A SEMILINEAR ELLIPTIC EQUATION WITH COMPETING POWERS AND A RADIAL POTENTIAL

By

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Abstract. We verify the existence of radial positive solutions for the semilinear equation

$$-\Delta u = u^p - V(y)u^q, \quad u > 0, \text{ in } \mathbb{R}^N$$

where $N \geq 3$, p is close to $p^* := (N+2)/(N-2)$, and V is a radial smooth potential. If q is super-critical, namely $q > p^*$, we prove that this problem has a radial solution behaving like a superposition of bubbles blowing-up at the origin with different rates of concentration, provided $V(0) < 0$. On the other hand, if $N/(N-2) < q < p^*$, we prove that this problem has a radial solution behaving like a super-position of flat bubbles with different rates of concentration, provided $\lim_{r\to\infty} V(r) < 0$.

1 Introduction

Let $N \geq 3$ and consider

$$-\Delta u = u^p - V(y)u^q, \quad u > 0, \text{ in } \mathbb{R}^N \tag{1}$$

where $V \in L^\infty(\mathbb{R}^N)$, $N \geq 3$, $q > p^s$, $p > p^*$, with

$$p^s = \frac{N}{N-2}, \quad p^* = \frac{N+2}{N-2}.$$

In this paper, we are interested in the case p slightly supercritical,

$$\begin{cases} -\Delta u = u^{p^*+\epsilon} - V(y)u^q, & \text{in } \mathbb{R}^N \\ u(y) \to 0, & \text{as } |y| \to \infty \end{cases} \tag{2}$$

where $\epsilon > 0$.

For $q = 1$, problem (2) was treated in the critical case ($\epsilon = 0$) in [3] and in the sub-critical case ($\epsilon < 0$) in [9] . The supercritical analogue ($\epsilon > 0$) was addressed in [12], where the existence of a radial positive solution to (2) when V is a radial

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smooth function with $V(0) < 0$ was proved. A previous construction can also be found in [13].

In [2], the authors consider problem (2) for any fixed q satisfying $p^s < q < p^*$. The existence of an increasing number of rapidly decaying ground states was proved, that is, solutions u of (2) such that $\lim_{|x|\to\infty} u(x) = 0$. The result in [2] is obtained via tools in geometrical dynamical systems. The same equation was also treated in [5] and in [8], using a different approach, which also provided precise asymptotics for the solutions. In bounded domains, the class of radial solutions behaving like a superposition of spikes was treated in the setting of supercritical exponents, in [14, 15].

Let us now consider problem (2) in the supercritical case ($\epsilon > 0$). In the case of a single power, i.e., $p^* + \epsilon = q$, and when $V(y) \equiv -1$, equation (2) is equivalent to

$$\begin{cases} \Delta u + u^{p^*} = 0, & \text{in } \mathbb{R}^N \\ u > 0, & \text{in } \mathbb{R}^N \end{cases} \tag{3}$$

if we let ϵ go to zero. It is well known that all bounded solutions of (3) are of the form

$$w_{\lambda,\xi}(x) = \gamma_N \left(\frac{\lambda}{\lambda^2 + |y - \xi|^2} \right)^{\frac{N-2}{2}}, \qquad \gamma_N = (N(N-2))^{\frac{N-2}{4}}$$

where λ is a positive parameter and $\xi \in \mathbb{R}^N$ [1, 16, 4]. These functions are known in the literature as **bubbles**.

We want to prove the existence of a solution whose shape resembles a superposition of bubbles around the origin 0 with different blow-up orders. This class of concentration phenomena is known as a **bubble-tower**. In the setting of semilinear elliptic equations with radial symmetry, these solutions were detected in a few situations, as we can see, for instance, in [14, 6, 7, 12, 5].

Bubble-towers highly concentrated around the origin exist for (2) under the assumption that $V(0) < 0$. This is the content of our first result.

