ON THE EQUIVALENCE OF HEAT KERNELS OF SECOND-ORDER PARABOLIC OPERATORS

By

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Abstract. Let P be a second-order, symmetric, and nonnegative elliptic operator with real coefficients defined on noncompact Riemannian manifold M, and let V be a real valued function which belongs to the class of small perturbation potentials with respect to the heat kernel of P in M. We prove that under some further assumptions (satisfied by large classes of P and M) the positive minimal heat kernels of $P - V$ and of P on M are equivalent. Moreover, the parabolic Martin boundary is stable under such perturbations, and the cones of all nonnegative solutions of the corresponding parabolic equations are affine homeomorphic.

1 Introduction

Let M be a smooth, noncompact, connected Riemannian manifold of dimension N. Let P be a second-order elliptic linear operator defined on M, and let V be a real valued potential. Denote the cone of all positive solutions of the equation $Pu = 0$ in M by $\mathcal{C}_P(M)$. The **generalized principal eigenvalue** of the operator P and a potential V is defined by

$$\lambda_0(P, V, M) := \sup\{\lambda \in \mathbb{R} \mid \mathcal{C}_{P-\lambda V}(M) \neq \emptyset\}.$$

We say that P is **nonnegative in** M (and denote it by $P \geq 0$) if $\lambda_0 := \lambda_0(P, \mathbf{1}, M) \geq 0$, where $\mathbf{1}$ is the constant function on M taking at any point $x \in M$ the value 1. Throughout the paper we always assume that $\lambda_0 \geq 0$, that is, $P \geq 0$ in M. So, let $P \geq 0$ in M, and consider the parabolic operator

$$Lu(x, t) := \partial_t u(x, t) + Pu(x, t), \quad (x, t) \in M \times (0, \infty). \tag{1.1}$$

Let $k_P^M(x, y, t)$ be the positive minimal (Dirichlet) heat kernel of the parabolic operator L on the manifold M. By definition, for a fixed $y \in M$, the function $(x, t) \mapsto k_P^M(x, y, t)$ is the minimal positive solution of the equation

$$Lu = 0 \quad \text{in } M \times (0, \infty), \tag{1.2}$$

subject to the initial data δ_y, the Dirac distribution at $y \in M$.

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Let g_1, g_2 be two positive functions defined in a domain D. We say that g_1 is **equivalent** to g_2 in D (and use the notation $g_1 \asymp g_2$ in D) if there exists a positive constant C such that

$$C^{-1}g_2(x) \leq g_1(x) \leq Cg_2(x) \quad \text{for all } x \in D.$$

The main aim of this article is to study the equivalence of two heat kernels associated with two parabolic operators in M. We are motivated by the following conjecture raised in [15].

Conjecture 1.1 (cf. [15])**.** Let P_1 and P_2 be two subcritical elliptic operators of the form either (2.1) or (2.2) which are defined on a Riemannian manifold M such that both P_1 and P_2 have the same principal part. Assume that $P_1 = P_2$ outside a compact set in M and that the generalized principal eigenvalues $\lambda_0(P_1, \mathbf{1}, M)$, $\lambda_0(P_2, \mathbf{1}, M)$ of P_1 and P_2 respectively in M are equal. Then $k_{P_1}^M \asymp k_{P_2}^M$ in $M \times M \times (0, \infty)$.

An important aspect of Conjecture 1.1 is towards understanding the stability of the large time behaviour of heat kernels, and of the parabolic Martin boundary under perturbations. We also remark that Conjecture 1.1 is related to strong ratio limit properties of the quotients of heat kernels of subcritical and critical operators, and to Davies' Conjecture (see [15]).

In the past four decades there has been extensive research in obtaining optimal sufficient conditions under which two second-order elliptic operators have equivalent positive minimal Green functions, and the elliptic case is pretty much well understood (see, for example, [2, 28, 32, 33, 34], and references therein). On the other hand, in spite of the huge literature dealing with two-sided heat kernel estimates, the question of the equivalence of heat kernels is far from being understood. In fact, there are only very few papers dealing with sufficient conditions that guarantee the equivalence of the heat kernels; see [7, 8, 15, 24, 26, 42, 43]. Moreover, most of these works study the particular case of a perturbation of the Laplace operator on $\mathbb{R}^N$ by a potential V that is either a signed potential, or satisfies additional smoothness assumptions.

Note that the explicit form of the heat kernel of the Laplacian on $\mathbb{R}^N$ is given by the Gauss–Weierstrass heat kernel

$$k_{-\Delta}^{\mathbb{R}^N}(x, y, t) := \left(\frac{1}{4\pi t}\right)^{\frac{N}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \quad x, y \in \mathbb{R}^N,\ t > 0, \tag{1.3}$$

and this explicit formula plays a crucial role in almost all the aforementioned papers (except [15]). Unfortunately, for general operators and manifolds such an expression is not available, despite the fact that in many cases the short and large time behaviour of the heat kernel is known. However, we should mention the recent paper by Chen and Hassell [10], where it is proved that under natural assumptions, the heat kernel of an asymptotically hyperbolic Cartan–Hadamard manifold is equivalent to the heat kernel of the hyperbolic space.

We provide a positive answer to Conjecture 1.1 in the case where P is symmetric and satisfies some further assumptions. We prove in Theorem 2.5 the equivalence of two heat kernels of two parabolic operators that differ by a compactly supported potential. This result is extended in Theorem 2.6 to a larger class of potentials known as the class of small perturbations with respect to the given heat kernel (see Definition 5.1). As an application we prove that the parabolic Martin boundary is stable under such perturbations, and the cones of all nonnegative solutions of the corresponding parabolic equations are affine homeomorphic.

Our study is based on the method used by M. Murata and Y. Pinchover in the study of the equivalence of the Green functions of elliptic operators (see [28, 32, 33]). In this approach one should obtain pointwise estimates for the iterated Green kernel, called the $3G$-inequality which implies sharp two-sided estimates for the corresponding Neumann series. To understand the difficulty in applying this method to the parabolic case, assume for simplicity that V has a compact support in M. In contrast to the elliptic case [28, 32], where the iterated kernel is given by integrations over a fixed compact set ($\operatorname{supp} V$), in the parabolic case the domain of integration is $\operatorname{supp}[V \times (0,t)]$ which grows as $t \to \infty$. Hence, the parabolic case requires a new and a different technique in order to prove the so-called $3k$-inequality. We refer to Section 3 for the definition of the $3k$-inequality.

The paper is organized as follows. In Section 2 we briefly review the theory of positive solutions of elliptic and parabolic equations and state our main results. Section 3 is devoted to several preparatory lemmas and propositions. In Section 4 we prove the aforementioned Theorem 2.5 concerning compactly supported perturbations, while in Section 5 we introduce the notion of small perturbations with respect to the given heat kernel and prove the aforementioned Theorem 2.6. Section 6 is devoted to the stability of the Martin boundary under small perturbations. We conclude our paper in Section 7 which is divided into three short subsections. In the first subsection we briefly extend our results to the class of quasi-symmetric heat kernels, in the second part we present some examples of manifolds and operators for which our results applies, and finally, a subsection devoted to a short discussion of some open problems ends the paper.

2 The setting and statements of the main results

The present section is devoted to the statements of our main theorems. Before going further we must introduce some notations, technical assumptions and definitions.

Let M be a smooth, noncompact, connected manifold of dimension N. We consider a second-order elliptic operator P with real coefficients which (in any coordinate system $(U; x_1, \ldots, x_N)$) is either of the form

$$Pu = -\sum_{i,j=1}^{N} a^{ij}(x)\partial_i\partial_j u + b(x)\cdot\nabla u + c(x)u, \tag{2.1}$$

or in the divergence form

$$Pu = -\mathrm{div}[(A(x)\nabla u + u\tilde{b}(x))] + b(x)\cdot\nabla u + c(x)u. \tag{2.2}$$

We assume that for every $x \in \Omega$ the matrix $A(x) := [a^{ij}(x)]$ is symmetric and that the real quadratic form

$$\xi\cdot A(x)\xi := \sum_{i,j=1}^{N} \xi_i a^{ij}(x)\xi_j, \qquad \xi \in \mathbb{R}^N$$

is positive definite. Moreover, it is assumed that P is locally uniformly elliptic. Hence, the principal part of the operator P induces a Riemannian metric $\mathfrak{g}$ on M. Throughout the paper we consider the Riemannian manifold $(M, \mathfrak{g})$. In particular, when $P = -\Delta_{\mathfrak{h}}$ is the Laplace–Beltrami on a given Riemannian manifold $(M, \mathfrak{h})$, then the induced metric $\mathfrak{g}$ on M coincides with the given metric $\mathfrak{h}$. We assume that $\mathrm{d}x$ is a given positive measure on M, satisfying $\mathrm{d}x = f\,\mathrm{vol}$, where f is a positive function, and vol is the Riemannian volume form of M with respect to the metric $\mathfrak{g}$ (which is just the Lebesgue measure in the case of a domain of $\mathbb{R}^N$ and the operator $P = -\Delta$, Euclidean Laplacian). Further, the minus divergence is the formal adjoint of the gradient with respect to the measure $\mathrm{d}x$.

Throughout the paper we assume that the coefficients of P are either C^∞-smooth or locally sufficiently regular in M such that the standard parabolic regularity theory holds true. For example, such sufficient conditions for P of the form (2.2) are: f and A are locally Hölder continuous, the vector fields b and $\tilde{b}$ are Borel measurable in M of class $L^p_{\mathrm{loc}}(M)$, and $c \in L^{p/2}_{\mathrm{loc}}(M)$ for some $p > N$. We denote by $P^\star$ the formal adjoint operator of P on its natural space $L^2(M, \mathrm{d}x)$.

When P is in divergence form (2.2) and $b = \tilde{b}$, the operator

$$Pu = -\mathrm{div}[(A\nabla u + ub)] + b\cdot\nabla u + cu \tag{2.3}$$

is **symmetric** in the space $L^2(M, \mathrm{d}x)$. Throughout the paper, we call this setting the **symmetric case**. We note that if P is symmetric and b is smooth enough, then P is in fact a Schrödinger-type operator of the form

$$Pu = -\mathrm{div}(A\nabla u) + (c - \mathrm{div} b)u.$$

Assume that $\lambda_0 \geq 0$, and let $k_P^M(x, y, t)$ be the **positive (minimal) heat kernel** of the parabolic operator L on the manifold M. It can be easily checked that for $\lambda \leq \lambda_0$, the heat kernel $k_{P-\lambda}^M$ of the operator $P - \lambda$ on M satisfies the identity

$$k_{P-\lambda}^M(x, y, t) = \mathrm{e}^{\lambda t} k_P^M(x, y, t) \quad \text{on } M \times M \times (0, \infty). \tag{2.4}$$

Definition 2.1. Suppose that $\lambda_0 = \lambda_0(P, \mathbf{1}, M) \geq 0$, and let k_P^M be the corresponding heat kernel. We say that the operator P is **subcritical** (respectively, **critical**) in M if for some $x \neq y$ (and therefore for any $x \neq y$), $x, y \in M$, we have

$$\int_0^\infty k_P^M(x, y, \tau)\,\mathrm{d}\tau < \infty \quad \left(\text{respectively, } \int_0^\infty k_P^M(x, y, \tau)\,\mathrm{d}\tau = \infty\right). \tag{2.5}$$

If P is subcritical in M, then

$$G_P^M(x, y) := \int_0^\infty k_P^M(x, y, \tau)\,\mathrm{d}\tau, \quad x, y \in M \tag{2.6}$$

is called the **positive minimal Green function** of the operator P in M.

Clearly, P is critical in M if and only if $P^\star$ is critical in M. Moreover, it is known that P is critical in M if and only if the equation $Pu = 0$ in M admits a unique (up to a multiplicative constant) positive supersolution [28, 32, 37]. In this case the corresponding unique (super)solution of the equation $Pu = 0$ in M is called the **(Agmon) ground state**.

Suppose that P is a critical operator in M and let ϕ and ϕ^* be the ground states of P and $P^\star$, respectively. Then P is said to be **positive-critical** (**null-critical**) in M with respect to the measure $\mathrm{d}x$ if $\phi^*\phi \in L^1(M, \mathrm{d}x)$ (resp., $\phi^*\phi \notin L^1(M, \mathrm{d}x)$).

Remark 2.2. We recall some general results concerning the large time behaviour of the heat kernel.

Let P be an elliptic operator of the form either (2.1) or (2.2), and assume that $\lambda_0 = \lambda_0(P, \mathbf{1}, M) \geq 0$. Then

$$-\lim_{t\to\infty} \frac{\log k_P^M(x, y, t)}{t} = \lambda_0 \tag{2.7}$$

(see [15, Remark 4] and references therein). Moreover,

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} k_P^M(x, y, t) = 0 \quad \text{locally uniformly in } M \times M,$$

unless $P-\lambda_0$ is positive-critical, and in this case,

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} k_P^M(x,y,t) = \frac{\phi(x)\phi^*(y)}{\int_M \phi^*(z)\phi(z)\mathrm{d}z}$$

locally uniformly in $M\times M$, where ϕ and ϕ^* are the ground states of $P-\lambda_0$ and $P^*-\lambda_0$, respectively (see [36, Theorem 1.2], and references therein).

Definition 2.3. Let P_i, $i=1,2$, be two elliptic operators of the form either (2.1) or (2.2) that are defined on M, and suppose that $\lambda_0(P_i,\mathbf{1},M)\geq 0$ for $i=1,2$. We say that the corresponding heat kernels $k_{P_1}^M(x,y,t)$ and $k_{P_2}^M(x,y,t)$ are **equivalent** (respectively, **semi-equivalent**) if

$$k_{P_1}^M \asymp k_{P_2}^M \quad \text{on } M\times M\times(0,\infty)$$
$$(\text{resp., } k_{P_1}^M(\cdot,y_0,\cdot)\asymp k_{P_2}^M(\cdot,y_0,\cdot) \text{ on } M\times(0,\infty) \text{ for some fixed } y_0\in M).$$

Similarly, we define the equivalence and the semi-equivalence of the Green functions $G_{P_i}^M(x,y)$, where $i=1,2$.

Remark 2.4. It follows that if $k_{P_1}^M \asymp k_{P_2}^M$, then P_1 is subcritical in M if and only if P_2 is subcritical in M, and in this case, (2.4) and (2.6) imply that $G_{P_1-\lambda}^M \asymp G_{P_2-\lambda}^M$ for any $\lambda\leq\lambda_0$ with the same equivalence constant. Moreover, $\lambda_0(P_1,\mathbf{1},M) := \lambda_0(P_2,\mathbf{1},M)$.

