quad \text{whenever } F', F \in \mathcal{F} \text{ with } F' \subsetneqq F, \\ E_I^\mu|f| &\le C_0 E_F^\mu|f| \quad \text{for } I \in \mathcal{C}_{\mathcal{F}}(F). \end{aligned}$$

For a cube $I \in \mathcal{D}$ let $\pi_{\mathcal{D}} I$ be the $\mathcal{D}$-parent of I in the grid $\mathcal{D}$, and let $\pi_{\mathcal{F}} I$ be the smallest member of $\mathcal{F}$ that contains I. For $F, F' \in \mathcal{F}$, we say that F' is an $\mathcal{F}$-child of F if $\pi_{\mathcal{F}}(\pi_{\mathcal{D}} F') = F$ (it could be that $F = \pi_{\mathcal{D}} F'$), and we denote by $\mathfrak{C}_{\mathcal{F}}(F)$ the set of $\mathcal{F}$-children of F.

For $F \in \mathcal{F}$, define the projection $\mathsf{P}_{\mathcal{C}_{\mathcal{F}}(F)}^\mu$ onto the linear span of the Alpert functions $\{h_{I;\kappa}^{\mu,a}\}_{I\in\mathcal{C}_F, a\in\Gamma_{I,n.\kappa}}$ by

$$\mathsf{P}_{\mathcal{C}_{\mathcal{F}}(F)}^\mu f = \sum_{I\in\mathcal{C}_{\mathcal{F}}(F)} \triangle_{I;\kappa}^\mu f = \sum_{I\in\mathcal{C}_F, a\in\Gamma_{I,n.\kappa}} \langle f, h_{I;\kappa}^{\mu,a}\rangle_{L^2(\sigma)} h_{I;\kappa}^{\mu,a}.$$

The standard properties of these projections are

$$f = \sum_{F\in\mathcal{F}} \mathsf{P}_{\mathcal{C}_{\mathcal{F}}(F)}^\mu f, \quad \int (\mathsf{P}_{\mathcal{C}_{\mathcal{F}}(F)}^\mu f) d\mu = 0, \quad \|f\|_{L^2(\mu)}^2 = \sum_{F\in\mathcal{F}} \|\mathsf{P}_{\mathcal{C}_{\mathcal{F}}(F)}^\mu f\|_{L^2(\mu)}^2.$$

There is also a μ-Carleson condition satisfied by the stopping cubes, namely

$$\sum_{F'\in\mathcal{F}:\ F'\subset F} |F'|_\mu \le C_0 |F|_\mu \quad \text{for all } F \in \mathcal{F}.$$

Thus with $\alpha_{\mathcal{F}} \equiv E_F^\mu|f|$, the triple $(C_0, \mathcal{F}, \alpha_{\mathcal{F}})$ constitutes stopping data for f in the sense of [LaSaShUr3], i.e., Definition 33 above.

Important restriction: In the proof of Theorem 6 we only use the Calderón–Zygmund corona decomposition, and in this case, property (1) can be improved to

$$\mathbb{E}_F^\mu |f| \approx \alpha_{\mathcal{F}}(F) \quad \text{for all } F \in \mathcal{F},$$

which we assume for the remainder of the proof.

6.6 Form splittings and decompositions. Let $(C_0, \mathcal{A}, \alpha_{\mathcal{A}})$ constitute stopping data for $f \in L^2(\sigma)$, and let $(C_0, \mathcal{B}, \alpha_{\mathcal{B}})$ constitute stopping data for $g \in L^2(\omega)$ as in the previous subsubsection. We now organize the bilinear form,

$$\begin{aligned}
\langle T_\sigma^\alpha f, g\rangle_\omega &= \left\langle T_\sigma^\alpha \left(\sum_{I\in\mathcal{D}} \triangle_{I;\kappa_1}^\sigma f \right), \left(\sum_{J\in\mathcal{D}} \triangle_{J;\kappa_2}^\omega g \right) \right\rangle_\omega \\
&= \sum_{I\in\mathcal{D} \text{and } J\in\mathcal{D}} \langle T_\sigma^\alpha (\triangle_{I;\kappa_1}^\sigma f), (\triangle_{J;\kappa_2}^\omega g)\rangle_\omega \\
&= \sum_{(A,B)\in\mathcal{A}\times\mathcal{B}} \ \sum_{I\in\mathcal{C}_{\mathcal{A}}(A) \text{ and } J\in\mathcal{C}_{\mathcal{B}}(B)} \langle T_\sigma^\alpha (\triangle_{I;\kappa_1}^\sigma f), (\triangle_{J;\kappa_2}^\omega g)\rangle_\omega \\
&= \sum_{(A,B)\in\mathcal{A}\times\mathcal{B}} \langle T_\sigma^\alpha (\mathsf{P}_{\mathcal{C}_{\mathcal{A}}(A)}^\sigma f), \mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B)}^\omega g\rangle_\omega,
\end{aligned}$$

as a sum over the families of Calderón–Zygmund stopping cubes $\mathcal{A}$ and $\mathcal{B}$, and then decompose this sum by the Parallel Corona decomposition, in which the 'diagonal cut' in the bilinear form is made according to the relative positions of intersecting coronas, rather than the traditional way of making the 'diagonal cut' according to relative side lengths of cubes. The parallel corona as used here was introduced in an unpublished manuscript on the arXiv [LaSaShUr4] by Lacey, Sawyer, Shen and Uriarte-Tuero that proved the Indicator/Interval Testing characterization for the Hilbert transform, just before Michael Lacey's breakthrough in controlling the local form [Lac]. This manuscript was referenced in the survey article [Lac2, see page 21], and subsequently used in at least [Hyt2], [Tan] and [LaSaShUrWi].

We have

$$\begin{aligned}
&\langle T_\sigma^\alpha f, g\rangle_\omega \\
&\quad = \sum_{(A,B)\in\mathcal{A}\times\mathcal{B}} \langle T_\sigma^\alpha (\mathsf{P}_{\mathcal{C}_{\mathcal{A}}(A)}^\sigma f), \mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B)}^\omega g\rangle_\omega \\
&\quad = \left\{ \sum_{(A,B)\in\text{Near}(\mathcal{A}\times\mathcal{B})} + \sum_{(A,B)\in\text{Disjoint}(\mathcal{A}\times\mathcal{B})} + \sum_{(A,B)\in\text{Far}(\mathcal{A}\times\mathcal{B})} \right\} \qquad (6.17) \\
&\qquad \times \langle T_\sigma^\alpha (\mathsf{P}_{\mathcal{C}_{\mathcal{A}}(A)}^\sigma f), \mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B)}^\omega g\rangle_\omega \\
&\quad \equiv \text{Near}(f,g) + \text{Disjoint}(f,g) + \text{Far}(f,g).
\end{aligned}$$

Here, $\mathrm{Near}(\mathcal{A}\times\mathcal{B})$ is the set of pairs $(A,B)\in\mathcal{A}\times\mathcal{B}$ such that one of A,B is contained in the other, and there is no $A_1\in\mathcal{A}$ with $B\subset A_1\subsetneqq A$, nor is there $B_1\in\mathcal{B}$ with $A\subset B_1\subsetneqq B$. The set $\mathrm{Disjoint}(\mathcal{A}\times\mathcal{B})$ is the set of pairs $(A,B)\in\mathcal{A}\times\mathcal{B}$ such that $A\cap B=\emptyset$. The set $\mathrm{Far}(\mathcal{A}\times\mathcal{B})$ is the complement of $\mathrm{Near}(\mathcal{A}\times\mathcal{B})\cup\mathrm{Disjoint}(\mathcal{A}\times\mathcal{B})$ in $\mathcal{A}\times\mathcal{B}$:

$$\mathrm{Far}(\mathcal{A}\times\mathcal{B})=(\mathcal{A}\times\mathcal{B})\setminus\{\mathrm{Near}(\mathcal{A}\times\mathcal{B})\cup\mathrm{Disjoint}(\mathcal{A}\times\mathcal{B})\}.$$

Note that if $(A,B)\in\mathrm{Far}(\mathcal{A}\times\mathcal{B})$, then either $B\subset A'$ for some $A'\in\mathfrak{C}_{\mathcal{A}}(A)$, or $A\subset B'$ for some $B'\in\mathfrak{C}_{\mathcal{B}}(B)$.

6.6.1 Disjoint form. By Lemma 30, the disjoint form $\mathrm{Disjoint}(f,g)$ is controlled by the A_2^α condition, the κ-cube testing conditions (6.3), and the κ-weak boundedness property (6.4):

$$|\,\mathrm{Disjoint}(f,g)|\lesssim\left(\mathfrak{T}_\alpha^{(\kappa)}+\mathfrak{T}_\alpha^{(\kappa),*}+\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}+\sqrt{A_2^\alpha}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}. \tag{6.18}$$

6.6.2 Far form. Next we control the far form

$$\mathrm{Far}(f,g)=\sum_{(A,B)\in\mathrm{Far}(\mathcal{A}\times\mathcal{B})}\left\langle T_\sigma^\alpha(\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f),\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g\right\rangle_\omega,$$

which we first decompose into 'far below' and 'far above' pieces,

$$\begin{aligned}\mathrm{Far}(f,g)&=\sum_{\substack{(A,B)\in\mathrm{Far}(\mathcal{A}\times\mathcal{B})\\ B\subset A}}\left\langle T_\sigma^\alpha(\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f),\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g\right\rangle_\omega\\&\quad+\sum_{\substack{(A,B)\in\mathrm{Far}(\mathcal{A}\times\mathcal{B})\\ A\subset B}}\left\langle T_\sigma^\alpha(\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f),\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g\right\rangle_\omega\\&=\mathop{\mathrm{Far}}_{\text{below}}(f,g)+\mathop{\mathrm{Far}}_{\text{above}}(f,g),\end{aligned}$$

where, as we noted above, if $(A,B)\in\mathrm{Far}(\mathcal{A}\times\mathcal{B})$ and $B\subset A$, then B is actually 'far below' the cube A in the sense that $B\subset A'$ for some $A'\in\mathfrak{C}_{\mathcal{A}}(A)$.

At this point we recall that the Intertwining Proposition 35 was built on the shifted corona decomposition,

$$\langle T_\sigma^\alpha f,g\rangle_\omega=\sum_{A,A'\in\mathcal{A}}\left\langle T_\sigma^\alpha(\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f),\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{A}}^{\tau-\mathrm{shift}}(A')}g\right\rangle_\omega,$$

in which the shifted $\mathcal{A}$-coronas $\{\mathcal{C}_{\mathcal{A}}^{\tau-\mathrm{shift}}(A')\}_{A'\in\mathcal{A}}$ are used in place of the parallel $\mathcal{B}$-coronas $\{\mathcal{C}_{\mathcal{B}}(B)\}_{B\in\mathcal{B}}$ in defining a complete set of projections in $L^2(\omega)$. In fact, using that

$$\bigcup_{A'\in\mathcal{A}:\,A'\subsetneqq A}\mathcal{C}_{\mathcal{A}}^{\tau\text{-}\mathrm{shift}}(A')=\{J\in\mathcal{D}:J\Subset_\tau A\},$$

the conclusion of the Intertwining Proposition 35 can be written

$$|\operatorname{Shift}(f,g)| \lesssim (\mathcal{V}_2^{\alpha,\kappa_1} + \sqrt{A_2^\alpha} + \mathfrak{T}_{T^\alpha}^{(\kappa_1)}) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)},$$
$$\text{where } \operatorname{Shift}(f,g) \equiv \sum_{A\in\mathcal{A}} \sum_{I:\, I\supsetneqq A} \sum_{J\in\mathcal{D}:\, J\Subset_\tau A} \langle T_\sigma^\alpha \triangle_{I;\kappa_1}^\sigma f, \triangle_{J;\kappa_2}^\omega g\rangle_\omega.$$

We now wish to apply this estimate to the far below form $\operatorname{Far}_{\text{below}}(f,g)$ in the parallel corona decomposition, and for this we write

$$\begin{aligned}
\operatorname*{Far}_{\text{below}}(f,g) &= \sum_{(A,B)\in\operatorname{Far}(\mathcal{A}\times\mathcal{B}):\, B\subset A} \langle T_\sigma^\alpha(\mathsf{P}_{\mathcal{C}_\mathcal{A}(A)}^\sigma f), \mathsf{P}_{\mathcal{C}_\mathcal{B}(B)}^\omega g\rangle_\omega \\
&= \sum_{A\in\mathcal{A}} \Big\langle T_\sigma^\alpha(\mathsf{P}_{\mathcal{C}_\mathcal{A}(A)}^\sigma f), \sum_{B\in\mathcal{B}:\, (A,B)\in\operatorname{Far}(\mathcal{A}\times\mathcal{B}) \text{ and } B\subset A} \mathsf{P}_{\mathcal{C}_\mathcal{B}(B)}^\omega g\Big\rangle_\omega \\
&= \sum_{A\in\mathcal{A}} \Big\langle T_\sigma^\alpha(\mathsf{P}_{\mathcal{C}_\mathcal{A}(A)}^\sigma f), \sum_{A'\in\mathcal{A}:\, A'\subsetneqq A} \sum_{B\in\mathcal{B}\cap\mathcal{C}_\mathcal{A}(A')} \mathsf{P}_{\mathcal{C}_\mathcal{A}(A')\cap\mathcal{C}_\mathcal{B}(B)}^\omega g\Big\rangle_\omega \\
&= \sum_{A'\in\mathcal{A}} \sum_{I:\, I\supsetneqq A'} \sum_{B\in\mathcal{B}\cap\mathcal{C}_\mathcal{A}(A')} \langle T_\sigma^\alpha \triangle_{I;\kappa_1}^\sigma f, \mathsf{P}_{\mathcal{C}_\mathcal{A}(A')\cap\mathcal{C}_\mathcal{B}(B)}^\omega g\rangle_\omega \\
&= \sum_{A'\in\mathcal{A}} \sum_{I:\, I\supsetneqq A'} \sum_{J\in\mathcal{C}_\mathcal{A}(A'):\, J\subset B\subset A' \text{ for some } B\in\mathcal{B}} \langle T_\sigma^\alpha \triangle_{I;\kappa_1}^\sigma f, \triangle_{J;\kappa_2}^\omega g\rangle_\omega .
\end{aligned}$$