Theorem 1.1. *Let $N \geq 3$ and $p^s < q < p^* < p$. Assume that $V \in L^\infty(\mathbb{R}^N)$ and $V(0) < 0$. Then for every integer $k \geq 1$ there exists $\epsilon_k > 0$ such that, for any $\epsilon \in (0, \epsilon_k)$, a solution u_ϵ of (2) exists and it has the form*

$$u_\epsilon(x) = \gamma_N \sum_{j=1}^{k} \left(\frac{1}{1 + \alpha_j^{\frac{4}{N-2}} \epsilon^{-(j-1+\frac{1}{p^*-q})\frac{4}{N-2}} |x|^2} \right)^{\frac{N-2}{2}} \alpha_j \epsilon^{-(j-1+\frac{1}{p^*-q})} (1 + o(1)) \tag{4}$$

with $o(1) \to 0$ uniformly on compact sets of $\mathbb{R}^N$, as $\epsilon \to 0$. The constants α_i have *explicit expressions and depend only on* k, N, q *and* $V(0)$,

$$\alpha_j = \Big[-\frac{a_5 V(0)(p^*-q)}{a_3 k}\Big]^{\frac{1}{p^*-q}} \Big(\frac{a_2}{a_3}\Big)^{j-1} \frac{(k-j)!}{(k-1)!}, \quad j = 1, \dots, k, \tag{5}$$

while a_2, a_3, a_5 *are the positive constants defined in* (19).

Also in the case in which $p^s < p^* < p < q$ bubble-towers do exist, but they are of a different nature, and their existence depends on the behavior of the potential V at infinity.

Theorem 1.2. *Let* $N \geq 3$ *and* $q > p^*$. *Assume that* $V \in L^\infty(\mathbb{R}^N)$ *and* $V_\infty := \lim_{|x|\to\infty} V(x) < 0$. *Then for every integer* $k \geq 1$ *there exists* $\epsilon_k > 0$ *such that, for any* $\epsilon \in (0, \epsilon_k)$, *a solution* u_ϵ *of* (2) *exists and it has the form*

$$\hat{u}_\epsilon(x) = \gamma_N \sum_{j=1}^{k} \Bigg(\frac{1}{1 + \hat{\alpha}_j^{\frac{4}{N-2}} \epsilon^{(j-1+\frac{1}{q-p^*})\frac{4}{N-2}} |x|^2}\Bigg)^{\frac{N-2}{2}} \hat{\alpha}_j \epsilon^{(j-1+\frac{1}{q-p^*})} (1 + o(1)) \tag{6}$$

with $o(1) \to 0$ *uniformly on compact sets of* $\mathbb{R}^N$, *as* $\epsilon \to 0$. *The constants* $\hat{\alpha}_i$ *have explicit expressions and depend only on* k, N, q *and* V_∞,

$$\hat{\alpha}_j = \Big[\frac{\hat{a}_5 V_\infty (p^*-q)}{a_3 k}\Big]^{\frac{1}{p^*-q}} \Big(\frac{a_2}{a_3}\Big)^{j-1} \frac{(k-j)!}{(k-1)!}, \quad j = 1, \dots, k, \tag{7}$$

while a_2, a_3, $\hat{a}_5$ *are the positive constants defined in* (19) *and* (37).

The bubble-tower in (6) describes a superposition of k **flat bubbles**.

In order to prove our results, we start by reducing the problem to a non-autonomous ordinary differential equation, using the so-called Emden–Fowler transformation, [11]. Then we perform a Lyapunov–Schmidt reduction, as in [10], to reduce the procedure of construction of solutions to a finite-dimensional variational problem.

The paper is organized as follows. In Section 2, we provide an asymptotic expansion of the energy functional associated to the ODE problem. The finite-dimensional reduction argument is discussed in Section 3. We prove Theorems 1.1 and 1.2 in Sections 4 and 5, respectively.

2 The energy asymptotic expansion

Since we are seeking a solution u of (2) with fast decay, we can assume that u is radial around the origin. Then we arrive at the following equivalent problem:

$$\begin{cases} u''(r) + \frac{N-1}{r} u'(r) + u^{p^*+\epsilon}(r) - V(r) u^q(r) = 0, \\ u(r) \to 0, \ \text{as } r \to \infty. \end{cases} \tag{8}$$

By introducing the so-called Emden–Fowler transformation,

(9) $$v(x) = r^{\frac{2}{p^*-1}} u(r), \quad \text{with } r = e^{-\frac{p^*-1}{2}x},$$

for $x \in \mathbb{R}$, the problem (8) becomes

(10) $$\begin{cases} v''(x) - v(x) + \beta[e^{\epsilon x} v^{p^*+\epsilon}(x) - V(e^{-\frac{p^*-1}{2}x}) e^{-(p^*-q)x} v^{q}(x)] = 0, \\ 0 < v(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases}$$

in $\mathbb{R}$, where $\beta = (\frac{2}{N-2})^2$. We henceforth denote $\omega(x) = V(e^{-\frac{p^*-1}{2}x})$.

