$\ell^P(\mathbb{Z}^D)$-IMPROVING PROPERTIES AND SPARSE BOUNDS FOR DISCRETE SPHERICAL MAXIMAL AVERAGES

By

ROBERT KESLER

Abstract. We exhibit a range of $\ell^p(\mathbb{Z}^d)$-improving properties for the discrete spherical maximal average in every dimension $d \geq 5$. These improving properties are then used to establish sparse bounds, which extend the discrete maximal theorem of Magyar, Stein, and Wainger to weighted spaces. In particular, the sparse bounds imply that in every dimension $d \geq 5$ the discrete spherical maximal average is a bounded map from $\ell^2(w)$ into $\ell^2(w)$ provided $w^{\frac{d}{d-4}}$ belongs to the Muckenhoupt class A_2.

1 Introduction

Let $\mathbf{A}^d_\lambda$ denote the continuous spherical averaging operator on $\mathbb{R}^d$ at radius λ, i.e.,

$$\mathbf{A}^d_\lambda f(x) = \int_{S^{d-1}} f(x - \lambda y) d\sigma(y),$$

where $d \geq 2$, S^{d-1} denotes the unit $d-1$ dimensional sphere in $\mathbb{R}^d$ and σ is the unit surface measure on S^{d-1}. Stein establishes the spherical maximal theorem for $d \geq 3$ in [19], which states that $\| \sup_\lambda |\mathbf{A}^d_\lambda| : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)\| < \infty$ for all $\frac{d}{d-1} < p \leq \infty$. Bourgain examines the $d = 2$ case in [1] and shows that $\| \sup_\lambda |\mathbf{A}^2_\lambda| : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)\| < \infty$ for all $2 < p \leq \infty$. The sharp $L^p(\mathbb{R}^d)$-$L^q(\mathbb{R}^d)$ result for $\sup_{1\leq\lambda<2} |\mathbf{A}^d_\lambda|$ is shown by Schlag in [18]:

Theorem 1. *Let $d \geq 2$. Define $\mathcal{T}(d)$ to be the interior convex hull of $\{T_{d,j}\}_{j=1}^4$, where*

$$T_{d,1} = (0, 1), \qquad T_{d,2} = \Big(\frac{d-1}{d}, \frac{1}{d}\Big),$$
$$T_{d,3} = \Big(\frac{d-1}{d}, \frac{d-1}{d}\Big), \quad T_{d,4} = \Big(\frac{d^2-d}{d^2+1}, \frac{d^2-d+2}{d^2+1}\Big).$$

Then for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{T}(d)$ there exists a constant $A = A(d, p, r)$ such that

$$\Big\| \sup_{1\leq\lambda<2} |\mathbf{A}^d_\lambda| : L^p(\mathbb{R}^d) \to L^{r'}(\mathbb{R}^d)\Big\| \leq A.$$

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 143 (2021)
DOI 10.1007/s11854-021-0150-y

By rescaling, for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{T}(d)$, *and* $\Lambda \in 2^{\mathbb{Z}}$,

$$\left\| \sup_{\Lambda \leq \lambda < 2\Lambda} |\boldsymbol{A}_\lambda^d| : L^p(\mathbb{R}^d) \to L^{r'}(\mathbb{R}^d) \right\|_{L^{r'}(\mathbb{R}^d)} \leq A\Lambda^{d(1/r'-1/p)}.$$

Lacey obtains a sparse extension of the continuous spherical maximal theorem in [11]. To state his result properly, we first need to set some notation for sparse bounds. Recall that a collection of cubes $\mathcal{S}$ in $\mathbb{R}^d$ is called ρ-sparse if for each $Q \in \mathcal{S}$, there is a subset $E_Q \subset Q$ such that (a) $|E_Q| > \rho|Q|$, and (b) $\| \sum_{Q \in \mathcal{S}} 1_{E_Q} \|_{L^\infty(\mathbb{R}^d)} \leq \rho^{-1}$. For a sparse collection $\mathcal{S}$, a sparse bilinear (p, r)-form Λ is defined by

$$\Lambda_{\mathcal{S},p,r}(f, g) := \sum_{Q \in \mathcal{S}} \langle f \rangle_{Q,p} \langle g \rangle_{Q,r} |Q|,$$

where $\langle h \rangle_{Q,t} := (\frac{1}{|Q|} \sum_{x \in Q} |f(x)|^t)^{1/t}$ for any $t : 1 \leq t < \infty$, cube $Q \subset \mathbb{Z}^d$ and $h : \mathbb{Z}^d \to \mathbb{C}$. Each ρ-sparse collection $\mathcal{S}$ can be split into $O(\rho^{-2})$ many $\frac{1}{2}$-sparse collections. As long as $\rho^{-1} = O(1)$, its exact value is not relevant. To simplify some of the arguments, we use the following definition introduced in [4]: for an operator T acting on measurable, bounded, and compactly supported functions $f : \mathbb{R}^n \to \mathbb{C}$ and $1 \leq p, r < \infty$, define its sparse norm $\|T : (p, r)\|$ to be the infimum over all $C > 0$ such that for all pairs of measurable, bounded and compactly supported functions $f, g : \mathbb{R}^n \to \mathbb{C}$,

$$|\langle Tf, g \rangle| \leq C \sup_{\mathcal{S}} \Lambda_{\mathcal{S},p,r}(f, g),$$

where the supremum is taken over all $\frac{1}{2}$-sparse forms. A collection $\mathcal{C}$ of "cubes" in $\mathbb{Z}^d$ is ρ-sparse provided there is a collection $\mathcal{S}$ of ρ-sparse cubes in $\mathbb{R}^d$ with the property that $\{R \cap \mathbb{Z}^d : R \in \mathcal{S}\} = \mathcal{C}$. For a discrete operator T, define the sparse norm $\|T : (p, r)\|$ to be the infimum over all $C > 0$ such that for all pairs of bounded and finitely supported functions $f, g : \mathbb{Z}^d \to \mathbb{C}$,

$$|\langle Tf, g \rangle| \leq C \sup_{\mathcal{S}} \Lambda_{\mathcal{S},p,r}(f, g),$$

where the supremum is taken over all $\frac{1}{2}$-sparse collections $\mathcal{S}$ consisting of discrete "cubes." The sparse bounds obtained for continuous spherical maximal averages by Lacey in [11] are given by

Theorem 2. *Let* $d \geq 2$ *and define* $\mathcal{R}_T(d)$ *as in Theorem 1. Then for all* $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{R}_T(d)$

$$\left\| \sup_{\lambda > 0} |\boldsymbol{A}_\lambda^d| \ : (p, r) \right\| < \infty.$$

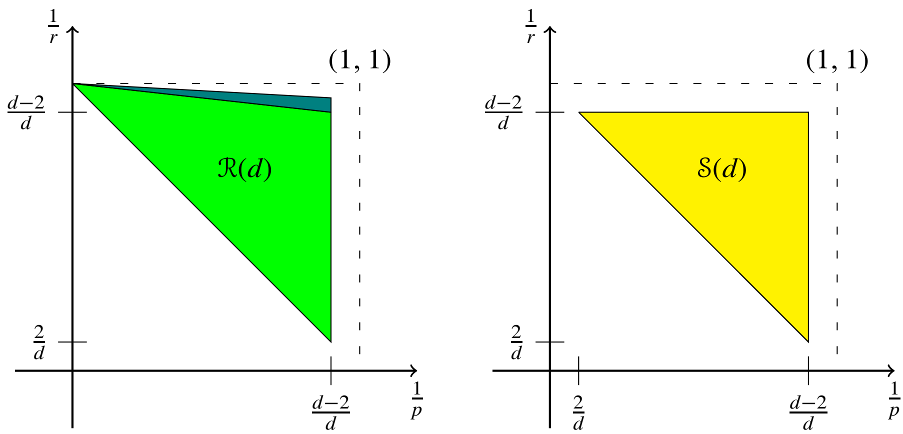

Figure 1. The green region (lighter triangle in left diagram) $\mathcal{R}(d)$ represents the range of uniform improving properties for $\sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{A}|$ and sparse bounds for $\sup_{\lambda \in \tilde{\Lambda}} |\mathscr{A}_\lambda|$ that we are able to prove. The teal region (darker triangle in left diagram) adjacent to $\mathcal{R}(d)$ represents the range of improving properties for $\sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{A}_\lambda|$ and sparse bounds for $\sup_{\lambda \in \tilde{\Lambda}} |\mathscr{A}_\lambda|$ that we cannot prove or disprove. The yellow region (triangel in right diagram) $\mathcal{S}(d)$ represents the range of improving properties for $\sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda|$ and $\sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{R}_\lambda|$ as well as sparse bounds for $\sup_{\lambda \in \tilde{\Lambda}} |\mathscr{C}_\lambda|$ and $\sup_{\lambda \in \tilde{\Lambda}} |\mathscr{R}_\lambda|$ that we are able to prove, where $\mathscr{C}_\lambda$ is the "major arc" piece of $\mathscr{A}_\lambda$ and $\mathscr{R}_\lambda = \mathscr{A}_\lambda - \mathscr{C}_\lambda$ is the residual.

Magyar, Stein, and Wainger prove their discrete spherical maximal theorem in [14]:

Theorem 3. *For each $\lambda \in \tilde{\Lambda} := \{\lambda > 0 : \lambda^2 \in \mathbb{N}\}$ define the discrete spherical average*

$$\mathscr{A}_\lambda f(x) = \frac{1}{|\{|y| = \lambda\}|} \sum_{y \in \mathbb{Z}^d : |y| = \lambda} f(x - y).$$

Then for all $d \geq 5$ and $\frac{d}{d-2} < p \leq \infty$

$$\left\| \sup_{\lambda \in \tilde{\Lambda}} |\mathscr{A}_\lambda| : \ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d) \right\| < \infty.$$

Our first theorem establishes a discrete analogue of Theorem 1:

Theorem 4. *Let $d \geq 5$. Define $\mathcal{R}(d)$ to be the interior convex hull of*

$$\mathcal{R}_{d,1} = (0, 1). \quad \mathcal{R}_{d,2} = \Big(\frac{d-2}{d}, \frac{2}{d}\Big), \quad \mathcal{R}_{d,3} = \Big(\frac{d-2}{d}, \frac{d-2}{d}\Big).$$

Then for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{R}(d)$ *there exists* $A = A(d,p,r)$ *such that, for every* $f \in \ell^p(\mathbb{Z}^d)$ *and* $\Lambda \in 2^{\mathbb{N}}$,

$$\left\| \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{A}_\lambda| : \ell^p(\mathbb{Z}^d) \to \ell^{r'}(\mathbb{Z}^d) \right\| \le A \Lambda^{d(1/r'-1/p)}. \tag{1}$$

A necessary condition for (1) *to hold for all* $\Lambda \in 2^{\mathbb{N}}$ *is* $\max\{\frac{1}{p} + \frac{2}{d}, \frac{1}{r} + \frac{2}{pd}\} \le 1$.

Our second theorem establishes the following discrete analogue of Theorem 2:

Theorem 5. *Let* $d \ge 5$ *and define* $\mathcal{R}(d)$ *as in Theorem 4. Then for all* $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{R}(d)$

$$\left\| \sup_{\lambda \in \tilde{\Lambda}} |\mathscr{A}_\lambda| : (p, r) \right\| < \infty. \tag{2}$$

A necessary condition for (2) *to hold is* $\max\{\frac{1}{p} + \frac{2}{d}, \frac{1}{r} + \frac{2}{pd}\} \le 1$.

2 Discussion of Results

While the study of improving properties for discrete maximal averages is new, much effort has focused on obtaining $\ell^p(\mathbb{Z}^d)$-estimates for discrete operators in harmonic analysis since the foundational work of Bourgain on ergodic theorems concerning polynomial averages. For instance, a number of delicate $\ell^p(\mathbb{Z}^d)$-bounds are obtained in the setting of radon transforms in [16, 5, 15], fractional variants in [21, 17], and Carleson operators in [9]. A well-known technique in this setting is the circle method of Hardy, Littlewood, and Ramanujan, which Magyar, Stein, and Wainger apply for the discrete spherical maximal averages in [14] to prove Theorem 3 by decomposing $\mathscr{A}_\lambda = \mathscr{C}_\lambda + \mathscr{R}_\lambda$, where $\mathscr{C}_\lambda$ is the "major arcs" consisting of a sum of modulated and fourier-localized copies of the continuous spherical averaging operator and $\mathscr{R}_\lambda$ is the residual term. We shall define $\mathscr{C}_\lambda$ in §2.