If we now replace A' with A in the last line, then the difference between forms is given by

$$\begin{aligned}
&\operatorname*{Far}_{\text{below}}(f,g) - \operatorname{Shift}(f,g) \\
&= \sum_{A\in\mathcal{A}} \sum_{I:I\supsetneqq A} \Big\{ \sum_{J\in\mathcal{C}_\mathcal{A}(A):J\subset B\subset A \text{ for some } B\in\mathcal{B}} - \sum_{J\in\mathcal{D}:J\Subset_\tau A} \Big\} \langle T_\sigma^\alpha \triangle_{I;\kappa_1}^\sigma f, \triangle_{J;\kappa_2}^\omega g\rangle_\omega \qquad (6.19)\\
&= \sum_{A\in\mathcal{A}} \sum_{I:\, I\supsetneqq A} \Big\{ \sum_{J\in\mathcal{W}_A} - \sum_{J\in\mathcal{X}_A} \Big\} \langle T_\sigma^\alpha \triangle_{I;\kappa_1}^\sigma f, \triangle_{J;\kappa_2}^\omega g\rangle_\omega \equiv S - T,
\end{aligned}$$

where

$$S = \sum_{A\in\mathcal{A}} \sum_{I:\, I\supsetneqq A} \sum_{J\in\mathcal{W}_A} \langle T_\sigma^\alpha \triangle_{I;\kappa_1}^\sigma f, \triangle_{J;\kappa_2}^\omega g\rangle_\omega \quad \text{and}$$
$$T = \sum_{A\in\mathcal{A}} \sum_{I:\, I\supsetneqq A} \sum_{J\in\mathcal{X}_A} \langle T_\sigma^\alpha \triangle_{I;\kappa_1}^\sigma f, \triangle_{J;\kappa_2}^\omega g\rangle_\omega,$$

and

$$\begin{aligned}
\mathcal{W}_A &\equiv \{J\in\mathcal{D} :\ J\in\mathcal{C}_\mathcal{A}(A), \ell(J)\ge 2^{-\tau}\ell(A), \\
&\qquad \text{and } J\subset B\subset A \text{ for some } B\in\mathcal{B}\}, \\
\mathcal{X}_A &\equiv \{J\in\mathcal{D} :\ J\in\mathcal{C}_\mathcal{A}(A), \ell(J) < 2^{-\tau}\ell(A), \\
&\qquad \text{and there is no } B\in\mathcal{B} \text{ with } J\subset B\subset A\}.
\end{aligned}$$

The sum T can be estimated directly by the Intertwining Proposition 35 using the Alpert projection

$$\widehat{g} = \sum_{A\in\mathcal{A}} \sum_{\substack{J\in\mathcal{C}_{\mathcal{A}}(A):\ \ell(J)<2^{-\tau}\ell(A) \\ \text{and there exists } B\in\mathcal{B} \text{ with } J\subset B\subset A}} \triangle^{\omega}_{J;\kappa_2} g.$$

Indeed, we then have

$$\sum_{J\in\mathcal{X}_A} \triangle^{\omega}_{J;\kappa_2} g = \sum_{J\in\mathcal{C}^{\tau-\text{shift}}_{\mathcal{A}}(A)} \triangle^{\omega}_{J;\kappa_2} \widehat{g}$$

and so we obtain

$$\begin{aligned} |T| &= \left| \sum_{A\in\mathcal{A}} \sum_{I:\ I\supsetneqq A} \sum_{J\in\mathcal{X}_A} \langle T^{\alpha}_{\sigma} \triangle^{\sigma}_{I;\kappa_1} f, \triangle^{\omega}_{J;\kappa_2} g\rangle_{\omega} \right| \\ &= \left| \sum_{A\in\mathcal{A}} \sum_{I:\ I\supsetneqq A} \sum_{J\in\mathcal{C}^{\tau-\text{shift}}_{\mathcal{A}}(A)} \langle T^{\alpha}_{\sigma} \triangle^{\sigma}_{I;\kappa_1} f, \triangle^{\omega}_{J;\kappa_2} \widehat{g}\rangle_{\omega} \right| \\ &\lesssim (\mathcal{V}^{\alpha,\kappa_1}_2 + \sqrt{A^{\alpha}_2} + \mathfrak{T}^{(\kappa_1)}_{T^{\alpha}}) \|f\|_{L^2(\sigma)} \|\widehat{g}\|_{L^2(\omega)} \\ &\leq (\mathcal{V}^{\alpha,\kappa_1}_2 + \sqrt{A^{\alpha}_2} + \mathfrak{T}^{(\kappa_1)}_{T^{\alpha}}) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}. \end{aligned}$$

Now we claim that S satisfies

$$|S| \lesssim (\mathfrak{T}^{(\kappa)}_{\alpha} + \mathfrak{T}^{(\kappa),*}_{\alpha} + \mathcal{WBP}^{(\kappa_1,\kappa_2)}_{T^{\alpha}} + \sqrt{A^{\alpha}_2}) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}. \tag{6.20}$$

To see (6.20), momentarily fix $A\in\mathcal{A}$ and $J\in\mathcal{W}_A$ and write

$$\begin{aligned} &\sum_{I:\ I\supsetneqq A} \langle T^{\alpha}_{\sigma} \triangle^{\sigma}_{I;\kappa_1} f, \triangle^{\omega}_{J;\kappa_2} g\rangle_{\omega} \\ &\qquad = \sum_{I:\ A\subsetneqq I\subset \pi^{(\tau)}_{\mathcal{D}} A} \langle T^{\alpha}_{\sigma} \triangle^{\sigma}_{I;\kappa_1} f, \triangle^{\omega}_{J;\kappa_2} g\rangle_{\omega} + \sum_{I:\ I\supsetneqq \pi^{(\tau)}_{\mathcal{D}} A} \langle T^{\alpha}_{\sigma} \triangle^{\sigma}_{I;\kappa_1} f, \triangle^{\omega}_{J;\kappa_2} g\rangle_{\omega} \\ &\qquad \equiv S^1_{A,J} + S^2_{A,J}. \end{aligned}$$

We have

$$\left| \sum_{A\in\mathcal{A}} \sum_{J\in\mathcal{W}_A} S^1_{A,J} \right| \lesssim (\mathfrak{T}^{(\kappa)}_{\alpha} + \mathfrak{T}^{(\kappa),*}_{\alpha} + \mathcal{WBP}^{(\kappa_1,\kappa_2)}_{T^{\alpha}} + \sqrt{A^{\alpha}_2}) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}$$

by Lemma 30, since $J\subset I$ and $\frac{\ell(I)}{\ell(J)} = \frac{\ell(I)}{\ell(A)}\frac{\ell(A)}{\ell(J)} \leq 2^{\tau}2^{\tau} < 2^{\rho}$. For the remaining sum,

$$\text{Parallel}(f,g) \equiv \sum_{A\in\mathcal{A}} \sum_{J\in\mathcal{W}_A} S^2_{A,J} = \sum_{A\in\mathcal{A}} \left\langle T^{\alpha}_{\sigma}\left(\sum_{I:\ I\supsetneqq \pi^{(\tau)}_{\mathcal{D}} A} \triangle^{\sigma}_{I;\kappa_1} f \right), \sum_{J\in\mathcal{W}_A} \triangle^{\omega}_{J;\kappa_2} g \right\rangle_{\omega},$$

we apply the Alternate Intertwining Corollary 36 to obtain

$$|\operatorname{Parallel}(f,g)| \lesssim (\mathcal{V}_2^{\alpha,\kappa_1} + \sqrt{A_2^\alpha} + \mathfrak{T}_{T^\alpha}^{(\kappa_1)}) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$

Altogether then we have

$$\begin{aligned} &|\operatorname{Far}(f,g)| \\ &\lesssim (\mathfrak{T}_{T^\alpha}^{(\kappa_1)} + \mathfrak{T}_{T^\alpha}^{(\kappa_2),*} + \mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)} + \sqrt{A_2^\alpha} + \mathcal{V}_2^{\alpha,\kappa_1} + \mathcal{V}_2^{\alpha,\kappa_2,*}) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}. \end{aligned} \tag{6.21}$$

6.6.3 Near form. It remains to control the near form $\operatorname{Near}(f,g)$ either by the Indicator/Cube Testing conditions and the classical Muckenhoupt condition A_2^α, or in the case the measures σ and ω are comparable, by the κ-Cube Testing conditions, Bilinear Indicator/Cube Testing property, and A_2^α. We first further decompose $\operatorname{Near}(f,g)$ into

$$\begin{aligned} \operatorname{Near}(f,g) &= \left\{ \sum_{\substack{(A,B)\in \operatorname{Near}(\mathcal{A}\times\mathcal{B}) \\ B\subset A}} + \sum_{\substack{(A,B)\in \operatorname{Near}(\mathcal{A}\times\mathcal{B}) \\ A\subset B}} \right\} \langle T_\sigma^\alpha (\mathsf{P}_{\mathcal{C}_\mathcal{A}(A)}^\sigma f), \mathsf{P}_{\mathcal{C}_\mathcal{B}(B)}^\omega g\rangle_\omega \\ &= \operatorname*{Near}_{\text{below}}(f,g) + \operatorname*{Near}_{\text{above}}(f,g). \end{aligned}$$

To control $\operatorname{Near}_{\text{below}}(f,g)$ we define projections

$$\mathsf{Q}_A^\omega g \equiv \sum_{\substack{B\in\mathcal{B}:\ (A,B)\in \operatorname{Near}(\mathcal{A}\times\mathcal{B}) \\ B\subset A}} \mathsf{P}_{\mathcal{C}_\mathcal{B}(B)}^\omega g,$$

and observe that, while the Alpert support of Q_A^ω need not be contained in the corona $\mathcal{C}_\mathcal{A}(A)$, these projections are nevertheless mutually orthogonal in the index $A\in\mathcal{A}$.

It is now an easy exercise to use the Indicator/Cube Testing condition (2.14) to control $\operatorname{Near}_{\text{below}}(f,g)$,

$$\begin{aligned} &|\operatorname*{Near}_{\text{below}}(f,g)| \\ &= \sum_{A\in\mathcal{A}} |\langle T_\sigma^\alpha \mathsf{P}_{\mathcal{C}_\mathcal{A}(A)}^\sigma f, \mathsf{Q}_A^\omega g\rangle_\omega| \\ &\le \sum_{A\in\mathcal{A}} \|T_\sigma^\alpha \mathsf{P}_{\mathcal{C}_\mathcal{A}(A)}^\sigma f\|_{L^2(\omega)} \|\mathsf{Q}_A^\omega g\|_{L^2(\omega)} \lesssim \mathfrak{T}_{T^\alpha}^{IC} \sum_{A\in\mathcal{A}} \alpha_\mathcal{A}(A)\sqrt{|A|_\sigma}\, \|\mathsf{Q}_A^\omega g\|_{L^2(\omega)} \\ &\le \mathfrak{T}_{T^\alpha}^{IC} \left(\sum_{A\in\mathcal{A}} \alpha_\mathcal{A}(A)^2 |A|_\sigma\right)^{\frac12} \left(\sum_{A\in\mathcal{A}} \|\mathsf{Q}_A^\omega g\|_{L^2(\omega)}^2\right)^{\frac12} \lesssim \mathfrak{T}_{T^\alpha}^{IC} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}, \end{aligned} \tag{6.22}$$

by quasi-orthogonality and the fact that the projections Q_A^ω are mutually orthogonal in the index $A\in\mathcal{A}$. Note that we have not used comparability of measures here since that is only needed for the Bilinear Carleson Embedding Theorem 20. This will give the first inequality (3.1) in Theorem 6 after we have removed the weak boundedness constant $\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega)$ in the next subsection.

But we must work harder to obtain control by the Bilinear Indicator/Cube Testing property and κ-Cube Testing in the presence of the comparability assumption on σ and ω. For this we proceed instead as follows. For fixed $A \in \mathcal{A}$ write

$$\begin{aligned}
&\sum_{\substack{B\in\mathcal{B}:\ (A,B)\in\mathrm{Near}(\mathcal{A}\times\mathcal{B})\\ B\subset A}} \langle T_\sigma^\alpha \mathsf{P}_{\mathcal{C}_{\mathcal{A}}(A)}^\sigma f, \mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B)}^\omega g\rangle_\omega \\
&\qquad = \sum_{B\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)} \langle T_\sigma^\alpha \mathsf{P}_{\mathcal{C}_{\mathcal{A}}(A)}^\sigma f, \mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B)}^\omega g\rangle_\omega \\
&\qquad = \sum_{B,B'\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)} \langle T_\sigma^\alpha (\mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B')}^\sigma \mathsf{P}_{\mathcal{C}_{\mathcal{A}}(A)}^\sigma f), \mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B)}^\omega g\rangle_\omega \\
&\qquad\quad + \sum_{B\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)} \langle T_\sigma^\alpha (\mathsf{P}_{\mathcal{C}_{\mathcal{B}}(\pi_{\mathcal{B}}A)}^\sigma \mathsf{P}_{\mathcal{C}_{\mathcal{A}}(A)}^\sigma f), \mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B)}^\omega g\rangle_\omega \\
&\qquad = \left\{ \sum_{\substack{B,B'\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)\\ B\cap B'=\emptyset}} + \sum_{\substack{B,B'\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)\\ B=B'}} + \sum_{\substack{B,B'\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)\\ B\subsetneqq B'}} + \sum_{\substack{B,B'\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)\\ B'\subsetneqq B}} \right\} \\
&\qquad\quad \times \langle T_\sigma^\alpha (\mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B')}^\sigma \mathsf{P}_{\mathcal{C}_{\mathcal{A}}(A)}^\sigma f), \mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B)}^\omega g\rangle_\omega \\
&\qquad\quad + \sum_{B\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)} \langle T_\sigma^\alpha (\mathsf{P}_{\mathcal{C}_{\mathcal{B}}(\pi_{\mathcal{B}}A)}^\sigma \mathsf{P}_{\mathcal{C}_{\mathcal{A}}(A)}^\sigma f), \mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B)}^\omega g\rangle_\omega \\
&\qquad \equiv I^A + II^A + III^A + IV^A + V^A,
\end{aligned}$$

where $\pi_{\mathcal{B}}A$ denotes the smallest cube $B \in \mathcal{B}$ that contains A.[10] Then term I^A is handled immediately by Lemma 6.5 to yield

$$\sum_{A\in\mathcal{A}} |I^A| \lesssim \left(\sqrt{A_2^\alpha} + \mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega) + \mathfrak{T}_{T^{\alpha,*}}^{(\kappa)}(\omega,\sigma) + \mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega)\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.$$

The sum $\sum_{A\in\mathcal{A}} |II^A|$ of terms II^A will be handled by the bilinear Carleson Embedding Theorem 20, using the Bilinear Indicator/Cube Testing property

$$\mathcal{BICT}_{T^\alpha}(\sigma,\omega) < \infty$$

as follows.