The energy functional related to (10) is

(11) $$E_\epsilon(\psi) = I_\epsilon(\psi) + \frac{\beta}{q+1} \int_{\mathbb{R}} \omega(x) e^{-(p^*-q)x} |\psi|^{q+1} dx$$

where

$$I_\epsilon(\psi) = \frac{1}{2} \int_{\mathbb{R}} (|\psi'|^2 + |\psi|^2) dx - \frac{\beta}{p^* + \epsilon + 1} \int_{\mathbb{R}} e^{\epsilon x} |\psi|^{p^*+\epsilon+1} dx.$$

Let us consider the positive radial solution of

(12) $$\Delta w + w^{p^*} = 0, \quad w(0) = \gamma_N$$

given by $w(r) = \gamma_N (\frac{1}{1+r^2})^{\frac{N-2}{2}}$. Now we set U to be the Emden–Fowler transformation of w

(13) $$U(x) = \gamma_N e^{-x} (1 + e^{-(p^*-1)x})^{-\frac{N-2}{2}}.$$

Then U satisfies

(14) $$U'' - U + \beta U^{p^*} = 0, \quad 0 < U(x) \to 0, \text{ as } |x| \to \infty$$

It is then natural to look for a solution of (10) of the form

$$v(x) = \sum_{i=1}^{k} U(x - \xi_i) + \phi(x)$$

for a certain choice of points $0 < \xi_1 < \xi_2 < \cdots < \xi_k$ and ϕ is small. We set

(15) $$U_i(x) = U(x - \xi_i), \quad \bar{U} = \sum_{i=1}^{k} U_i(x).$$

and choose the points ξ_i as follows:

(16) $$\begin{aligned} \xi_1 &= -\frac{1}{p^* - q} \log \epsilon - \log \Lambda_1, \\ \xi_{i+1} - \xi_i &= -\log \epsilon - \log \Lambda_{i+1}, \quad i = 1, \ldots, k-1, \end{aligned}$$

where the Λ_i's are positive parameters. This choice of the ξ_i's turns out to be convenient in the proof of the following asymptotic expansion of $E_\epsilon(\bar{U})$. We set $\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_k)$.

Lemma 2.1. *Let $N \geq 3$, $\delta > 0$ fixed, $k \in \mathbb{N}$. Assume that*

$$\delta < \Lambda_i < \delta^{-1}, \quad i = 1, 2, \dots, k. \tag{17}$$

Then there exist positive numbers a_i, $i = 1, \dots, 5$, depending on N, p and q, such that

$$E_\epsilon(\bar{U}) = ka_1 + \epsilon\Psi_k(\Lambda) + k\epsilon\beta a_4 + \epsilon\theta_\epsilon(\Lambda) - \frac{a_3 k}{2(p^* - p)}((1-k)(p^* - q) - 2)\epsilon\log\epsilon$$

where

$$\Psi_k(\Lambda) = a_3 k \log\Lambda_1 + a_5 V(0)\Lambda_1^{(p^*-q)} + \sum_{i=1}^{k}[(k-i+1)a_3\log\Lambda_i - a_2\Lambda_i] \tag{18}$$

and $\theta_\epsilon(\Lambda) \to 0$ as $\epsilon \to 0$ uniformly in C^1-sense on the set of Λ_i's satisfying (17).

Proof. We estimate

$$\begin{aligned}
I_\epsilon(\bar{U}) &= \frac{1}{2}\int_{\mathbb{R}}(|\bar{U}'|^2 + |\bar{U}|^2)dx - \frac{\beta}{p^* + \epsilon + 1}\int_{\mathbb{R}} e^{\epsilon x}|\bar{U}|^{p^*+\epsilon+1}dx \\
&= I_0(\bar{U}) - \frac{\beta}{p^*+1}\int_{\mathbb{R}}(e^{\epsilon x} - 1)|\bar{U}|^{p^*+\epsilon+1}dx \\
&\quad + \Big(\frac{1}{p^*+1} - \frac{1}{p^*+1+\epsilon}\Big)\beta\int_{\mathbb{R}} e^{\epsilon x}|\bar{U}|^{p^*+\epsilon+1}dx \\
&\quad + \frac{\beta}{p^*+1}\int_{\mathbb{R}}(|\bar{U}|^{p^*+1} - |\bar{U}|^{p^*+\epsilon+1})dx \\
&= I_0(\bar{U}) - \frac{\beta}{p^*+1}\int_{\mathbb{R}}(e^{\epsilon x} - 1)|\bar{U}|^{p^*+\epsilon+1} + A_\epsilon,
\end{aligned}$$