Throughout the paper we consider a perturbation of an elliptic operator P by a real valued potential V. We introduce the following one-parameter family of operators:

$$P_\varepsilon := P-\varepsilon V, \quad \varepsilon\in\mathbb{R}, \tag{2.8}$$

where P is a given elliptic operator of the form either (2.1) or (2.2) , and V is a given potential satisfying the above regularity assumption.

Now we are in a situation to state the main results of the paper. In fact, we provide a positive answer to Conjecture 1.1 under further assumptions.

Theorem 2.5. *Let $(M,\mathfrak{g})$ be a connected and noncompact Riemannian manifold of dimension N. Let P be a symmetric subcritical operator with C^∞-coefficients, such that the induced Riemannian metric by P is equal to $\mathfrak{g}$. Let $V\in L^p_{\mathrm{loc}}(M)$ be a nonzero real valued potential with compact support, where $p>\frac{N}{2}$.*

Assume that for some $x_0\in M$ and $T>0$ there exists $C:=C(T,x_0)>0$ such that the following doubling condition holds:

$$k_P^M\left(x_0,x_0,\frac{t}{2}\right)\leq C k_P^M(x_0,x_0,t) \quad \text{for all } t>T. \tag{2.9}$$

Then:

(1) *There exists* $\varepsilon_0 > 0$ *such that* $k^M_{P-\lambda} \asymp k^M_{P_\varepsilon-\lambda}$ *for all* $|\varepsilon| < \varepsilon_0$ *and all* $\lambda \leq 0$.
(2) *Suppose further that* $V \geq 0$. *Then* $k^M_{P-\lambda} \asymp k^M_{P_\varepsilon-\lambda}$ *for all* $-\infty < \varepsilon < \varepsilon_0$ *and all* $\lambda \leq 0$.
(3) *Suppose further that* $P - V$ *is subcritical in* M *and satisfies* (2.9). *Then* $k^M_P \asymp k^M_{P-V}$.
(4) *Assume that* P *is a symmetric subcritical operator with locally regular coefficients, and that* (2.9) *is satisfied. Then assertions (1)–(3) hold true (without the* C^∞*-assumption and the assumption on the metric) provided* V *is a bounded measurable potential with compact support.*

The following theorem extends Theorem 2.5 from the class of compactly supported potentials to the class of small perturbations (see Definition 5.1).

Theorem 2.6. *Suppose that the Riemannian manifold* $(M, \mathfrak{g})$*, the operator* P*, and its kernel* k^M_P *satisfy the assumptions of Theorem 2.5. Let* $V \in L^p_{\mathrm{loc}}(M)$ *be a small perturbation with respect to* k^M_P *in* M*, where* $p > N/2$.
(1) *Then there exists* $\varepsilon_0 > 0$ *such that* $k^M_{P-\lambda} \asymp k^M_{P_\varepsilon-\lambda}$ *for all* $|\varepsilon| < \varepsilon_0$ *and all* $\lambda \leq 0$.
(2) *Suppose further that* $V \geq 0$. *Then* $k^M_{P-\lambda} \asymp k^M_{P_\varepsilon-\lambda}$ *for all* $-\infty < \varepsilon < \varepsilon_0$ *and all* $\lambda \leq 0$.
(3) *Suppose further that* $P - V$ *is subcritical in* M *and satisfies the doubling condition* (2.9) *(without any sign assumption on* V*). Then* $k^M_P \asymp k^M_{P-V}$.

Moreover, if V *is only a semismall perturbation, then* (1) *and* (2) *hold true with the semi-equivalence replacing the equivalence assertion.*

Remark 2.7. Assumption (2.9) necessarily implies that $\lambda_0(P, \mathbf{1}, M) = 0$. Indeed, if $\lambda_0 > 0$, then (2.7) implies that k^M_P decays exponentially as $t \to \infty$, and this contradicts (2.9).

On the other hand, if $P \geq 0$ in M, and $k^M_P \asymp k^M_{P_\varepsilon}$ for all $|\varepsilon| < \varepsilon_0$, then (2.4) implies that $k^M_{P-\lambda} \asymp k^M_{P_\varepsilon-\lambda}$ for all $\lambda \leq \lambda_0$ and $|\varepsilon| < \varepsilon_0$ (and in particular, $P_\varepsilon - \lambda_0$ is subcritical in M, see Proposition 2.10 below).

Remark 2.8. If $\lambda_0 > 0$ and $P - \lambda_0$ satisfies the assumptions of Theorem 2.5 or Theorem 2.6, then the conclusions of these theorems hold true for $P - \lambda$ for all $\lambda \leq \lambda_0$ (see, e.g., Example 7.12).

Remark 2.9. The doubling condition (2.9) is not very restrictive. Clearly, the positive minimal heat kernels of the Laplacian on $\mathbb{R}^N$ with $N \geq 3$, and on the upper half-space $\mathbb{R}^N_+$ with $N \geq 1$ satisfy (2.9) (see [20, 39]). In Subsection 7.2 we provide further examples of manifolds M and operators P satisfying (2.9).

On the other hand, A. Grigor'yan kindly pointed out to us that for some model subcritical manifolds M with $\lambda_0 = 0$ and with exponential volume growth $V(r) = \exp(r^\alpha)$, where $0 < \alpha < 1$, the heat kernel satisfies the on-diagonal estimates

$$k^M_{-\Delta}(x_0, x_0, t) \asymp \exp(-ct^{\alpha/(2-\alpha)}).$$

So, the doubling condition (2.9) is not satisfied (see Example 5.36 and Theorem 5.42 in [16]).

In the critical case we have the following result.

Proposition 2.10. *Assume that P is critical in M, and let V be a nonzero potential.*

Then for any $\lambda \le 0$ there does not exist any $\varepsilon_0 > 0$ such that $k^M_{P-\lambda} \asymp k^M_{P_\varepsilon-\lambda}$ for all $|\varepsilon| < \varepsilon_0$. Moreover, the corresponding heat kernel k^M_P does not satisfy the $3k$-inequality (3.2) *with respect to any nonzero potential V.*

Proof. It follow from [33, Theorem 3.1], that if P is critical, then there exists at most one $\varepsilon_1 \neq 0$ such that P_ε is also critical in M. Hence, $k^M_P \not\asymp k^M_{P_\varepsilon}$ for all $\varepsilon \neq \varepsilon_1$. In light of (2.4), we conclude the result for all $\lambda \le 0$. The last part of the proposition follows from the proof of the first part and Theorem 3.5. □

Remark 2.11. Proposition 2.10 is counter intuitive, since in the context of the Green function, even if $P - \lambda_0$ is critical in M, yet for any nonzero potential V with a compact support, and any $\lambda < \lambda_0$, there exists $\varepsilon_0 = \varepsilon_0(V, \lambda) > 0$ such that $G^M_{P-\lambda} \asymp G^M_{P_\varepsilon-\lambda}$ for any $|\varepsilon| < \varepsilon_0$.

Remark 2.12. Let P be a subcritical operator in M and V a nonzero potential. Then $P - \lambda_0$ is subcritical if there exists ε_0 such that $k^M_{P-\lambda} \asymp k^M_{P_\varepsilon-\lambda}$ for all $|\varepsilon| < \varepsilon_0$ and for some $\lambda \le \lambda_0$.

The proof of Theorem 2.5 relies on a suitable $3k$-inequality (see Definition 3.3 below). We note that an analogous $3G$-inequality is used frequently for proving the equivalence of Green functions (see, for example, [28, 33, 34]).

3 Preparatory results

In the present section we recall some basic properties of the heat kernel, define the notion of $3k$-inequality, and prove some basic general results concerning the equivalence of heat kernels. The lemma below summarizes some fundamental properties of the heat kernel.

Lemma 3.1. *Let P be an elliptic operator of the form either* (2.1) *or* (2.2)*, which is nonnegative in M. Then the positive minimal heat kernel $k_P^M(x,y,t)$ satisfies the following properties:*

(1) *$k_P^M(x,y,t)$ satisfies the Chapman–Kolmogorov equation (the semigroup property)*

$$k_P^M(x,y,s+t) = \int_M k_P^M(x,z,s)k_P^M(z,y,t)\mathrm{d}z \quad \forall s,t > 0 \text{ and } \forall\, x,y \in M.$$

(2) $k_P^M(x,y,t) \geq 0, \quad k_{P^\star}^M(x,y,t) = k_P^M(y,x,t) \quad \forall t > 0$ *and* $\forall\, x,y \in M$.

(3) *The heat kernel is monotone increasing as a function of the domain.*

(4) *If $V \geq 0$, then $k_{P+V}^M \leq k_P^M$.*
Suppose further that P is symmetric. Then:

(5) $k_P^M(x,y,t) \leq k_P^M(x,x,t)^{\frac{1}{2}} k_P^M(y,y,t)^{\frac{1}{2}} \quad \forall\, t > 0$ *and* $\forall\, x,y \in M$.

(6) *The function $t \mapsto k_P^M(x,x,t)$ is positive, monotone decreasing and log-convex for all $x \in M$.*

(7) *Assume that P is a nonnegative selfadjoint operator on $L^2(M,\mathrm{d}x)$, then*

$$\mathrm{e}^{-Pt}f(x) = \int_M k_P^M(x,y,t)f(y)\mathrm{d}y$$

for all $t > 0$ and $f \in L^2(M,\mathrm{d}x)$.

For the proof of the above lemma we refer to [13, Lemma 1].

In the sequel we need the following log-convexity property of the heat kernels with respect to a perturbation by a potential W (see, for example, [40, Lemma B.7.73]).

Proposition 3.2. *Suppose that the elliptic operators P_0 and $P_1 := P_0 + W$ both admit positive minimal heat kernels k_0 and k_1, respectively, in M. Then for any $0 \leq \alpha \leq 1$, the operator $P_\alpha := (1-\alpha)P_0 + \alpha P_1$ admits a positive minimal heat kernel k_α in M, and k_α satisfies*

$$k_\alpha(x,y,t) \leq (k_0(x,y,t))^{(1-\alpha)}(k_1(x,y,t))^\alpha \quad \forall\, x,y \in M, \text{ and } t > 0. \tag{3.1}$$

Definition 3.3. Let P be a subcritical operator defined on M. We say that the heat kernel k_P^M satisfies the $3k$**-inequality** with respect to V if there exists a constant $C > 0$ such that the following inequality holds true:

$$\int_0^t \int_M k_P^M(x,z,t-s)|V(z)|k_P^M(z,y,s)\mathrm{d}z\mathrm{d}s \leq Ck_P^M(x,y,t) \qquad \forall x,y \in M, \text{ and } t > 0. \tag{3.2}$$

We say that the heat kernel k_P^M satisfies the **restricted** $3k$**-inequality** with respect to V if for any $T > 0$ there exists a constant $C(T) > 0$ such that the following inequality holds true:

$$\int_0^t \int_M k_P^M(x, z, t-s)|V(z)|k_P^M(z, y, s)\mathrm{d}z\mathrm{d}s \leq C(T)k_P^M(x, y, t) \tag{3.3}$$

for all $x, y \in M$ and $0 < t \leq T$.

If V is a bounded potential, then the heat kernel satisfies the restricted $3k$-inequality. Indeed, the Chapman–Kolmogorov equation clearly implies:

Proposition 3.4. *Let P be an elliptic operator of the form either* (2.1) *or* (2.2)*, which is nonnegative in M, and let V be a bounded potential. Then the following restricted* $3k$*-inequality holds true:*

$$\int_0^t \int_M k_P^M(x, z, t-s)|V(z)|k_P^M(z, y, s)\mathrm{d}z\mathrm{d}s \leq T\|V\|_\infty k_P^M(x, y, t)$$

for all $x, y \in M$ and $0 < t \leq T$.

The next theorem asserts that if k_P^M satisfies the $3k$-inequality, then for small $|\varepsilon|$, we have $k_{P_\varepsilon}^M \asymp k_P^M$ (cf. [15, Theorem 5.3]).

Theorem 3.5. (1) *Let V be a potential such that k_P^M satisfies the $3k$-inequality* (3.2)*. Then there exists $\varepsilon_0 > 0$ such that $k_{P_\varepsilon}^M \asymp k_P^M$ for all $|\varepsilon| < \varepsilon_0$.*

(2) *If k_P^M satisfies the restricted $3k$-inequality* (3.3)*, then for any $T > 0$ there exists positive $\varepsilon_0(T)$ such that for all $\varepsilon < \varepsilon(T)$*

$$k_{P_\varepsilon}^M \asymp k_P^M \quad \text{on } M \times M \times (0, T]. \tag{3.4}$$

(3) *Under the assumptions of either* (1) *or* (2)*, let ε be such that* (3.4) *holds true with $0 < T \leq \infty$. Then the heat kernel $k_{P_\varepsilon}^M$ satisfies the resolvent equations*

$$\begin{aligned} k_{P_\varepsilon}^M(x, y, t) &= k_P^M(x, y, t) + \varepsilon \int_0^t \int_M k_P^M(x, z, t-s)V(z)k_{P_\varepsilon}^M(z, y, s)\mathrm{d}z\mathrm{d}s \\ &= k_P^M(x, y, t) + \varepsilon \int_0^t \int_M k_{P_\varepsilon}^M(x, z, t-s)V(z)k_P^M(z, y, s)\mathrm{d}z\mathrm{d}s \end{aligned} \tag{3.5}$$

for all $(x, y, t) \in M \times M \times (0, T)$.

Proof of (1) and (2). Fix $0 < T \leq \infty$ and $y \in M$. Consider the iterated kernel

$$k_P^{(i)}(x, y, t) := \begin{cases} k_P^M(x, y, t), & i = 0, \\ \int_0^t \int_M k_P^{(i-1)}(x, z, t-s)V(z)k_P^M(z, y, s)\mathrm{d}z\mathrm{d}s, & i \geq 1. \end{cases}$$

It follows from the $3k$-inequality (3.2) (or the restricted $3k$-inequality (3.3)) that for all $0 < t < T$ we have

$$k_P^{(i)}(x, y, t) \le C^i k_P^M(x, y, t). \tag{3.6}$$

Hence,

$$\sum_{i=0}^{\infty} |\varepsilon|^i |k_P^{(i)}(x, y, t)| \le \frac{1}{1 - C|\varepsilon|} k_P^M(x, y, t), \tag{3.7}$$

provided $|\varepsilon| < C^{-1}$.

Fix such ε. Using a standard parabolic regularity argument, it follows that the Neumann series

$$H_P^{\varepsilon}(x, y, t) := \sum_{i=0}^{\infty} \varepsilon^i k_P^{(i)}(x, y, t)$$

converges locally uniformly in $M \times (0, T)$ to a positive fundamental solution of the equation $(u_t + P_\varepsilon)u = 0$. Hence, $k_{P_\varepsilon}^M(x, y, t)$ exists, and by the minimality of the heat kernel and (3.7) we obtain

$$k_{P_\varepsilon}^M(x, y, t) \le H_P^{\varepsilon}(x, y, t) \le \frac{1}{1 - C|\varepsilon|} k_P^M(x, y, t). \tag{3.8}$$

Let M_j be an exhaustion of M, i.e., a sequence of smooth, relatively compact subdomains of M such that $y \in M_1$, $M_j \Subset M_{j+1}$ and $\bigcup_{j=1}^{\infty} M_j = M$.