Note that for σ and ω doubling measures, we have the following two properties,

$$\|\mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B)}^\sigma \mathsf{P}_{\mathcal{C}_{\mathcal{A}}(A)}^\sigma f\|_{L^\infty(\sigma)} \lesssim \alpha_{\mathcal{A}}(A) \quad \text{and} \quad \|\mathsf{P}_{\mathcal{C}_{\mathcal{B}}(B)}^\omega g\|_{L^\infty(\sigma)} \lesssim \alpha_{\mathcal{B}}(B),$$

[10] We thank Ignacio Uriarte-Tuero for pointing out that term V^A was missing from the argument.

since our coronas are Calderón–Zygmund, and thus if $A' \in \mathfrak{C}_{\mathcal{A}}(A)$, then

$$\frac{1}{|A'|_\sigma}\int_{A'}|f|d\sigma \lesssim \frac{1}{|\pi_{\mathcal{D}}A'|_\sigma}\int_{\pi_{\mathcal{D}}A'}|f|d\sigma \lesssim \alpha_{\mathcal{A}}(A),$$

and so

$$|\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{B}}(B)\cap\mathcal{C}_{\mathcal{A}}(A)}f| \lesssim \sup_{I\in[\mathcal{C}_{\mathcal{B}}(B)\cap\mathcal{C}_{\mathcal{A}}(A)]\cup\mathfrak{C}_{\mathcal{A}}(A)} \frac{1}{|I|_\sigma}\int_I |f|d\sigma \lesssim \alpha_{\mathcal{A}}(A).$$

In the first inequality in the above display we have used the telescoping identities for Alpert wavelets.

We then have, using the Bilinear Indicator/Cube Testing property, that

$$\begin{aligned}
&|II^A| \\
&= \Big|\sum_{B\in\mathcal{B}\cap\mathcal{C}_A}\langle T^\alpha_\sigma(\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{B}}(B)}\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f),\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g_B\rangle_\omega\Big| \\
&\lesssim \alpha_{\mathcal{A}}(A)\sum_{B\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)}\alpha_{\mathcal{B}}(B)\Big|\Big\langle T^\alpha_\sigma\Big(\frac{\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{B}}(B)}\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f}{\|\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{B}}(B)}\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f\|_{L^\infty(\sigma)}}\Big),\frac{\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g_B}{\|\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g_B\|_{L^\infty(\sigma)}}\Big\rangle_\omega\Big| \\
&\le \alpha_{\mathcal{A}}(A)\sum_{B\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)}\mathcal{BICT}_{T^\alpha}(\sigma,\omega)\alpha_{\mathcal{B}}(B)\sqrt{|B|_\sigma}\sqrt{|B|_\omega}.
\end{aligned}$$

Now we use

$$\begin{aligned}
\alpha_{\mathcal{A}}(A) &\lesssim \frac{1}{|A|_\sigma}\int_A|f|d\sigma \le \sup_{K\in\mathcal{D}:\ K\supset I}\frac{1}{|K|_\sigma}\int_K|f|d\sigma, \quad \text{for } I\in\mathcal{C}_{\mathcal{A}}(A),\\
\alpha_{\mathcal{B}}(B) &\lesssim \frac{1}{|B|_\sigma}\int_B|g|d\sigma \le \sup_{L\in\mathcal{D}:\ L\supset J}\frac{1}{|L|_\omega}\int_L|g|d\omega, \quad \text{for } J\in\mathcal{C}_{\mathcal{B}}(B),
\end{aligned}$$

and apply the bilinear Carleson Embedding Theorem 20 with

$$a_I \equiv \begin{cases}\sqrt{|I|_\sigma|I|_\omega} & \text{if } I\in\mathcal{C}_{\mathcal{A}}(A)\cap\mathcal{B}\\ 0 & \text{if } I\notin\mathcal{C}_{\mathcal{A}}(A)\cap\mathcal{B}\end{cases}$$

to conclude that

$$\begin{aligned}
\sum_{A\in\mathcal{A}}|II^A| &\lesssim \mathcal{BICT}_{T^\alpha}(\sigma,\omega)\sum_{A\in\mathcal{A}}\sum_{B\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)}\sqrt{|B|_\sigma}\sqrt{|B|_\omega}\Big(\sup_{K\in\mathcal{D}:\ K\supset B}\frac{1}{|K|_\sigma}\int_K|f|d\sigma\Big)\\
&\qquad\qquad\times\Big(\sup_{L\in\mathcal{D}:\ L\supset B}\frac{1}{|L|_\omega}\int_L|g|d\omega\Big)\\
&\lesssim \mathcal{BICT}_{T^\alpha}(\sigma,\omega)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.
\end{aligned}$$

Finally, observe that the Carleson condition (4.18) holds here for $J \in \mathcal{B}$ since the geometric decay in the ω-Carleson condition for $\mathcal{B}$ gives

$$\begin{aligned}\sum_{I\in\mathcal{D}:\ I\subset J} a_I &= \sum_{B\in\mathcal{B}:\ B\subset J} \sqrt{|B|_\sigma |B|_\omega} = \sum_{m=0}^{\infty} \sum_{B\in\mathfrak{C}_{\mathcal{B}}^{(m)}(J)} \sqrt{|B|_\sigma |B|_\omega} \\ &\le \sum_{m=0}^{\infty} \sqrt{\sum_{B\in\mathfrak{C}_{\mathcal{B}}^{(m)}(J)} |B|_\sigma} \sqrt{\sum_{B\in\mathfrak{C}_{\mathcal{B}}^{(m)}(J)} |B|_\omega} \\ &\le \sum_{m=0}^{\infty} \sqrt{|J|_\sigma}\sqrt{2^{-m\delta}|J|_\omega} \le C'\sqrt{|J|_\sigma |J|_\omega},\end{aligned}$$

and now the case of general J follows as usual.

Remark 37. This is the only place in the proof where the Bilinear Indicator/Cube Testing property (2.15) is used, and also the only place requiring the comparability of the measures (through the use of the bilinear Carleson Embedding Theorem 20). It is the Parallel Corona that permits this relatively simple application of a bilinear testing property.

To handle term III^A we decompose it into two terms,

$$\begin{aligned}III^A &= \sum_{\substack{B,B'\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)\\ B\subsetneqq B'\subset A}} \langle T_\sigma^\alpha(\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{B}}(B')}\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f), \mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g\rangle_\omega \\ &\quad + \sum_{\substack{B\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A), B'\in\mathcal{B}\\ B'\supsetneqq A}} \langle T_\sigma^\alpha(\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{B}}(B')}\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f), \mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g\rangle_\omega \\ &\equiv III_1^A + III_2^A.\end{aligned}$$

Then we proceed with

$$\begin{aligned}III_1^A &= \sum_{\substack{B,B'\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)\\ B\subsetneqq B'}} \langle T_\sigma^\alpha(\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{B}}(B')}\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f), \mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g\rangle_\omega \\ &= \sum_{\substack{B,B'\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)\\ B\subsetneqq B'}} \langle (\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{B}}(B')}\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f), T_\omega^{\alpha,*}(\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g)\rangle_\sigma \\ &= \sum_{B\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)} \left\langle \left(\sum_{\substack{B'\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)\\ B\subsetneqq B'}} \mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{B}}(B')}\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f\right), T_\omega^{\alpha,*}(\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g)\right\rangle_\sigma .\end{aligned}$$

As in our treatment of the $\mathrm{Far}_{\mathrm{below}}$ form above, we now apply an argument analogous to that surrounding (6.19), in order to control the sum $\sum_{A\in\mathcal{A}} III_1^A$ using

Lemma 30, and the dual forms of the Intertwining Proposition 35, and the Alternate Intertwining Corollary 36. This results in the estimate

$$\left|\sum_{A\in\mathcal{A}} III_1^A\right| \lesssim \left(\sqrt{\mathcal{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega)+\mathfrak{T}_{T^{\alpha,*}}^{(\kappa)}(\omega,\sigma)+\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega)\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.$$

For the sum of terms III_2^A, we also apply an argument analogous to that surrounding (6.19) using Lemma 30 and the dual forms of Proposition 35 and Corollary 36. This also results in the estimate

$$\left|\sum_{A\in\mathcal{A}} III_2^A\right| \lesssim \left(\sqrt{\mathcal{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega)+\mathfrak{T}_{T^{\alpha,*}}^{(\kappa)}(\omega,\sigma)+\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega)\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.$$

In the same way, for the sum of terms V^A, we first write

$$V^A=\sum_{B\in\mathcal{B}\cap\mathcal{C}_{\mathcal{A}}(A)}\left\langle(\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{B}}(\pi_{\mathcal{B}}A)}\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f),T_\omega^{\alpha,*}(\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g)\right\rangle_\sigma,$$

and then apply once more an argument analogous to that surrounding (6.19) using Lemma 30 and the dual forms of Proposition 35 and Corollary 36, that results in the estimate

$$\left|\sum_{A\in\mathcal{A}} V^A\right| \lesssim \left(\sqrt{\mathcal{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega)+\mathfrak{T}_{T^{\alpha,*}}^{(\kappa)}(\omega,\sigma)+\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega)\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.$$

The bound for the sum $\sum_{A\in\mathcal{A}}|IV_2^A|$ is essentially dual to that for $\sum_{A\in\mathcal{A}}|III_2^A|$, and so altogether, since $\mathfrak{T}_{T^\alpha}\le\mathfrak{T}_{T^\alpha}^{(\kappa)}$, we have shown that

$$\begin{aligned}&|\operatorname*{Near}_{\text{below}}(f,g)|\\&\quad\lesssim\left(\sqrt{\mathcal{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega)+\mathfrak{T}_{T^{\alpha,*}}^{(\kappa)}(\omega,\sigma)+\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega)+\mathcal{BICT}_{T^\alpha}(\sigma,\omega)\right)\\&\qquad\times\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.\end{aligned}$$

By symmetry, we also have that the form

$$\begin{aligned}\operatorname*{Near}_{\text{above}}(f,g)&=\sum_{\substack{(A,B)\in\operatorname{Near}(\mathcal{A}\times\mathcal{B})\\A\subset B}}\left\langle T_\sigma^\alpha(\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f),\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g\right\rangle_\omega\\&=\sum_{\substack{(A,B)\in\operatorname{Near}(\mathcal{A}\times\mathcal{B})\\A\subset B}}\left\langle\mathsf{P}^\sigma_{\mathcal{C}_{\mathcal{A}}(A)}f,T_\omega^{\alpha,*}(\mathsf{P}^\omega_{\mathcal{C}_{\mathcal{B}}(B)}g)\right\rangle_\omega\end{aligned}$$

satisfies

$$\begin{aligned}&|\operatorname*{Near}_{\text{above}}(f,g)|\\&\quad\lesssim\left(\sqrt{\mathcal{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega)+\mathfrak{T}_{T^{\alpha,*}}^{(\kappa)}(\omega,\sigma)+\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega)+\mathcal{BICT}_{T^\alpha}(\sigma,\omega)\right)\\&\qquad\times\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.\end{aligned}$$

Combining these estimates completes our control of the near form Near(f, g):

$$\begin{aligned}
&|\operatorname{Near}(f,g)| \\
&\lesssim \Big(\sqrt{A_2^\alpha} + \mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega) + \mathfrak{T}_{T^{\alpha,*}}^{(\kappa)}(\omega,\sigma) + \mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)} + \mathcal{BICT}_{T^\alpha}(\sigma,\omega)\Big) \\
&\quad \times \|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.
\end{aligned} \tag{6.23}$$

The three inequalities (6.18), (6.21) and (6.23) finish the proof of (6.16), thus yielding the inequality

$$\begin{aligned}
\mathfrak{N}_{T^\alpha} &\lesssim C_{\kappa_1,(\beta_1,\gamma_1),\kappa_2,(\beta_2,\gamma_2)} \\
&\quad \times \Big(\mathfrak{T}_{T^\alpha}^{(\kappa_1)} + \mathfrak{T}_{T^\alpha}^{(\kappa_2),*} + \mathcal{BICT}_{T^\alpha} + \sqrt{A_2^\alpha} + \mathcal{V}_2^{\alpha,\kappa_1} + \mathcal{V}_2^{\alpha,\kappa_2,*} + \mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}\Big),
\end{aligned} \tag{6.24}$$

after taking the supremum over f and g in their respective unit balls. It now remains only to remove $\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}$ from the right-hand side of (6.24) in order to finish the proof of Theorem 6.

6.7 Eliminating the weak boundedness property by doubling. Here we show that the weak boundedness constant $\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega)$ can be easily eliminated from the right-hand side of (6.16) or (6.24) using the doubling properties of the measures. We first use Corollary 25 to obtain the inequality

$$\begin{aligned}
&\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega) \\
&= \sup_{\mathcal{D}\in\Omega} \sup_{\substack{Q,Q'\in\mathcal{D} \\ Q\subset 3Q'\setminus Q' \text{ or } Q'\subset 3Q\setminus Q}} \frac{1}{\sqrt{|Q|_\sigma |Q'|_\omega}} \\
&\quad \times \sup_{\substack{f\in(\mathcal{P}_Q^{\kappa_1})_{\text{norm}}(\sigma) \\ g\in(\mathcal{P}_{Q'}^{\kappa_2})_{\text{norm}}(\omega)}} \left| \int_{Q'} T_\sigma^\alpha(\mathbf{1}_Q f) g \, d\omega \right| < \infty \\
&\le \sup_{\mathcal{D}\in\Omega} \sup_{\substack{Q,Q'\in\mathcal{D} \\ Q\subset 3Q'\setminus Q' \text{ or } Q'\subset 3Q\setminus Q}} \frac{1}{\sqrt{|Q|_\sigma |Q'|_\omega}} \sup_{f\in(\mathcal{P}_Q^{\kappa_1})_{\text{norm}}(\sigma)} \int_{Q'} (T_\sigma^\alpha(\mathbf{1}_Q f))^2 d\omega \\
&\le \mathfrak{F}\mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega)^2 \le C_{\kappa,\varepsilon}\mathfrak{T}_{T^\alpha}(\sigma,\omega) + C_{\kappa,\varepsilon}\mathcal{A}_2^\alpha(\sigma,\omega) + \varepsilon\mathfrak{N}_{T^\alpha}(\sigma,\omega),
\end{aligned} \tag{6.25}$$

valid for doubling measures σ and ω.