In the case where the supremum is taken only over discrete spherical averages with radii belonging to a thin set, for example a lacunary sequence, one can expand the range of sparse and ℓ^p-ℓ^q improving estimate beyond $\mathcal{R}(d)$ by using Kloosterman and Ramanujan sum refinements, and a good $L^\infty(\mathbb{T}^d)$ estimate on the symbol of $\mathscr{R}_\lambda$, namely $O_\epsilon(\lambda^{-\frac{d-3}{2}+\epsilon})$ from [14]. However, if the radii appearing in the supremum cluster too closely together, then one cannot reduce the argument to an estimate that is uniform in λ. It is for this reason that our analysis of the residual term $\mathscr{R}_\lambda$ in this paper is substantially more involved than in the lacunary case [12]. Moreover, as this paper only considers the full set of radii, Kloosterman and Ramanujan sums along with a good $L^\infty(\mathbb{T}^d)$ bound on the symbol of the residual operator $\mathscr{R}_\lambda$ are not able to improve our results and are therefore omitted from the analysis.

More than half of the paper is devoted to obtaining sparse bounds for discrete maximal spherical averages in the full supremum case. Pointwise sparse domination for Calderón–Zygmund operators is obtained by Conde–Alonso and Rey in [3] and is recently obtained as a consequence of work by Lacey in [10] on martingale transforms using a stopping time argument. Sparse form domination is a relaxation of the pointwise approach and holds in many settings, including Bochner–Riesz operators in [6] and oscillatory integrals in [13] to name but a few.

Recent work of Lacey establishes sparse form domination for the continuous spherical maximal averages using the improving estimates in Theorem 1 and thereby shows a variety of weighted inequalities. The underlying method of proof relies on Theorem 1, a certain continuity property derived by interpolating against a favorable $\ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ estimate, and a carefully applied Calderón–Zygmund decomposition in a manner related to Christ and Stein's analysis in [2]. Moreover, there are several recent sparse results in the discrete setting involving random Carleson operators in [8], the cubic Hilbert transform in [4], and a family of quadratically modulated Hilbert transforms in [7].

The proof of Theorem 4 reduces to showing that for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$ there exists $A = A(d, p, r)$ such that for every $f \in \ell^p(\mathbb{Z}^d)$ and $\Lambda \in 2^{\mathbb{N}}$

$$\Big\| \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{C}_\lambda| : \ell^p(\mathbb{Z}^d) \to \ell^{r'}(\mathbb{Z}^d) \Big\| \le A \Lambda^{d(1/r' - 1/p)}, \tag{3}$$

$$\Big\| \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{R}_\lambda| : \ell^p(\mathbb{Z}^d) \to \ell^{r'}(\mathbb{Z}^d) \Big\| \le A \Lambda^{d(1/r' - 1/p)}, \tag{4}$$

where $\mathcal{S}(d)$ is the interior convex hull of

$$S_{d,1} = \Big(\frac{2}{d}, \frac{d-2}{d}\Big), \quad S_{d,2} = \Big(\frac{d-2}{d}, \frac{2}{d}\Big), \quad S_{d,3} = \Big(\frac{d-2}{d}, \frac{d-2}{d}\Big).$$

Indeed, estimate (1) is an immediate consequence of interpolating estimates close to $(\frac{d-2}{d}, \frac{d-2}{d})$ with the trivial endpoint estimate at $(0, 1)$. Furthermore, the arguments for (3) and (4) rely on interpolating between favorable $\ell^2 \to \ell^2$ bounds and boundary estimates arising from pointwise control of various kernels. See Figure 1 for a depiction of $\mathcal{S}(d)$.

The proof of Theorem 5 is reduced to showing that for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$,

$$\Big\| \sup_{\lambda \in \tilde{\Lambda}} |\mathscr{C}_\lambda| : (p, r) \Big\| < \infty, \tag{5}$$

$$\Big\| \sup_{\lambda \in \tilde{\Lambda}} |\mathscr{R}_\lambda| : (p, r) \Big\| < \infty, \tag{6}$$

in conjunction with a restricted weak-type interpolation argument from [12]. The arguments for (6) and (5) rely on the improving properties in (4) and (3) respectively.

Once we obtain sparse bounds for $\sup_{\lambda\in\tilde{\Lambda}} |\mathscr{C}_\lambda|$ and $\sup_{\lambda\in\tilde{\Lambda}} |\mathscr{R}_\lambda|$ throughout $\mathcal{S}(d)$, we extend these estimates to (p, r)-sparse bounds for $(\frac{1}{p}, \frac{1}{r})$ arbitrarily close to $(0, 1)$ by reducing the problem to obtaining restricted weak-type sparse bounds via Theorem 16 and then applying a localized variant of Theorem 5 near $(\frac{d-2}{d}, \frac{d-2}{2})$ as described in Theorem 22.

A weighted consequence of the sparse bounds in Theorem 5 is

Corollary 6. *Let $d \geq 5$ and $w : \mathbb{Z}^d \to [0, \infty)$ satisfy $w^{\frac{d}{d-4}} \in A_2$. Then*

$$\left\| \sup_{\lambda\in\tilde{\Lambda}} |\mathscr{A}_\lambda| : \ell^2(w) \to \ell^2(w) \right\| < \infty.$$

As we may choose $r < 2$ so that $w \in A_2 \cap RH_r \implies w^{\frac{d}{d-4}} \in A_2$, it suffices for the weight w to be in the intersection of the Muckenhoupt class A_2 and the reverse Hölder class RH_r.

To the author's knowledge, no $\ell^p(\mathbb{Z}^d)$-improving properties, sparse bounds, or weighted inequalities were previously known for the discrete spherical maximal averages in the full supremum case. We leave open the question of whether the ranges for $\ell^p(\mathbb{Z}^d)$-improving properties in Theorem 4 and sparse bounds in Theorem 5 are sharp.

This paper is structured as follows: §3 introduces relevant background from the proof of the discrete spherical maximal theorem in [14], §4 contains the proof of estimate (3), §5 contains the proof of estimate (4), §6 contains the proof of estimate (5), §7 contains the proof of estimate (6), §8 contains the proof of estimate (2), and §9 contains the counterexamples for the negative content of Theorems 4 and 5.

The letter A is always used in the mathematical expressions of this paper to denote a positive constant, which depends only on inessential parameters and whose precise value is allowed to change from line to line.

3 Decomposition and transference of discrete spherical averages

We now introduce the decomposition of the discrete spherical average $\mathscr{A}_\lambda = \mathscr{C}_\lambda + \mathscr{R}_\lambda$ and a transference lemma, both from [14]. The symbol of the multiplier $\mathscr{A}_\lambda$ for $\Lambda \leq \lambda < 2\Lambda$ and $\Lambda \in 2^{\mathbb{N}}$ can be written as

$$a_\lambda(\xi) = \sum_{q=1}^{\Lambda} \sum_{(a,q)=1; 1\leq a\leq q} a_\lambda^{a/q}(\xi), \tag{7}$$

where

$$a_\lambda^{a/q}(\xi) = e^{-2\pi i\lambda^2 a/q} \sum_{\ell\in\mathbb{Z}^d} G(a/q,\ell) J_\lambda(a/q, \xi - \ell/q), \tag{8}$$

$$G(a/q,\ell) = \frac{1}{q^d} \sum_{n\in\mathbb{Z}^d/q\mathbb{Z}^d} e^{2\pi i|n|^2 a/q} e^{-2\pi i n\cdot l/q}, \tag{9}$$

$$J_\lambda(a/q,\xi) = \frac{e^{2\pi}}{\lambda^{d-2}} \int_{I(a,q)} e^{-2\pi i\lambda^2\tau}(2(\epsilon - i\tau))^{-d/2} e^{\frac{-\pi|\xi|^2}{2(\epsilon-i\tau)}} d\tau, \qquad \epsilon = \frac{1}{\lambda^2},$$

and

$$I(a,q) = [a/q, -\beta/(q\Lambda), a/q + \alpha/(q\Lambda)]$$

for $\alpha = \alpha(a/q,\Lambda) \simeq 1$, $\beta = \beta(a/q,\Lambda) \simeq 1$. An important fact is the Gauss sum estimate

$$|G(a/q,\ell)| \le Aq^{-d/2}, \tag{10}$$

which holds uniformly in a, q, and ℓ; this is well-known in the $d=1$ case from which the $d \ge 2$ case immediately follows. Next, we shall pick $\Phi \in \mathscr{C}^\infty([-1/4, 1/4]^d)$ such that $\Phi \equiv 1$ on $[-1/8, 1/8]^d$ and for $q \in \mathbb{N}$, set

$$\Phi_q(\xi) = \frac{1}{q^d}\Phi\Big(\frac{\xi}{q}\Big),$$

and define

$$\begin{aligned} b_\lambda(\xi) &= \sum_{q=1}^{\Lambda} \sum_{1\le a\le q:(a,q)=1} b_\lambda^{a/q}(\xi), \\ b_\lambda^{a/q}(\xi) &= e^{-2\pi i\lambda^2 a/q} \sum_{\ell\in\mathbb{Z}^d/q\mathbb{Z}^d} G(a/q,\ell)\Phi_q(\xi - \ell/q) J_\lambda(a/q, \xi - \ell/q), \end{aligned} \tag{11}$$

along with

$$\mathscr{B}_\lambda^{a/q} : f \mapsto f * \check{b}_\lambda^{a/q} \quad \text{and} \quad \mathscr{B}_\lambda : f \mapsto f * \check{b}_\lambda.$$

Therefore, $b_\lambda^{a/q}$ is constructed from $a_\lambda^{a/q}$ by inserting cutoff factors into each summand of $a_\lambda^{a/q}$ at length scale $\frac{1}{q}$. We subsume the difference $b_\lambda - a_\lambda$ into the residual term $\mathscr{R}_\lambda$. Lastly, it is convenient to extend the domain of integration in the definition of J_λ to all of $\mathbb{R}$ and subsume this difference as part of the residual term $\mathscr{R}_\lambda$. To this end, we introduce

$$I_\lambda(a/q,\xi) = \frac{e^{2\pi}}{\lambda^{d-2}} \int_{-\infty}^{\infty} e^{-2\pi i\lambda^2\tau}(2(\epsilon - i\tau))^{-d/2} e^{\frac{-\pi|\xi|^2}{2(\epsilon-i\tau)}} d\tau$$

and let

$$(12)\qquad \begin{aligned} c_\lambda(\xi) &= \sum_{q=1}^{\Lambda} \sum_{1\le a\le q:(a,q)=1} c_\lambda^{a/q}(\xi), \\ c_\lambda^{a/q}(\xi) &= e^{-2\pi i\lambda^2 a/q} \sum_{\ell\in\mathbb{Z}^d/q\mathbb{Z}^d} G(a/q,\ell)\Phi_q(\xi-\ell/q)I_\lambda(\xi-\ell/q), \end{aligned}$$

along with

$$\mathscr{C}_\lambda^{a/q} : f \mapsto f * \check{c}_\lambda^{a/q} \quad \text{and} \quad \mathscr{C}_\lambda : f \mapsto f * \check{c}_\lambda.$$

Since $I_\lambda = c_d\widehat{d\sigma_\lambda}$, where c_d is a dimensional constant and $d\sigma_\lambda$ is the unit surface measure of the sphere in $\mathbb{R}^d$ of radius λ,

$$(13)\qquad c_\lambda^{a/q}(\xi) = c_d e^{-2\pi i\lambda^2 a/q} \sum_{\ell\in\mathbb{Z}^d/q\mathbb{Z}^d} G(a/q,\ell)\Phi_q(\xi-\ell/q)\widehat{d\sigma_\lambda}(\xi-\ell/q).$$

It follows that

$$\begin{aligned} c_\lambda(\xi) &= \sum_{q=1}^{\Lambda} \sum_{(a,q)=1;1\le a\le q} c_\lambda^{a/q}(\xi) \\ &= c_d \sum_{q=1}^{\Lambda} \sum_{(a,q)=1;1\le a\le q} e^{-2\pi i\lambda^2 a/q} \sum_{\ell\in\mathbb{Z}^d} G(a/q,\ell)\Phi_q(\xi-\ell/q)\widehat{d\sigma_\lambda}(\xi-\ell/q), \end{aligned}$$