Using the resolvent equation (Duhamel's principle) in M_j

$$k_{P_\varepsilon}^{M_j}(x, y, t) = k_P^{M_j}(x, y, t) + \varepsilon \int_0^t \int_{M_j} k_P^{M_j}(x, z, t - s) V(z) k_{P_\varepsilon}^{M_j}(z, y, s) \mathrm{d}z \mathrm{d}s, \tag{3.9}$$

and by the dominated convergence theorem we obtain that $k_{P_\varepsilon}^M$ satisfies the resolvent equation

$$k_{P_\varepsilon}^M(x, y, t) = k_P^M(x, y, t) + \varepsilon \int_0^t \int_M k_P^M(x, z, t - s) V(z) k_{P_\varepsilon}^M(z, y, s) \mathrm{d}z \mathrm{d}s.$$

Moreover, by the resolvent equation and inequality (3.2), we have

$$\begin{aligned} k_{P_\varepsilon}^M(x, y, t) &= k_P^M(x, y, t) + \varepsilon \int_0^t \int_M k_P^M(x, z, t - s) V(z) k_{P_\varepsilon}^M(z, y, s) \mathrm{d}z \mathrm{d}s \\ &\ge k_P^M(x, y, t) - \frac{|\varepsilon|}{1 - C|\varepsilon|} \int_0^t \int_M k_P^M(x, z, t - s) |V(z)| k_P^M(z, y, s) \mathrm{d}z \mathrm{d}s \\ &\ge k_P^M(x, y, t) - \Big(\frac{C|\varepsilon|}{1 - C|\varepsilon|}\Big) k_P^M(x, y, t) = \Big(\frac{1 - 2C|\varepsilon|}{1 - C|\varepsilon|}\Big) k_P^M(x, y, t). \end{aligned}$$

Hence, for $|\varepsilon| < 1/(2C)$ we have $k^M_{P_\varepsilon} \asymp k^M_P$, which in turn implies that $H^\varepsilon_P \asymp k^M_{P_\varepsilon}$. The minimality of $k^M_{P_\varepsilon}$ implies now that

$$k^M_{P_\varepsilon}(x, y, t) = H^\varepsilon_P(x, y, t) := \sum_{i=0}^{\infty} \varepsilon^i k^{(i)}_P(x, y, t)$$

for any $|\varepsilon| < 1/(2C)$. This proves parts (1) and (2) of the theorem.

Part (3) of the theorem follows from the resolvent equation (3.9) in M_j, the $3k$-inequality, and the dominated convergence theorem. □

Remark 3.6. It is evident from Theorem 3.5 that in order to prove Theorems 2.5 and 2.6, it is enough to establish the $3k$-inequality (3.2).

For a perturbation by a nonnegative potential V, we have

Lemma 3.7 ([15, Corollary 2])**.** *Let P be a subcritical operator, and let V be a nonnegative potential. Suppose that $k^M_{P_{\varepsilon_0}} \asymp k^M_P$ for some $\varepsilon_0 > 0$. Then $k^M_{P_\varepsilon} \asymp k^M_P$ for any $\varepsilon < \varepsilon_0$.*

We provide here a detailed proof of the above lemma.

Proof. By the generalized maximum principle, if $\varepsilon_1 < \varepsilon_2$, then

$$k^M_{P_{\varepsilon_1}} \leq k^M_{P_{\varepsilon_2}}. \tag{3.10}$$

So, by our assumption, there exists $C > 0$ such that

$$k^M_P \leq k^M_{P_{\varepsilon_0}} \leq C k^M_P.$$

Let $0 \leq \varepsilon \leq \varepsilon_0$. Then, by (3.10) and Proposition 3.2 with $0 \leq \alpha := \frac{\varepsilon}{\varepsilon_0} \leq 1$, we have

$$k^M_{P_\varepsilon} \leq (k^M_P)^{1-\alpha}(k^M_{P_{\varepsilon_0}})^\alpha \leq C^\alpha k^M_P = C^{\varepsilon/\varepsilon_0} k^M_P.$$

On the other hand, if $\varepsilon < 0$, then by (3.10) and Proposition 3.2 we have with $\alpha = -\varepsilon/(\varepsilon_0 - \varepsilon)$

$$k^M_{P_\varepsilon} \leq k^M_P \leq (k^M_{P_{\varepsilon_0}})^\alpha (k^M_{P_\varepsilon})^{1-\alpha} \leq C^\alpha (k^M_P)^\alpha (k^M_{P_\varepsilon})^{1-\alpha},$$

and hence,

$$k^M_{P_\varepsilon} \leq k^M_P \leq C^{\alpha/(1-\alpha)} k^M_{P_\varepsilon} = C^{-\varepsilon/\varepsilon_0} k^M_{P_\varepsilon}.$$

So,

$$C^{\varepsilon/\varepsilon_0} k^M_P \leq k^M_{P_\varepsilon} \leq k^M_P,$$

and this completes the proof of the lemma. □

Remark 3.8. In [7, 8, 43], the authors consider the special case of the Laplacian on $\mathbb{R}^N$ and signed potential perturbations. It is proved there that for $V \geq 0$ which is in a certain L^p subspace, $k^{\mathbb{R}^N}_{-\Delta} \asymp k^{\mathbb{R}^N}_{-\Delta-\varepsilon V}$ for any $\varepsilon \leq 0$. Our Lemma 3.7 and Theorem 2.6, applied to this particular case, extend these results even for signed potentials V, since in this case, by our results, the interval of equivalence is $(-\infty, \varepsilon_0)$, where $\varepsilon_0 > 0$ (and not only $\mathbb{R}_-$).

Recall that by [33], the set

$$S = S(P, V, M) := \{\varepsilon \in \mathbb{R} \mid P_\varepsilon \geq 0 \text{ in } M\}$$

is a closed convex set, which contains the convex set

$$S_+ = S_+(P, V, M) := \{\varepsilon \in \mathbb{R} \mid P_\varepsilon \text{ is subcritical}\},$$

and

$$\{\varepsilon \in \mathbb{R} \mid P_\varepsilon \text{ is critical}\} \subset \partial S.$$

Moreover, if V is a small perturbation of P in the sense of Green functions, then $S_+ = \operatorname{int} S$, and $G^M_{P_\varepsilon} \asymp G^M_P$ for any $\varepsilon \in S_+$ (see [33]).

We note that, in general, the convexity of the set

$$\{(\lambda, \varepsilon) \in \mathbb{R}^2 \mid \lambda \leq \lambda_0,\ \varepsilon \in S_+(P-\lambda, V, M)\}$$

implies that for any $\lambda \leq \lambda_0$, we have

$$\{\varepsilon \in \mathbb{R} \mid k^M_{P_\varepsilon-\lambda} \asymp k^M_{P-\lambda}\} \subset S_+(P-\lambda_0, V, M). \tag{3.11}$$

The following lemma shows that under some conditions we have

$$\{\varepsilon \in \mathbb{R} \mid k^M_{P_\varepsilon} \asymp k^M_P\} = S_+(P-\lambda_0, V, M). \tag{3.12}$$

Lemma 3.9. *Let P and $P-V$ be two subcritical elliptic operators such that for some $0 < \alpha < \beta < 1$, $k^M_{P_\alpha} \asymp k^M_P$ and $k^M_{P_\beta} \asymp k^M_{P-V}$. Then*

$$k^M_P \asymp k^M_{P-V}.$$

Proof. By (3.11), we may assume that $\lambda_0(P, \mathbf{1}, M) = 0$. Proposition 3.2 and the lemma's hypothesis $k^M_P \asymp k^M_{P_\alpha}$ imply that

$$k^M_P \asymp k^M_{P_\alpha} \leq (k^M_P)^{1-\alpha}(k^M_{P-V})^\alpha.$$

This implies $C_1 k^M_P \leq k^M_{P-V}$. Similarly,

$$k^M_{P-V} \asymp k^M_{P_\beta} \leq (k^M_P)^{1-\beta}(k^M_{P-V})^\beta$$

implies $k^M_{P-V} \leq C_2 k^M_P$. Hence, the lemma is proved. □

In the study of equivalence of heat kernels, one would expect that as in the elliptic case (see, for example, [33, 34]), the local Harnack inequality should play a pivotal role. Unfortunately, the parabolic Harnack inequality for nonnegative solutions is weaker than the elliptic one. Nevertheless, in the symmetric case, the heat kernel satisfies the following elliptic-type Harnack inequality due to E. B. Davies [13, Theorem 10].

Lemma 3.10 (Davies–Harnack inequality for the heat kernel)**.** *Fix a compact subset A of M, and $T > 0$. Then there exists a positive constant $C := C(T, A, P)$ such that*

$$\sup_{x,y\in A} k_P^M(x, y, t) \leq C \inf_{x,y\in A} k_P^M(x, y, t) \quad \forall\, t \geq T. \tag{3.13}$$

4 Proof of Theorem 2.5

The proof of Theorem 2.5 hinges on the following key proposition.

Proposition 4.1. *Assume that P, V and k_P^M satisfy the assumptions of the first part of Theorem 2.5. In addition, assume that the diameter of* $\operatorname{supp} V$ *is small enough. Then the corresponding heat kernel k_P^M satisfies the $3k$-inequality* (3.2)*. Consequently, there exists $\varepsilon_0 > 0$ such that $k_{P_\varepsilon}^M \asymp k_P^M$ for all $|\varepsilon| < \varepsilon_0$.*

4.1 Short time asymptotic.

One of the key steps of the proof of the $3k$-inequality of Proposition 4.1 relies on the local short time asymptotic of the heat kernel $k_P^M(x, y, t)$. Recall that two-sided short time estimates of the heat kernel have been extensively studied in the past forty years. However, for our purpose, we need the local short time **asymptotic** of the heat kernel which is given by the following theorem of Y. Kannai [21] (see also [27] for the result in the compact case). For the global analogue result see Section 4 of [21, Theorem 4.1], and for subsequent developments of these results see [6, 9, 41].

Lemma 4.2 ([21])**.** *Assume that an elliptic operator P of the form either* (2.1) *or* (2.2) *with C^∞-coefficients is defined on a smooth noncompact manifold M. Let $d(x, y)$ be the Riemannian distance induced by the principal part of the operator P.*

For any relatively compact set $K \subset M \times M$, there is a $\delta > 0$ and smooth functions $H_n(x, y)$, $n = 0, 1, \ldots$, defined on K such that the following asymptotic expansion

$$k_P^M(x, y, t) \sim \left(\frac{1}{4\pi t}\right)^{\frac{N}{2}} \exp\left(-\frac{d(x, y)^2}{4t}\right) \sum_{n=0}^{\infty} H_n(x, y) t^n \tag{4.1}$$

holds locally uniformly as $t \to 0$ in K, whenever $d(x, y) < \delta$. Moreover,

$$H_0(x, y) > 0 \quad \text{and} \quad H_0(x, x) = 1.$$

In particular, for small enough $t > 0$, and for x, y in a small compact set in $M \times M$, we have

$$k_P^M(x, y, t) \asymp \Big(\frac{1}{4\pi t}\Big)^{\frac{N}{2}} \exp\Big(-\frac{d(x, y)^2}{4t}\Big).$$

Next, we state and prove another key ingredient for the proof of the $3k$-inequality.

Lemma 4.3. *Let $V \in L^p(M), p > \frac{N}{2}$ be a potential with compact support K, and let A be a bounded domain with a smooth boundary such that $K \Subset A$. Assume that there exists a constant $C > 0$ such that*

$$\int_0^t \int_K k_P^M(x, z, t-s)|V(z)|k_P^M(z, y, s)\mathrm{d}z\mathrm{d}s \le Ck_P^M(x, y, t), \tag{4.2}$$

for any $x, y \in A$, and $t > 0$. Then

$$\int_0^t \int_K k_P^M(x, z, t-s)|V(z)|k_P^M(z, y, s)\mathrm{d}z\mathrm{d}s \le Ck_P^M(x, y, t), \tag{4.3}$$

for any $x, y \in M$, and $t > 0$.

Proof. Following common practice, in the sequel, the letter C will denote an irrelevant positive constant, the value of which might change from line to line, and even in the same line.

Fix $y \in A$, and define

$$U_y(x, t) := \int_0^t \int_K k_P^M(x, z, t-s)|V(z)|k_P^M(z, y, s)\mathrm{d}z\mathrm{d}s. \tag{4.4}$$

By (4.2),

$$U_y(x, t) \le Ck_P^M(x, y, t) \quad \forall x \in \partial A \text{ and } t > 0.$$

Moreover, U_y is a solution of the equation

$$\frac{\partial}{\partial t}U_y + PU_y = |V(x)|k_P^M(x, y, t), \quad x \in M \text{ and } t > 0.$$

In particular, $\frac{\partial}{\partial t}U_y + PU_y = 0$ for all x outside K and $t > 0$.

Let $\{M_n\}_{n=0}^{\infty}$ be an exhaustion of M such that $A \subset M_0$, and set

$$U_{y,n}(x, t) := \int_0^t \int_A k_P^{M_n}(x, z, t-s)|V(z)|k_P^{M_n}(z, y, s)\mathrm{d}z\mathrm{d}s,$$

where $k_P^{M_n}(x, y, t)$ is the Dirichlet heat kernel of P on M_n.

Recall that as a function of x, the heat kernel $k_P^{M_n}(x, y, t)$ satisfies the equation $\frac{\partial}{\partial t}k_P^{M_n} + Pk_P^{M_n} = 0$ in $M_n \times (0, \infty)$. Moreover, since $U_{y,n}$ and $k_P^{M_n}$ converge locally uniformly to U_y and k_P^M, respectively, it follows that for any $\varepsilon > 0$ there is N_ε, such that for any $n \geq N_\varepsilon$

$$U_{y,n}(x) \leq (C + \varepsilon)k_P^{M_n}(x, y, t) \quad \forall x \in \partial A \text{ and } t > 0.$$

Therefore, for such n, we have

$$\begin{cases} \frac{\partial}{\partial t}U_{y,n} + PU_{y,n} = 0 & \text{in } (M_n \setminus A) \times (0, \infty), \\ U_{y,n} \leq (C + \varepsilon)k_P^{M_n} & \text{on } \partial A \times (0, \infty), \\ U_{y,n} = 0 & \text{on } \partial M_n \times (0, \infty), \\ U_{y,n} = 0 & \text{on } (M_n \setminus A) \times \{0\}. \end{cases}$$

The generalized maximum principle implies that

$$U_{y,n} \leq (C + \varepsilon)k_P^{M_n} \quad \text{on } (M_n \setminus A) \times (0, \infty).$$

Letting $n \to \infty$ we arrive at

$$U_y(x, t) \leq Ck_P^M(x, y, t) \quad \forall x \in M, y \in A \text{ and } t > 0. \tag{4.5}$$

Next, we fix $x \in M$ and define for $y \in M$

$$U_x^\star(y, t) := \int_0^t \int_A k_P^M(x, z, t - s)|V(z)|k_P^M(z, y, s)\mathrm{d}z\mathrm{d}s.$$

Then as a function of y, $U_x^\star$ is a solution of the equation

$$\frac{\partial}{\partial t}U_x^\star + P^\star U_x^\star = |V(y)|k_P^M(x, y, t), \quad y \in M \text{ and } t > 0.$$

In particular, $\frac{\partial}{\partial t}U_x^\star + P^\star U_x^\star = 0$ for all y outside A.