Now we plug (6.25) into inequality (6.24) to obtain

$$\begin{aligned}
\mathfrak{N}_{T^\alpha} &\lesssim C_{\kappa_1,(\beta_1,\gamma_1),\kappa_2,(\beta_2,\gamma_2)}\{\mathfrak{T}_{T^\alpha}^{(\kappa_1)} + \mathfrak{T}_{T^\alpha}^{(\kappa_2),*} + \mathcal{BICT}_{T^\alpha} + \sqrt{A_2^\alpha} + \mathcal{V}_2^{\alpha,\kappa_1} + \mathcal{V}_2^{\alpha,\kappa_2,*}\} \\
&\quad + C_{\kappa_1,(\beta_1,\gamma_1),\kappa_2,(\beta_2,\gamma_2)} C_{\kappa,\varepsilon}\{\mathfrak{T}_{T^\alpha}(\sigma,\omega) + \mathcal{A}_2^\alpha(\sigma,\omega)\} \\
&\quad + C_{\kappa_1,(\beta_1,\gamma_1),\kappa_2,(\beta_2,\gamma_2)}\varepsilon\mathfrak{N}_{T^\alpha}(\sigma,\omega).
\end{aligned}$$

If we now choose $\varepsilon > 0$ so small that the term $C_{\kappa_1,(\beta_1,\gamma_1),\kappa_2,(\beta_2,\gamma_2)}\varepsilon\mathfrak{N}_{T^\alpha}(\sigma,\omega)$ can be absorbed into the left-hand side, we obtain the desired inequality (6.15). This completes the proof of both inequalities (3.1) and (3.2) in Theorem 6.

7 Proof of Theorem 12 on restricted weak type

Here we prove Theorem 12 which, we remind the reader, does not assume comparability of measures. The proof of the theorem is a standard application of an idea originating four and a half decades ago, namely the 1973 good $-\lambda$-inequality of Burkholder [Bur], and specifically the 1974 inequality of R. Coifman and C. Fefferman [CoFe], and the related 1974 inequality of B. Muckenhoupt and R. L. Wheeden [MuWh]. We also introduce an α-fractional analogue A_∞^α of the A_∞ condition, and use it to improve the inequality in [MuWh] when $\alpha > 0$. We begin by briefly recalling the inequality of Coifman and Fefferman that relates maximal truncations of a Calderón–Zygmund singular integral to the maximal operator M.

7.1 Good-λ inequalities. Given an α-fractional Calderón–Zygmund operator T^α, define the maximal truncation operator $T_\flat^\alpha$ by

$$T_\flat^\alpha(f\sigma)(x) \equiv \sup_{0<\varepsilon<R<\infty} \left| \int_{\{\varepsilon<|y|<R\}} K^\alpha(x,y)f(y)d\sigma(y) \right|, \quad x \in \mathbb{R}^n,$$

for any locally finite positive Borel measure σ on $\mathbb{R}^n$, and $f \in L^2(\sigma)$. Define the α-fractional Hardy–Littlewood maximal operator M^α by

$$M^\alpha(f\sigma)(x) \equiv \sup_{Q\in\mathcal{P}^n:\ x\in Q} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f| d\sigma, \quad x \in \mathbb{R}^n,$$

where here we may take the cubes Q in the supremum to be closed.

Let ω be an A_∞ weight. Suppose first that $\alpha = 0$. Then the Coifman–Fefferman good $-\lambda$ inequality in [CoFe, see inequality (7) on page 245] is

$$\begin{aligned} &|\{x \in Q : T_\flat(f\sigma)(x) > 2\lambda \text{ and } M(f\sigma)(x) \le \beta\lambda\}|_\omega \\ &\qquad \le C\beta^\varepsilon |\{x \in Q : T_\flat(f\sigma)(x) > \lambda\}|_\omega, \end{aligned} \tag{7.1}$$

for all $\lambda > 0$, where $\varepsilon > 0$ is the A_∞ exponent in (2.2). The kernels considered in [CoFe] are convolution kernels with order 1 smoothness and bounded Fourier transform. However, since we are assuming here that T is bounded on unweighted $L^2(\mathbb{R}^n)$, standard Calderón–Zygmund theory [Ste2, Corollary 2 on page 36] implies that $T_\flat$ is weak type $(1,1)$ on Lebesgue measure. This estimate is the key to the proof in [CoFe, see pages 245–246 where the weak type $(1,1)$ inequality for $T_\flat$ is used], and this proof shows that the kernel of the operator T may be taken to be a standard kernel in the sense used here.

In the case $0 < \alpha < n$, this good-λ inequality for an A_∞ weight ω was extended in [MuWh] (by essentially the same proof) when $T_\flat$ and M are replaced by I^α

and M^α respectively:

$$\begin{aligned}(7.2)\qquad &|\{x \in Q : I^\alpha(f\sigma)(x) > \gamma\lambda \text{ and } M^\alpha(f\sigma)(x) \le \beta\lambda\}|_\omega \\ &\qquad\qquad \le \circ\left(\frac{1}{\gamma}\right)|\{x \in Q : I^\alpha(f\sigma)(x) > \lambda\}|_\omega,\end{aligned}$$

for all $\lambda > 0$, for some $\beta > 0$ chosen sufficiently small. Here the fractional integral I^α is given by

$$I^\alpha \nu(x) \equiv \int_{\mathbb{R}^n} |x-y|^{\alpha-n} d\nu(y),$$

and we will use below the obvious fact that $|T^\alpha_\flat \nu(x)| \le CI^\alpha \nu(x)$ for $d\nu \ge 0$. (I_α is a positive operator satisfying the weak type $(1, \frac{n}{n-\alpha})$ inequality on Lebesgue measure, and this is why there is no need to assume any additional unweighted boundedness of T^α when $\alpha > 0$.)

However, it is possible to enlarge the collection of weights that satisfy (7.2) by using a relative α-capacity of E in Q in place of the ratio $\frac{|E|}{|Q|}$ appearing in the definition of the A_∞ condition. We introduce this next.

7.1.1 A fractional good-λ inequality. Let $\mathcal{D}$ be a dyadic grid on $\mathbb{R}^n$. Suppose that Ω is an open subset of $\mathbb{R}^n$ with compact closure. We define the **Whitney collection** $\mathcal{W}_\Omega$ to be the set $\{Q_j\}_j$ of maximal dyadic cubes $Q_j \in \mathcal{D}$ such that $3Q_j \subset \Omega$. The following three properties are then immediate:

$$\begin{cases} \text{(disjoint cover)} & \Omega = \bigcup_j Q_j \text{ and } Q_j \cap Q_i = \emptyset \text{if } i \neq j, \\ \text{(Whitney condition)} & 3Q_j \subset \Omega \text{ and } 9Q_j \cap \Omega^c \neq \emptyset \text{ for all } j, \\ \text{(bounded overlap)} & \sum_j \mathbf{1}_{2Q_j} \le C_n \mathbf{1}_\Omega. \end{cases}$$

Definition 38. Define the **Whitney decomposition** $\boldsymbol{W}_{I_\alpha f}$ of the fractional integral $I_\alpha f$ of a positive measure f to be the set whose elements are the Whitney collections $\mathcal{W}_{\Omega_k}$ for the open sets $\Omega_k \equiv \{x \in \mathbb{R}^n : I_\alpha f(x) > 2^k\}$, $k \in \mathbb{Z}$, i.e.,

$$\boldsymbol{W}_{I_\alpha f} \equiv \{\mathcal{W}_{\Omega_k}\}_{k \in \mathbb{Z}},$$

which we can identify with $\{Q^k_j\}_{k,j}$ if $\mathcal{W}_{\Omega_k} = \{Q^k_j\}_j$.

The nested property is immediate,

$$\text{(nested property)} \quad Q^k_j \subsetneqq Q^\ell_i \text{ implies } k > \ell,$$

and the maximum principle is proved in [Saw5]: there is N sufficiently large that

$$\text{(maximum principle)} \quad I_\alpha(\mathbf{1}_{2Q^{k-N}_i} f)(x) > 2^{k-1} \text{ for } x \in \Omega_k \cap Q^{k-N}_i, \quad \text{all } k, j.$$

Consider now a positive measure f and the Whitney decomposition of $I_\alpha f$ where

$$\Omega_k = \{I_\alpha f > 2^k\} = \bigcup_j Q_j^k \quad \text{for } k \in \mathbb{Z}.$$

We claim that if

$$E_i^{k-N}(\beta) \equiv Q_i^{k-N} \cap \{I_\alpha f > 2^k \text{ and } M_\alpha f \le \beta 2^{k-N}\},$$

then we have

$$\mathbf{Cap}_\alpha(E_i^{k-N}(\beta); Q_i^{k-N}) \le \beta 2^{1-N}, \quad 0 < \beta \le 1. \tag{7.3}$$

To see this we note that the cubes Q_j^k and Q_i^{k-N} above satisfy

$$|9Q_i^{k-N}|^{\frac{\alpha}{n}-1} \int_{9Q_i^{k-N}} f \le \beta 2^{k-N},$$
$$I_\alpha(\mathbf{1}_{2Q_i^{k-N}} f)(x) > 2^{k-1} \quad \text{for } x \in Q_j^k,$$

where the first inequality follows from the Whitney condition, and the second inequality from the maximum principle for fractional integrals. This then shows that the nonnegative function $h \equiv \frac{1}{2^{k-1}} \mathbf{1}_{2Q_i^{k-N}} f$ satisfies $I_\alpha h \ge 1$ on E and

$$|2Q_i^{k-N}|^{\frac{\alpha}{n}-1} \int_{2Q_i^{k-N}} h \le \left(\frac{2}{9}\right)^{n-\alpha} |9Q_i^{k-N}|^{\frac{\alpha}{n}-1} \int_{9Q_i^{k-N}} \frac{1}{2^{k-1}} f \le \beta 2^{1-N},$$

which proves (7.3).

Using the relative capacity inequality (7.3), we can now prove the good-λ inequality for the pair (I_α, M_α) with respect to an A_∞^α measure ω.

Lemma 39. *If $\omega \in A_\infty^\alpha$, or $\omega \in C_q$ for some $q > 2$, then there are positive constants C, ε such that*

$$|\{I_\alpha f > \gamma\lambda \text{ and } M_\alpha f \le \beta\lambda\}|_\omega \le C\left(\frac{\beta}{\gamma}\right)^\varepsilon |\{I_\alpha f > \lambda\}|_\omega. \tag{7.4}$$

Proof. The case where $\omega \in C_q$ for some $q > 2$ is in [CeLiPeRi, see Remark 6 on page 13] with an even smaller constant on the right, so we turn to the case $\omega \in A_\infty^\alpha$. It clearly suffices to consider the special cases where $\lambda = 2^{k-N}$, $\gamma = 2^N$ and $0 < \beta \le 1$ for all $k \in \mathbb{Z}$ and all sufficiently large $N \in \mathbb{N}$. Now with

$$\{I_\alpha f > 2^k\} = \bigcup_j Q_j^k$$

as above we have, from the A_∞^α condition,

$$
\begin{aligned}
&|\{I_\alpha f > 2^k \text{ and } M_\alpha f \le \beta 2^{k-N}\}|_\omega \\
&\qquad \le \sum_i \Bigg| \bigcup_{\substack{j:\ Q_j^k \subset Q_i^{k-N} \\ Q_j^k \cap \{M_\alpha f \le \beta 2^{k-N}\} \neq \emptyset}} Q_j^k \Bigg|_\omega \\
&\qquad \le \sum_i |2Q_i^{k-N}|_\omega C \mathbf{Cap}_\alpha (E_i^{k-N}(\beta); Q_i^{k-N})^\varepsilon \\
&\qquad \le \sum_i C(\beta 2^{1-N})^\varepsilon |Q_i^{k-N}|_\omega = C_n C(\beta 2^{1-N})^\varepsilon |\{I_\alpha f > 2^{k-N}\}|_\omega,
\end{aligned}
$$

where C_n is the bounded overlap constant in the Whitney collection. This completes the proof of (7.4). □

7.2 Control of restricted weak type. From such good-λ inequalities for A_∞, A_∞^α and C_q weights ω, standard arguments in [CoFe], [MuWh], [Saw1] and [CeLiPeRi] show that $\|T_\flat^\alpha(f\sigma)\|_{L^2(\omega)} \lesssim \|M^\alpha(f\sigma)\|_{L^2(\omega)}$ for $0 \le \alpha < n$ and $f \in L^2(\sigma)$. We will use a weak type variant of this latter inequality, together with the equivalence of $\mathfrak{N}_{M^\alpha}^{\text{weak}}(\sigma,\omega)$ and $A_2^\alpha(\sigma,\omega)$, to prove the theorem.