$\mathscr{C}_\lambda : f \mapsto f * \check{c}_\lambda$, and $\mathscr{C}_\lambda^{a/q} : f \mapsto f * \check{c}_\lambda^{a/q}$. We now recall two estimates:

$$\begin{aligned} \Big\| \sup_{\Lambda\le\lambda<2\Lambda} |\mathscr{C}_\lambda^{a/q} f| \Big\|_{\ell^2(\mathbb{Z}^d)} &\le A q^{-d/2} \|f\|_{\ell^2(\mathbb{Z}^d)}, \\ \Big\| \sup_{\Lambda\le\lambda<2\Lambda} |\mathscr{R}_\lambda f| \Big\|_{\ell^2(\mathbb{Z}^d)} &\le A \Lambda^{2-d/2} \|f\|_{\ell^2(\mathbb{Z}^d)}, \end{aligned}$$

which are Propositions 3.1 and 4.1 from [14] respectively. These favorable ℓ^2 bounds are intimately related to the decay of the Gauss sum in (10). Furthermore, from the fact that for $d \ge 5$ and $\lambda \in \tilde{\Lambda}$

$$\frac{1}{A}\lambda^{d-2} \le |\{x\in\mathbb{Z}^d : |x| = \lambda\}| \le A\lambda^{d-2},$$

the basic pointwise estimate

$$|\mathscr{A}_\lambda f(x)| \le A\Lambda^2 \left[\frac{1}{\Lambda^d} \sum_{|y|\le\Lambda} |f(x-y)|\right]$$

holds and so

$$(14)\qquad \Big\| \sup_{\Lambda\le\lambda<2\Lambda} |\mathscr{A}_\lambda| \Big\|_{\ell^1(\mathbb{Z}^d)} \le A\Lambda^2 \|f\|_{\ell^1(\mathbb{Z}^d)}.$$

The transference lemma from [14] can be stated as follows:

Lemma 7. *For $d \geq 1$ and an integer $q \geq 1$ suppose that $m : [-1/2, 1/2)^d \to B$ is supported on $[-1/(2q), 1/(2q))^d$, where B is any Banach space. Set*

$$m^q_{per}(\xi) = \sum_{\ell \in \mathbb{Z}^d} m(\xi - \ell/q)$$

and let T^q_{dis} be the convolution operator on $\mathbb{Z}^d$ with m^q_{per} as its multiplier, i.e., for all $f \in \ell^1(\mathbb{Z}^d)$

$$\widehat{T^q_{dis}f}(\xi) = m^q_{per}(\xi)\hat{f}(\xi) \quad \forall \xi \in [-1/2, 1/2)^d.$$

Moreover, let T be the convolution operator on $\mathbb{R}^d$ with m as its multiplier. Then there is a constant A such that for any $1 \leq p < \infty$

$$\|T^q_{dis}\|_{\ell^p(\mathbb{Z}^d) \to \ell^p_B(\mathbb{Z}^d)} \leq A \|T\|_{L^p(\mathbb{R}^d) \to L^p_B(\mathbb{R}^d)}. \tag{15}$$

As an application of Lemma 7, we shall show the following estimate, which will be used in §4 to obtain the sparse bound (5). First, we need to set a bit more notation. Let $\{\psi_{2^k}\}_{k \in \mathbb{Z}}$ be a standard Littlewood–Paley decomposition where each ψ_k is supported in $\{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$. For all $q \in \mathbb{N}$ and $N, \Lambda \in 2^{\mathbb{Z}_+}$ such that $N \leq \frac{\Lambda}{q}$ define $P^q_{N/\Lambda}$ for all $f \in \ell^1(\mathbb{Z}^d)$ according to

$$\widehat{P^q_{N/\Lambda}f}(\xi) = \sum_{\ell \in \mathbb{Z}^d/q\mathbb{Z}^d} \psi_{N/\Lambda}(\xi - \ell/q)\hat{f}(\xi) \quad \forall \xi \in [-1/2, 1/2)^d.$$

Moreover, for any $\# \in \mathbb{R}_{>0}$ let $P_{\leq \#}$ be the operator defined by

$$\widehat{P^d_{\leq \#}(f)}(\xi) = \sum_{\ell \in \mathbb{Z}^d/q\mathbb{Z}^d} \sum_{2^k \leq \#} \psi_k(\xi - \ell/q)\widetilde{\Phi}_q(\xi - \ell/q)\hat{f}(\xi) \quad \forall \xi \in [-1/2, 1/2)^d$$

where $\widetilde{\Phi}_q$ is given in (18). For convenience, we will just write $P_{N/\Lambda}$ and $P_{<\#}$ instead of $P^q_{N/\Lambda}$ and $P^d_{\leq \#}$; the dependence on q will be implicit and always clear from the context.

Lemma 8. *For every $d \geq 5$, $\epsilon > 0$, $\frac{d}{d-2} < p \leq 2, N \in 2^{\mathbb{N}}, q \in \mathbb{N}$, and $a : 1 \leq a \leq q$ and $(a, q) = 1$, there exists $A = A(d, p, \epsilon)$ such that*

$$\left\| \sup_{\Lambda \geq Nq} \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}^{a/q}_\lambda P_{N/\Lambda}| : \ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d) \right\| \leq AN^{1-d(1-1/p)+\epsilon} q^{-d(1-1/p)+\epsilon}, \tag{16}$$

$$\left\| \sup_{\Lambda \geq q} \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}^{a/q}_\lambda P_{\leq \frac{1}{\Lambda}}| : \ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d) \right\| \leq Aq^{-d(1-1/p)+\epsilon}. \tag{17}$$

We shall need (17) for the proof of Theorem 17.

Proof. We first choose $\tilde{\Phi}\in\mathscr{C}^\infty([-3/8,3/8]^d)$ satisfying $\tilde{\Phi}\equiv 1$ on $[-1/4,1/4]^d$ so that

$$\begin{aligned}c_\lambda^{a/q}(\xi) &= e^{-2\pi i\lambda^2 a/q}\Big[\sum_{\ell\in\mathbb{Z}^d} G(a/q,\ell)\tilde{\Phi}_q(\xi-\ell/q)\Big] \\ &\qquad\times\Big[\sum_{\ell\in\mathbb{Z}^d}\Phi_q(\xi-\ell/q)\widehat{d\sigma_\lambda}(\xi-\ell/q)\Big] \\ &=: e^{-2\pi i\lambda^2 a/q}c_{\lambda,1}^{a/q}(\xi)\cdot c_{\lambda,2}^{a/q}(\xi).\end{aligned}\tag{18}$$

For $m\in L^\infty(\mathbb{T}^d)$, let T_m denote the convolution operator with symbol m. Now take $B=\ell^\infty(\tilde{\Lambda})$, and apply Lemma 7 to the family of symbols

$$c_{\lambda,N}^q(\xi) := \psi_{N/\Lambda}(\xi)\Phi_q(\xi)\widehat{d\sigma_\lambda}(\xi),$$

where $N\in 2^{\mathbb{N}}$ is arbitrary and $\Lambda\in 2^{\mathbb{N}}$ satisfies $\Lambda\le\lambda<2\Lambda$, to deduce

$$\begin{aligned}&\Big\|\sup_{\Lambda\ge Nq}\sup_{\Lambda\le\lambda<2\Lambda}|\mathscr{C}_\lambda^{a/q}(P_{N/\Lambda}f)|\Big\|_{\ell^2(\mathbb{Z}^d)} \\ &\quad\le A\Big\|\sup_{\Lambda\ge Nq}\sup_{\Lambda\le\lambda<2\Lambda}|T_{c_{\lambda,N}^q}|: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)\Big\|\cdot\|T_{c_{\lambda,1}^{a/q}}f\|_{\ell^2(\mathbb{Z}^d)}.\end{aligned}\tag{19}$$

By Plancherel and the Gauss sum estimate (10),

$$\|T_{c_{\lambda,1}^{a/q}}:\ell^2(\mathbb{Z}^d)\to\ell^2(\mathbb{Z}^d)\|\le Aq^{-d/2}.\tag{20}$$

Moreover,

$$\Big\|\sup_{\Lambda\ge Nq}\sup_{\Lambda\le\lambda<2\Lambda}|T_{c_{\lambda,N}^q}|: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)\Big\|\le AN^{1-d/2}.\tag{21}$$

Indeed, for fixed λ, $\|T_{c_{\lambda,N}^q}: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)\|\le AN^{1/2-d/2}$ on account of the decay of $\widehat{d\sigma_\lambda}$ on the support of $\psi_{N/\Lambda}$. The additional factor of $N^{1/2}$ appearing on the right side of (21) arises from the supremum over λ and can be justified using standard techniques. See, for example, [20] for details. Combining (19), (20) and (21) yields

$$\Big\|\sup_{\Lambda\ge Nq}\sup_{\Lambda\le\lambda<2\Lambda}|\mathscr{C}_\lambda^{a/q}(P_{N/\Lambda}f)|\Big\|_{\ell^2(\mathbb{Z}^d)}\le AN^{1-d/2}q^{-d/2}\|f\|_{\ell^2(\mathbb{Z}^d)}.\tag{22}$$

Moreover, from the pointwise estimate

$$|T_{c_{\lambda,N}^q}f|\le ANM_{HL}f,\tag{23}$$

it again follows from Lemma 7 that for every $\epsilon > 0$

$$\Big\| \sup_{\Lambda \geq Nq} \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q}(P_{N/\Lambda} f)| \Big\|_{\ell^{1+\epsilon}(\mathbb{Z}^d)} \leq AN \| T_{c_{\lambda,1}^{a/q}} : \ell^{1+\epsilon}(\mathbb{Z}^d) \to \ell^{1+\epsilon}(\mathbb{Z}^d) \| \cdot \|f\|_{\ell^{1+\epsilon}(\mathbb{Z}^d)}.$$

We may use the easily checked fact that

(24) $$\sum_{\ell \in \mathbb{Z}^d / q\mathbb{Z}^d} G(a/q, \ell) e^{-2\pi i y \cdot \ell / q} = e^{2\pi i |y|^2 a/q} \quad \forall y \in \mathbb{Z}^d / q\mathbb{Z}^d$$

to obtain the pointwise estimate $|\check{c}_{\lambda,1}^{a/q}| \leq A |\check{\tilde{\Phi}}_q|$ and

(25) $$\| T_{c_{\lambda,1}^{a/q}} : \ell^1(\mathbb{Z}^d) \to \ell^1(\mathbb{Z}^d) \| \leq A.$$

Interpolating between (20) and (25) yields, for every $1 \leq p \leq 2$,

(26) $$\| T_{c_{\lambda,1}^{a/q}} : \ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d) \| \leq A q^{-d(1-1/p)}.$$

Plugging (26) into (19) gives, for every $0 < \epsilon \leq 1$,

(27) $$\Big\| \sup_{\Lambda \geq Nq} \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q}(P_{N/\Lambda} f)| \Big\|_{\ell^{1+\epsilon}(\mathbb{Z}^d)} \leq ANq^{-\frac{d\epsilon}{1+\epsilon}} \|f\|_{\ell^{1+\epsilon}(\mathbb{Z}^d)}.$$

Interpolating between (22) and (27) shows estimate (16). Using the facts that

(28) $$\Big\| \sup_{\Lambda} \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\leq 1/\Lambda}| : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d) \Big\| \leq A q^{-d/2}$$

and $\sup_\Lambda \sup_{\Lambda \leq \lambda < 2\Lambda} |P_{\leq 1/\Lambda}(f * \check{\Phi}_q * d\sigma_\lambda)| \leq AM_{HL} f$, estimate (17) is similarly obtained, and so the details are omitted. □

4 Improving properties for $\sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda|$

Our goal in this section is to obtain estimate (3), which is the improving property for the "major arc" term. The argument relies on interpolating between the $\ell^2 \to \ell^2$ bound (22) and straightforward boundary estimates related to (23). We begin with an elementary lemma, which will also be used later in showing estimates (5) and (6).