Since $U_x^\star(y, t) = U_y(x, t)$, estimate (4.5) implies

$$U_x^\star(y, t) \leq Ck_P^M(x, y, t) \quad \forall y \in A \text{ and } x \in M.$$

Hence, the above exhaustion and comparison arguments finally imply

$$\int_0^t \int_A k_P^M(x, z, t - s)|V(z)|k_P^M(z, y, s)\mathrm{d}z\mathrm{d}s \leq Ck_P^M(x, y, t), \tag{4.6}$$

for any $x, y \in M$, and $t > 0$. □

Having proven Lemma 4.3, we turn to the proof of the $3k$-inequality.

Proof of Proposition 4.1. By Lemma 4.3, it is sufficient to prove the $3k$-inequality for all $x, y \in A$ and all $t > 0$. So, it is enough to prove the existence of a constant $C > 0$ such that

$$S(V, x, y, t) := \int_0^t \int_M \frac{k_P^M(x, z, t-s)k_P^M(z, y, s)}{k_P^M(x, y, t)} |V(z)| \mathrm{d}z\mathrm{d}s \leq C \tag{4.7}$$

for all $x, y \in A$ and all $t > 0$.

The proof is divided into several steps. We fix an arbitrary small $\delta_0 > 0$ (to be chosen later), and prove the boundedness of $S(V, x, y, t)$ in two separate regions; $t \geq \delta_0$ and $0 < t < \delta_0$.

Step 1. In this step we estimate (4.7) when $t \geq \delta_0$ and $x, y \in A$, where A is a smooth compact subset of M containing $K = \operatorname{supp} V$. Fix $0 < \delta < \frac{\delta_0}{2}$. Fubini's theorem yields

$$\begin{aligned} &\int_0^t \int_M k_P^M(x, z, t-s)k_P^M(z, y, s)|V(z)|\mathrm{d}z\mathrm{d}s \\ &\qquad = \int_A \left(\int_0^\delta k_P^M(x, z, t-s)k_P^M(z, y, s)\mathrm{d}s \right) |V(z)|\mathrm{d}z \\ &\qquad\quad + \int_A \left(\int_\delta^t k_P^M(x, z, t-s)k_P^M(z, y, s)\mathrm{d}s \right) |V(z)|\mathrm{d}z. \end{aligned} \tag{4.8}$$

Consider the first term of (4.8), namely,

$$I_1^\delta := \int_A \left(\int_0^\delta k_P^M(x, z, t-s)k_P^M(z, y, s)\mathrm{d}s \right) |V(z)|\mathrm{d}z.$$

Since $t > \delta_0$, we have for $0 < s < \delta$

$$\delta < \frac{t}{2} < t - \delta < t - s < t.$$

Hence, in light of parts (5) and (6) of Lemma 3.1, and Davies–Harnack inequality (3.13), we obtain

$$\begin{aligned} I_1^\delta &\leq \int_A \left(\int_0^\delta (k_P^M(x, x, t-s))^{\frac{1}{2}} (k_P^M(z, z, t-s))^{\frac{1}{2}} k_P^M(z, y, s)\mathrm{d}s \right) |V(z)|\mathrm{d}z \\ &\leq C\left(k_P^M\left(x, x, \frac{t}{2}\right)\right)^{\frac{1}{2}} \left(k_P^M\left(y, y, \frac{t}{2}\right)\right)^{\frac{1}{2}} \int_A \left(\int_0^\infty k_P^M(z, y, s)\mathrm{d}s \right) |V(z)|\mathrm{d}z. \end{aligned}$$

Using our assumption that P is subcritical in M, the Davies–Harnack inequality (3.13), and the doubling condition (2.9), we get

$$I_1^\delta \leq C(k_P^M(x_0, x_0, t))^{\frac{1}{2}} (k_P^M(x_0, x_0, t))^{\frac{1}{2}} \int_A G_P^M(z, y)|V(z)|\mathrm{d}z, \tag{4.9}$$

where G_P^M is the Green function of the operator P in M. Consequently, the Davies–Harnack inequality (3.13) for the heat kernel implies

$$I_1^\delta \leq C(\delta, A) k_P^M(x, y, t) \int_A G_P^M(z, y)|V(z)|\mathrm{d}z \quad \forall x, y \in A, \ t > \delta. \tag{4.10}$$

On the other hand, the well-known behaviour of the Green function near a singularity and the local elliptic Harnack inequality imply that there exists a positive constant C such that

$$C^{-1}|z-y|^{2-N} \leq G(z, y) \leq C|z-y|^{2-N} \quad \forall z, y \in A.$$

Hence, the Hölder inequality with $p > N/2$ and $p' = p/(p-1)$ yields

$$\begin{aligned}\int_A G(z, y)|V(z)|\mathrm{d}z &\leq C\left(\int_A |y-z|^{(2-N)p'}\mathrm{d}z\right)^{\frac{1}{p'}}\left(\int_A |V|^p\mathrm{d}z\right)^{\frac{1}{p}} \\ &\leq C(K, p, N)\|V\|_p \quad \forall y \in A.\end{aligned} \tag{4.11}$$

Hence, by substituting (4.11) into (4.10) we obtain

$$I_1^\delta \leq C k_P^M(x, y, t) \quad \forall x, y \in A \text{ and } t \geq \delta_0, \tag{4.12}$$

where the constant C depends on δ, A, p, N, and $\|V\|_p$.

Next, consider the second term of (4.8), namely,

$$I_2^\delta := \int_A \left(\int_\delta^t k_P^M(x, z, t-s) k_P^M(z, y, s)\mathrm{d}s\right)|V(z)|\mathrm{d}z.$$

Acting as for I_1^δ we obtain

$$\begin{aligned}I_2^\delta \leq& \int_A \left(\int_\delta^{t/2} k_P^M(x, x, t-s)^{\frac{1}{2}} k_P^M(z, z, t-s)^{\frac{1}{2}} k_P^M(z, y, s)\mathrm{d}s\right)|V(z)|\mathrm{d}z \\ &+ \int_A \left(\int_{t/2}^t (k_P^M(z, z, s))^{\frac{1}{2}} (k_P^M(y, y, s))^{\frac{1}{2}} k_P^M(x, z, t-s)\mathrm{d}s\right)|V(z)|\mathrm{d}z \\ \leq& C\left(k_P^M\left(x, x, \frac{t}{2}\right)\right)^{\frac{1}{2}}\left(k_P^M\left(z, z, \frac{t}{2}\right)\right)^{\frac{1}{2}} \int_A G(z, y)|V(z)|\mathrm{d}z \\ &+ C\left(k_P^M\left(z, z, \frac{t}{2}\right)\right)^{\frac{1}{2}}\left(k_P^M\left(y, y, \frac{t}{2}\right)\right)^{\frac{1}{2}} \int_A G(x, z)|V(z)|\mathrm{d}z \\ \leq& C k_P^M(x, y, t) \int_A (G(z, y) + G(x, z))|V(z)|\mathrm{d}z.\end{aligned}$$

In light of (4.11), we obtain

$$I_2^\delta \leq C k_P^M(x, y, t) \quad \forall x, y \in A \text{ and } t \geq \delta_0. \tag{4.13}$$

Hence, by adding estimates (4.12) and (4.13) we obtain

$$S(V, x, y, t) \le C \quad \forall x, y \in A \text{ and } t \ge \delta_0,$$

where the constant C depends on $\delta, A, p, N,$ and $\|V\|_p$.

Step 2. In this step we use our assumption that the diameter of $K := \operatorname{supp} V$ is 'small enough', and estimate $S(V, x, y, t)$ for $t < \delta_0$ and $x, y \in A$, where A is a 'small' bounded domain with a smooth boundary containing K. We use the short time behaviour of the heat kernel (see Lemma 4.2).

Denote by $g(x, y, t)$ the Gauss–Weierstrass type kernel

$$g(x, y, t) := \Big(\frac{1}{4\pi t}\Big)^{\frac{N}{2}} \exp\Big(-\frac{d(x, y)^2}{4t}\Big). \tag{4.14}$$

Due to our assumptions on the smallness of K and the smoothness of P and M, Lemma 4.2 implies that there exist $\delta_0 > 0$ and $C > 0$ such that

$$C^{-1} g(x, y, t) \le k_P^M(x, y, t) \le C g(x, y, t) \quad \forall x, y \in A \text{ and } t < \delta_0. \tag{4.15}$$

Note that

$$g(x, y, t)^p = g\Big(x, y, \frac{t}{p}\Big)(4\pi t)^{\frac{(1-p)N}{2}}(p)^{-\frac{N}{2}} \quad \forall x, y \in M \text{ and } t > 0. \tag{4.16}$$

Following [7], and using (4.15), and (4.16) for $x, y \in A$ and $0 < t < \delta_0$, we obtain

$$\begin{aligned} S(V, x, y, t) &:= \int_0^t \int_A \frac{k_P^M(x, z, t-s) k_P^M(z, y, s)}{k_P^M(x, y, t)} |V(z)| \mathrm{d}z \mathrm{d}s \\ &\le \int_0^t \int_A \frac{[(g(x, z, t-s))^{p'}(g(z, y, s))^{p'}]^{1/p'} |V(z)|}{g(x, y, t)} \mathrm{d}z \mathrm{d}s \\ &\le C \int_0^t \Big[\frac{s(t-s)}{t}\Big]^{-\frac{N}{2p}} \int_A \frac{[g(x, z, \frac{t-s}{p'}) g(z, y, \frac{s}{p'})]^{1/p'} |V(z)| \mathrm{d}z}{g(x, y, \frac{t}{p'})^{1/p'}} \mathrm{d}s. \end{aligned}$$

Consequently, the Hölder inequality, (4.15), the Chapman–Kolmogorov equation, and our assumption that $p > N/2$ imply that for all $x, y \in A$ and $t < \delta_0$

$$\begin{aligned} S(V, x, y, t) &\le C||V||_p \int_0^t \Big[\frac{s(t-s)}{t}\Big]^{-\frac{N}{2p}} \frac{[\int_A g(x, z, \frac{t-s}{p'}) g(z, y, \frac{s}{p'}) \mathrm{d}z]^{1/p'}}{g(x, y, \frac{t}{p'})^{1/p'}} \mathrm{d}s \\ &\le C||V||_p \int_0^t \Big[\frac{s(t-s)}{t}\Big]^{-\frac{N}{2p}} \frac{[\int_A k_P^M(x, z, \frac{t-s}{p'}) k_P^M(z, y, \frac{s}{p'}) \mathrm{d}z]^{1/p'}}{k_P^M(x, y, \frac{t}{p'})^{1/p'}} \mathrm{d}s \\ &\le C||V||_p \int_0^t \Big[\frac{s(t-s)}{t}\Big]^{-\frac{N}{2p}} \mathrm{d}s = C||V||_p \, t^{1-\frac{N}{2p}} \int_0^1 \sigma^{-\frac{N}{2p}} (1-\sigma)^{-\frac{N}{2p}} \mathrm{d}\sigma \\ &\le C||V||_p \, B\Big(1 - \frac{N}{2p}, 1 - \frac{N}{2p}\Big) t^{1-\frac{N}{2p}} \le C, \end{aligned}$$

where B denotes the beta function.

Step 3. Steps 1 and 2 imply that the $3k$-inequality holds for all $x, y \in A$ and all $t > 0$. Hence, Lemma 4.3 implies that the $3k$-inequality holds for all $x, y \in M$ and all $t > 0$. Consequently, Theorem 3.5 implies that there exists $\varepsilon_0 > 0$ such that $k^M_{P_\varepsilon} \asymp k^M_P$ for all $|\varepsilon| < \varepsilon_0$. □

Proof of Theorem 2.5. (1) Let V be the given potential with a compact support, and let $\{A_i\}_{i=1}^m$ be a finite open covering of $\operatorname{supp} V$ by smooth bounded sufficiently 'small' domains A_i such that k^M_P satisfies

$$C^{-1}g(x, y, t) \leq k^M_P(x, y, t) \leq Cg(x, y, t) \quad \forall x, y \in A_i,\ t < \delta_0 \text{ and } 1 \leq i \leq m,$$

where g is the Gauss–Weierstrass type kernel (4.14).

Let $\{\chi_i\}_{i=1}^m$ be a smooth partition of unity subordinated to this covering, and let $V_i(x) := \chi_i(x)V(x)$. Then $V(x) := \sum_{i=1}^m \chi_i(x)V(x) = \sum_{i=1}^m V_i(x)$.

Using Proposition 4.1 m-times with ε small enough, we obtain that

$$k^M_P \asymp k^M_{P-\varepsilon V_1} \asymp \cdots \asymp k^M_{P-\varepsilon(\sum_{i=1}^{m-1} V_i)} \asymp k^M_{P_\varepsilon}.$$

(2) The proof follows immediately from assertion (1) and Lemma 3.7.

(3) Since $P-V$ is a subcritical operator with a heat kernel satisfying the doubling condition (2.9), we may apply part (1) of the theorem to the operator $P - V$ to conclude that there exists some $\tilde{\varepsilon}_0$ such that $k^M_{P-V} := k^M_{P_1} \asymp k^M_{P_{(1-\varepsilon)}}$ for all $|\varepsilon| < \tilde{\varepsilon}_0$ holds true. Therefore, there exist α and β such that the hypotheses of Lemma 3.9 are satisfied, and hence $k^M_P \asymp k^M_{P-V}$.