Proof of Theorem 12. Since the restricted weak type inequality is self-dual, we can assume without loss of generality that ω is an A_∞^α or $C_{2+\varepsilon}$ weight. We begin by assuming that $\omega \in A_\infty^\alpha$ and showing that the good-λ inequalities for A_∞^α weights ω imply weak type control, exercising care in absorbing terms. Indeed, for $t > 0$, we obtain from (7.1) and (7.4) that

$$
\begin{aligned}
&\sup_{0<\lambda\le t} \lambda^2 |\{T_\flat^\alpha(f\sigma) > \lambda\}|_\omega \\
&\qquad = 4 \sup_{0<\lambda\le \frac{t}{2}} \lambda^2 |\{T_\flat^\alpha(f\sigma) > 2\lambda\}|_\omega \\
&\qquad \le 4 \sup_{0<\lambda\le \frac{t}{2}} \lambda^2 \Big\{ |\{M^\alpha(f\sigma) > \beta\lambda\}|_\omega + \frac{C}{\beta} |\{T_\flat^\alpha(f\sigma) > \lambda\}|_\omega \Big\} \\
&\qquad = \frac{4}{\beta^2} \sup_{0<\lambda\le \frac{\beta}{2}t} \lambda^2 |\{M^\alpha(f\sigma) > \beta\lambda\}|_\omega + 4 \sup_{0<\lambda\le\frac{t}{2}} \frac{C}{\beta} |\{T_\flat^\alpha(f\sigma) > \lambda\}|_\omega \\
&\qquad \le \frac{4}{\beta^2} \|M^\alpha(f\sigma)\|_{L^{2,\infty}(\omega)}^2 + \frac{4C}{\beta} \sup_{0<\lambda\le t} \lambda^2 |\{T_\flat^\alpha(f\sigma) > \lambda\}|_\omega.
\end{aligned}
$$

Now choose β so that $\frac{4C}{\beta} = \frac{1}{2}$. Provided that $\sup_{0<\lambda\le t} \lambda^2 |\{T_\flat^\alpha(f\sigma) > \lambda\}|_\omega$ is finite for each $t > 0$, we can absorb the final term on the right-hand into the left-hand side to obtain

$$
\sup_{0<\lambda\le t} \lambda^2 |\{T_\flat^\alpha(f\sigma) > \lambda\}|_\omega \le \frac{8}{\beta^2} \|M^\alpha \nu\|_{L^{2,\infty}(\omega)}^2, \quad t > 0,
$$

which gives

$$\|T_\flat^\alpha(f\sigma)\|_{L^{2,\infty}(\omega)}^2 = \sup_{0<\lambda<\infty} \lambda^2 |\{T_\flat^\alpha(f\sigma) > \lambda\}|_\omega \leq \frac{8}{\beta^2} \|M^\alpha(f\sigma)\|_{L^{2,\infty}(\omega)}^2.$$

Suppose first that $\alpha = 0$. In order to obtain finiteness of the supremum over $0 < \lambda \leq t$, we take $f \in L^2(\sigma)$ wtih $|f| \leq 1$ and $\operatorname{supp} f \subset B(0, r)$ with $1 \leq r < \infty$ and $|B(0, r)|_\sigma > 0$. Then if $x \notin B(0, 2r)$, we have $|K(x, y)| \leq C_{CZ} r^{-n}$ and hence

$$\begin{aligned} T_\flat(f\sigma)(x) &= \sup_{0<\varepsilon<R<\infty} \left| \int_{\{\varepsilon<|y|<R\}\cap B(0,r)} K(x, y) f(y) d\sigma(y) \right| \\ &\leq C_{CZ} \left(\frac{2}{|x|}\right)^n |B(0, r)|_\sigma. \end{aligned}$$

This shows that

$$\begin{aligned} &\sup_{0<\lambda\leq t} \lambda^2 |\{T_\flat \nu > \lambda\}|_\omega \\ &\quad\leq t^2 |B(0, 2r)|_\omega + \sup_{0<\lambda<C_{CZ} r^{-n} |B(0,r)|_\sigma} \lambda^2 |\{T_\flat \nu > \lambda\} \setminus B(0, 2r)|_\omega \\ &\quad\leq t^2 |B(0, 2r)|_\omega + \sup_{0<\lambda\leq t} \lambda^2 \left| \left\{ C_{CZ} \left(\frac{2}{|x|}\right)^n |B(0, r)| > \lambda \right\} \right|_\omega \\ &\quad= t^2 |B(0, 2r)|_\omega + \sup_{0<\lambda\leq t} \lambda^2 |B(0, \gamma_\lambda r)|_\omega, \end{aligned}$$

with $\gamma_\lambda \equiv 2\sqrt[n]{\frac{C_{CZ} c}{\lambda}}$, since

$$\left\{ C_{CZ} \left(\frac{2}{|x|}\right)^n |B(0, r)| > \lambda \right\} = B\left(0, 2\sqrt[n]{\frac{C_{CZ} c}{\lambda}} r\right)$$

where $|B(0, r)| = cr^n$.

On the other hand, the A_2 condition implies that for $\lambda \leq \lambda_0 \equiv C_{CZ} c$, we have $\gamma_\lambda \geq \gamma_{\lambda_0} = 2$ so that

$$\begin{aligned} |B(0, \gamma_\lambda r)|_\omega &\lesssim A_2(\sigma, \omega) \frac{|B(0, \gamma_\lambda r)|^2}{|B(0, \gamma_\lambda r)|_\sigma} \leq A_2(\sigma, \omega) \frac{(\gamma_\lambda r)^{2n}}{|B(0, 2r)|_\sigma} A_2(\sigma, \omega) \\ &= \frac{4^n (\frac{C_{CZ} c}{\lambda})^2}{|B(0, 2r)|_\sigma}, \end{aligned}$$

and hence

$$\lambda^2 |B(0, \gamma_\lambda r)|_\omega \leq \lambda^2 \frac{4^n (\frac{C_{CZ} c}{\lambda})^2}{|B(0, 2r)|_\sigma} = \frac{4^n C_{CZ} c^2}{|B(0, 2r)|_\sigma}, \quad \text{for } \lambda \leq \lambda_0.$$

Finally we have

$$\sup_{\lambda_0<\lambda\leq t} \lambda^2 |B(0, \gamma_\lambda r)|_\omega \leq t^2 |B(0, \gamma_{\lambda_0} r)|_\omega = t^2 |B(0, 2r)|_\omega,$$

and altogether then

$$\sup_{0<\lambda\leq t}\lambda^2|\{T_\flat\nu>\lambda\}|_\omega\leq t^2|B(0,2r)|_\omega+\frac{4^nC_{CZ}c^2}{|B(0,2r)|_\sigma}+t^2|B(0,2r)|_\omega$$

which is finite for $0<t<\infty$.

Thus we conclude that

$$\mathfrak{N}_T^{\text{restricted weak}}(\sigma,\omega)\leq\mathfrak{N}_T^{\text{weak}}(\sigma,\omega)\leq\mathfrak{N}_{T_\flat}^{\text{weak}}(\sigma,\omega)\lesssim\mathfrak{N}_M^{\text{weak}}(\sigma,\omega)\approx A_2(\sigma,\omega),$$

where the final equivalence is well known, and can be obtained by averaging over dyadic grids $\mathcal{D}$ the inequality $\mathfrak{N}_{M_{\mathcal{D}}^\alpha}^{\text{weak}}(\sigma,\omega)\lesssim A_2(\sigma,\omega)$ for dyadic operators

$$M_{\mathcal{D}}^\alpha f(x)\equiv\sup_{Q\in\mathcal{P}^n:\ x\in Q}\frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int_Q|f|d\sigma.$$

The dyadic inequality is in turn an immediate consequence of the dyadic covering lemma. Conversely, if T^α is elliptic, then $A_2(\sigma,\omega)\lesssim\mathfrak{N}_T^{\text{restricted weak}}(\sigma,\omega)$ (see [LiTr] and [SaShUr7]).

The same sort of arguments give the analogous inequality when $0<\alpha<n$,

$$\begin{aligned}\mathfrak{N}_{T^\alpha}^{\text{restricted weak}}(\sigma,\omega)\leq\mathfrak{N}_{I^\alpha}^{\text{restricted weak}}(\sigma,\omega)\leq\mathfrak{N}_{I^\alpha}^{\text{weak}}(\sigma,\omega)\lesssim\mathfrak{N}_{M^\alpha}^{\text{weak}}(\sigma,\omega)\\ \approx A_2^\alpha(\sigma,\omega),\end{aligned}$$

and conversely, $A_2^\alpha(\sigma,\omega)\lesssim\mathfrak{N}_{T^\alpha}^{\text{restricted weak}}(\sigma,\omega)$ if T^α is elliptic.

Finally, when $\omega\in C_{2+\varepsilon}$, the proof for strong type norms of $T_\flat f$ and Mf in [Saw1] is easily adapted to weak type norms, while the proof for strong type norms of $I_\alpha f$ and $M_\alpha f$ in [CeLiPeRi, see Subsubsection 7.2.1 Lemmata on pages 33–35.]—which follows closely the arguments in [Saw1]—is easily adapted to weak type norms as was just done above. This completes the proof of Theorem 12. □

8 Proof of Theorem 9, a $T1$ theorem

Inequality (3.4) in Theorem 9 follows immediately from Theorems 6 and 12. On the other hand, if T^α is strongly elliptic, then

$$\sqrt{\mathcal{A}_2^\alpha(\sigma,\omega)+\mathcal{A}_2^\alpha(\omega,\sigma)}\lesssim\mathfrak{N}_{T^\alpha}(\sigma,\omega),$$

by [SaShUr7, Lemma 4.1 on page 92.]. This completes the proof of Theorem 9.

Remark 40. If we drop the assumption that one of the weights is A_∞^α or $C_{2+\varepsilon}$, then inequality (3.4) remains true if we include on the right-hand side the Bilinear Indicator Cube Testing constant $\mathcal{BICT}_{T^\alpha}(\sigma,\omega)$ defined in (2.15) above.

9 Proof of Theorem 10 on optimal cancellation conditions

Here we follow very closely the treatment in [Ste2, Section 3 of Chapter VII] to show how Theorem 10 follows from Theorem 9. The argument we follow in [Ste2, Section 3 of Chapter VII] uses balls instead of cubes, and we thus adapt the argument to cubes by using the distance function

$$\|y\| \equiv \max_{1\leq k\leq n} |y_k|$$

instead of the Euclidean distance $|y| = \sqrt{\sum_{1\leq k\leq n} |y_k|^2}$, which results in minimal cosmetic changes. Thus the corresponding balls $B(x,r) = \{y \in \mathbb{R}^n : \|x-y\| < r\}$ are familiar cubes with sides parallel to the axes, centered at x with side length $2r$. In order to free up superscripts for other uses, we will drop the fractional superscript α from both the kernel K^α and its associated operator T^α. Finally, we will need the following result on truncations, which extends the case $q = 2$ of Proposition 1 in Stein [Ste2, page 31] to a pair of doubling measures σ and ω.

9.1 Boundedness of truncations. For $\varepsilon > 0$, and a smooth α-fractional Calderón–Zygmund kernel $K(x,y)$, define the truncated kernels

$$K_\varepsilon(x,y) \equiv \begin{cases} K(x,y) & \text{if } \varepsilon < \|x-y\| \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$T_\varepsilon(x) \equiv \int K_\varepsilon(x,y) f(y) d\sigma(y), \quad \text{for } x \in \mathbb{R}^n \text{ and } f \in L^2(\sigma).$$

Proposition 41. Suppose that σ and ω are positive locally finite Borel measures on $\mathbb{R}^n$ satisfying the classical A_2^α condition, and that $K(x,y)$ is a smooth α-fractional Calderón–Zygmund kernel on $\mathbb{R}^n$. Suppose moreover that there is a bounded operator $T : L^2(\sigma) \to L^2(\omega)$, i.e.,

$$\|T(f\sigma)\|_{L^2(\omega)} \leq A\|f\|_{L^2(\sigma)}, \quad \text{for all } f \in L^2(\sigma),$$

associated with the kernel $K(x,y)$ in the sense that (3.6) holds. Then there is a positive constant A' such that the truncations T_ε satisfy

$$\|T_\varepsilon(f\sigma)\|_{L^2(\omega)} \leq A'\|f\|_{L^2(\sigma)}, \quad \text{for all } f \in L^2(\sigma) \text{ and } \varepsilon > 0. \tag{9.1}$$

Moreover, $A' \approx A + \sqrt{A_2^\alpha}$.

Proof. The proof is virtually identical to that of Stein in [Ste2, page 31] (which treated a doubling measure μ in place of Lebesgue measure dx) upon including appropriate use of the classical $A_2^\alpha(\sigma,\omega)$ condition to handle the extension to two otherwise arbitrary weights, and we now sketch the details.

For each x, the function $K_\varepsilon(x,\cdot)$ is in $L^2(\sigma)$ and so T_ε is well-defined on $L^2(\sigma)$ by Cauchy–Schwarz. Let $\widetilde{T}_\varepsilon \equiv T - T_\varepsilon$ be the 'near' part of T. Fix $\overline{x} \in \mathbb{R}^n$ and $f \in L^2(\sigma)$. All estimates in what follows are independent of ε, $\overline{x}$ and f. The crux of the proof is then to show that there are positive numbers C and $0 < a < \frac{1}{3}$ so that

$$\|\mathbf{1}_{B(\overline{x},a\varepsilon)}\widetilde{T}_\varepsilon(f\sigma)\|_{L^2(\omega)} \le \left(A + C\left(1+\frac{1}{a}\right)^n\sqrt{A_2^\alpha}\right)\|\mathbf{1}_{B(\overline{x},(a+1)\varepsilon)}f\|_{L^2(\sigma)}, \tag{9.2}$$

where the balls $B(\overline{x},r)$ are actually cubes in the new distance function, and we will often refer to them as cubical balls.

Note that $\widetilde{T}_\varepsilon(f\sigma)(x) = 0$ if $\mathrm{Supp} f \subset B(x,\varepsilon)^c$ and that $\widetilde{T}_\varepsilon(f\sigma)(x) = T(f\sigma)(x)$ if $\mathrm{Supp} f \subset B(x,\varepsilon)$, so that

$$\mathbf{1}_{B(\overline{x},a\varepsilon)}\widetilde{T}_\varepsilon(f\sigma) = \mathbf{1}_{B(\overline{x},a\varepsilon)}\widetilde{T}_\varepsilon(\mathbf{1}_{B(\overline{x},(a+1)\varepsilon)}f\sigma).$$

Next we split the right-hand side into two pieces:

$$\begin{aligned}&\mathbf{1}_{B(\overline{x},a\varepsilon)}\widetilde{T}_\varepsilon(\mathbf{1}_{B(\overline{x},(a+1)\varepsilon)}f\sigma)\\ &\qquad = \mathbf{1}_{B(\overline{x},a\varepsilon)}\widetilde{T}_\varepsilon(\mathbf{1}_{B(\overline{x},d\varepsilon)}f\sigma) + \mathbf{1}_{B(\overline{x},a\varepsilon)}\widetilde{T}_\varepsilon([\mathbf{1}_{B(\overline{x},(a+1)\varepsilon)} - \mathbf{1}_{B(\overline{x},d\varepsilon)}]f\sigma),\end{aligned} \tag{9.3}$$

where we choose $2a < d < 1-a$. In particular, $B(\overline{x},d\varepsilon) \subset B(x,\varepsilon)$ whenever $x \in B(\overline{x},a\varepsilon)$. This gives

$$\mathbf{1}_{B(\overline{x},a\varepsilon)}\widetilde{T}_\varepsilon(\mathbf{1}_{B(\overline{x},d\varepsilon)}f\sigma) = \mathbf{1}_{B(\overline{x},a\varepsilon)}T(\mathbf{1}_{B(\overline{x},d\varepsilon)}f\sigma),$$

and

$$\begin{aligned}\left\|\mathbf{1}_{B(\overline{x},a\varepsilon)}\widetilde{T}_\varepsilon(\mathbf{1}_{B(\overline{x},d\varepsilon)}f\sigma)\right\|_{L^2(\omega)} &= \left\|\mathbf{1}_{B(\overline{x},a\varepsilon)}T(\mathbf{1}_{B(\overline{x},d\varepsilon)}f\sigma)\right\|_{L^2(\omega)}\\ &\le A\|\mathbf{1}_{B(\overline{x},d\varepsilon)}f\|_{L^2(\sigma)} \le A\|\mathbf{1}_{B(\overline{x},(a+1)\varepsilon)}f\|_{L^2(\sigma)}.\end{aligned}$$