Lemma 9. *Fix $d \geq 1$. For every $\Lambda \in 2^{\mathbb{N}}$ let ϕ_Λ^1 be given by*

(29) $$\phi_\Lambda^1(x) := \frac{1}{\Lambda^d} \left[\frac{1}{1 + \frac{|x|}{\Lambda}} \right]^{2d}.$$

Then for every $\frac{1}{p} + \frac{1}{r} \geq 1$,

(30) $$\|f * \phi_\Lambda^1\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A \Lambda^{d(1/r' - 1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}.$$

Proof. Estimate (30) is trivial when $r' = p$ as the kernel belongs to $\ell^1(\mathbb{Z}^d)$ uniformly in Λ. The estimate when $r' = \infty$ follows immediately from Hölder's inequality. Interpolating between these two cases yields the conclusion of Lemma 9. □

We now use Lemma 9 to deduce the following improving property.

Lemma 10. *Let $d \geq 5$ and $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$. Then there exists $A = A(d,p,r)$ and $\delta = \delta(d,p,r) > 0$ such that, for every $f \in \ell^p(\mathbb{Z}^d)$ together with $\Lambda, N \in 2^{\mathbb{N}}$ such that $1 \leq N \leq \frac{\Lambda}{q}$,*

$$\Big\| \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{N/\Lambda} f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A N^{-\delta} q^{-2-\delta} \Lambda^{d(1/r'-1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}, \tag{31}$$

$$\Big\| \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\leq 1/\Lambda} f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A q^{-2-\delta} \Lambda^{d(1/r'-1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}. \tag{32}$$

Proof. The proof is by interpolation. Estimates (22) and (28) immediately yield

$$\Big\| \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{N/\Lambda}| : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d) \Big\| \leq A N^{1-d/2} q^{-d/2}, \tag{33}$$

$$\Big\| \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\leq 1/\Lambda}| : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d) \Big\| \leq A q^{-d/2}. \tag{34}$$

We next invoke the pointwise estimates valid for all $M \geq 1$,

$$\sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{N/\Lambda} f| \leq \frac{A_M}{\Lambda^d} N |f| * [\Lambda^d \phi_\Lambda^1]^M, \tag{35}$$

$$\sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\leq 1/\Lambda} f| \leq \frac{A_M}{\Lambda^d} |f| * [\Lambda^d \phi_\Lambda^1]^M, \tag{36}$$

and Lemma 9 to deduce that for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{B} := \{(\frac{1}{p}, \frac{1}{r}) \in [0,1]^2 : \max\{\frac{1}{p}, \frac{1}{r}\} = 1\}$

$$\Big\| \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{N/\Lambda} f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A N \Lambda^{d(1/r'-1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}, \tag{37}$$

$$\Big\| \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\leq 1/\Lambda} f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A \Lambda^{d(1/r'-1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}. \tag{38}$$

Interpolating (33) and (37) yields (31), while interpolating (34) and (38) yields (32). □

A direct consequence of Lemma 10 is estimate (3), which we record separately as

Proposition 11. *Let $d \geq 5$ and $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$. Then there exists $A = A(d,p,r)$ such that for all $\Lambda \in 2^{\mathbb{N}}$ and $f \in \ell^p(\mathbb{Z}^d)$,*

$$\Big\| \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{C}_\lambda f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A \Lambda^{d(1/r'-1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}.$$

Proof. Sum estimate (31) over all $N \in 2^{\mathbb{N}}$, $(a, q) = 1$ such that $1 \le a < q$ and $1 \le q \le \Lambda$. Sum estimate (31) over all $(a, q) = 1$ such that $1 \le a < q$ and $1 \le q \le \Lambda$. □

5 Improving properties for $\sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{R}_\lambda|$

In this section we obtain estimate (4) by showing improving properties for $\sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{A}_\lambda - \mathscr{B}_\lambda|$ and $\sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{B}_\lambda - \mathscr{C}_\lambda|$ separately. Recall that

$$\mathscr{A}_\lambda : f \mapsto f * \check{a}_\lambda, \quad \mathscr{B}_\lambda : g \mapsto g * \check{b}_\lambda, \quad \mathscr{C}_\lambda : h \mapsto h * \check{c}_\lambda,$$

where the symbols a_λ, b_λ, and c_λ are defined in (7), (11), and (12), respectively. The following result is needed to obtain improving properties for $\sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{A}_\lambda - \mathscr{B}_\lambda|$.

Lemma 12. *Fix $d \ge 5$, $q \in \mathbb{N}$, $(a, q) = 1$ such that $1 \le a < q$. For $\tau \in \mathbb{R}$ let*

$$\mu_\tau(\xi) = \sum_{\ell \in \mathbb{Z}^d} G(a/q, \ell)(1 - \Phi_q(\xi - \ell/q))e^{-\pi|\xi - \ell/q|^2/2(\epsilon - i\tau)}.$$

Then for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$, there exists $A = A(d, p, r)$ and $\delta = \delta(d, p, r) > 0$ such that for all $k \in \mathbb{Z}_+$, $\Lambda \in 2^{\mathbb{N}}$, $\tau \in I_k(a, q) := \{\tau \in \mathbb{R} : \frac{2^k - 1}{\Lambda^2} \le |\tau| \le \frac{2^k}{\Lambda^2}\}$, and $f \in \ell^p(\mathbb{Z}^d)$

$$\|f * \check{\mu}_\tau\|_{\ell^{r'}(\mathbb{Z}^d)} \le A 2^{dk/2} \Lambda^{-2-\delta} \Lambda^{d(1/r' - 1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}. \tag{39}$$

Proof. First observe that $\|\mu_\tau\|_{L^\infty(\mathbb{T}^d)} \le A[\frac{\epsilon^2 + \tau^2}{\epsilon}]^{d/4}$, so that

$$\|f * \check{\mu}_\tau\|_{\ell^2(\mathbb{Z}^d)} \le A \left[\frac{\epsilon^2 + \tau^2}{\epsilon}\right]^{d/4} \|f\|_{\ell^2(\mathbb{Z}^d)}. \tag{40}$$

Consequently, estimate (39) holds at $(p, r) = (2, 2)$. We may also observe the kernel bound

$$|\check{\mu}_\tau(x)| \le A_M \left[\frac{\epsilon^2 + \tau^2}{\epsilon^2}\right]^{d/4} [\phi_\Lambda^1(x)]^M \quad \forall M \ge 1 \tag{41}$$

where ϕ_Λ^1 is given by (29). By Lemma 9, it follows that for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{B}$

$$\|f * \check{\mu}_\tau\|_{\ell^{r'}(\mathbb{Z}^d)} \le A \left[\frac{\epsilon^2 + \tau^2}{\epsilon^2}\right]^{d/4} \Lambda^{d(1/r' - 1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}. \tag{42}$$

Interpolating (40) and (42) yields for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$,

$$\|f * \check{\mu}_\tau\|_{\ell^{r'}(\mathbb{Z}^d)} \le A \Lambda^{-\delta} \left[\frac{\epsilon^2 + \tau^2}{\epsilon}\right] \left[\frac{\epsilon^2 + \tau^2}{\epsilon^2}\right]^{d/4 - 1} \Lambda^{d(1/r' - 1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}.$$

Using $\epsilon = \frac{1}{\Lambda^2}$ and $|\tau| \simeq \frac{2^k}{\Lambda^2}$ then yields (39). □

The next result is used to obtain improving properties for $\sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{B}_\lambda - \mathscr{C}_\lambda|$:

Lemma 13. *Fix $d \geq 5$, $q \in \mathbb{N}$, (a, q) such that $1 \leq a < q$. For $\tau \in \mathbb{R}$ let*

$$\gamma_\tau(\xi) = \sum_{\ell \in \mathbb{Z}^d/q\mathbb{Z}^d} G(a/q, \ell)\Phi_q(\xi - \ell/q)e^{-\pi|\xi-\ell/q|^2/2(\epsilon - i\tau)}.$$

Then for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$, there exist $A = A(d, p, r)$ and $\delta = \delta(d, p, r) > 0$ such that, for all $k \geq 0$, $\Lambda \in 2^{\mathbb{N}}$, $\tau \in I_k(a, q)$, and $f \in \ell^p(\mathbb{Z}^d)$,

$$\|f * \check{\gamma}_\tau\|_{\ell^{r'}(\mathbb{Z}^d)} \leq Aq^{-2-\delta}2^{dk(1/2-2/d)}\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(\mathbb{Z}^d)}. \tag{43}$$

Proof. First note that $\|\gamma_\tau\|_{L^\infty(\mathbb{T}^d)} \leq Aq^{-d/2}$ so that

$$\|f * \check{\gamma}_\tau\|_{\ell^2(\mathbb{Z}^d)} \leq Aq^{-d/2}\|f\|_{\ell^2(\mathbb{Z}^d)}. \tag{44}$$

We also have the kernel bound

$$|\check{\gamma}_\tau(x)| \leq A_M\Big[\frac{\epsilon^2 + \tau^2}{\epsilon^2}\Big]^{d/4}\frac{[\Lambda^d\phi^1_\Lambda(x)]^M}{\Lambda^d} \quad \forall M \geq 1 \tag{45}$$

where ϕ^1_Λ is again given by (29). Estimate (45) and Lemma 9 imply that, for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{B}$,

$$\|f * \check{\gamma}_\tau\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A\Big[\frac{\epsilon^2 + \tau^2}{\epsilon^2}\Big]^{d/4}\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(\mathbb{Z}^d)}. \tag{46}$$

Interpolating estimates (44) and (46) gives that, for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$,

$$\|f * \check{\gamma}_\tau\|_{\ell^{r'}(\mathbb{Z}^d)} \leq Aq^{-2-\delta(d,p,r)}\Big[\frac{\epsilon^2 + \tau^2}{\epsilon^2}\Big]^{d/4-1}\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(\mathbb{Z}^d)}.$$

Using $\epsilon = \frac{1}{\Lambda^2}$ and $|\tau| \simeq \frac{2^k}{\Lambda^2}$ quickly yields (43). □

We now prove estimate (4) in the following result:

Proposition 14. *Let $d \geq 5$ and $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$. Then there is $A = A(d, p, r)$ such that, for all $\Lambda \in 2^{\mathbb{N}}$ and $f \in \ell^p(\mathbb{Z}^d)$,*

$$\Big\|\sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{R}_\lambda f|\Big\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(\mathbb{Z}^d)}. \tag{47}$$

Proof. To verify (47), it is enough to show that

$$\Big\|\sup_{\Lambda \leq \lambda < 2\Lambda} |(\mathscr{A}_\lambda - \mathscr{B}_\lambda)| : \ell^p(\mathbb{Z}^d) \to \ell^{r'}(\mathbb{Z}^d)\Big\| < \infty,$$

$$\Big\|\sup_{\Lambda \leq \lambda < 2\Lambda} |(\mathscr{B}_\lambda - \mathscr{C}_\lambda)f|\Big\|_{\ell^{r}(\mathbb{Z}^d)} \leq A\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(\mathbb{Z}^d)}.$$

To this end we observe that, for every $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$,

$$\begin{aligned}\Big\| \sup_{\Lambda \le \lambda < 2\Lambda} |(\mathscr{A}_\lambda^{a/q} - \mathscr{B}_\lambda^{a/q})f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \\ \le A\Lambda^{-d+2} \sum_{k=0}^{\log_2(\Lambda/q)+A} \int_{I_k(a,q)} (\epsilon^2 + \tau^2)^{-d/4} \|f * \check{\mu}_\tau\|_{\ell^{r'}(\mathbb{Z}^d)} d\tau.\end{aligned}$$

By Lemma 12, the last line of the above display can be bounded by

$$\begin{aligned} A\Lambda^{-d+2} \sum_{k=0}^{\log_2(\Lambda/q)+A} \Big[\frac{2^k}{\Lambda^2}\Big]^{1-d/2} 2^{dk/2} \Lambda^{-2} [\Lambda^{d(1/r'-1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}] \\ \le A\Lambda^{-d+2}\Lambda^{d-4}\frac{\Lambda}{q}[\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(\mathbb{Z}^d)}] \\ = A\frac{1}{q\Lambda}\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(\mathbb{Z}^d)}.\end{aligned}$$

Summing on a and then q then yields

$$\begin{aligned}\Big\| \sup_{\Lambda \le \lambda < 2\Lambda} |(\mathscr{A}_\lambda - \mathscr{B}_\lambda)f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \le \sum_{q=1}^{\Lambda} \sum_{1 \le a \le q : (a,q)=1} \Big\| \sup_{\Lambda \le \lambda < 2\Lambda} |(\mathscr{A}_\lambda^{a/q} - \mathscr{B}_\lambda^{a/q})f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \\ \le A\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(\mathbb{Z}^d)}.\end{aligned}$$