(4) Proposition 3.4, Step 1 of the proof of Proposition 4.1, and Lemma 4.3 imply the $3k$-inequality. Hence, by Theorem 3.5 there exists $\varepsilon_0 > 0$ such that $k^M_{P_\varepsilon} \asymp k^M_P$ for all $|\varepsilon| < \varepsilon_0$. Consequently, assertions (2) and (3) for a bounded compactly supported potential V follow exactly as above. □

Conversely, it turns out that if $V \geq 0$, and $k^M_P \asymp k^M_{P+V}$, then the $3k$-inequality holds true. Indeed

Proposition 4.4. *Let $V \geq 0$ and $k^M_P \asymp k^M_{P+V}$. Then the heat kernel k^M_P satisfies the $3k$-inequality* (3.2).

Proof. Since $k^M_P \asymp k^M_{P+V}$, part (3) of Theorem 3.5 implies that k^M_{P+V} satisfies the resolvent equation

$$k^M_{P+V}(x, y, t) = k^M_P(x, y, t) - \int_0^t \int_M k^M_P(x, z, t - s)V(z)k^M_{P+V}(z, y, s)\mathrm{d}z\mathrm{d}s.$$

Hence,

$$\int_0^t \int_M k_P^M(x,z,t-s)V(z)k_P^M(z,y,s)\mathrm{d}z\mathrm{d}s$$
$$\leq C\int_0^t \int_M k_P^M(x,z,t-s)V(z)k_{P+V}^M(z,y,s)\mathrm{d}z\mathrm{d}s \leq Ck_P^M(x,y,t)$$

for all $x, y \in M$ and $t > 0$. □

5 Small perturbations and the proof of Theorem 2.6

In the present section we introduce the class of small perturbations (see Definition 5.1), and prove Theorem 2.6 that extends Theorem 2.5 from the class of compactly supported perturbations to the class of small perturbations. In the context of Green functions the notion of small perturbations was introduced in [33] and was then extended to the notion of semismall perturbations by M. Murata in [28] (see [25, 28, 35] and references therein for some applications). Similarly to the elliptic case, we study here the properties of small perturbations with respect to the heat kernel k_P^M.

Let $\{M_n\}_{n=0}^{\infty}$ be an exhaustion of M as in the proof of Theorem 3.5, and denote $M_n^* := M \setminus \overline{M_n}$. Let V be a given potential, and $\{\Phi_n\}_{n=0}^{\infty}$ be a sequence of smooth cutoff functions subordinated to the exhaustion $\{M_n\}$ satisfying

$$\Phi_n(x) = \begin{cases} 1 & \text{if } x \in M_n, \\ 0 & \text{if } x \in M_{n+1}^*, \end{cases}$$

and $0 \leq \Phi_n \leq 1$. Set $V_n(x) := \Phi_n(x)V(x)$ and $W_n(x) := V(x) - V_n(x)$.

Definition 5.1. We say that V is a **small** (resp., **semismall**) **perturbation** with respect to the heat kernel k_P^M if

$$\lim_{n\to\infty}\left\{\sup_{\substack{x,y\in M_n^*\\ t>0}} \int_0^t \int_{M_n^*} \frac{k_P^M(x,z,t-s)|V(z)|k_P^M(z,y,s)}{k_P^M(x,y,t)}\mathrm{d}z\mathrm{d}s\right\} = 0 \tag{5.1}$$

(resp.,

$$\lim_{n\to\infty}\left\{\sup_{\substack{y\in M_n^*\\ t>0}} \int_0^t \int_{M_n^*} \frac{k_P^M(x_0,z,t-s)|V(z)|k_P^M(z,y,s)}{k_P^M(x_0,y,t)}\mathrm{d}z\mathrm{d}s\right\} = 0, \tag{5.2}$$

where x_0 is a fixed reference point in M).

Clearly, if V is a small perturbation with respect to k_P^M, then it is also a semismall perturbation with respect to k_P^M (see Subsection 7.3 for further discussions).

Example 5.2. Suppose that P is a subcritical operator in M. Then a real valued function $V \in L^p(M)$, $p > \frac{N}{2}$ with **compact support** is a small perturbation of P with respect to k_P^M.

Example 5.3. Let $P := -\Delta$ in $\mathbb{R}^N$, $N \geq 3$ and suppose that $V \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, where $q < \frac{N}{2} < p$. It follows from [7] that V is a small perturbation with respect to the Gauss–Weierstrass heat kernel (1.3). For example, for $t > 2$ and $x, y \in \mathbb{R}^N$ we have

$$\int_0^t \int_M \frac{k_P^M(x,z,t-s)|W_n(z)|k_P^M(z,y,s)}{k_P^M(x,y,t)} \mathrm{d}z\mathrm{d}s \\ \leq c_1 \|W_n\|_p \int_0^1 s^{-N/(2p)}\mathrm{d}s + c_2 \|W_n\|_q \int_1^\infty s^{-N/(2q)}\mathrm{d}s,$$

and the dominated convergence theorem implies that V satisfies (5.1).

It turns out that under some further assumptions, if V is a small perturbation with respect to k_P^M, then k_P^M satisfies the $3k$-inequality with respect to V. We have.

Lemma 5.4. *Suppose that the Riemannian manifold $(M, \mathfrak{g})$, the operator P, and its kernel k_P^M satisfy the assumptions of Theorem 2.5. Let $V \in L^p_{\mathrm{loc}}(M)$ be a small perturbation with respect to the heat kernel k_P^M, where $p > N/2$. Then k_P^M satisfies the $3k$-inequality (3.2) with respect to V.*

Proof. Theorem 2.5 and Lemma 3.7 imply that $k_P^M \asymp k_{P+|V_{n+1}|}^M$ for any $n \in \mathbb{N}$. Therefore, by Proposition 4.4, for each $n \in \mathbb{N}$ there exists $C_n > 0$ such that

$$\int_0^t \int_{M_n} k_P^M(x,z,t-s)|V(z)|k_P^M(z,y,s)\mathrm{d}z\mathrm{d}s \\ \leq \int_0^t \int_M k_P^M(x,z,t-s)|V_{n+1}(z)|k_P^M(z,y,s)\mathrm{d}z\mathrm{d}s \leq C_n k_P^M(x,y,t) \tag{5.3}$$

for all $x, y \in M$ and $t > 0$.

On the other hand, by the definition of a small perturbation we have

$$\int_0^t \int_{M_n^*} k_P^M(x,z,t-s)|V(z)|k_P^M(z,y,s)\mathrm{d}z\mathrm{d}s \leq C k_P^M(x,y,t) \tag{5.4}$$

for any $x, y \in M_n^*$ and $t > 0$. So, by adding (5.3) and (5.4), we see that the $3k$-inequality (3.2) holds true for $x, y \in M_n^*$ and $t > 0$.

Fix $y \in M_n^*$ and for $x \in M_n$ define

$$U_y(x,t) := \int_0^t \int_{M_{n+1}^*} k_P^M(x,z,t-s)|W_{n+1}(z)|k_P^M(z,y,s)\mathrm{d}z\mathrm{d}s.$$

By (5.4) and continuity we have

$$U_y(x,t) \le \varepsilon k_P^M(x,y,t) \quad \forall x \in \partial M_n,\ t > 0.$$

On the other hand,

$$U_y(x,0) = k_P^M(x,y,0) = 0 \quad \forall x \in M_n.$$

Moreover, U_y satisfies the equation

$$\frac{\partial}{\partial t}U_y + PU_y = |W_{n+1}(x)| k_P^M(x,y,t), \quad x \in M,\ t > 0.$$

In particular,

$$\frac{\partial}{\partial t}U_y + PU_y = 0$$

for all $x \in M_n$. Furthermore, the heat kernel $k_P^M(x,y,t)$, as a function x, also satisfies the equation

$$\frac{\partial}{\partial t}k_P^M + Pk_P^M = 0$$

for all $x \in M_n$. The generalized maximum principle in M_n implies that for any $y \in M_n^*$

$$U_y(x,t) \le \varepsilon k_P^M(x,y,t) \quad \forall x \in M_n,\ t > 0. \tag{5.5}$$

Hence, taking into account (5.3) it follows that the $3k$-inequality (3.2) holds true for $x \in M_n$ and $y \in M_n^*$. The same argument shows that the $3k$-inequality holds true for $y \in M_n$ and $x \in M_n^*$.

Suppose that $x, y \in M_n$. Then a similar comparison argument in M_n shows that

$$\int_0^t \int_{M_{n+1}^*} k_P^M(x,z,t-s)|W_{n+1}(z)|k_P^M(z,y,s)\mathrm{d}z\mathrm{d}s \le Ck_P^M(x,y,t) \tag{5.6}$$

for all $x, y \in M_n$ and $t > 0$. Once again, taking into account (5.3) it follows that the $3k$-inequality (3.2) holds true also for $x \in M_n$ and $y \in M_n$. Thus, the lemma is proved. □

In light of Part (3) of Theorem 3.5 we obtain

Corollary 5.5. *Suppose that V is a (semi)small perturbation with respect to P in M. If $k_{P_\varepsilon}^M \asymp k_P^M$ for some $\varepsilon \in \mathbb{R}$, then the heat kernel $k_{P_\varepsilon}^M$ satisfies the resolvent equations* (3.5).

Next, we prove Theorem 2.6.

Proof of Theorem 2.6. Part (1) follows from Lemma 5.4 and Theorem 3.5.

(2) The proof follows immediately from part (1) and Lemma 3.7.

(3) Since W_n is a small perturbation of k_P^M, it follows from part (1) that for n large enough the $3k$-inequality holds true with respect to W_n with a constant $C < 1$, and therefore

$$k_P^M \asymp k_{P-W_n}^M. \tag{5.7}$$

On the other hand, since $P - V = (P - W_n) - V_n$ and V_n has compact support in M, it follows from (5.7) and part (3) of Theorem 2.5 that

$$k_{P-V}^M \asymp k_{P-W_n}^M \asymp k_P^M. \qquad \square$$

Remark 5.6. We note that if the heat kernels k_P^M and k_{P-V}^M are semi-equivalent for each fixed $x \in M$, then by the Davies–Harnack inequality, and either by the short-time asymptotics of the heat kernels or under the additional assumption that $V \in L^\infty(M)$, we get the equivalence of this heat kernels in $K \times M \times (0, \infty)$ for any $K \Subset M$.

Corollary 5.7. *Suppose that the operator P and the potential V satisfy the assumptions of Theorem 2.6, and that $k_{P_\varepsilon}^M$ satisfies* (2.9) *for all $\varepsilon \in S_+$. Then*

$$S_+ = \{\varepsilon \in \mathbb{R} \mid k_{P_\varepsilon}^M \asymp k_P^M\}.$$

6 Stability of the parabolic Martin boundary

In this section we study the behaviour of $\mathcal{C}_L(D)$, the cone of all nonnegative solutions of the parabolic equation

$$Lu := \partial_t u + Pu = 0 \quad \text{in } D := M \times (a, b) \tag{6.1}$$

under small perturbations, where P is of the form either (2.1) or (2.2) and $-\infty \le a < b \le \infty$.

Our discussion is along the lines of the study of the elliptic case in [28, 32], but the parabolic case needs special care since the cone $\mathcal{C}_L(D)$ does not have a compact base. Before formulating the main result of the present section, we introduce some useful definitions and notations.

Definition 6.1. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two convex cones embedded in topological spaces $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are said to be **affine equivalent** (and we denote it by $\mathcal{C}_1 \cong \mathcal{C}_2$) if there exists homeomorphism $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ which preserves convex combinations. Such a Φ is called an **affine homeomorphism**.

Particularly, we are interested in the question whether $k_P^M \asymp k_{P-V}^M$ implies $\mathcal{C}_L(D) \cong \mathcal{C}_{L-V}(D)$, where V is a small perturbation.

We recall some of the basic facts concerning the parabolic Martin boundary and the parabolic Martin representation theorem (for more details see [14, 29]).

Let x_0 be a fixed reference point in M. Consider a nonnegative continuous function ψ on M such that $\psi(x) = 1$ on $B(x_0, r)$ and $\psi(x) = 0$ outside $B(x_0, 2r)$, for some $r > 0$ small enough. Also choose a nonnegative continuous h on (a, b) such that $h(t) = 0$ on $(a, a_1]$ and $h(t) > 0$ on (a_1, b), where $a < a_1 < b$.

Define a measure ρ on D by $\mathrm{d}\rho(x,t) := \psi(x)h(t)\mathrm{d}x\mathrm{d}t$. For any nonnegative measurable function u on D, we define

$$\rho(u) := \int_a^b \int_M u(x,t)\mathrm{d}\rho(x,t). \tag{6.2}$$

In the literature $\mathrm{d}\rho$ is known as a **reference measure**.

Define

$$\mathcal{C}_{\rho,L}(D) := \{u \in \mathcal{C}_L(D) \mid \rho(u) < \infty\}, \quad \mathcal{C}^1_{\rho,L}(D) := \{u \in \mathcal{C}_L(D) \mid \rho(u) \le 1\}.$$

Clearly, for every $u \in \mathcal{C}_L(D)$, there exists h as defined above such that $\rho(u) < \infty$. Hence, $\mathcal{C}_L(D) = \bigcup_\rho \mathcal{C}_{\rho,L}(D)$. Moreover, the parabolic Harnack inequality implies that if $u \in \mathcal{C}_{\rho,L}(D)$ and $\rho(u) = 0$, then $u = 0$. Recall that a nonnegative solution $u \in \mathcal{C}_L(D)$ is said to be minimal if for any nonnegative solution $v \in \mathcal{C}_L(D)$ such that $v \le u$, there exists a nonnegative constant c such that $v = cu$. Denote by $\mathcal{C}^m_L(D)$ the set of all minimal solutions in $\mathcal{C}_L(D)$. By the Harnack principle, $\mathcal{C}^1_{\rho,L}(D)$ is a compact convex set in the compact-open topology, and by the Choquet theorem, the set of all extreme points of $\mathcal{C}^1_{\rho,L}(D)$ is equal to the union of the zero function and $\mathcal{C}^m_L(D) \cap \{u \in \mathcal{C}^1_{\rho,L}(D) \mid \rho(u) = 1\}$.

We now introduce the Martin kernels. Let

$$k^M_P((x,t),(y,s)) := \begin{cases} k^M_P(x,y,t-s), & a < s < t < b, \text{ and } x, y \in M, \\ 0, & a < t \le s < b, \text{ and } x, y \in M. \end{cases}$$

Fix a reference measure ρ, and define $\mathcal{K}^\rho_P((x,t),(y,s))$ the (**parabolic**) **ρ-Martin kernel** on $D \times D$ by

$$\mathcal{K}^\rho_P((x,t),(y,s)) := \begin{cases} \dfrac{k^M_P((x,t),(y,s))}{\rho(k^M_P(\cdot\,,(y,s)))}, & a < s < t < b, \text{ and } x, y \in M, \\ 0, & a < t \le s < b, \text{ and } x, y \in M. \end{cases}$$

It follows that up to a homeomorphism, there exists a unique metrizable compactification D^ρ_L of D with the following properties (for details see [29, Section 2]):

(1) The function $\mathcal{K}^\rho_P$ has a continuous extension to $D \times D^\rho_L$ such that for each $(x,t) \in D$, the function $\mathcal{K}^\rho_P((x,t),\cdot)$ is finite valued and continuous on $D^\rho_L \setminus \{(x,t)\}$.