To estimate the second term on the right hand side of (9.3), we use $a < d$ and the association of T with K given in (3.6) to obtain

$$\begin{aligned}&\mathbf{1}_{B(\overline{x},a\varepsilon)}(x)\widetilde{T}_\varepsilon([\mathbf{1}_{B(\overline{x},(a+1)\varepsilon)} - \mathbf{1}_{B(\overline{x},d\varepsilon)}]f\sigma)(x)\\ &\qquad = \int_{B(x,\varepsilon)\cap\{B(\overline{x},(a+1)\varepsilon)\setminus B(\overline{x},d\varepsilon)\}} K(x,y)f(y)d\sigma(y),\\ &\qquad\qquad \text{for } \sigma\text{-a.e. } x \in B(\overline{x},a\varepsilon),\end{aligned}$$

since the cubical annulus $B(\overline{x},(a+1)\varepsilon)\setminus B(\overline{x},d\varepsilon)$ is disjoint from the cubical ball $B(\overline{x},a\varepsilon)$. For y in the above range of integration, we have

$$d\varepsilon < \|\overline{x}-y\| \le \|\overline{x}-x\| + \|x-y\| \le a\varepsilon + \|x-y\|,$$

and using $2a < d$, we conclude that $\|x-y\| \ge (d-a)\varepsilon \ge a\varepsilon$. Thus

$$|K(x,y)| \le \frac{C}{(a\varepsilon)^n} = C\Big(1+\frac{1}{a}\Big)^n \frac{1}{|B(\overline{x},(a+1)\varepsilon)|},$$

and so

$$\begin{aligned}
&\big\|\mathbf{1}_{B(\overline{x},a\varepsilon)}\widetilde{T}_\varepsilon([\mathbf{1}_{B(\overline{x},(a+1)\varepsilon)}-\mathbf{1}_{B(\overline{x},d\varepsilon)}]f\sigma)\big\|_{L^2(\omega)}\\
&\quad= \bigg\|\int_{B(x,\varepsilon)\cap\{B(\overline{x},(a+1)\varepsilon)\setminus B(\overline{x},d\varepsilon)\}} K(x,y)f(y)d\sigma(y)\bigg\|_{L^2(\omega)}\\
&\quad\le C\Big(1+\frac{1}{a}\Big)^n \frac{1}{|B(\overline{x},(a+1)\varepsilon)|}\sqrt{|B(\overline{x},(a+1)\varepsilon)|_\omega}\int_{B(\overline{x},(a+1)\varepsilon)}|f(y)|d\sigma(y)\\
&\quad\le C\Big(1+\frac{1}{a}\Big)^n \frac{\sqrt{|B(\overline{x},(a+1)\varepsilon)|_\omega}\sqrt{|B(\overline{x},(a+1)\varepsilon)|_\sigma}}{|B(\overline{x},(a+1)\varepsilon)|}\|\mathbf{1}_{B(\overline{x},(a+1)\varepsilon)}f\|_{L^2(\sigma)}\\
&\quad\le C\Big(1+\frac{1}{a}\Big)^n\sqrt{A_2^\alpha(\sigma,\omega)}\|\mathbf{1}_{B(\overline{x},(a+1)\varepsilon)}f\|_{L^2(\sigma)}.
\end{aligned}$$

Plugging our two estimates into (9.3), we obtain (9.2).

As in [Ste2, page 31], we now add up the inequalities in (9.2) for a suitable collection of cubical balls covering $\mathbb{R}^n$ to obtain (9.1) with

$$A' = 2^n\Big(1+\frac{1}{a}\Big)^n\Big(A + C\Big(1+\frac{1}{a}\Big)^n\sqrt{A_2^\alpha}\Big)^2.$$

Indeed, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}|\widetilde{T}_\varepsilon(f\sigma)|^2d\omega &\le \sum_{k=1}^\infty\int_{B(\overline{x},a\varepsilon)}|\widetilde{T}_\varepsilon(f\sigma)|^2d\omega\\
&\le (A + C\sqrt{A_2^\alpha})^2\sum_{k=1}^\infty\int_{B(\overline{x},(a+1)\varepsilon)}|f|^2d\sigma\\
&\le (A + C\sqrt{A_2^\alpha})^2 N\int_{\mathbb{R}^n}|f|^2d\sigma
\end{aligned}$$

provided

$$\bigcup_k B(\overline{x}^k,a\varepsilon) = \mathbb{R}^n \quad \text{and} \quad \sum_k \mathbf{1}_{B(\overline{x}^k,(a+1)\varepsilon)} \le N.$$

But these two properties are achieved for any $N > 2^n(1+\frac{1}{a})^n - 1$ by letting $\{B(\overline{x}^k,\frac{a}{2}\varepsilon)\}_{k=1}^\infty$ be a maximal pairwise disjoint collection:

(1) If $z \in \mathbb{R}^n \setminus \bigcup_k B(\overline{x}^k, a\varepsilon)$, then $B(z, \frac{a}{2}\varepsilon) \cap [\bigcup_k B(\overline{x}^k, \frac{a}{2}\varepsilon)] = \emptyset$ since if there is w in $B(z, \frac{a}{2}\varepsilon) \cap B(\overline{x}^k, \frac{a}{2}\varepsilon)$, then $\|z - \overline{x}^k\| \leq \|z - w\| + \|w - \overline{x}^k\| < a\varepsilon$, contradicting pairwise disjointedness of the collection $\{B(\overline{x}^k, \frac{a}{\sqrt{n}}\varepsilon)\}_{k=1}^{\infty}$. But then $B(z, \frac{a}{2}\varepsilon)$ could be included in the collection $\{B(\overline{x}^k, \frac{a}{2}\varepsilon)\}_{k=1}^{\infty}$, contradicting its maximality.

(2) If $z \in \bigcap_{j=1}^{N+1} B(\overline{x}^{k_j}, (a+1)\varepsilon)$, then $B(\overline{x}^{k_j}, a\varepsilon) \subset B(z, 2(a+1)\varepsilon)$ and so

$$c(2(a+1)\varepsilon)^n = |B(z, 2a\varepsilon)| \geq \left| \bigcup_{j=1}^{N+1} B(\overline{x}^{k_j}, a\varepsilon) \right| = \sum_{j=1}^{N+1} |B(\overline{x}^{k_j}, a\varepsilon)|$$
$$= (N+1)c(a\varepsilon)^n,$$

which is a contradiction if $N + 1 > 2^n(1 + \frac{1}{a})^n$. □

9.2 The cancellation theorem. Now we turn to the proof of Theorem 10, where we follow Stein [Ste2, Section 3 of Chapter VII], but subtracting a higher order Taylor polynomial to control estimates for doubling measures.

Proof of Theorem 10. Recall the cancellation condition (3.7),

$$\int_{\|x-x_0\|<N} \left| \int_{\varepsilon<\|x-y\|<N} K(x,y)d\sigma(y) \right|^2 d\omega(x) \leq \mathfrak{A}_K(\sigma,\omega)|B(x_0,N)|_\sigma, \tag{9.4}$$

for all ε, N, x_0. By the previous proposition, together with the Independence of Truncations at the end of Subsubsection 2.2.1, the roughly truncated operators $T_{\varepsilon,N}$, with kernel $K_{\varepsilon,N}(x,y) = K(x,y)\mathbf{1}_{\{\varepsilon<|x-y|<N\}}$, are bounded from $L^2(\sigma)$ to $L^2(\omega)$ by a multiple of $\|T\|_{L^2(\sigma)\to L^2(\omega)}$ uniformly in $0 < \varepsilon < N < \infty$. Thus we have the following Cube Testing condition for $T_{\varepsilon,N}$ uniformly in $0 < \varepsilon < N < \infty$, i.e.

$$\begin{aligned} \int_{B(x_0,N)} \left| \int_{\varepsilon<\|x-y\|<N} K(x,y)\mathbf{1}_{B(x_0,N)}(y)d\sigma(y) \right|^2 d\omega(x) \\ \leq \|T\|^2_{L^2(\sigma)\to L^2(\omega)}|B(x_0,N)|_\sigma, \end{aligned} \tag{9.5}$$

for all cubical balls $B(x_0,N)$. However, the inner integrals with respect to σ in (9.4) and (9.5) don't match up. On the other hand, their difference is an integral in σ supported outside the cubical ball $B(x_0,N)$ where ω is supported. This fact is exploited in the following argument of Stein [Ste2, Section 3 of Chapter VII].

We begin by proving the necessity of (3.7) for the norm inequality, i.e.,

$$\mathfrak{A}_K(\sigma,\omega) \lesssim \|T\|^2_{L^2(\sigma)\to L^2(\omega)} + A_2^\alpha(\sigma,\omega).$$

Set

$$I_{\varepsilon,N}(x) \equiv \int_{\varepsilon<\|x-y\|<N} K(x,y)d\sigma(y).$$

First observe that it suffices to show

$$\int_{\|x-x_0\|<\frac{N}{2}} |I_{\varepsilon,N}(x)|^2 d\omega(x) \le \|T\|^2_{L^2(\sigma)\to L^2(\omega)} |B(x,N)|_\sigma, \tag{9.6}$$

since every cubical ball $B(x_0,N)$ of radius N can be covered by a bounded number J of cubical balls of radius $\frac{N}{2}$ (2^n such cubical balls suffice). Indeed if

$$B(x_0,N) \subset \dot{\bigcup_{j=1}^{J}} B\left(x_j, \frac{N}{2}\right),$$

then

$$\begin{aligned}
&\int_{\|x-x_0\|<N} \left| \int_{\varepsilon<\|x-y\|<N} K(x,y)d\sigma(y)\right|^2 d\omega(x) \\
&\qquad \le \sum_{j=1}^{J} \int_{\|x-x_j\|<\frac{N}{2}} \left| \int_{\varepsilon<\|x-y\|<N} K(x,y)d\sigma(y)\right|^2 d\omega(x) \\
&\qquad \le \sum_{j=1}^{J} \|T\|^2_{L^2(\sigma)\to L^2(\omega)} \left| B\left(x_j, \frac{N}{2}\right)\right|_\sigma \lesssim \|T\|^2_{L^2(\sigma)\to L^2(\omega)} |B(x_0,N)|_\sigma,
\end{aligned}$$

since σ is doubling.

As before, define the truncated kernels

$$K_\varepsilon(x,y) \equiv \begin{cases} K(x,y) & \text{if } \varepsilon < \|x-y\| \\ 0 & \text{if not,} \end{cases}$$

and set

$$T_\varepsilon(x) \equiv \int K_\varepsilon(x,y)f(y)d\sigma(y), \quad \text{for } x \in \mathbb{R}^n \text{ and } f \in L^2(\sigma).$$

By the previous proposition, the operators T^α_ε are uniformly bounded from $L^2(\sigma)$ to $L^2(\omega)$.

Continuing to follow Stein [Ste2, Section 3 of Chapter VII], we compare $I_{\varepsilon,N}(x)$ with $T_\varepsilon(\mathbf{1}_{B(x_0,N)})(x)$. Since

$$\{B(x,N) \setminus B(x_0,N)\} \cup \{B(x_0,N) \setminus B(x,N)\} \subset B\left(x, \frac{3N}{2}\right) \setminus B\left(x, \frac{N}{2}\right),$$

provided $\|x-x_0\| < \frac{N}{2}$, and since

$$\begin{aligned}
&I^E_{\varepsilon,N}(x) - T_\varepsilon(\mathbf{1}_{B(x_0,N)}\sigma)(x) \\
&\qquad = \int_{B(x,N)\setminus B(x,\varepsilon)} K(x,y)d\sigma(y) - \int_{B(x_0,N)\setminus B(x,\varepsilon)} K(x,y)d\sigma(y),
\end{aligned}$$

it follows that

$$|I^E_{\varepsilon,N}(x) - T_\varepsilon(\mathbf{1}_{B(x_0,N)})(x)| \le \int_{B(x,\frac{3N}{2})\setminus B(x,\frac{N}{2})} |K(x,y)| d\sigma(y) \lesssim \frac{1}{N^n}\left|B\left(x,\frac{3N}{2}\right)\right|_\sigma,$$

when $\|x-x_0\| < \frac{N}{2}$. Then

$$\begin{aligned}
&\int_{\|x-x_0\|<\frac{N}{2}} |I_{\varepsilon,N}(x)|^2 d\omega(x)\\
&\lesssim \int_{B(x_0,\frac{N}{2})} |T_\varepsilon(\mathbf{1}_{B(x_0,N)}\sigma)(x)|^2 d\omega(x) + \int_{B(x_0,\frac{N}{2})} |I_{\varepsilon,N}(x) - T^\alpha_\varepsilon(\mathbf{1}_{B(x_0,N)})(x)|^2 d\omega(x)\\
&\lesssim \sup_{\varepsilon>0} \|T_\varepsilon\|^2_{L^2(\sigma)\to L^2(\omega)} |B(x_0,N)|_\sigma + \left|B\left(x_0,\frac{N}{2}\right)\right|_\omega \frac{1}{N^{2n}}\left|B\left(x_0,\frac{3N}{2}\right)\right|^2_\sigma\\
&\lesssim \{\|T\|^2_{L^2(\sigma)\to L^2(\omega)} + A^\alpha_2(\sigma,\omega)\}\left|B\left(x_0,\frac{3N}{2}\right)\right|_\sigma\\
&\lesssim \{\|T\|^2_{L^2(\sigma)\to L^2(\omega)} + A^\alpha_2(\sigma,\omega)\}|B(x_0,N)|_\sigma,
\end{aligned}$$

since σ is doubling. This proves (9.6), and hence the necessity of (3.7) with

$$\mathfrak{A}_K(\sigma,\omega) \lesssim \|T\|^2_{L^2(\sigma)\to L^2(\omega)} + A^\alpha_2(\sigma,\omega).$$

The proof of necessity of the dual condition to (3.7) is similar using that ω is doubling.