where

$$\begin{aligned}
A_\epsilon =& \Big(\frac{1}{p^*+1} - \frac{1}{p^*+1+\epsilon}\Big)\beta\int_{\mathbb{R}} e^{\epsilon x}|\bar{U}|^{p^*+\epsilon+1}dx \\
&+ \frac{\beta}{p^*+1}\int_{\mathbb{R}}(|\bar{U}|^{p^*+1} - |\bar{U}|^{p^*+\epsilon+1})dx.
\end{aligned}$$

As in [14], we can prove that

$$A_\epsilon = k\epsilon\beta\Big(\frac{1}{(1+p^*)^2}\int_{\mathbb{R}}|U|^{p^*+1}dx - \frac{1}{(1+p^*)}\int_{\mathbb{R}}|U|^{p^*+1}\log U dx\Big) + o(\epsilon).$$

Also, by reasoning in a similar manner we have

$$\int_{\mathbb{R}}(e^{\epsilon x}-1)|\bar{U}|^{p^*+\epsilon+1}dx=\epsilon\int_{\mathbb{R}}x|\bar{U}|^{p^*+\epsilon+1}dx+o(\epsilon)=\epsilon\left(\sum_{l=1}^{k}\xi_l\right)\int_{\mathbb{R}}U^{p^*+1}dy+o(\epsilon)$$

and

$$I_0(\bar{U})=kI_0(U)-\beta C_N\int_{\mathbb{R}}U^{p^*}e^x dx\left(\sum_{l=2}^{k}e^{\xi_l-\xi_{l-1}}\right)+o(\epsilon).$$

Now we need to evaluate $\int_{\mathbb{R}}\omega(x)e^{-(p^*-q)x}|\bar{U}|^{q+1}dx$. By following the argument in [12] and using our choice of ξ_l's, we have

$$\begin{aligned}\int_{\mathbb{R}}\omega(x)e^{-(p^*-q)x}|\bar{U}|^{q+1}dx&=\sum_{i=1}^{k}\int_{\mathbb{R}}\omega(x)e^{-(p^*-q)x}|U_i|^{q+1}dx+o(\epsilon)\\&=\int_{\mathbb{R}}\omega(x)e^{-(p^*-q)x}|U_1|^{q+1}dx+o(\epsilon).\end{aligned}$$

On the other hand, the following holds:

$$\begin{aligned}\int_{\mathbb{R}}\omega(x)e^{-(p^*-q)x}|U_1|^{q+1}dx&=e^{-(p^*-q)\xi_1}\int_{\mathbb{R}}\omega(x+\xi_1)e^{-(p^*-q)x}|U_1(x+\xi_1)|^{q+1}dx\\&=e^{-(p^*-q)\xi_1}\int_{\mathbb{R}}\omega(x+\xi_1)e^{-(p^*-q)x}|U|^{q+1}dx\\&=e^{-(p^*-q)\xi_1}V(0)\int_{\mathbb{R}}e^{-(p^*-q)x}|U|^{q+1}dx+o(1).\end{aligned}$$

We thus have the following:

$$\begin{aligned}E_\epsilon(\bar{U})&=I_\epsilon(\bar{U})+\frac{\beta}{q+1}\int_{\mathbb{R}}\omega(x)e^{-(p^*-q)x}|\bar{U}|^{q+1}dx\\&=I_0(\bar{U})-\frac{\beta}{p^*+1}\int_{\mathbb{R}}(e^{\epsilon x}-1)|\bar{U}|^{p^*+\epsilon+1}dx+A_\epsilon\\&\quad+\frac{\beta}{q+1}\int_{\mathbb{R}}\omega(x)e^{-(p^*-q)x}|\bar{U}|^{q+1}dx\\&=kI_0(U)-\beta C_N\int_{\mathbb{R}}U^{p^*}e^x dx\left(\sum_{l=2}^{k}e^{\xi_l-\xi_{l-1}}\right)\\&\quad-\frac{\beta}{p^*+1}\left(\epsilon\left(\sum_{l=1}^{k}\xi_l\right)\int_{\mathbb{R}}U^{p^*+1}dy\right)\\&\quad+k\epsilon\beta\left(\frac{1}{(1+p^*)^2}\int_{\mathbb{R}}|U|^{p^*+1}dx-\frac{1}{1+p^*}\int_{\mathbb{R}}|U|^{p^*+1}\log U dx\right)\\&\quad+\frac{\beta}{q+1}\left(e^{-(p^*-q)\xi_1}V(0)\int_{\mathbb{R}}e^{-(p^*-q)x}|U|^{q+1}dx\right)+o(\epsilon)\end{aligned}$$