(2) For $\sigma_1, \sigma_2 \in D_L^\rho$ we have $\mathcal{K}_P^\rho(\cdot\,, \sigma_1) = \mathcal{K}_P^\rho(\cdot\,, \sigma_2)$ if and only if $\sigma_1 = \sigma_2$.

We write $\partial_L^\rho D := D_L^\rho \setminus D$, and we call it the **parabolic ρ-Martin boundary of D with respect to L and a reference measure ρ**. A sequence

$$\{Y_n\} := \{(y_n, \tau_n)\} \subset D \times D$$

is said to be **a fundamental sequence** if $\{Y_n\}$ has no accumulation point in $D\times D$, $Y_n \to \sigma \in \partial_L^\rho D$. In particular, $\mathcal{K}_P^\rho((x,t),(y_n,\tau_n)) \to \mathcal{K}_P^\rho((x,t),\sigma)$ locally uniformly in D, and $\mathcal{K}_P^\rho(\cdot\,,\sigma)$ is a nonnegative solution to (6.1). Note that by Fatou's Lemma, we have $\rho(\mathcal{K}_P^\rho(\cdot\,,\sigma)) \leq 1$. So, $\mathcal{K}_P^\rho(\cdot\,,\sigma) \in \mathcal{C}^1_{\rho,L}(D)$ for any $\sigma \in \partial_L^\rho D$.

We recall the parabolic Martin representation theorem. Define

$$\partial_{L,1}^{\rho,m} D := \{\sigma \in \partial_L^\rho D \mid \mathcal{K}_P^\rho(\cdot\,,\sigma) \in \mathcal{C}_L^m(D),\, \rho(\mathcal{K}_P^\rho(\cdot\,,\sigma)) = 1\}. \tag{6.3}$$

We call $\partial_{L,1}^{\rho,m} D$ the **(nontrivial) minimal parabolic ρ-Martin boundary**.

The parabolic Martin representation theorem states: $u \in \mathcal{C}_{\rho,L}(D)$, if and only if there exists a unique Borel measure λ on $\partial_L^\rho D$ supported on $\partial_{L,1}^{\rho,m} D$, such that

$$u(x,t) = \int_{\partial_L^\rho D} \mathcal{K}_P^\rho((x,t),\sigma)\mathrm{d}\lambda(\sigma), \tag{6.4}$$

and $\rho(u) = \lambda(\partial_{L,1}^\rho D)$.

Next, we formulate our main result of the present section.

Theorem 6.2. *Let P and $P - V$ be two subcritical operators such that V is a small perturbation with respect to the heat kernel k_P^M, and $k_P^M \asymp k_{P-V}^M$ in $M \times M \times (0,\infty)$ with an equivalence constant C.*

Then there exists an affine homeomorphism $\mathcal{T} : \mathcal{C}_L(D) \to \mathcal{C}_{L-V}(D)$ such that

$$(\mathcal{T}u)(x,t) := u(x,t) + \int_0^t \int_M k_{P-V}^M(x,z,t-s)V(z)u(z,s)\mathrm{d}z\mathrm{d}s \quad \forall u \in \mathcal{C}_L(D). \tag{6.5}$$

Moreover, for each $u \in \mathcal{C}_L(D)$, we have $\mathcal{T}u \asymp u$ with equivalence constant C^2.

Remark 6.3. In Theorem 6.2 we do not assume that P is symmetric.

Remark 6.4. For the sake of brevity we present only the proof in the case $a = 0$ and $b = \infty$. So, we prove the case $D = M \times (0,\infty)$. It will be evident from the proof that all other cases follow along similar lines (see Remark 6.8).

Remark 6.5. Let $D = M \times (0,\infty)$. Then any fundamental sequence $\{(y_n,\tau_n)\}$ converging to $\sigma \in \partial_L^\rho D$ satisfies (up to a subsequence) $\tau_n \to T$, where $0 \leq T \leq \infty$.

Therefore, if $T = \infty$, i.e., $\tau_n \to \infty$, then for a fixed $x \in M$ and $t > 0$

$$k_P^M(x, y_n, t-\tau_n) = k_{P-V}^M(x, y_n, t-\tau_n) = 0$$

for n large enough, and therefore

$$\lim_{n\to\infty} \mathcal{K}_P^\rho((x,t),(y_n,\tau_n)) = \lim_{n\to\infty} \mathcal{K}_{P-V}^\rho((x,t),(y_n,\tau_n)) = 0.$$

On the other hand, if $\tau_n \to T$, where $0 < T < \infty$, then the Martin kernel $\mathcal{K}_P^\rho((x,t),\sigma)$ satisfies $\mathcal{K}_P^\rho((x,t),\sigma) = 0$ for all $t \le T$. Hence, if the uniqueness of the positive Cauchy problem holds true, then $\mathcal{K}_P^\rho(\cdot,\sigma) = 0$ in D.

The proof of Theorem 6.2 hinges on the following key proposition.

Proposition 6.6. *Let P and $\tilde{P}$ be two subcritical operators such that $k_P^M \asymp k_{\tilde{P}}^M$ in $M \times M \times (0,\infty)$. Then there exists a homeomorphism $\alpha_\rho : \partial_{L,1}^{\rho,m} D \to \partial_{\tilde{L},1}^{\rho,m} D$ and $C > 0$ such that*

$$C^{-1}\mathcal{K}_P^\rho((x,t),\sigma) \le \mathcal{K}_{\tilde{P}}^\rho((x,t),\alpha_\rho(\sigma)) \le C\mathcal{K}_P^\rho((x,t),\sigma) \tag{6.6}$$

for every $\sigma \in \partial_{L,1}^{\rho,m} D$ and $(x,t) \in M \times (0,\infty)$.

For the proof of the above proposition we need the following lemma.

Lemma 6.7. *Suppose that $k_P^M \asymp k_{\tilde{P}}^M$. Then for every $u \in C^1_{\rho,L}(D)$ there exists $\tilde{u} \in C^1_{\rho,\tilde{L}}(D)$ that satisfies*

$$C^{-2}u(x,t) \le \tilde{u}(x,t) \le C^2 u(x,t), \quad (x,t) \in M \times (0,\infty)$$

where C is the equivalent constant for k_P^M and $k_{\tilde{P}}^M$.

The proof of Lemma 6.7 is similar to the proof of [32, Lemma 2.4], and therefore it is omitted.

Proof of Proposition 6.6. Let $\sigma \in \partial_{L,1}^{\rho,m} D$, and for $k = 1,2$, let $\{(y_n^k,\tau_n^k)\}$ be two fundamental subsequences of a fundamental sequence $\{(y_n,\tau_n)\} \subset D$ such that

$$(y_n^k,\tau_n^k) \to \sigma \quad \text{in } D_L^\rho, \qquad \text{and} \qquad (y_n^k,\tau_n^k) \to \tilde{\sigma}_k \quad \text{in } D_{\tilde{L}}^\rho.$$

We claim that $\tilde{\sigma}_1 = \tilde{\sigma}_2$, and $\tilde{\sigma}_1 \in \partial_{\tilde{L},1}^{\rho,m} D$. In particular, the mapping $\sigma \mapsto \tilde{\sigma}_1$ is a well defined mapping $\alpha_\rho : \partial_{L,1}^{\rho,m} D \to \partial_{\tilde{L},1}^{\rho,m} D$, defined by $\alpha_\rho(\sigma) := \tilde{\sigma}$, if $(y_n,\tau_n) \to \sigma \in \partial_{L,1}^{\rho,m} D$, and $(y_n,\tau_n) \to \tilde{\sigma} \in \partial_{\tilde{L},1}^{\rho,m} D$.

Indeed, from our assumption that $k_P^M \asymp k_{\tilde{P}}^M$ it follows that

$$C^{-2}\mathcal{K}_P^\rho((x,t),\sigma) \le \mathcal{K}_{\tilde{P}}^\rho((x,t),\tilde{\sigma}_k) \le C^2\mathcal{K}_P^\rho((x,t),\sigma) \quad \forall (x,t) \in D, \tag{6.7}$$

where C is the equivalence constant. Using (6.7), we obtain

$$\mathcal{K}_{\tilde{P}}^\rho((x,t),\tilde{\sigma}_1) - C^{-4}\mathcal{K}_{\tilde{P}}^\rho((x,t),\tilde{\sigma}_2) \ge 0.$$

We now use the maximal ε trick. Define

$$\varepsilon_0 := \max\{\varepsilon > 0 : \mathcal{K}^\rho_{\tilde{P}}((x,t),\tilde{\sigma}_1) - \varepsilon\mathcal{K}^\rho_{\tilde{P}}((x,t),\tilde{\sigma}_2) \geq 0\},$$

and let

$$\tilde{v_\rho}(x,t) := \mathcal{K}^\rho_{\tilde{P}}((x,t),\tilde{\sigma}_1) - \varepsilon_0\mathcal{K}^\rho_{\tilde{P}}((x,t),\tilde{\sigma}_2).$$

Clearly $\tilde{v_\rho} \geq 0$, and we may assume that $\rho(\tilde{v_\rho}) > 0$, since otherwise, $\tilde{\sigma}_1 = \tilde{\sigma}_2$. Lemma 6.7 implies that there exists $u \in C^1_{\rho,L}(D)$ such that

$$C^{-2}u(x,t) \leq \frac{\tilde{v_\rho}(x,t)}{\rho(\tilde{v_\rho})} \leq C^2 u(x,t). \tag{6.8}$$

Therefore, $0 \leq u(x,t) \leq C^4(\rho(\tilde{v_\rho}))^{-1}\mathcal{K}^\rho_P((x,t),\sigma)$. Since $\mathcal{K}^\rho_P((x,t),\sigma)$ is a minimal solution, we have $u(x,t) = \mu\mathcal{K}^\rho_P((x,t),\sigma)$ for some $\mu > 0$. By substituting this in (6.8), we obtain

$$C^{-4}\mu\rho(\tilde{v_\rho})\mathcal{K}^\rho_{\tilde{P}}((x,t),\tilde{\sigma}_2) \leq C^{-2}\mu\rho(\tilde{v_\rho})\mathcal{K}^\rho_P((x,t),\sigma) \leq \tilde{v_\rho}(x,t).$$

Thus, by letting $\mu_0 := C^{-4}\mu\rho(\tilde{v_\rho}) > 0$, we obtain

$$0 \leq \tilde{v_\rho}(x,t) - \mu_0\mathcal{K}^\rho_{\tilde{P}}((x,t),\tilde{\sigma}_2) = \mathcal{K}^\rho_{\tilde{P}}((x,t),\tilde{\sigma}_1) - (\varepsilon_0 + \mu_0)\mathcal{K}^\rho_{\tilde{P}}((x,t),\tilde{\sigma}_2),$$

which contradicts the definition of ε_0. Hence, $\tilde{\sigma}_1 = \tilde{\sigma}_2$, and therefore, α_ρ is well defined. Moreover, (6.7) and Lemma 6.7, and the maximal ε trick imply that $\tilde{\sigma}_1 \in \partial^{\rho,m}_{\tilde{L},1}D$, so $\alpha_\rho : \partial^{\rho,m}_{L,1}D \to \partial^{\rho,m}_{\tilde{L},1}D$. By similar arguments, α_ρ is injective, surjective and homeomorphism. □

We can now prove Theorem 6.2.

Proof of Theorem 6.2. Let $\{M_j\}_{j=0}^\infty$ be an exhaustion of M, and denote $M_j^* := M \setminus \overline{M_j}$. Let $Y_n = \{(y_n,\tau_n)\}$ be a fundamental sequence converging to $\sigma \in \partial^{\rho,m}_{L,1}D$, and $\tau_n \to T$.

Fix $\varepsilon > 0$, and x in M and $t > 0$. Since V is a small perturbation with respect to k^M_P, and since k^M_P is equivalent to k^M_{P-V}, it follows from (5.5) that there exists $j(\varepsilon)$ and $n(\varepsilon)$ such that for $j > j(\varepsilon)$ and $n > n(\varepsilon)$, we have $y_n \in M^*_{j(\varepsilon)}$, and for $t > \tau_n$ the following inequality holds:

$$\begin{aligned} &\int_{\tau_n}^t \int_{M_j^*} \frac{k^M_{P-V}(x,z,t-s)|V(z)|k^M_P(z,y_n,s-\tau_n)}{k^M_P(x,y_n,t-\tau_n)}\,\mathrm{d}z\mathrm{d}s \\ &\quad = \int_0^{t-\tau_n}\int_{M_j^*} \frac{k^M_{P-V}(x,z,t-\tau_n-\tilde{s})|V(z)|k^M_P(z,y_n,\tilde{s})}{k^M_P(x,y_n,t-\tau_n)}\,\mathrm{d}z\,\mathrm{d}\tilde{s} < \varepsilon. \end{aligned} \tag{6.9}$$

Since

$$\lim_{n\to\infty} \frac{k^M_P(x,y_n,t-\tau_n)}{\rho(k^M_P(\cdot\,,(y_n,\tau_n)))} = \mathcal{K}^\rho_P((x,t),\sigma),$$

it follows that

$$\int_{\tau_n}^{t}\int_{M_j^*}\frac{k_{P-V}^M(x,z,t-s)|V(z)|k_P^M(z,y_n,s-\tau_n)}{\rho(k_P^M(\cdot\,,(y_n,\tau_n)))}\mathrm{d}z\mathrm{d}s \le \varepsilon M.$$

Hence, the sequence of functions

$$\Big\{f_n(z,s) := k_{P-V}^M(x,z,t-s)V(z)\frac{k_P^M(z,y_n,s-\tau_n)}{\rho(k_P^M(\cdot\,,(y_n,\tau_n)))}\Big\}$$

is uniformly integrable and tight, and

$$\lim_{n\to\infty} f_n(z,s) = k_{P-V}^M(x,z,t-s)V(z)\mathcal{K}_P^\rho((z,s),\sigma) \quad \text{locally uniformly.}$$

In light of Corollary 5.5, the resolvent equation implies

$$\begin{aligned}\frac{k_{P-V}^M(x,y_n,t-\tau_n)}{\rho(k_P^M(\cdot\,,(y_n,\tau_n)))} =& \frac{k_P^M(x,y_n,t-\tau_n)}{\rho(k_P^M(\cdot\,,(y_n,\tau_n)))}\\ &+\int_{\tau_n}^{t}\int_M \frac{k_{P-V}^M(x,z,t-s)V(z)k_P^M(z,y_n,s-\tau_n)}{\rho(k_P^M(\cdot\,,(y_n,\tau_n)))}\mathrm{d}z\mathrm{d}s.\end{aligned} \tag{6.10}$$

Hence, by the Vitali convergence theorem ([38, p. 98]) we may pass to the limit to obtain

$$\begin{aligned}&\lim_{n\to\infty}\frac{k_{P-V}^M(x,y_n,t-\tau_n)}{\rho(k_P^M(\cdot\,,(y_n,\tau_n)))}\\ &\qquad = \mathcal{K}_P^\rho((x,t),\sigma)+\int_T^t\int_M k_{P-V}^M(x,z,t-s)V(z)\mathcal{K}_P^\rho((z,s),\sigma)\mathrm{d}z\mathrm{d}s.\end{aligned} \tag{6.11}$$

Furthermore, since k_P^M is equivalent to $\asymp k_{P-V}^M$, we may define (up to a subsequence)

$$\mathcal{K}_{P-V}^\rho((x,t),\alpha_\rho(\sigma)) := \lim_{n\to\infty}\frac{k_{P-V}^M(x,y_n,t-\tau_n)}{\rho(k_{P-V}^M(\cdot\,,(y_n,\tau_n))}\in \partial_{L-V,1}^{\rho,m}D,$$
$$\text{and } \lambda_\rho(\sigma) := \lim_{n\to\infty}\frac{\rho(k_{P-V}^M(\cdot\,,(y_n,\tau_n))}{\rho(k_P^M(\cdot\,,(y_n,\tau_n))},$$

where $C^{-1}\le\lambda_\rho(\sigma)\le C$. Moreover, Proposition 6.6 implies that $\alpha_\rho(\sigma)$ is well defined, and consequently, the sequence $\{(y_n,\tau_n)\}$ converges in D_{L-V}^ρ to the point $\alpha_\rho(\sigma)\in\partial_{L-V,1}^{\rho,m}D$. Therefore, $\lambda_\rho(\sigma)$ does not depend on the subsequence.