It remains to handle the estimate for $\mathscr{B}_\lambda - \mathscr{C}_\lambda$. To this end, observe that

$$\begin{aligned}\Big\| \sup_{\Lambda \le \lambda < 2\Lambda} |(\mathscr{B}_\lambda^{a/q} - \mathscr{C}_\lambda^{a/q})f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \\ \le A\Lambda^{-d+2} \sum_{k=\log_2(\Lambda/q)-A}^{\infty} \int_{I_k(a,q)} (\epsilon^2 + \tau^2)^{-d/4} \|f * \check{\gamma}_\tau\|_{\ell^{r'}(\mathbb{Z}^d)} d\tau\end{aligned}$$

By Lemma 13, the last line of the above display can be bounded by

$$\begin{aligned} A\Lambda^{-d+2} \sum_{k=\log_2(\Lambda/q)-A}^{\infty} \Big[\frac{2^k}{\Lambda^2}\Big]^{1-d/2} q^{-2} 2^{dk(1/2-2/d)} [\Lambda^{d(1/r'-1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}] \\ = A \sum_{k=\log_2(\Lambda/q)-A}^{\infty} 2^{-k} q^{-2} [\Lambda^{d(1/r'-1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}] \\ \le \frac{A}{q\Lambda}[\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(\mathbb{Z}^d)}].\end{aligned}$$

Summing on $a : 1 \le a \le q$ and $(a, q) = 1$ as well as $q : 1 \le q \le \Lambda$ yields an upper bound $O([\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(\mathbb{Z}^d)}])$. □

Proposition 15. *Let $d \geq 5$ and $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{R}(d)$. Then there is $A = A(d, p, r)$ such that, for all $\Lambda \in 2^{\mathbb{N}}$ and $f \in \ell^p(\mathbb{Z}^d)$,*

$$\Big\| \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{A}_\lambda f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A\Lambda^{d(1/r'-1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}.$$

Proof. By Propositions 11 and 14, it follows that for all $d \geq 5$, $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$ there is $A = A(d, p, r)$ such that, for all $\Lambda \in 2^{\mathbb{N}}$ and $f \in \ell^p(\mathbb{Z}^d)$,

$$\Big\| \sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{A}_\lambda f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A\Lambda^{d(1/r'-1/p)} \|f\|_{\ell^p(\mathbb{Z}^d)}. \tag{48}$$

Interpolating estimate (48) with the trivial $\ell^\infty \to \ell^\infty$ bound for $\sup_{\Lambda \leq \lambda < 2\Lambda} |\mathscr{A}_\lambda|$ yields the Proposition. □

6 Sparse domination for $\sup_\lambda |\mathscr{C}_\lambda|$

Our goal in this section is to prove estimate (5), where

$$\mathscr{C}_\lambda : f \mapsto f * \check{c}_\lambda$$

and c_λ is given in (12). To this end, we need to state a restricted weak-type sparse result, which first appears in [12]. We include an original, self-contained proof for convenience.

Theorem 16. *Let T be an operator on $\mathbb{Z}^d$ satisfying the property that for some $p, r : \frac{1}{p} + \frac{1}{r} > 1$ there is an A such that, for all finite sets $E_1, E_2 \subset \mathbb{Z}^d$ and $|f| \leq 1_{E_1}$, $|g| \leq 1_{E_2}$, there is a sparse collection $\mathcal{S}$ such that*

$$|\langle Tf, g\rangle| \leq A\Lambda_{\mathcal{S},p,r}(1_{E_1}, 1_{E_2}).$$

Then for every $\tilde{p} > p$, $\tilde{r} > r$ such that $\frac{1}{\tilde{p}} + \frac{1}{\tilde{r}} > 1$ there is A such that for all finitely supported $f, g : \mathbb{Z}^d \to \mathbb{C}$ there is a sparse collection $\mathcal{S}$ such that

$$|\langle Tf, g\rangle| \leq A\Lambda_{\mathcal{S},\tilde{p},\tilde{r}}(f, g).$$

The assumption of Theorem 16 is referred to as a restricted weak-type sparse bound on T. The conclusion allows us to upgrade the restricted weak-type bound to a standard sparse bound, at the cost of raising the averaging exponents p, r by an arbitrarily small amount.

Proof. Fix $f, g : \mathbb{Z}^d \to \mathbb{C}$ supported on a cube $3E$ where E is dyadic. Without loss of generality, suppose $|f|, |g| \leq 1$ and decompose

$$f = \sum_{k \geq 0} 2^{-k} f_k, \quad g = \sum_{l \geq 0} 2^{-l} g_l,$$

where

$$f_k = 2^k f 1_{\{2^{-k+1} < |f| \le 2^{-k}\}}, \quad g_l = 2^l g 1_{\{2^{-l+1} < |g| \le 2^{-l}\}}.$$

Then by assumption

$$\begin{aligned} |\langle Tf, g\rangle| &\le \sum_{k,l\ge 0} 2^{-k-l} \langle |T 1_{\{2^{-k+1} < f \le 2^{-k}\}}|, 1_{\{2^{-l+1} < g \le 2^{-l}\}}\rangle \\ &\le A \sum_{k,l\ge 0} 2^{-k-l} \Lambda_{\mathcal{S}_{k,l},p,r}(1_{\{2^{-k+1} < f \le 2^{-k}\}}, 1_{\{2^{-l+1} < g \le 2^{-l}\}}). \end{aligned}$$

For $\mu_1, \mu_2 \ge 0$, let $\mathcal{Q}_{\mu_1,\mu_2}(k,l) := \mathcal{Q}^1_{\mu_1}(k,l) \cap \mathcal{Q}^2_{\mu_2}(k,l)$, where

$$\begin{aligned} \mathcal{Q}^1_{\mu_1}(k,l) &:= \Big\{Q \in \mathcal{S}_{k,l} : 2^{-\mu_1-1} < \frac{|Q \cap \{2^{-k+1} < |f| \le 2^{-k}\}|}{|Q|} \le 2^{-\mu_1}\Big\}, \\ \mathcal{Q}^2_{\mu_2}(k,l) &:= \Big\{Q \in \mathcal{S}_{k,l} : 2^{-\mu_2-1} < \frac{|Q \cap \{2^{-l+1} < |f| \le 2^{-l}\}|}{|Q|} \le 2^{-\mu_2}\Big\}. \end{aligned}$$

It suffices to produce a sparse collection $\mathcal{S}(f,g)$ such that, for every $\mu_1, \mu_2 \ge 0$ and $\tilde{p} > p, \tilde{r} > r$,

$$\sum_{Q \in \mathcal{Q}_{\mu_1,\mu_2}} \langle f\rangle_{Q,\tilde{p}} \langle g\rangle_{Q,\tilde{r}} |Q| \le A \sum_{Q \in \mathcal{S}(f,g)} \langle f\rangle_{Q,\tilde{p}} \langle g\rangle_{Q,\tilde{r}} |Q|.$$

The first generation is denoted by $\mathcal{S}_1(f,g)$ and is set equal to the maximal shifted dyadic cubes $Q \subset 3E$ such that

$$\langle f\rangle_{Q,\tilde{p}} \ge A_0 \langle f\rangle_{3E,\tilde{p}} \quad \text{or} \quad \langle g\rangle_{Q,\tilde{r}} \ge A_0 \langle g\rangle_{3E,\tilde{r}}.$$

For large enough constant A_0, $|\bigcup_{\mathcal{S}(f,g)} Q| \le \frac{|E|}{100}$. For each $Q \in \mathcal{S}_1(f,g)$, we choose $R \in \mathcal{S}_2(f,g)$ provided it is a maximal shifted dyadic cube inside Q such that

$$\langle f\rangle_{R,\tilde{p}} \ge A_0 \langle f\rangle_{Q,\tilde{p}} \quad \text{or} \quad \langle g\rangle_{R,\tilde{r}} \ge A_0 \langle g\rangle_{Q,\tilde{r}}.$$

For large enough constant A_0, $|\bigcup_{R \in \mathcal{S}_2(f,g) \subset Q} R| \le \frac{|Q|}{100}$ for all $Q \in \mathcal{S}_1(f,g)$. Iterating this procedure a finite number of times yields the desired sparse collection of cubes

$$\mathcal{S}(f,g) = \{3E\} \cup \bigcup_{m=1}^{k_0(E)} \mathcal{S}_m(f,g).$$

Next, we may suppose without loss of generality that the cubes $\mathcal{Q}_{\mu_1,\mu_2}$ are dyadic and set, for each $m \ge 1$,

$$\mathcal{Q}_{\mu_1,\mu_2,m} = \{Q \in \mathcal{Q}_{\mu_1,\mu_2} : \min\{l : \exists R \in \mathcal{S}_l(f,g) : R \supset Q\} = m\}.$$

If there is no $R \in \mathcal{S}(f, g)$ for which $R \supset Q$, then assign $Q \in \mathcal{Q}_{\mu_1,\mu_2,0}$. By construction,

$$\begin{aligned}\sum_{Q\in\mathcal{Q}_{\mu_1,\mu_2}} &\langle f\rangle_{Q,\bar{p}}\langle g\rangle_{Q,\bar{r}}|Q| \\ &= \sum_{m=0}^{k_0}\sum_{Q\in\mathcal{Q}_{\mu_1,\mu_2,m}} \langle f\rangle_{Q,\bar{p}}\langle f\rangle_{Q,\bar{r}}|Q| \\ &\le A2^{-\mu_1/\bar{p}}2^{-\mu_2/\bar{r}}\sum_{m=0}^{k_0}\sum_{R\in\mathcal{S}_m(f,g)} \sum_{\substack{k,l\ge 0\\ \mathcal{Q}^1_{\mu_1}(k,l)\cap\mathcal{Q}^2_{\mu_2}(k,l)\cap\mathcal{Q}_{\mu_1,\mu_2,m}\neq\emptyset}} 2^{-k}2^{-l} \\ &\qquad\times \sum_{\substack{Q\subset R\\ Q\in\mathcal{Q}^1_{\mu_1}(k,l)\cap\mathcal{Q}^2_{\mu_2}(k,l)}} |Q|.\end{aligned}$$

Note that because $\mathcal{S}_{k,l}$ is a sparse collection for each $k, l \geq 0$,

$$\sum_{\substack{Q\subset R\\ Q\in\mathcal{Q}^1_{\mu_1}(k,l)\cap\mathcal{Q}^2_{\mu_2}(k,l)}} |Q| \le A|R|.$$

If $\mathcal{Q}^1_{\mu_1}(k, l) \cap \mathcal{Q}^2_{\mu_2}(k, l) \cap \mathcal{Q}_{\mu_1,\mu_2,m} \neq \emptyset$ for some $m \geq 1$, then any cube

$$Q \in \mathcal{Q}^1_{\mu_1}(k, l) \cap \mathcal{Q}^2_{\mu_2}(k, l) \cap \mathcal{Q}_{\mu_1,\mu_2,m}$$

such that $Q \subset R$ for $R \in \mathcal{S}_m(f, g)$ satisfies

$$\begin{aligned}2^{-k}2^{-\mu_1/\bar{p}} &\le A\langle f_k\rangle_{Q,\bar{p}} \le A\langle f\rangle_{Q,\bar{p}} \le A\langle f\rangle_{R,\bar{p}}, \\ 2^{-l}2^{-\mu_2/\bar{r}} &\le A\langle g_l\rangle_{Q,\bar{r}} \le A\langle g\rangle_{Q,\bar{r}} \le A\langle g\rangle_{R,\bar{r}}.\end{aligned}$$

If $\mathcal{Q}^1_{\mu_1}(k, l) \cap \mathcal{Q}^2_{\mu_2}(k, l) \cap \mathcal{Q}_{\mu_1,\mu_2,0} \neq \emptyset$, then any cube

$$Q \in \mathcal{Q}^1_{\mu_1}(k, l) \cap \mathcal{Q}^2_{\mu_2}(k, l) \cap \mathcal{Q}_{\mu_1,\mu_2,0}$$

satisfies

$$\begin{aligned}2^{-k}2^{-\mu_1/\bar{p}} &\le A\langle f_k\rangle_{Q,\bar{p}} \le A\langle f\rangle_{Q,\bar{p}} \le A\langle f\rangle_{3E,\bar{p}}, \\ 2^{-l}2^{-\mu_2/\bar{r}} &\le A\langle g_l\rangle_{Q,\bar{r}} \le A\langle g\rangle_{Q,\bar{r}} \le A\langle g\rangle_{3E,\bar{r}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{Q\in\mathcal{Q}_{\mu_1,\mu_2}} &\langle f\rangle_{Q,\bar p}\langle g\rangle_{Q,\bar r}|Q| \\ &\le A2^{-\mu_1/\bar p}2^{-\mu_2/\bar r}\sum_{m=0}^{k_0}\sum_{R\in\mathcal{S}_m(f,g)}\sum_{\substack{k,l\ge 0\\ 2^{-k}\le A2^{\mu_1/\bar p}\\ 2^{-l}\le A2^{\mu_2/\bar r}}} 2^{-k}2^{-l}\Bigg[\sum_{\substack{Q\subset R\\ Q\in\mathcal{Q}_{\mu_1,\mu_2,m}}}|Q|\Bigg] \\ &\le A\sum_{R\in\mathcal{S}(f,g)}\langle f\rangle_{R,\bar p}\langle g\rangle_{R,\bar r}|R|. \end{aligned}$$

□

We now restate estimate (5) as a stand-alone result and then prove it.