Conversely, as in Stein [Ste2, Section 3 of Chapter VII], let $K^\varepsilon(x,y)$ be a smooth truncation of K given by

$$K^\varepsilon(x,y) \equiv \eta\left(\frac{x-y}{\varepsilon}\right)K(x,y),$$

where $\eta(x)$ is smooth, vanishes if $\|x\| \le \frac{1}{2}$ and equals 1 if $\|x\| \ge 1$. Note that the kernels $K^\varepsilon(x,y)$ satisfy(3.5) uniformly in $\varepsilon > 0$, and can be used as truncations in defining the weighted norm inequality as in Subsubsection 2.2.1—see Independence of Truncations 2.2.1. We will show that the operators T^ε corresponding to K^ε satisfy the κ-Cube Testing conditions, also uniformly in $\varepsilon > 0$. For this we begin by controlling the full κ-Cube Testing condition for T^ε by the following polynomial variant of (9.4):

$$\begin{aligned}
&\int_{\|x-x_0\|<N}\left|\int_{\varepsilon<\|x-y\|<N} K^\alpha(x,y)\frac{p(y)}{\|\mathbf{1}_{B(x_0,N)}p\|_\infty} d\sigma(y)\right|^2 d\omega(x)\\
&\qquad\qquad \le \mathfrak{A}^{(\kappa)}_{K^\alpha}(\sigma,\omega)\int_{\|x_0-y\|<N} d\sigma(y),
\end{aligned} \tag{9.7}$$

for all polynomials p of degree less than κ, all $0 < \varepsilon < N$ and $x_0 \in \mathbb{R}^n$, where $\mathfrak{A}_{K^\alpha}^{(\kappa)}(\sigma,\omega)$ denotes the smallest constant for which (9.7) holds, and where $\kappa \in \mathbb{N}$.

To see this, fix a positive integer $\kappa > n-\alpha$, and define

$$I_{\varepsilon,R}(x) \equiv \int_{\varepsilon<\|x-y\|<R} K(x,y)\mathbf{1}_{B(x_0,R)}(y)d\sigma(y).$$

If ϕ_κ^{R,x_0} is a $B(x_0,R)$-normalized polynomial of degree less than κ as in Definition 17, and $\|x-x_0\| < 2R$, and if we denote by $\operatorname{Tay} f(x)$ the $(\kappa-1)^{st}$-degree Taylor polynomial of f at x, then

$$\begin{aligned}
&T^\varepsilon(\phi_\kappa^{R,x_0}\mathbf{1}_{B(x_0,R)}\sigma)(x)\\
&\qquad = \int K^\varepsilon(x,y)\phi_\kappa^{R,x_0}(y)\mathbf{1}_{B(x_0,R)}(y)d\sigma(y)\\
&\qquad = \int K^\varepsilon(x,y)[\phi_\kappa^{R,x_0}(y) - \operatorname{Tay}\phi_\kappa^{R,x_0}(x)]\mathbf{1}_{B(x,3R)}(y)\mathbf{1}_{B(x_0,R)}(y)d\sigma(y)\\
&\qquad\quad + \phi_\kappa^{R,x_0}(x)\int_{B(x,3R)} \operatorname{Tay} K^\varepsilon(x,y)\mathbf{1}_{B(x_0,R)}(y)d\sigma(y).
\end{aligned}$$

The first integral is estimated by

$$A\int_{B(x,3R)} |x-y|^{\alpha-n}\left(\frac{|x-y|}{R}\right)^\kappa \mathbf{1}_{B(x_0,R)}(y)d\sigma(y) \lesssim A\frac{1}{R^n}|B(x,3R)|_\sigma,$$

since we chose $\kappa > n-\alpha$. On the other hand, the integral

$$\int_{B(x,3R)} \operatorname{Tay} K^\varepsilon(x,y)\mathbf{1}_{B(x_0,R)}(y)d\sigma(y)$$

differs from $I_{\varepsilon,R}(x)$ by

$$\int_{B(x,3R)\setminus B(x,\varepsilon)} \{\operatorname{Tay} K^\varepsilon(x,y) - K(x,y)\}\mathbf{1}_{B(x_0,R)}(y)d\sigma(y),$$

whose modulus is again at most

$$\int_{B(x,3R)\setminus B(x,\varepsilon)} |x-y|^{\alpha-n}\left(\frac{|x-y|}{R}\right)^\kappa \mathbf{1}_{B(x_0,R)}(y)d\sigma(y) \lesssim A\frac{1}{R^n}|B(x,3R)|_\sigma.$$

Thus (9.6) implies that

$$\begin{aligned}
\int_{B(x_0,2R)} |T^\varepsilon(\phi_\kappa^{R,x_0}\mathbf{1}_{B(x_0,R)}\sigma)|^2 d\omega(x) &\lesssim \{A_2^\alpha + \mathfrak{A}_K^{(\kappa)}(\sigma,\omega)\}|B(x_0,5R)|_\sigma\\
&\lesssim \{A_2^\alpha + \mathfrak{A}_K^{(\kappa)}(\sigma,\omega)\}|B(x_0,R)|_\sigma,
\end{aligned}$$

since σ is doubling. Taking the supremum over cubical balls $B(x_0,R)$ yields

$$\mathfrak{F}\mathfrak{T}_T^{(\kappa)}(\sigma,\omega) \lesssim \sqrt{A_2^\alpha} + \mathfrak{A}_K^{(\kappa)}(\sigma,\omega).$$

Similarly we have $\mathfrak{F}\mathfrak{T}_{T^*}^{(\kappa)}(\sigma,\omega) \lesssim \sqrt{A_2^\alpha} + \mathfrak{A}_{K^*}^{(\kappa)}(\omega,\sigma)$.

At this point we need the following $\mathfrak{A}_K^{(\kappa)}$-variant of Corollary 25: For every $\kappa \in \mathbb{N}$ and $0 < \delta < 1$, there is a positive constant $C_{\kappa,\delta}$ such that

$$\mathfrak{A}_{K^\alpha}^{(\kappa)}(\sigma, \omega) \leq C_{\kappa,\delta}\mathfrak{A}_{K^\alpha}(\sigma, \omega) + \delta\mathfrak{N}_T(\sigma, \omega),$$

and where the constants $C_{\kappa,\delta}$ depend only on κ and δ, and not on the operator norm $\mathfrak{N}_T(\sigma, \omega)$. The proof of this variant is similar to that of Theorem 22, Proposition 24 and Corollary 25, and is left to the reader. With this variant in hand, we now have

$$\mathfrak{F}\mathfrak{T}_T^{(\kappa)}(\sigma, \omega) \lesssim C_{\kappa,\delta}[\sqrt{A_2^\alpha} + \mathfrak{A}_K(\sigma, \omega)] + \delta\mathfrak{N}_T(\sigma, \omega),$$

for arbitrarily small $\delta > 0$.

In view of Theorem 9, and absorbing the term $\delta\mathfrak{N}_T(\sigma, \omega)$ for $\delta > 0$ sufficiently small, the operator norms of the truncated operators T^ε are now bounded uniformly in $\varepsilon > 0$. Thus there is a sequence $\{\varepsilon_k\}_{k=1}^\infty$ with

$$\lim_{k\to\infty} \varepsilon_k = 0$$

such that the operators T^{ε_k} converge weakly to a bounded operator T from $L^2(\sigma)$ to $L^2(\omega)$. Since the truncated kernels $K^{\varepsilon_k}(x, y)$ converge pointwise and dominatedly to $K(x, y)$, Lebesgue's Dominated Convergence theorem applies to show that for $x \notin \mathrm{Supp}(f\sigma)$, and where the doubling measure σ has no atoms and the function f has compact support, we have

$$\begin{aligned} T(f\sigma)(x) &= \lim_{k\to\infty} T^{\varepsilon_k}(f\sigma)(x) = \lim_{k\to\infty} \int K^{\varepsilon_k}(x, y)f(y)d\sigma(y) \\ &= \int K(x, y)f(y)d\sigma(y), \end{aligned}$$

which is the representation (3.6). This completes the proof of Theorem 10. □

10 Concluding remarks

The problem investigated in this paper is that of fixing a measure pair(σ, ω), and then asking for a characterization of the α-fractional Calderón–Zygmund operators T^α that are bounded from $L^2(\sigma)$ to $L^2(\omega)$—the first solution being the one weight case of Lebesgue measure with $\alpha = 0$ in [DaJo]. This problem of fixing a measure pair is in a sense 'orthogonal' to other recent investigations of two weight norm inequalities, in which one fixes the elliptic operator T^α, and asks for a characterization of the weight pairs (σ, ω) for which T^α is bounded.

This latter investigation for a fixed operator is extraordinarily difficult, with essentially just one Calderón–Zygmund operator T^α known to have a characterization of the weight pairs (σ, ω), namely the Hilbert transform on the line; see the two part paper [LaSaShUr3]; [Lac] and [Hyt3], and also [SaShUr10] for an extension to gradient elliptic operators on the line. In particular, matters appear to be very bleak in higher dimensions due to the example in [Saw] which shows that the energy side condition, used in virtually all attempted characterizations, fails to be necessary for even the most basic elliptic operators—the stronger pivotal condition is however shown in [LaLi] to be necessary for boundedness of the g-function, a Hilbert space valued Calderón–Zygmund operator with a strong gradient positivity property, and the weight pairs were then characterized in [LaLi] by a single testing condition.[11]

On the other hand, the problem for a fixed measure pair has proved somewhat more tractable. However, the techniques required for these results are taken largely from investigations of the problem where the operator is fixed. In particular, an adaptation of the 'pivotal' argument in [NTV4] to the weighted Alpert wavelets in [RaSaWi] and a Parallel Corona decomposition from [LaSaShUr4] are used.

The question of relaxing the side conditions of doubling, comparability of measures, and A_∞^α or C_q on the weights remains open, with the main stumbling blocks being (**1**) the limitations of weighted Alpert wavelets which require doubling, and (**2**) our bilinear Carleson Embedding Theorem which requires comparability of measures. There is in fact no known example of a α-fractional Calderón–Zygmund operator for which the $T1$ theorem fails.

For $0 < \alpha < n$ there ought to be a larger class C_q^α of measures that includes both A_∞^α and C_q, $q > 2$, and for which a weighted norm of the fractional integral I_α is controlled by that of the fractional maximal function M_α. One possibility for the definition of such a class C_p^α of measures for $0 < \alpha < n$ and $1 < p < \infty$ is

$$\frac{|E|_\omega}{\int_Q |M\mathbf{1}_Q|^p \omega} \leq \eta(\mathbf{Cap}_\alpha^Q(E)), \quad \text{for all compact subsets } E \text{ of a cube } Q,$$

$$\text{for some function } \eta : [0,1] \to [0,1] \text{ with } \lim_{t \searrow 0} \eta(t) = 0.$$

In the case $\alpha = 0$, there is the problem analogous to the 'A_2 conjecture' solved in general in [Hyt], of determining the optimal dependence of the above estimates on the A_2 characteristic. In particular, the dependence for the restricted weak type inequality should follow using the pigeonholing and corona construction introduced in [LaPeRe] and used in [Hyt].

We end by summarizing the drawbacks in the methodology used here. The $T1$ theorem here is proved for general Calderón–Zygmund operators, and thus in the

[11] The testing condition (1.3) in [LaLi] implies the weights share no common point masses, and then an argument in [LaSaUr1] using the asymmetric A_2 condition of Stein shows that the A_2 condition is implied by the testing condition. Thus (1.3) can be dropped from the statement of Theorem 1.2.

absence of any special positivity properties of the Calderón–Zygmund kernels K^α. As a consequence, there is no catalyst available to enable control of the difficult 'far below' and 'stopping' terms by 'goodness' of cubes in the NTV bilinear Haar decomposition (see, e.g., [NTV4]). In the case of the aforementioned Hilbert transform, the positivity of the derivative of the convolution kernel $\frac{1}{x}$ permitted the derivation of a strong catalyst, namely the energy condition, from the testing and Muckenhoupt conditions (see, e.g., [LaSaShUr3]), and in the case of Riesz transforms there is a partial reversal energy that yields the energy condition when the measures are both doubling (see, e.g., [LaWi] and [SaShUr9]). But the lack of a suitable catalyst for general Calderón–Zygmund operators (see [SaShUr11] and [Saw] for negative results) limits us to using the weighted Alpert wavelets in [RaSaWi]. The weighted Alpert wavelets in turn have two defects[12] that limit their use to doubling measures, and to situations that avoid the paraproduct/neighbor/stopping form decomposition of NTV in [NTV4]. This forces us to use the parallel corona, and ultimately to invoke comparability of measures and the A_∞^α or C_q, $q > 2$, assumption on one of the measures.

11 Reference List of conditions

For the reader's convenience we assemble here a Reference List of the conditions on weights and weight pairs arising in this paper in roughly the order of their appearance.

11.1 Conditions on a single measure.

(1) μ is doubling if

$$\int_{2Q} d\mu \lesssim \int_Q d\mu,$$

for all cubes $Q \subset \mathbb{R}^n$.

(2) ω is an A_∞ weight if $\frac{|E|_\omega}{|Q|_\omega} \leq C\left(\frac{|E|}{|Q|}\right)^\varepsilon$ for all compact subsets E of a cube Q.

(3) σ is a C_p weight if

$$\frac{|E|_\sigma}{\int_{\mathbb{R}^n} |M\mathbf{1}_Q|^p d\sigma} \leq C\left(\frac{|E|}{|Q|}\right)^\varepsilon$$

whenever E compact $\subset Q$ a cube.

(4) ω is an A_∞^α measure if $\frac{|E|_\omega}{|Q|_\omega} \leq \eta(\mathbf{Cap}_\alpha^Q(E))$, for all compact subsets E of a cube Q.

[12] Weighted L^2 projections fail to satisfy L^∞ bounds in general, and the size of an extension of a nonconstant polynomial is uncontrolled.

11.2 Conditions on a pair of measures.

(1) (σ,ω) comparable if $\frac{|E|_\sigma}{|Q|_\sigma}<\eta$ whenever E compact $\subset Q$ a cube with $\frac{|E|_\omega}{|Q|_\omega}<\varepsilon$.