Consequently, the following resolvent equation for minimal Martin functions holds true:

$$\begin{aligned}&\lambda_\rho(\sigma)\mathcal{K}_{P-V}^\rho((x,t),\alpha_\rho(\sigma))\\ &\qquad =\mathcal{K}_P^\rho((x,t),\sigma)+\int_T^t\int_M k_{P-V}^M(x,z,t-s)V(z)\mathcal{K}_P^\rho((z,s),\sigma)\mathrm{d}z\mathrm{d}s.\end{aligned} \tag{6.12}$$

But since $\mathcal{K}^{\rho}_{P}((z,s),\sigma)=0$ for $0\le s\le T$, we have

$$\begin{aligned}\lambda_\rho(\sigma)\mathcal{K}^{\rho}_{P-V}((x,t),\alpha_\rho(\sigma))\\ = \mathcal{K}^{\rho}_{P}((x,t),\sigma)+\int_0^t\int_M k^M_{P-V}(x,z,t-s)V(z)\mathcal{K}^{\rho}_{P}((z,s),\sigma)\mathrm{d}z\mathrm{d}s.\end{aligned}\tag{6.13}$$

Define

$$\mathcal{T}_\rho : \{\mathcal{K}^{\rho}_{P}(\cdot\,,\sigma)\mid \sigma\in\partial^{\rho,m}_{L,1}D\}\to \mathcal{C}_{\rho,L-V}(D)$$

by

$$\mathcal{T}_\rho\mathcal{K}^{\rho}_{P}((x,t),\sigma):=\lambda_\rho(\sigma)\mathcal{K}^{\rho}_{P-V}((x,t),\alpha_\rho(\sigma)).$$

Extend $\mathcal{T}_\rho$ to an affine transformation (with a slight abuse of notation)

$$\mathcal{T}_\rho : \mathrm{Conv}(\{\mathcal{K}^{\rho}_{P}(\cdot\,,\sigma)\mid \sigma\in\partial^{\rho,m}_{L,1}D\})\to \mathcal{C}_{\rho,L-V}(D),$$

where $\mathrm{Conv}(A)$ is the convex hull of a set A. Then, using the parabolic Martin representation theorem and a standard continuity argument (follows from continuity of the Martin kernel $\mathcal{K}^{\rho}_{P}(\cdot\,,\sigma)$), we extend $\mathcal{T}_\rho$ to a continuous affine transformation $\mathcal{T}_\rho:\mathcal{C}_{\rho,L}(D)\to\mathcal{C}_{\rho,L-V}(D)$ given by

$$(\mathcal{T}_\rho u)(x,t):=u(x,t)+\int_0^t\int_M k^M_{P-V}(x,z,t-s)V(z)u(z,s)\mathrm{d}z\mathrm{d}s.\tag{6.14}$$

Recall that $\mathcal{C}_L(D)=\bigcup_\rho\mathcal{C}_{\rho,L}(D)$. Moreover, the mapping $\mathcal{T}_\rho$ given by (6.14) does not depend on ρ. Therefore, we may extend the family of transformations $\{\mathcal{T}_\rho\}_\rho$ to a continuous affine transformation $\mathcal{T}:\mathcal{C}_L(D)\to\mathcal{C}_{L-V}(D)$ by $\mathcal{T}u:=\mathcal{T}_\rho u$ for $u\in\mathcal{C}_{\rho,L}(D)$, so, we get (6.5).

Analogously, define $\mathcal{S}:\mathcal{C}_{L-V}(D)\to\mathcal{C}_L(D)$ by

$$(\mathcal{S}v\ )(x,t):=v\ (x,t)-\int_0^t\int_M k^M_{P}(x,z,t-s)V(z)v\ (z,s)\mathrm{d}z\mathrm{d}s.\tag{6.15}$$

We claim that $\mathcal{S}\mathcal{T}=\mathrm{Id}_{\mathcal{C}_L(D)}$ and $\mathcal{T}\mathcal{S}=\mathrm{Id}_{\mathcal{C}_{L-V}(D)}$, where Id_A is the identity map on the set A. We show that $\mathcal{S}\mathcal{T}=\mathrm{Id}_{\mathcal{C}_L(D)}$ and the second assertion follows similarly.

For $u\in\mathcal{C}_L(D)$ we have

$$\begin{aligned}(\mathcal{S}\mathcal{T}u)(x,t)&=\mathcal{S}\left(u(x,t)+\int_0^t\int_M k^M_{P-V}(x,y,t-\alpha)V(y)u(y,\alpha)\mathrm{d}y\mathrm{d}\alpha\right)\\ &=u(x,t)+\int_0^t\int_M k^M_{P-V}(x,y,t-\alpha)V(y)u(y,\alpha)\mathrm{d}y\mathrm{d}\alpha\\ &\quad-\int_0^t\int_M k^M_{P}(x,y,t-\alpha)V(y)u(y,\alpha)\mathrm{d}y\mathrm{d}\alpha\\ &\quad-\int_0^t\int_M k^M_{P}(x,y,t-\alpha)V(y)\\ &\qquad\times\left(\int_0^\alpha\int_M k^M_{P-V}(y,z,\alpha-s)V(z)u(z,s)\mathrm{d}z\mathrm{d}s\right)\mathrm{d}y\mathrm{d}\alpha.\end{aligned}$$

Using Fubini's theorem and the resolvent equation for the heat kernel k_{P-V}^M, we obtain

$$\begin{aligned}
&\int_0^t \int_M k_{P-V}^M(x,z,t-\alpha)V(z)u(z,\alpha)\mathrm{d}z\mathrm{d}\alpha \\
&= \int_0^t \int_M k_P^M(x,y,t-\alpha)V(y)u(y,\alpha)\mathrm{d}y\mathrm{d}\alpha \\
&\quad + \int_0^t \int_M k_P^M(x,y,t-\alpha)V(y)\left(\int_0^\alpha \int_M k_{P-V}^M(y,z,\alpha-s)V(z)u(z,s)\mathrm{d}z\mathrm{d}s\right)\mathrm{d}y\mathrm{d}\alpha.
\end{aligned}$$

Thus, $(\mathcal{S}\mathcal{T}u)(x,t) = u(x,t)$. □

Remark 6.8. In the general case, where $D = M \times (a,b)$, with $-\infty \le a < b \le \infty$, the transformations $\mathcal{T}$ and $\mathcal{S}$, given by (6.14) and (6.15) (with a replacing 0), are well-defined affine homeomorphisms even if $a = -\infty$ (thanks to the $3k$-inequality (see Lemma 5.4)).

7 Concluding remarks

This section consists of three subsections. In the first one, we briefly extend our results to a certain class of nonsymmetric operators, while in Subsection 7.2 we provide several examples to illustrate our results. Finally, in Subsection 7.3 we pose some open problems.

7.1 Quasi-symmetric heat kernels. The positive minimal heat kernel k_P^M is said to be **quasi-symmetric** if

$$k_P^M(x,y,t) \asymp k_P^M(y,x,t) \quad \forall x,y \in M,\ t > 0. \tag{7.1}$$

Remark 7.1. In [3] A. Ancona introduced the notion of quasi-symmetric operators (with respect to its Naïm kernel). Clearly, if the heat kernel k_P^M is quasi-symmetric, and the operator P is subcritical, then P is quasi-symmetric in the sense of Ancona.

Lemma 7.2. *Suppose that the heat kernel k_P^M is quasi-symmetric. Then there exists a constant $C > 0$ such that*

$$k_P^M(x,y,t) \le C\left(k_P^M(x,x,t)\right)^{\frac{1}{2}}\left(k_P^M(y,y,t)\right)^{\frac{1}{2}} \quad \forall x,y \in M,\ t > 0. \tag{7.2}$$

Proof. Using the Chapman–Kolmogorov equation and the Hölder inequality, we see that

$$\begin{aligned}
k_P^M(x,y,t) &= \int_M k_P^M\Big(x,z,\frac{t}{2}\Big)k_P^M\Big(z,y,\frac{t}{2}\Big)\mathrm{d}z \\
&\le \bigg(\int_M \Big(k_P^M\Big(x,z,\frac{t}{2}\Big)\Big)^2\mathrm{d}z\bigg)^{\frac{1}{2}}\bigg(\int_M \Big(k_P^M\Big(z,y,\frac{t}{2}\Big)\Big)^2\mathrm{d}z\bigg)^{\frac{1}{2}} \\
&\le C\bigg(\int_M k_P^M\Big(x,z,\frac{t}{2}\Big)k_P^M\Big(z,x,\frac{t}{2}\Big)\mathrm{d}z\bigg)^{\frac{1}{2}}\bigg(\int_M k_P^M\Big(y,z,\frac{t}{2}\Big)k_P^M(z,y,\frac{t}{2})\mathrm{d}z\bigg)^{\frac{1}{2}} \\
&= C(k_P^M(x,x,t))^{\frac{1}{2}}(k_P^M(y,y,t))^{\frac{1}{2}}.
\end{aligned}$$

□

Definition 7.3. The heat kernel k_P^M is said to be **quasi-monotone** at $x_0 \in M$ if for any $T > 0$ there exists $C := C(x_0, T) > 0$ such that

$$k_P^M(x_0, x_0, t_2) \le C k_P^M(x_0, x_0, t_1), \quad \forall\, t_2 \ge t_1 > T.$$

Clearly, the heat kernel of a symmetric operator is quasi-symmetric and also quasi-monotone at all $x \in M$.

Remark 7.4. Suppose that k_P^M is quasi-symmetric and also quasi-monotone at a point $x_0 \in M$. Following the proof of Davies in [13, Theorem 10], it follows that such k_P^M satisfies the Davies–Harnack inequality (3.13). In light of Lemma 7.2, we can analogously deduce Theorems 2.5 and 2.6, (and hence also Theorem 6.2) for the class of quasi-symmetric heat kernels which are quasi-monotone (and satisfy (2.9)).

Remark 7.5. It should be noted that we are unaware of any example of a nonsymmetric operator whose heat kernel is quasi-symmetric but whose heat kernel is not equivalent to a symmetric one. Conversely, if the heat kernel of any nonsymmetric operator P is equivalent to the heat kernel of a symmetric operator in M, then the heat kernel of P is quasi-monotone at any point $x_0 \in M$, and quasi-symmetric (and P is quasi-symmetric as well).

7.2 Examples. In the present subsection we give various examples of Riemannian manifolds M and heat kernels k_P^M defined on M which satisfy our main assumption (2.9) of Theorems 2.5 and 2.6 (the doubling condition). Hence, our main results of the paper apply to these cases.

The study of heat kernel estimates has a long history (see, for example, [12, 17, 31]. In particular, proving pointwise two-sided Gaussian estimates for the heat kernel was a subject of intense research for the past few decades. It started with the celebrated works of Nash [30] and Aronson [4], where two-sided Gaussian estimates were obtained for the heat kernel of a uniformly elliptic operator in divergence form in $\mathbb{R}^N$. For such operators we obtain:

Example 7.6. Consider a parabolic equation of the form $\frac{\partial u}{\partial t} + Pu = 0$ on $\mathbb{R}^N \times (0, \infty)$, where $N \geq 3$ and

$$P = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big) \tag{7.3}$$

is a uniformly elliptic operator with real, bounded coefficients satisfying the assumptions of Theorem 2.5. Denote by $k_P^{\mathbb{R}^N}$ the corresponding positive minimal heat kernel. Aronson [4, Theorem 7] proved that $k_P^{\mathbb{R}^N}$ admits two-sided Gaussian estimates, i.e., there exist positive constants C_1, C_2, C_3, C_4 such that

$$\frac{C_3}{t^{N/2}} \exp\Big(-\frac{|x-y|^2}{C_4 t}\Big) \leq k_P^{\mathbb{R}^N}(x, y, t) \leq \frac{C_1}{t^{N/2}} \exp\Big(-\frac{|x-y|^2}{C_2 t}\Big) \tag{7.4}$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Estimate (7.4) readily implies that

$$k_P^{\mathbb{R}^N}(x, x, \frac{t}{2}) \leq C2^{\frac{N}{2}} k_P^{\mathbb{R}^N}(x, x, t) \quad \forall x \in \mathbb{R}^N \text{ and } t > 0,$$

and hence $k_P^{\mathbb{R}^N}$ satisfies the doubling condition (2.9). Therefore, if V is a small perturbation of $k_P^{\mathbb{R}^N}$, then there exists $\varepsilon_0 > 0$ such that $k_{P_\varepsilon}^{\mathbb{R}^N} \asymp k_P^{\mathbb{R}^N}$ for all $|\varepsilon| < \varepsilon_0$.