Theorem 17. *Let $d \ge 5$ and $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$. Then*

$$\Big\| \sup_{\lambda\in\tilde\Lambda} |\mathscr{C}_\lambda| : (p, r)\Big\| < \infty.$$

Proof. It suffices to prove the Theorem under the additional restriction $\frac{d}{d-2} < p \le 2$. In particular, it is enough to prove the conclusion of Theorem 17 for $(\frac{1}{p}, \frac{1}{r})$ near $(\frac{d-2}{d}, \frac{d-2}{d})$ because the result is strongest there. To proceed, we recall that for any $\# \in 2^{\mathbb{Z}}$, the operator $P_{\le\#}$ is defined by

$$\widehat{P_{\le\#}(f)}(\xi) = \sum_{\ell\in\mathbb{Z}^d/q\mathbb{Z}^d}\sum_{2^k\le\#} \psi_k(\xi-\ell/q)\widetilde{\Phi}_q(\xi-\ell/q)\hat f(\xi) \quad \forall\xi\in[-1/2,1/2)^d$$

where $\widetilde{\Phi}_q$ is given in (18). Then we obtain by the triangle inequality

$$\begin{aligned}\sup_{\Lambda\in 2^{\mathbb{N}}}\sup_{\Lambda\le\lambda<2\Lambda} &|\mathscr{C}_\lambda f| \\ &\le \sup_{\Lambda\in 2^{\mathbb{N}}}\sup_{\Lambda\le\lambda<2\Lambda}|\mathscr{C}_\lambda P_{\le 1/\Lambda} f| + \sum_{N\in 2^{\mathbb{N}}}\sup_{\Lambda\ge Nq}\sup_{\Lambda\le\lambda<2\Lambda}|\mathscr{C}_\lambda P_{N/\Lambda} f|.\end{aligned}$$

We first focus our attention on obtaining $\delta = \delta(d, p, r) > 0$ such that, for all $q \in \mathbb{N}$, $a : (a, q) = 1$ and $1 \le a < q$,

$$\Big\|\sup_{\Lambda}\sup_{\Lambda\le\lambda<2\Lambda}|\mathscr{C}_\lambda^{a/q}P_{\le 1/\Lambda}| : (p, r)\Big\| \le Aq^{-2-\delta}. \tag{49}$$

By Theorem 16, it suffices to obtain for all $(\frac{1}{p}, \frac{1}{r})$ satisfying $\max\{\frac{1}{p}, \frac{1}{q}\} < \frac{d-2}{d}$ and arbitrarily close to $(\frac{d-2}{d}, \frac{d-2}{d})$ some $\delta = \delta(d, p, r) > 0$ such that

$$\Big\|\sup_{\Lambda}\sup_{\Lambda\le\lambda<2\Lambda}|\mathscr{C}_\lambda^{a/q}P_{\le 1/\Lambda}| : (p, r)\Big\|_{restricted} \le Aq^{-2-\delta}, \tag{50}$$

where the sparse restricted norm $\|T : (p,q)\|_{restricted}$ is defined to be the infimum over all $C > 0$ such that $\forall f, g : \mathbb{Z}^d \to \mathbb{C}$ s.t. $|f| \le 1_{E_1}$, $|g| \le 1_{E_2}$, $\max\{|E_1|, |E_2|\} < \infty$, the estimate $|\langle Tf, g\rangle| \le C \sup_{\mathcal{S}} \Lambda_{\mathcal{S},p,r}(1_{E_1}, 1_{E_2})$ holds. To this end, let $f, g : \mathbb{Z}^d \to \mathbb{C}$ be finitely supported on $3E$ where E is a dyadic cube. Now let $\mathcal{Q}(E)$ be the maximal dyadic cubes satisfying the condition

$$\langle f\rangle_{3Q,1} \ge A_0 \langle f\rangle_{3E,1}$$
$$\Big\langle \sup_{\Lambda} \sup_{\Lambda \le \lambda < 2\Lambda} \Big|\mathscr{C}_\lambda^{a/q} P_{\le 1/\Lambda} f\Big| \Big\rangle_{3Q,p} \ge A_0 q^{-2-\delta} \langle f\rangle_{3E,p},$$

so that $|\bigcup_{J\in\mathcal{Q}(E)} J| < \frac{|E|}{100}$ for a large enough constant A_0. We first majorize

$$\begin{aligned}
&\Big\langle \sup_{\Lambda} \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\le 1/\Lambda} f|, g\Big\rangle \\
&\quad \le \sum_{Q\in\mathcal{Q}(E)} \Big\langle 1_Q \sup_{\Lambda \le \ell(Q)} \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\le 1/\Lambda}(1_{3Q} f)|, g\Big\rangle \\
&\qquad + \sum_{Q\in\mathcal{Q}(E)} \Big\langle 1_Q \sup_{\Lambda \le \ell(Q)} \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\le 1/\Lambda}(1_{(3Q)^c} f)|, g\Big\rangle \\
&\qquad + \sum_{Q\in\mathcal{Q}(E)} \Big\langle 1_Q \sup_{\Lambda > \ell(Q)} \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\le 1/\Lambda} f|, g\Big\rangle \\
&\qquad + \langle 1_{(\bigcup_{\mathcal{Q}(E)} Q)^c} \sup_{\Lambda} \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\le 1/\Lambda} f|, g\rangle \\
&\quad = \sum_{Q\in\mathcal{Q}(E)} I_Q + \sum_{Q\in\mathcal{Q}(E)} II_Q + \sum_{Q\in\mathcal{Q}(E)} III_Q + IV
\end{aligned}$$

and proceed to obtain satisfactory bounds for each of the above terms separately. First note the pointwise bound

$$1_{(\bigcup_{\mathcal{Q}(E)} Q)^c} \sup_{\Lambda} \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\le 1/\Lambda} f| \le A q^{-2-\delta} \langle f\rangle_{3E,p}$$

by construction of the stopping time. Therefore, $IV \le Aq^{-2-\delta}\langle f\rangle_{3E,p}\langle g\rangle_{3E,1}|E|$. Next, we may observe from (36) and the stopping conditions the pointwise bound

$$\sum_{Q\in\mathcal{Q}(E)} 1_Q \sup_{\Lambda > \ell(Q)} \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\le 1/\Lambda} f| \le A q^2 \langle f\rangle_{3E,1}. \tag{51}$$

From estimate (28), it follows that

$$\Big\langle \sum_{Q\in\mathcal{Q}(E)} 1_Q \sup_{\Lambda > \ell(Q)} \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\le 1/\Lambda} f| \Big\rangle_{3E,2} \le A q^{-d/2} \langle f\rangle_{3E,2}. \tag{52}$$

From (51) and (52), we may observe that

$$\Big\langle \sum_{Q\in\mathcal{Q}(E)} 1_Q \sup_{\Lambda > \ell(Q)} \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\le 1/\Lambda} f| \Big\rangle_{3E,r'} \le A q^{-2-\delta} \langle 1_{E_1}\rangle_{3E,p}. \tag{53}$$

Estimate (53) combined with Hölder's inequality implies

$$\sum_{Q\in\mathfrak{Q}(E)} III_Q \leq Aq^{-2-\delta}\langle 1_{E_1}\rangle_{3E,p}\langle g\rangle_{3E,r}|E|.$$

As we shall be able to recurse on $\sum_{Q\in\mathfrak{Q}(E)} I_Q$ by letting each $Q \in \mathfrak{Q}(E)$ play the role that E played in the initial stage, it suffices to obtain

$$\sum_{Q\in\mathfrak{Q}(E)} II_Q \leq Aq^{-2-\delta}\langle 1_{E_1}\rangle_{3E,p}\langle g\rangle_{3E,r}|E|. \tag{54}$$

To this end, we observe from the pointwise bound (32) and stopping conditions that

$$\sum_{Q\in\mathfrak{Q}(E)} 1_Q \sup_{\Lambda>\ell(Q)} \sup_{\Lambda\leq\lambda<2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\leq 1/\Lambda}(1_{(3Q)^c} f)| \leq Aq^2\langle f\rangle_{3E,1}. \tag{55}$$

Furthermore, estimate (28) ensures

$$\begin{aligned}
&\Big\langle \sum_{Q\in\mathfrak{Q}(E)} 1_Q \sup_{\Lambda>\ell(Q)} \sup_{\Lambda\leq\lambda<2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\leq 1/\Lambda}(1_{(3Q)^c} f)| \Big\rangle_{3E,2} \\
&\quad\leq \Big\langle \sum_{Q\in\mathfrak{Q}(E)} 1_Q \sup_{\Lambda>\ell(Q)} \sup_{\Lambda\leq\lambda<2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\leq 1/\Lambda}(1_{(3Q)} f)| \Big\rangle_{3E,2} \\
&\qquad + \Big\langle \sum_{Q\in\mathfrak{Q}(E)} 1_Q \sup_{\Lambda>\ell(Q)} \sup_{\Lambda\leq\lambda<2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\leq 1/\Lambda} f| \Big\rangle_{3E,2} \\
&\quad\leq Aq^{-d/2}\langle f\rangle_{3E,2}.
\end{aligned} \tag{56}$$

From (55) and (56), it follows that

$$\Big\langle \sum_{Q\in\mathfrak{Q}(E)} 1_Q \sup_{\Lambda>\ell(Q)} \sup_{\Lambda\leq\lambda<2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\leq 1/\Lambda} f| \Big\rangle_{3E,r'} \leq Aq^{-2-\delta}\langle 1_{E_1}\rangle_{3E,p}. \tag{57}$$

Estimate (57) combined with Hölder's inequality implies (54). Recursing on $\sum_{Q\in\mathfrak{Q}(E)} I_Q$ then yields (50). That

$$\Big\| \sup_{\Lambda} \sup_{\Lambda\leq\lambda<2\Lambda} |\mathscr{C}_\lambda^{a/q} P_{\leq 1/\Lambda}| : (p,r) \Big\|_{restricted} \leq AN^{-\delta}q^{-2-\delta}, \tag{58}$$

for all $N \in 2^{\mathbb{N}}$, $q \in \mathbb{N}$, and $(\frac{1}{p}, \frac{1}{r})$ satisfying $\max\{\frac{1}{p}, \frac{1}{q}\} < \frac{d-2}{d}$ and arbitrarily close to $(\frac{d-2}{d}, \frac{d-2}{d})$ and some $\delta = \delta(d,p,r) > 0$, follows a very similar argument, and so the details are omitted. Summing (54) on a, q and (58) on a, q, and N concludes the proof of Theorem 17. □

7 Sparse Domination for $\sup_\lambda |\mathscr{R}_\lambda|$

Our goal is now to obtain estimate (6), which is the sparse bound for $\sup_\lambda |\mathscr{R}_\lambda|$. We proceed by first proving

Lemma 18. *Let $\frac{1}{A}\frac{2^k}{\Lambda^2} \leq |\tau| \leq A\frac{2^k}{\Lambda^2}$. Then for all $d \geq 5$ and $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$ there exist $A = A(d, p, r)$ and $\delta = \delta(d, p, r) > 0$ such that, for all $\Lambda \in 2^{\mathbb{N}}$,*

$$\|T_{\check{\mu}_\tau} : (p, r)\| \leq A2^{dk/2}\Lambda^{-2-\delta}, \tag{59}$$

$$\|T_{\check{\gamma}_\tau} : (p, r)\| \leq Aq^{-2-\delta}2^{dk(1/2-2/d)}. \tag{60}$$

Here, as elsewhere, $T_m : f \mapsto f * \check{m}$ for all symbols $m \in L^\infty(\mathbb{T}^d)$.