(2) For $0\leq\alpha<n$, the α-fractional Muckenhoupt conditions for the weight pair (σ,ω) are

$$A_2^\alpha(\sigma,\omega)\equiv\sup_{Q\in\mathcal{P}^n}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}}<\infty,$$

$$\mathcal{A}_2^\alpha(\sigma,\omega)\equiv\sup_{Q\in\mathcal{Q}^n}\mathcal{P}^\alpha(Q,\sigma)\frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}}<\infty,$$

$$\mathcal{A}_2^{\alpha,*}(\sigma,\omega)\equiv\sup_{Q\in\mathcal{Q}^n}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^\alpha(Q,\omega)<\infty,$$

$$\mathcal{P}^\alpha(Q,\mu)\equiv\int_{\mathbb{R}^n}\left(\frac{|Q|^{\frac{1}{n}}}{(|Q|^{\frac{1}{n}}+|x-x_Q|)^2}\right)^{n-\alpha}d\mu(x).$$

(3) The **κ-cube testing conditions** for T^α are

$$(\mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega))^2\equiv\sup_{Q\in\mathcal{P}^n}\max_{0\leq|\beta|<\kappa}\frac{1}{|Q|_\sigma}\int_Q|T_\sigma^\alpha(\mathbf{1}_Q m_Q^\beta)|^2\omega<\infty,$$

$$(\mathfrak{T}_{(T^\alpha)^*}^{(\kappa)}(\omega,\sigma))^2\equiv\sup_{Q\in\mathcal{P}^n}\max_{0\leq|\beta|<\kappa}\frac{1}{|Q|_\omega}\int_Q|(T_\sigma^\alpha)^*(\mathbf{1}_Q m_Q^\beta)|^2\sigma<\infty,$$

with $m_Q^\beta(x)\equiv(\frac{x-c_Q}{\ell(Q)})^\beta$ for any cube Q and multiindex β, where c_Q is the center of the cube Q.

(4) The **full κ-cube testing conditions** for T^α are

$$(\mathfrak{F}\mathfrak{T}_{T^\alpha}^{(\kappa)}(\sigma,\omega))^2\equiv\sup_{Q\in\mathcal{P}^n}\max_{0\leq|\beta|<\kappa}\frac{1}{|Q|_\sigma}\int_{\mathbb{R}^n}|T_\sigma^\alpha(\mathbf{1}_Q m_Q^\beta)|^2\omega<\infty,$$

$$(\mathfrak{F}\mathfrak{T}_{(T^\alpha)^*}^{(\kappa)}(\omega,\sigma))^2\equiv\sup_{Q\in\mathcal{P}^n}\max_{0\leq|\beta|<\kappa}\frac{1}{|Q|_\omega}\int_{\mathbb{R}^n}|(T^\alpha)_\omega^*(\mathbf{1}_Q m_Q^\beta)|^2\sigma<\infty.$$

(5) The weak boundedness constant is

$$\mathcal{WBP}_{T^\alpha}^{(\kappa_1,\kappa_2)}(\sigma,\omega)$$
$$=\sup_{\mathcal{D}\in\Omega}\sup_{\substack{Q,Q'\in\mathcal{D}\\ Q\subset 3Q'\setminus Q'\text{ or }Q'\subset 3Q\setminus Q}}\frac{1}{\sqrt{|Q|_\sigma|Q'|_\omega}}\sup_{\substack{f\in(\mathcal{P}_Q^{\kappa_1})_{\text{norm}}\\ g\in(\mathcal{P}_Q^{\kappa_2})_{\text{norm}}}}\left|\int_{Q'}T_\sigma^\alpha(\mathbf{1}_Q f)g\,d\omega\right|<\infty,$$

where $(\mathcal{P}_\kappa^Q)_{\text{norm}}$ is the space of Q-normalized polynomials of degree less than κ (Definition 17).

(6) The Bilinear Indicator/Cube Testing property is

$$\mathcal{BICT}_{T^\alpha}(\sigma,\omega)\equiv\sup_{Q\in\mathcal{P}^n}\sup_{E,F\subset Q}\frac{1}{\sqrt{|Q|_\sigma|Q|_\omega}}\left|\int_F T_\sigma^\alpha(\mathbf{1}_E)\omega\right|<\infty,$$

where the second supremum is taken over all compact sets E and F contained in a cube Q.

(7) The κ^{th}-order fractional Pivotal Conditions $\mathcal{V}_2^{\alpha,\kappa}, \mathcal{V}_2^{\alpha,\kappa,*} < \infty$, $\kappa \geq 1$, are

$$(\mathcal{V}_2^{\alpha,\kappa})^2 = \sup_{Q\supset\dot{\cup}Q_r} \frac{1}{|Q|_\sigma} \sum_{r=1}^{\infty} \mathrm{P}_\kappa^\alpha(Q_r, \mathbf{1}_Q\sigma)^2 |Q_r|_\omega,$$

$$(\mathcal{V}_2^{\alpha,\kappa,*})^2 = \sup_{Q\supset\dot{\cup}Q_r} \frac{1}{|Q|_\omega} \sum_{r=1}^{\infty} \mathrm{P}_\kappa^\alpha(Q_r, \mathbf{1}_Q\omega)^2 |Q_r|_\sigma,$$

$$\mathrm{P}_\kappa^\alpha(Q, \mu) = \int_{\mathbb{R}^n} \frac{\ell(Q)^\kappa}{(\ell(Q) + |y - c_Q|)^{n+\kappa-\alpha}} d\mu(y), \quad \kappa \geq 1,$$

where the supremum is taken over all subdecompositions of a cube $Q \in \mathcal{P}^n$ into pairwise disjoint subcubes Q_r.

References

[AdHe] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Springer, Berlin–Heidelberg, 1999.

[Bur] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. Probab. **1** (1973), 19–42.

[CeLiPeRi] M. E. Cejasa, K. Li, C. Pérez and I. P. Rivera-Ríos, *Vector-valued operators, optimal weighted estimates and the C_p condition*, arXiv:1712.05781 [math.CA].

[CoFe] R. R. Coifman and C. L. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.

[DaJo] G. David and J.-L. Journé, *A boundedness criterion for generalized Calderón–Zygmund operators*, Ann. of Math. (2) **120** (1984), 371–397.

[GaKiSc] J. Garnett, R. Killip and R. Schul, *A doubling measure on $\mathbb{R}^d$ can charge a rectifiable curve*, Proc. Amer. Math. Soc. **138** (2010), 1673–1679.

[GrPa] C. Grigoriadis and M. Paparizos, *Counterexample to the off-testing condition in two dimensions*, arXiv:2004.06207 [math.CA].

[HuMuWh] R. Hunt, B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251.

[Hyt] T. Hytönen, *The sharp weighted bound for general Calderón–Zygmund operators*, Ann. of Math. (2) **175** (2012), 1473–1506.

[Hyt2] T. Hytönen, *The A2 theorem: remarks and complements*, in *Harmonic Analysis and Partial Differential Equations*, American Mathematical Society, Providence, RI, 2014, pp. 91–106.

[Hyt3] T. Hytönen, *The two weight inequality for the Hilbert transform with general measures*, Proc. Lond. Math. Soc. (3) **117** (2018), 483–526.

[HyPeTrVo] T. Hytönen, C. Perez, S. Treil and A. Volberg, *Sharp weighted estimates for dyadic shifts and the A_2 conjecture*, J. Reine Angew. Math. **687** (2014), 43–86.

[Lac] M. T. Lacey, *Two weight inequality for the Hilbert transform: A real variable characterization, II*, Duke Math. J. **163** (2014), 2821–2840.

[Lac2] M. T. Lacey, *The two weight inequality for the Hilbert transform: a primer*, in *Harmonic Analysis, Partial Differential Equations, Banach Spaces and Operator Theory. Vol. 2*, Springer, Cham, 2017, pp. 11–84,

[LaLi] M. T. Lacey and K. Li, *Two weight norm inequalities for the g function*, Math. Res. Lett. **21** (2014), 521–536.

[LaPeRe] M. T. Lacey, S. Petermichl and M. C. Reguera, *Sharp A_2 inequality for Haar shift operators*, Math. Ann. **348** (2010), 127–141.

[LaSaUr1] M. T. Lacey, E. T. Sawyer and I. Uriarte-Tuero, *A characterization of two weight norm inequalities for maximal singular integrals with one doubling measure*, Anal. PDE **5** (2012), 1–60.

[LaSaUr2] M. T. Lacey, E. T. Sawyer and I. Uriarte-Tuero, *Two weight inequalities for discrete positive operators*, arXiv:0911.3437 [math.CA].

[LaSaShUr3] M. T. Lacey, E. T. Sawyer, C.-Y. Shen and I. Uriarte-Tuero, *Two weight inequality for the Hilbert transform: A real variable characterization I*, Duke Math. J. **163** (2014), 2795–2820.

[LaSaShUr4] M. T. Lacey, E. T. Sawyer, C.-Y. Shen and I. Uriarte-Tuero, *Two weight inequality for the Hilbert transform: A real variable characterization, I*, arXiv:1201.4319v9 [math.CA] .

[LaSaShUrWi] M. T. Lacey, E. T. Sawyer, C.-Y. Shen, I. Uriarte-Tuero and B. D. Wick, *Two weight inequalities for the Cauchy transform from $\mathbb{R}$ to $\mathbb{C}_+$*, arXiv:1310.4820 [math.CV].

[LaWi] M. T. Lacey and B. D. Wick, *Two weight inequalities for Riesz transforms: uniformly full dimension weights*, arXiv:1312.6163 [math.CA].

[Ler] A. K. Lerner, *A characterization of the weighted weak type Coifman–Fefferman and Fefferman–Stein Inequalities*, Math. Ann. **378** (2020), 425–446.

[LiTr] C. Liaw and S. Treil, *Regularizations of general singular integral operators*, Rev. Mat. Iberoam. **29** (2013), 53–74.

[Muc] B. Muckenhoupt, *Norm inequalities relating the Hilbert transform to the Hardy–Littlewood maximal function*, in *Functional Analysis and Approximation (Oberwolfach,1980)*, Birkhäuser, Basel–Boston, MA, 1981, pp. 219–23.

[MuWh] B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. **192** (1974), 261–274.

[Naz] F. Nazarov, *A counterexample to a problem of Sarason on boundedness of the product of two Toeplitz operators*, preprint, 1996.

[NTV] F. Nazarov, S. Treil and A. Volberg, *The Bellman functions and two-weight inequalities for Haar multipliers*, J. Amer. Math. Soc. **12** (1999), 909–928.

[NTV1] F. Nazarov, S. Treil and A. Volberg, *Weak type estimates and Cotlar inequalities for Calderón–Zygmund operators on nonhomogeneous spaces*, Int. Math. Res. Not. IMRN **1998** (1998), 463–487

[NTV3] F. Nazarov, S. Treil and A. Volberg, *Accretive system Tb-theorems on nonhomogeneous spaces*, Duke Math. J. **113** (2002), 259–312.

[NTV4] F. Nazarov, S. Treil and A. Volberg, *Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures*, arxiv:1003.1596 [math.AP].

[NaVo] F. Nazarov and A. Volberg, *Bellman function, two-weighted Hilbert transform, and embeddings of the model spaces K_θ*, J. Anal. Math. **87** (2002), 385-414 .

[PeTrVo] C. Pérez, S. Treil and A. Volberg, *On A_2 conjecture and corona decomposition of weights*, arXiv:1006.2630 [math.AP].

[RaSaWi] R. Rahm, E. T. Sawyer and B. D. Wick, *Weighted Alpert wavelets*, J. Fourier Anal. Appl. **27** (2021), Article no. 1

[Saw] E. T. Sawyer, *Energy conditions and twisted localizations of operators*, arXiv:1801.03706 [math.CA].

[Saw0] E. T. Sawyer, *Weighted norm inequalities for fractional maximal operators*, in *1980 Seminar on Harmonic Analysis*, American Mathematical Society, Providence, RI, 1981, pp. 283–309.

[Saw1] E. T. Sawyer, *Norm inequalities relating singular integrals and the maximal function*, Studia Math. **75** (1983), 253–263.

[Saw2] E. T. Sawyer, *A $T1$ theorem with BICT for fractional singular integrals with doubling measures: cancellation conditions for Calderón–Zygmund operators*, arXiv:1906.05602 [math.CA].

[Saw3] E. T. Sawyer, *A restricted weak type inequality with application to a Tp theorem and cancellation conditions for CZO's*, arXiv:1907.07571 [math.CA].

[Saw4] E. T. Sawyer, *$T1$ implies Tp polynomial testing: optimal cancellation conditions for CZO's*, arXiv:1907.10734 [math.CA].

[Saw5] E. Sawyer, *A characterization of two weight norm inequalities for fractional and Poisson integrals*, Trans. Amer. Math. Soc. **308** (1988), 533–545.

[SaShUr4] E. T. Sawyer, C.-Y. Shen and I. Uriarte-Tuero, *A note on failure of energy reversal for classical fractional singular integrals*, Int. Math. Res. Not. IMRN **2015** (2015), 9888–9920.

[SaShUr7] E. T. Sawyer, C. -Y. Shen I. Uriarte-Tuero, *A two weight theorem for α-fractional singular integrals with an energy side condition*, Rev. Mat. Iberoam. **32** (2016), 79–174.

[SaShUr9] E. T. Sawyer, C. -Y. Shen and I. Uriarte-Tuero, *A two weight fractional singular integral theorem with side conditions, energy and k-energy dispersed*, in *Harmonic Analysis, Partial Differential Equations, Complex Analysis, Banach Spaces, and Operator Theory, Vol. 2*, Springer, Cham, 2017, pp. 305–372.

[SaShUr10] E. T. Sawyer, C.-Y. Shen and I. Uriarte-Tuero, *A good-λ lemma, two weight $T1$ theorems without weak boundedness, and a two weight accretive global Tb theorem*, in *Harmonic Analysis, Partial Differential Equations and Applications*, Birkhäuser, Cham, 2017, pp. 125–164.

[SaShUr11] E. T. Sawyer, C.-Y. Shen and I. Uriarte-Tuero, *Energy counterexamples in two weight Calderón–Zygmund theory*, Int. Math. Res. Not. IMRN **2021** (2021), 16271–16356.

[SaShUr12] E. T. Sawyer, C.-Y. Shen and I. Uriarte-Tuero, *A two weight local Tb theorem for the Hilbert transform*, Rev. Mat. Iberoam. **37** (2021), 415–641.

[SaUr] E. Sawyer and I. Uriarte-Tuero, *Control of the bilinear indicator cube testing property*, Ann. Fenn. Math. **46** (2021), 1105–1122.

[SaWh] E. Sawyer and R. L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*, Amer. J. Math. **114** (1992), 813–874.

[Ste] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

[Ste2] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.

[StWe] E. M. Stein and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. **87** (1958), 159–172.

[StWe2] E. M. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, 1971.

[Tan] H. Tanaka, *A characterization of two-weight trace inequalities for positive dyadic operators in the upper triangle case*, Potential Anal. **41** (2014), 487–499.

Eric T. Sawyer
DEPARTMENT OF MATHEMATICS & STATISTICS
MCMASTER UNIVERSITY
1280 MAIN STREET WEST
HAMILTON, ONTARIO L8S 4K1, CANADA
email: sawyer@mcmaster.ca

(Received July 30, 2019 and in revised form November 3, 2021)