Example 7.7 (Periodic operator)**.** Consider a uniformly elliptic operator P on $\mathbb{R}^N$, $N \geq 3$ of the form

$$P = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big) + U(X).$$

Assume that $P \geq 0$ in $\mathbb{R}^N$, and that the coefficients of P satisfy the assumptions of Theorem 2.5. Suppose that the coefficients of P are periodic in $x_1, \ldots, x_n$ with period 1. Without loss of generality we may assume that $\lambda_0(P, \mathbf{1}, \mathbb{R}^N) = 0$. Then the equation $Pu = 0$ in $\mathbb{R}^N$ admits a unique (up to a multiplicative constant) positive solution ϕ. Moreover, (in the symmetric case) ϕ is periodic in $x_1, \ldots, x_n$ with period 1 [1].

Using the ground state transform we get the operator

$$P_\phi := (\phi)^{-1}P\phi = -(\phi)^{-2}(x)\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(\phi^2(x)a_{ij}(x)\frac{\partial}{\partial x_j}\Big),$$

whose heat kernel satisfies $k_{P_\phi}^{\mathbb{R}^N}(x, y, t) = (\phi)^{-1}(x)k_P^{\mathbb{R}^N}(x, y, t)\phi(y)$.

Consequently, P_ϕ is, in fact, of the form (7.3) on $L^2(\mathbb{R}^N, \varphi^2 \mathrm{d}x)$, and therefore, $k_{P_\phi}^{\mathbb{R}^N}$ satisfies assumption (2.9) which in turn implies that $k_P^{\mathbb{R}^N}$ also satisfies (2.9). Therefore, if V is a small perturbation of $k_P^{\mathbb{R}^N}$, then there exists $\varepsilon_0 > 0$ such that $k_{P_\varepsilon}^{\mathbb{R}^N} \asymp k_P^{\mathbb{R}^N}$ for all $|\varepsilon| < \varepsilon_0$.

Next, we consider perturbations of the Laplace–Beltrami operators on noncompact Riemannian manifolds. Following the seminal work of Aronson, the question of estimating the heat kernel on Riemannian manifolds was extensively studied by many authors. One of the most general estimates of heat kernels k_P^M for the Laplace–Beltrami operators was proved by P. Li and S. T. Yau [23, Corollary 3.1 and Theorem 4.1] under a suitable curvature assumption. We use these celebrated results in the following example.

Example 7.8. Let (M, g) be a complete, connected, noncompact Riemannian manifold of dimension N with nonnegative Ricci curvature. Let $P := -\Delta_g$ denote the (positive) Laplace–Beltrami operator on M and let k_P^M denote the corresponding heat kernel. Then by [23, Corollary 3.1 and Theorem 4.1] there exist positive constants C_1, C_2, C_3, C_4 such that

$$\frac{C_3}{V(x,\sqrt{t})}\exp\Big(-\frac{d^2(x,y)}{C_4 t}\Big) \le k_P^M(x,y,t) \le \frac{C_1}{V(x,\sqrt{t})}\exp\Big(-\frac{d^2(x,y)}{C_2 t}\Big) \tag{7.5}$$

for all $x, y \in M$ and $t > 0$, where $d(x, y)$ is the geodesic distance on M and $V(x, r)$ is the Riemannian volume of the geodesic ball $B(x, r) = \{y \in M : d(x, y) < r\}$. Moreover, under the above assumptions, M satisfies the doubling volume property (7.7) (see [16, Theorem 15.21]), and hence, (2.9) is satisfied.

Alternatively, under the above assumptions E. B. Davies proved [12, Corollary 5.3.6] that the positive minimal heat kernel k_P^M satisfies the following global exponential-type upper bound:

$$k_P^M(x,x,t+s) \le k_P^M(x,x,t) \le k_P^M(x,y,t+s)\Big(\frac{t+s}{t}\Big)^{\frac{N}{2}}\exp\Big(\frac{d(x,y)^2}{4s}\Big)$$

for all $t, s > 0$. In particular, for $t = s$, we have

$$k_P^M(x,x,t) \le 2^{\frac{N}{2}} k_P^M(x,x,2t) \quad \forall t > 0. \tag{7.6}$$

Hence, (2.9) is satisfied. Thus, if P is subcritical our main results hold true.

An interesting question is to find 'minimal' geometric assumptions on M that imply Gaussian estimates of the type (7.5). The upper bound in (7.5) is known to be equivalent to a certain Faber–Krahn type inequality (see [17, 18]). A well-known geometric condition related to the on-diagonal lower bound in (7.5) is the doubling volume property (7.7). In particular, in the next examples we do not assume any a priori curvature assumption on the manifold.

Example 7.9. Let (M, g) be a complete, connected, noncompact manifold of dimension N, and let $P := -\Delta_g$ denote the Laplace–Beltrami operator which satisfy the following properties:

(1) For some $x_0 \in M$, there exists $C > 0$ such that the following doubling volume property holds:

$$V(x_0, 2r) \leq CV(x_0, r) \quad \forall r > 0. \tag{7.7}$$

(2) P is subcritical in M.

(3) There exists $C_1 > 0$ such that the following on-diagonal upper bound estimate holds true:

$$k_P^M(x_0, x_0, t) \leq \frac{C_1}{V(x_0, \sqrt{t})} \quad \forall t > 0.$$

Then by [11] there exists $c > 0$ such that

$$k_P^M(x_0, x_0, t) \geq \frac{c}{V(x_0, \sqrt{t})} \quad \forall t > 0,$$

and in particular, there exists $C > 0$ such that

$$k_P^M(x_0, x_0, t/2) \leq Ck_P^M(x_0, x_0, t) \quad \forall t > 0.$$

Example 7.10. Let M be a complete, connected, noncompact weighted Riemannian manifold of dimension N. Consider the weighted Laplacian P on M, and denote by k_P^M the corresponding heat kernel. Then the two-sided Gaussian estimates (7.5) is equivalent to the validity of the uniform parabolic Harnack inequality (PHI) (see [17, 39]). We refer to [17, 20, 39] for examples of weighted manifolds satisfying (PHI).

Example 7.11. In stochastic processes, the transition density of the random motion naturally leads to the notion of the heat semigroup and hence to the heat kernel. In particular, Dirichlet forms of many families of fractals admit continuous heat kernels that satisfy sub-Gaussian estimates. By a **sub-Gaussian kernel** $\tilde{g}$, we mean

$$\tilde{g}(x, y, t) := \frac{C}{t^{\frac{\alpha}{\beta}}} \exp\Big(-c\Big(\frac{d^\beta(x, y)}{t}\Big)\Big)^{\frac{1}{\beta-1}}, \tag{7.8}$$

where $\alpha > 0$, $\beta > 1$ are two parameters that come from the geometric properties of the underlying fractal. The notion of sub-Gaussian estimates was introduced by M. T. Barlow, and E. A. Perkins in [5]. A. Grigor'yan and A. Telcs [18] developed sub-Gaussian estimates for the heat kernel on metric spaces under suitable assumptions. It follows that complete Riemannian manifolds which admit two-sided sub-Gaussian estimates for the corresponding heat kernels satisfy our assumption (2.9).

We give an example of a manifold with negative Ricci curvature, such that our assumption (2.9) holds true.

Example 7.12. Cartan–Hadamard manifolds, whose sectional curvatures are bounded above by a strictly negative constant, are known to admit a Poincaré type (or L^2-spectral gap) inequality. Namely, the generalized principal eigenvalue

$$\lambda_0 = \inf_{u \in C_c^\infty(M)\setminus\{0\}} \frac{\int_M |\nabla_g u|^2 \mathrm{d}v_g}{\int_M u^2 \mathrm{d}v_g}$$

is strictly positive.

The classical example of such a manifold is of course the hyperbolic space $\mathbb{H}^N$, where $\lambda_0 = (N-1)^2/4$. Let $M = \mathbb{H}^3$ be the hyperbolic space of dimension 3. Then the heat kernel of $P := -\Delta_{\mathbb{H}^3} - \lambda_0$ is given explicitly by

$$k_P^M(x, y, t) = \Big(\frac{1}{4\pi t}\Big)^{-\frac{3}{2}} \frac{d(x, y)}{\sinh d(x, y)} \exp\Big(-\frac{d(x, y)^2}{4t}\Big),$$

where $d(x, y)$ denotes the hyperbolic distance between x and y. Hence clearly, $k_P^M(x, x, \frac{t}{2}) \leq 2^{\frac{3}{2}} k_P^M(x, x, t)$ holds true for all $t > 0$ and $x \in \mathbb{H}^3$. For higher dimension $N > 3$, the heat kernel of the operator $P := -\Delta_{\mathbb{H}^N} - \lambda_0$ satisfies

$$k_P^M(x, y, t) \asymp \Big(\frac{1}{4\pi t}\Big)^{-\frac{N}{2}} \{(1 + d(x, y) + t)^{\frac{N-3}{2}} (1 + d(x, y))\} \\ \times \exp\Big(-\frac{(N-1)d(x, y)}{2} - \frac{d(x, y)^2}{4t}\Big),$$

and hence, $k_P^M(x, x, \frac{t}{2}) \leq C k_P^M(x, x, t)$ holds true for all $t > 0$ and $x \in \mathbb{H}^N$. Consequently, the results of the present paper hold true for such P, and $N \geq 3$. In particular, for any small perturbation potential V with respect to the heat kernel k_P^M, there exists $\varepsilon_0 > 0$ such that $k^{\mathbb{H}^N}_{-\Delta_{\mathbb{H}^N} - \varepsilon V} \asymp k^{\mathbb{H}^N}_{-\Delta_{\mathbb{H}^N}}$ for all $|\varepsilon| < \varepsilon_0$.

Example 7.13. Let P_i be a symmetric elliptic operator defined on M_i such that $\lambda_0(P_i, \mathbf{1}, M_i) = 0$, where $i = 1, 2$. Consider the skew product operator $P := P_1 \times I_1 + I_2 \times P_2$ on $M := M_1 \times M_2$, where I_i is the identity operator on M_i. Then

$$k_P^M(x, y, t) = k_{P_1}^M(x_1, y_1, t) k_{P_2}^M(x_2, y_2, t),$$

where $x = (x_1, x_2), y = (y_1, y_2) \in M$. If both operators are subcritical and satisfy (2.9), then clearly P is subcritical in M, and its heat kernel satisfies (2.9). Moreover, if P_1 is positive-critical in M_1, and P_2 is subcritical in M_2, and its heat kernel $k_{P_2}^M$ satisfies (2.9), then P is subcritical in M, and by Remark 2.2, k_P^M satisfies (2.9). We mention also the case of a twisted tube [19] (which is a perturbation of a product space), for which (2.9) is also satisfied.

An anonymous colleague has kindly pointed out to us that our results hold true for the case of universal cover of a compact manifold of negative curvature. Indeed, we have:

Example 7.14. Let M be the universal cover of a compact manifold of negative curvature. Ledrappier and Lim in [22] proved recently that the heat kernel of the Laplacian in M satisfies

$$\lim_{t\to\infty} t^{\frac{3}{2}} e^{\lambda_0 t} k_{-\Delta_g}^M(x, y, t) = C(x, y),$$

where $C(x, y)$ is a strictly positive formal eigenfunction of $-\Delta_g$ with an eigenvalue λ_0. In particular, the heat kernel of the shifted Laplacian $P := -\Delta_g - \lambda_0$ is subcritical in M and satisfies (2.9). Hence, our main results hold true for P on M.

7.3 Open problems. We conclude the paper with some problems that remain open.

(1) Do Theorems 2.5 and 2.6 remain true without assuming the doubling condition (2.9)? Note that affirmative answers in particular imply that in the class of small perturbations with respect to a subcritical heat kernel k_P^M such that $\lambda_0(P, \mathbf{1}, M) = 0$, the following holds true:

$$S_+(P, V, M) = \{\varepsilon \in \mathbb{R} \mid k_{P_\varepsilon}^M \asymp k_P^M\}.$$

(2) Prove or disprove Conjecture 1.1 in the general nonsymmetric case.

(3) Study the relationships between the notion of (semi)small perturbations with respect to the Green function and with respect to the heat kernel.

(4) Recall that in the context of (semi)small perturbations with respect to Green functions if G satisfies a certain quasi-metric property, then the semismallness of a perturbation implies smallness [35]. It would be interesting to find an analogous condition on semismall perturbations with respect to k_P^M that guarantees smallness. We remark that, as in the case of small perturbations with respect to Green functions, we are not aware of any example of a semismall perturbation with respect to a heat kernel which is not a small perturbation.

Apart from the above open problems related directly to the equivalence of heat kernels, we mention below a far reaching conjecture by M. Fraas, D. Krejčiřík and Y. Pinchover regarding the strong ratio limit of the quotients of heat kernels of subcritical and critical operators. Note that if P_+ and P_0 are subcritical and critical operators in M, respectively, then obviously $k_{P_+}^M \not\asymp k_{P_0}^M$, and

$$\liminf_{t\to\infty} \frac{k_{P_+}^M(x,y,t)}{k_{P_0}^M(x,y,t)} = 0.$$

Conjecture 7.15 ([15, Conjecture 1])**.** Let P_+ and P_0 be respectively subcritical and critical operators in M. Then

$$\lim_{t\to\infty} \frac{k_{P_+}^M(x,y,t)}{k_{P_0}^M(x,y,t)} = 0 \tag{7.9}$$

locally uniformly in $M \times M$.

It follows that for perturbations of the type studied in the present paper, Conjecture 7.15 holds true.

Lemma 7.16 (cf. [15, Theorem 5.4])**.** *Let P_0 be a symmetric critical operator in M. Assume that $V = V_+ - V_-$ is a potential such that $V_\pm \geq 0$ and $P_+ := P_0 + V$ is subcritical in M.*

Assume further that $k_{P_+}^M$ satisfies the $3k$-inequality with respect to V_-. Then there exists a positive constant C such that

$$k_{P_+}^M(x,y,t) \leq Ck_{P_0}^M(x,y,t) \quad \forall x,y \in M \text{ and } t > 0. \tag{7.10}$$

Moreover, we have

$$\lim_{t\to\infty} \frac{k_{P_+}^M(x,y,t)}{k_{P_0}^M(x,y,t)} = 0, \tag{7.11}$$

locally uniformly in $M \times M$.

In particular, Conjecture 7.15 holds true for $P_+ := P_0 + V$, where V is any nonzero nonnegative potential.

Proof. By Theorem 3.5 and Lemma 3.7, we have $k_{P_+}^M \asymp k_{P_++V_-}^M(x,y,t)$. Note that $P_+ + V_- = P_0 + V_+$. Therefore, we have

$$C^{-1}k_{P_+}^M(x,y,t) \leq k_{P_0+V_+}^M(x,y,t) \leq k_{P_0}^M(x,y,t) \quad \forall x,y \in M \text{ and } t > 0. \tag{7.12}$$

Using [15, Theorem 3.1], we conclude that (7.11) holds true. □

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