Proof. Fix $f, g : \mathbb{Z}^d \to \mathbb{C}$ finitely supported. Letting $\mathcal{D}_\Lambda$ denote the dyadic cubes with $\ell(Q) = \Lambda$, observe that

$$\begin{aligned} |\langle f * \check{\mu}_\tau, g\rangle| &\leq \sum_{Q\in\mathcal{D}_\Lambda} |\langle f * \check{\mu}_\tau, g1_Q\rangle| \\ &\leq \sum_{Q\in\mathcal{D}_\Lambda} \left[|\langle(1_Q f) * \check{\mu}_\tau, g1_Q\rangle| + \sum_{l\geq 1} |\langle(1_{3^lQ\cap(3^{l-1}Q)^c} f) * \check{\mu}_\tau, g1_Q\rangle|\right] \\ &= \sum_{Q\in\mathcal{D}_\Lambda} A_Q + B_Q. \end{aligned}$$

By Lemma 12, $A_Q \leq A2^{dk/2}\Lambda^{-2-\delta(d,p,r)}\langle f\rangle_{Q,p}\langle g\rangle_{Q,r}|Q|$. Moreover, by estimate (41) and Lemma 9, it holds that for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{B}$ and $M \geq 1$

$$\|(1_{3^lQ\cap(3^{l-1}Q)^c} f) * \check{\mu}_\tau\|_{\ell^{r'}(Q)} \leq A_M 3^{-Ml}\left[\frac{\epsilon^2+\tau^2}{\epsilon^2}\right]^{d/4}\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(3^lQ)}. \tag{61}$$

Interpolating between estimates (40) and (61) ensures that for all $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$

$$\|(1_{3^lQ\cap(3^{l-1}Q)^c} f) * \check{\mu}_\tau\|_{\ell^{r'}(Q)} \leq A3^{-20dl}2^{dk/2}\Lambda^{-2-\delta(d,p,r)}\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(3^lQ)}$$

provided we choose $M \geq M_0(d)$. From this estimate, it follows that

$$B_Q \leq A2^{dk/2}\Lambda^{-2} \ll f \gg_{Q,p} \langle g\rangle_{Q,r}|Q|.$$

Moreover, there is a sparse collection $\mathcal{S}$ for which

$$\sum_{Q\in\mathcal{D}_\Lambda} \ll f \gg_{Q,p} \langle g\rangle_{Q,r}|Q| \leq A\sum_{S\in\mathcal{S}}\langle f\rangle_{S,p}\langle g\rangle_{S,r}|S|.$$

The proof of the estimate involving $f * \check{\gamma}_\tau$ is very similar, except that Lemma 13 and estimate (44) are used in place of Lemma 12 and estimate (40). $\square$

We now use Lemma 18 to deduce

Lemma 19. *For all $d \geq 5$ and $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$, there exist $A = A(d, p, r)$ and $\delta = \delta(d, p, r) > 0$ such that, for all $\Lambda \in 2^{\mathbb{N}}$,*

$$\Big\| \sup_{\Lambda \leq \lambda < 2\Lambda} |(\mathscr{A}_\lambda - \mathscr{B}_\lambda)| : (p, r) \Big\| \leq A\Lambda^{-\delta}, \tag{62}$$

$$\Big\| \sup_{\Lambda \leq \lambda < 2\Lambda} |(\mathscr{B}_\lambda - \mathscr{C}_\lambda)| : (p, r) \Big\| \leq A\Lambda^{-\delta}. \tag{63}$$

Proof. Begin by using (59) to observe that for every $f, g : \mathbb{Z}^d \to \mathbb{C}$ finitely supported

$$\begin{aligned} &\Big| \Big\langle \sup_{\Lambda \leq \lambda < 2\Lambda} |(\mathscr{A}_\lambda^{a/q} - \mathscr{B}_\lambda^{a/q}) f|, g \Big\rangle \Big| \\ &\qquad \leq A\Lambda^{-d+2} \sum_{k=0}^{\log_2(\Lambda/q)+A} \int_{I_k(a,q)} (\epsilon^2 + \tau^2)^{-d/4} |\langle f * \check{\mu}_\tau, g\rangle| d\tau \\ &\qquad \leq A\Lambda^{-d+2} \sum_{k=0}^{\log_2(\Lambda/q)+A} \Big[\frac{2^k}{\Lambda^2}\Big]^{1-d/2} 2^{dk/2} \Lambda^{-2-\delta} \sup_{\mathcal{S}} \Lambda_{\mathcal{S},p,r}(f, g). \end{aligned}$$

However,

$$\Lambda^{-d+2} \sum_{k=0}^{\log_2(\Lambda/q)+A} \Big[\frac{2^k}{\Lambda^2}\Big]^{1-d/2} 2^{dk/2} \Lambda^{-2-\epsilon} \leq A\Lambda^{-\delta} \frac{1}{q\Lambda}.$$

Summing on $a : 1 \leq a \leq q$ and $(a, q) = 1$ followed by $q : 1 \leq q \leq \Lambda$ yields an upper bound $O(\Lambda^{-\delta} \sup_{\mathcal{S}} \Lambda_{\mathcal{S},p,r}(f, g))$. To finish, it suffices to note using (60) that for every $f, g : \mathbb{Z}^d \to \mathbb{C}$ finitely supported

$$\begin{aligned} &\Big| \Big\langle \sup_{\Lambda \leq \lambda < 2\Lambda} |(\mathscr{B}_\lambda^{a/q} - \mathscr{C}_\lambda^{a/q}) f|, g \Big\rangle \Big| \\ &\qquad \leq A\Lambda^{-d+2} \sum_{k=\log_2(\Lambda/q)-A}^{\infty} \int_{I_k(a,q)} (\epsilon^2 + \tau^2)^{-d/4} |\langle f * \check{\gamma}_\tau, g\rangle| d\tau \\ &\qquad \leq A\Lambda^{-d+2} \sum_{k=\log_2(\Lambda/q)-A}^{\infty} \Big[\frac{2^k}{\Lambda^2}\Big]^{1-d/2} q^{-2-\delta} 2^{dk(1/2-2/d)} \sup_{\mathcal{S}} \Lambda_{\mathcal{S},p,r}(f, g). \end{aligned}$$

However,

$$\Lambda^{-d+2} \sum_{k=\log_2(\Lambda/q)-A}^{\infty} \Big[\frac{2^k}{\Lambda^2}\Big]^{1-d/2} q^{-2-\delta} 2^{dk(1/2-2/d)} \leq A \frac{1}{\Lambda q^{1+\delta}}.$$

Summing on a and q yields an upper bound $O(\Lambda^{-\delta} \sup_{\mathcal{S}} \Lambda_{\mathcal{S},p,r}(f, g))$. $\square$

Summing (62) and (63) over $\Lambda \in 2^{\mathbb{N}}$ gives

Proposition 20. *For all $d \geq 5$ and $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$,*

$$\left\| \sup_{\lambda \in \tilde{\Lambda}} |\mathscr{R}_\lambda| : (p, r) \right\| < \infty.$$

Theorem 21. *For all $d \geq 5$ and $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{S}(d)$,*

$$\left\| \sup_{\lambda \in \tilde{\Lambda}} |\mathscr{A}_\lambda| : (p, r) \right\| < \infty.$$

Proof. Combine Theorem 17 and Proposition 20. □

8 Sparse domination for $\sup_\lambda |\mathscr{A}_\lambda|$

In addition to Theorem 16, we shall need a localized variant of Theorem 21:

Theorem 22. *Let $d \geq 5$ and $(\frac{1}{r}, \frac{1}{s}) \in \mathcal{S}(d)$. For any collection of cubes $\mathcal{C}$ and $f, g : \mathbb{Z}^d \to \mathbb{C}$ finitely supported, there is a sparse collection of cubes $\mathcal{S}$ such that*

$$\left\langle \sup_{S \supset \mathcal{C}} |\mathscr{A}_S f|, |g| \right\rangle \leq A \Lambda_{\mathcal{S},r,s}(f, g),$$

where the supremum is restricted to those spheres $S = \{y \in \mathbb{Z}^d : |x - y| = \lambda\}$ for which the corresponding ball $B_S = \{y \in \mathbb{Z}^d : |x - y| \leq \lambda\}$ satisfies $B_S \supset Q$ for some cube $Q \in \mathcal{C}$, and the sparse collection $\mathcal{S}$ satisfies the property that for every cube $Q \in \mathcal{S}$ there is a cube $Q_* \in \mathcal{C}$ such that $Q \supset Q_*$.

Proof. Retrace the arguments used to show Theorem 21. □

The rest of this section is dedicated to showing estimate (2), which we rewrite as

Theorem 23. *Let $d \geq 5$ and $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{R}(d)$. Then*

$$\left\| \sup_{\lambda \in \tilde{\Lambda}} |\mathscr{A}_\lambda| : (p, r) \right\| < \infty.$$

There are two difficulties in the sparse setting that complicate the proof of Theorem 23. The first is that there is no general sparse interpolation machinery. The second is that there is no sparse bound at $(0, 1)$, as this point does not break the duality condition. Any successful argument that extends sparse bounds from $\mathcal{S}(d)$ to $\mathcal{R}(d)$ must work with localized sparse bounds for $\sup_{\lambda \in \tilde{\Lambda}} |\mathscr{A}_\lambda|$ near $(\frac{d-2}{d}, \frac{d-2}{d})$ and appropriately leverage the trivial $\ell^\infty \to \ell^\infty$ estimate.

Proof of Theorem 23. By Theorem 16, it suffices to prove a restricted weak-type sparse bound in a small neighborhood of the line connecting $(0, 1)$ with $(\frac{d-2}{d}, \frac{d-2}{d})$ intersected with $\mathcal{R}(d)$. In particular, we shall fix $(\frac{1}{p_1}, \frac{1}{p_2})$ close to $(\frac{d-2}{d}, \frac{d-2}{d})$ and prove sparse esitmates along the line connecting $(\frac{d-2}{d}, \frac{d-2}{d})$ to $(0, 1)$. So fix $(\frac{1}{p}, \frac{1}{r}) \in \mathcal{R}(d)$ on this line and $f, g : \mathbb{Z}^d \to \mathbb{C}$, $|f| \leq 1_{E_1}$, $|g| \leq 1_{E_2}$ and supported in $3E$ for some dyadic cube E. Let the first sparse generation of cubes $\mathcal{Q}(E)$ be those maximal dyadic cubes with respect to the properties

$$\langle f\rangle_{3Q,p_1} \geq A_0\langle f\rangle_{3E,p_1}, \quad \langle g\rangle_{3Q,r_1} \geq A_0\langle g\rangle_{3E,r_1}, \quad \Big\langle \sup_{\lambda\in\tilde{\Lambda}} |\mathscr{A}_\lambda f|\Big\rangle_{3Q,p_1} \geq A_0\langle f\rangle_{3E,p_1}.$$

For large enough constant A_0, $|\bigcup_{Q\in\mathcal{Q}(E)} Q| \leq \frac{|E|}{100}$. The restricted weak-type sparse bound is therefore reduced to dominating

$$\begin{aligned}\Big\langle \sum_{Q\in\mathcal{Q}(E)} 1_Q \sup_{\lambda\in\tilde{\Lambda}} |\mathscr{A}_\lambda f|, g\Big\rangle &= \Big\langle \sum_{Q\in\mathcal{Q}(E)} 1_Q \sup_{S\not\subset 3Q} |\mathscr{A}_S f|, g\Big\rangle \\ &\quad + \Big\langle \sum_{Q\in\mathcal{Q}(E)} 1_Q \sup_{S\subset 3Q} |\mathscr{A}_S f|, g\Big\rangle \\ &= I + II.\end{aligned}$$

Since we are able to recurse on the term II, it suffices to bound term I by $A\langle 1_{E_1}\rangle_{3E,p}\langle 1_{E_2}\rangle_{3E,r}|E|$. To this end, estimate using Corollary 22 with $\mathcal{B}=\mathcal{Q}(E)$ that

$$\begin{aligned}\Big\langle \sum_{Q\in\mathcal{Q}(E)} 1_Q \sup_{S\not\subset 3Q} |\mathscr{A}_S f|, |g|\Big\rangle &\leq \Big\langle \sup_{S\supset\mathcal{Q}(E)} |\mathscr{A}_S f|, |g|\Big\rangle \\ &\leq A\sum_{Q\in\mathcal{S}} \langle f\rangle_{Q,p_1}\langle g\rangle_{Q,r_1}|Q|,\end{aligned}$$

where the supremum is restricted to those discrete spheres $S = \{y := |x - y| = \lambda\}$ for which the corresponding ball $B_S = \{y : |x - y| \leq \lambda\}$ satisfies $B_S \supset R$ for some ball $R \in \mathcal{Q}(E)$, and the sparse collection $\mathcal{S}$ satisfies the property that for all $Q \in \mathcal{S}$ there is $R \in \mathcal{Q}(E)$ such that $Q \supset R$. So, for each $Q \in \mathcal{S}$,

$$\langle f\rangle_{Q,p_1} \leq \langle f\rangle_{3E,p_1}, \quad \langle g\rangle_{Q,r_1} \leq \langle g\rangle_{3E,r_1}$$

and

$$\begin{aligned}\sum_{Q\in\mathcal{S}} \langle f\rangle_{Q,p_1}\langle g\rangle_{Q,p_2}|Q| &\leq A\sum_{Q\in\mathcal{S}:|Q|\leq|E|} \langle f\rangle_{Q,p_1}\langle g\rangle_{Q,p_2}|Q| + \sum_{Q\in\mathcal{S}:|Q|>|E|} \langle f\rangle_{Q,p_1}\langle g\rangle_{Q,p_2}|Q| \\ &\leq A\langle f\rangle_{3E,p_1}\langle g\rangle_{3E,p_2} \sum_{Q\in\mathcal{S}:|Q|\leq|E|, Q\cap E\neq\emptyset} |Q| \\ &\quad + \sum_{Q\in\mathcal{S}:|Q|>|E|, Q\cap E\neq\emptyset} \|f\|_{\ell^{p_1}(E)}\|g\|_{\ell^{r_1}(E)}|Q|^{1-\frac{1}{p_1}-\frac{1}{r_1}} \\ &\leq A\langle f\rangle_{3E,p_1}\langle g\rangle_{3E,r_1}|E|.\end{aligned}$$

However, we have the following trivial estimate:

$$\Big\langle \sum_{Q\in\mathcal{Q}(E)} 1_Q \sup_{S\subset 3Q} |\mathscr{A}_\lambda f|, |g| \Big\rangle \leq \langle f\rangle_{3E,\infty}\langle |g|\rangle_{3E,1}|E|.$$

The restricted weak-type estimate is finally obtained by noting that

$$\begin{aligned}\Big\langle \sum_{Q\in\mathcal{Q}(E)} 1_Q \sup_{S\subset 3Q} |\mathscr{A}_\lambda f|, |g| \Big\rangle &\\ &\leq \min\{\langle 1_{E_1}\rangle_{3E,\infty}\langle 1_{E_2}\rangle_{3E,1}, A\langle 1_{E_1}\rangle_{3E,p_1}\langle 1_{E_2}\rangle_{3E,r_1}\}|E|\\ &\leq A\langle 1_{E_1}\rangle_{3E,p}\langle 1_{E_2}\rangle_{3E,r}|E|.\end{aligned}$$

□

9 Counterexamples

We finish by showing the necessary statements at the ends of Theorems 4 and 5.

Proposition 24. *Let $d \geq 5$. A necessary condition for the estimate*

$$\Big\| \sup_{\Lambda\leq\lambda<2\Lambda} |\mathscr{A}_\lambda f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A\Lambda^{d(1/r'-1/p)}\|f\|_{\ell^p(\mathbb{Z}^d)}$$

to hold for all $\Lambda \in 2^{\mathbb{N}}$ is $\max\{\frac{1}{p}+\frac{2}{d}, \frac{1}{r}+\frac{2}{pd}\} \leq 1$.

Proof. The necessity of $\frac{1}{p}+\frac{2}{d} \leq 1$ follows by taking $f = 1_{\{0\}}$. Indeed, it is straightforward to see that for $d \geq 5$ and this choice of f,

$$\sup_{\Lambda\leq\lambda<2\Lambda} f(n) \geq A\Lambda^{2-d}1_{|n|\simeq\Lambda}(n),$$

and the uniform estimate

$$\Lambda^{-d/r'}\Big\| \sup_{\Lambda\leq\lambda<2\Lambda} |\mathscr{A}_\lambda f| \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A\Lambda^{-d/p}\|f\|_{\ell^p(\mathbb{Z}^d)}$$

implies $\Lambda^{2-d} \leq A\Lambda^{-d/p}$. That the condition $\frac{1}{p}+\frac{2}{d} \leq 1$ must hold follows by taking $\Lambda \in 2^{\mathbb{N}}$ arbitrarily large.

The necessity of $\frac{1}{r}+\frac{2}{pd} \leq 1$ follows from setting $f_\Lambda = 1_{|n|=\Lambda}(n)$. Then it is immediate that $\sup_{\Lambda\leq\lambda<2\Lambda} \mathscr{A}_\lambda f_\Lambda(0) = 1$ and

$$\Lambda^{-d/r'} \leq \Lambda^{-d/r'}\Big\| \sup_{\Lambda\leq\lambda<2\Lambda} \mathscr{A}_\lambda f_\Lambda \Big\|_{\ell^{r'}(\mathbb{Z}^d)} \leq A\Lambda^{-d/p}\|f_\Lambda\|_{\ell^p(\mathbb{Z}^d)} \leq A\Lambda^{-d/p}\Lambda^{(d-2)/p}.$$

The necessity of $\frac{1}{r}+\frac{2}{pd} \leq 1$ follows by again taking $\Lambda \in 2^{\mathbb{N}}$ arbitrarily large. □

Theorem 4 follows from Propositions 15 and 24. Lastly, we record

Proposition 25. *A necessary condition for*

$$\Big\| \sup_{\Lambda \le \lambda < 2\Lambda} |\mathscr{A}_\lambda f| : (p, r) \Big\| \le A\Lambda^{d(1/r' - 1/p)}$$

to hold for all $\Lambda \in 2^{\mathbb{N}}$ *is* $\max\{\frac{1}{p} + \frac{2}{d}, \frac{1}{r} + \frac{2}{pd}\} \le 1$.

Proof. Fix $\Lambda \in 2^{\mathbb{N}}$ and set $f(n) = 1_{\{0\}}$ and $g_\Lambda(n) = 1_{\Lambda \le |n| < 2\Lambda}(n)$. Then observe that

$$\Big\langle \sup_{\Lambda \le \lambda < 2\Lambda} \mathscr{A}_{\Lambda \le \lambda < 2\Lambda} f, g_\Lambda \Big\rangle \ge \frac{\Lambda^2}{A}.$$

However, $\sup_{\mathcal{S}} \Lambda_{\mathcal{S},p,r}(f_\Lambda, g) \le A\Lambda^{d(1-1/p)}$. Indeed, for any sparse collection $\mathcal{S}$,

$$\sum_{Q \in \mathcal{S}} \langle f \rangle_{Q,p} \langle g_\Lambda \rangle_{Q,r} |Q| = \sum_{\substack{Q \in \mathcal{S} : \ell(Q) \ge \frac{\Lambda}{\sqrt{d}} \\ Q \ni \{0\}}} \langle f \rangle_{Q,p} \langle g_\Lambda \rangle_{Q,r} |Q| \le A\Lambda^{d(1-1/p)}.$$

The claim that $\frac{1}{p} + \frac{2}{d} \le 1$ then follows by taking $\Lambda \in 2^{\mathbb{N}}$ arbitrarily large. Moreover, for a given $\Lambda \in 2^{\mathbb{N}}$, set $f_\Lambda(n) = 1_{|n| = \Lambda}(n)$ and $g(n) = 1_{\{0\}}(n)$. Then

$$\Big\langle \sup_{\Lambda \le \lambda < 2\Lambda} \mathscr{A}_{\Lambda \le \lambda < 2\Lambda} f_\Lambda, g \Big\rangle \ge \frac{1}{A}.$$

However, $\sup_{\mathcal{S}} \Lambda_{\mathcal{S},p,r}(f_\Lambda, g) \le A\Lambda^{-2/p}\Lambda^{d(1-1/r)}$. The necessity of $\frac{1}{r} + \frac{2}{pd} \le 1$ then follows from taking $\Lambda \in 2^{\mathbb{N}}$ arbitrarily large. □

Theorem 5 follows from Theorem 23 and Proposition 25.

References

[1] J. Bourgain, *Averages in the plane over convex curves and maximal operators*, J. Anal. Math. **47** (1986), 69–85.

[2] M. Christ and E. M. Stein, *A remark on singular Calderón–Zygmund theory*, Proc. Amer. Math. Soc. **99** (1987), 71–75.

[3] J. M. Conde-Alonso and G. Rey, *A pointwise estimate for positive dyadic shifts and some applications*, Math. Ann. **365** (2016), 1111–1135.

[4] A. Culiuc, R. Kesler and M. T. Lacey, *Sparse bounds for the discrete cubic Hilbert transform*, Anal. PDE **12** (2019), 1259–1272.

[5] A. D. Ionescu and S. Wainger, *L^p boundedness of discrete singular Radon transforms*, J. Amer. Math. Soc. **19** (2006), 357–383.

[6] R. Kesler and M. T. Lacey, *Sparse endpoint estimates for Bochner–Riesz multipliers on the plane*, Collect. Math. **69** (2018), 427–435.

[7] R. Kesler and D. M. Arias, *Uniform sparse bounds for discrete quadratic phase Hilbert transforms*, Anal. Math. Phys. **9** (2019), 263–274.

[8] B. Krause and M. T. Lacey, *Sparse bounds for random discrete Carleson theorems*, in *50 Years with Hardy Spaces*, Birkhäuser, Cham, 2018, pp. 317–332.

[9] B. Krause and M. T. Lacey, *A discrete quadratic Carleson theorem on ℓ^2 with a restricted supremum*, Int. Math. Res. Not. IMRN (2017), 3180–3208.

[10] M. T. Lacey, *An elementary proof of the A_2 bound*, Israel J. Math. **217** (2017), 181–195.

[11] M. T. Lacey, *Sparse bounds for spherical maximal functions*, J. Anal. Math. **139** (2019), 613–635.

[12] M. T. Lacey and R. Kesler, *ℓ^p-improving inequalities for discrete spherical averages*, Anal. Math. **46** (2020), 85–95.

[13] M. T. Lacey and S. Spencer, *Sparse bounds for oscillatory and random singular integrals*, New York J. Math. **23** (2017), 119–131.

[14] A. Magyar, *On the distribution of lattice points on spheres and level surfaces of polynomials*, J. Number Theory **122** (2007), 69–83.

[15] M. Mirek, E. M. Stein and B. Trojan, *$\ell^p(\mathbb{Z}^d)$-estimates for discrete operators of Radon type: variational estimates*, Invent. Math. **209** (2017), 665–748.

[16] Lillian B. Pierce, *A note on twisted discrete singular Radon transforms*, Math. Res. Lett. **17** (2010), 701–720.

[17] L. B. Pierce, *Discrete fractional Radon transforms and quadratic forms*, Duke Math. J. **161** (2012), 69–106.

[18] W. Schlag, *A generalization of Bourgain's circular maximal theorem*, J. Amer. Math. Soc. **10** (1997), 103–122.

[19] E. M. Stein, *Maximal functions. I. Spherical means*, Proc. Natl. Acad. Sci. USA **73** (1976), 2174–2175.

[20] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.

[21] E. M. Stein and S. Wainger, *Two discrete fractional integral operators revisited*, J. Anal. Math. **87** (2002), 451–479,

Robert Kesler
SCHOOL OF MATHEMATICS
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA GA 30332, USA
email: rkesler6@math.gatech.edu

(Received May 30, 2018 and in revised form May 31, 2018)