which lead us to the following expression:

$$E_\epsilon(\bar{U}) = ka_1 - a_2 \sum_{l=2}^{k} e^{-(\xi_l - \xi_{l-1})} - \epsilon a_3 \left(\sum_{i=1}^{k} \xi_i \right) + k\epsilon\beta a_4 + a_5 V(0) e^{-(p^*-q)\xi_1} + o(\epsilon).$$

By using our choice of ξ_i's

$$E_\epsilon(\bar{U}) = ka_1 + \epsilon\Psi_k(\Lambda) - \frac{a_3 k}{2(p^*-q)}((1-k)(p^*-q)-2)\epsilon\log\epsilon + k\epsilon\beta a_4\epsilon + o(\epsilon),$$

where $\Psi_k(\Lambda)$ is given by (18) and the constants a_i, $i = 1, \dots, 5$, are explicitly expressed as follows:

$$(19) \quad \begin{cases} a_1 = I_0(U), \quad a_2 = \beta C_N \int_{\mathbb{R}} U^{p^*}(x) e^x dx, \quad a_3 = \frac{\beta}{p^*+1} \int_{\mathbb{R}} U^{p^*+1}(x) dx, \\ a_4 = \frac{1}{(p^*+1)^2} \int_{\mathbb{R}} U^{p^*+1}(x) dx - \frac{1}{p^*+1} \int_{\mathbb{R}} U^{p^*+1}(x) \log U(x) dx, \\ a_5 = \frac{\beta}{q+1} \int_{\mathbb{R}} e^{-(p^*-q)x} U^{q+1}(x) dx. \end{cases}$$

Notice that the term $o(\epsilon)$ in the above expression for $E_\epsilon(\bar{U})$ is uniform in the set of the Λ_i's satisfying (17). A similar computation shows that differentiation with respect to the Λ_i's leaves the term $o(\epsilon)$ of the same order in the C^1-sense. □

3 The finite-dimensional reduction

We consider again points $0 < \xi_1 < \xi_2 < \cdots < \xi_k$ which are for now arbitrary and define

$$Z_i(x) = U_i'(x), \quad i = 1, \dots, k.$$

Next we consider the problem of finding a function ϕ for which there are constants c_i, $i = 1, \dots, k$, such that, in $\mathbb{R}$

$$(20) \quad \begin{cases} \sum_{i=1}^{k} c_i Z_i = -(\bar{U}+\phi)'' + (\bar{U}+\phi) \\ \qquad - \beta[e^{\epsilon x}(\bar{U}+\phi)_+^{p^*+\epsilon} - \omega(x) e^{-(p^*-q)x}(\bar{U}+\phi)^q], \\ \phi(x) \to 0, \quad |x| \to \infty, \\ \int_{\mathbb{R}} Z_i \phi dx = 0, \quad i = 1, \dots, k. \end{cases}$$

Let us consider the linearized operator around $\bar{U}$,

$$\mathcal{L}_\epsilon \phi = -\phi'' + \phi - \beta[(p^*+\epsilon) e^{\epsilon x} \bar{U}^{p^*+\epsilon-1} - q\omega(x) e^{-(p^*-q)x} \bar{U}^{q-1}]\phi.$$

Then (20) can be rewritten as

$$(21) \quad \begin{cases} \mathcal{L}_\epsilon \phi = N_\epsilon^1(\phi) + N_\epsilon^2(\phi) + R_\epsilon + \sum_{i=1}^{k} c_i Z_i, \\ \phi(x) \to 0, \quad |x| \to \infty, \\ \int_{\mathbb{R}} Z_i \phi dx = 0, \quad i = 1, \dots, k. \end{cases}$$

where

$$N_\epsilon^1 = \beta e^{\epsilon x}[(\bar{U}+\phi)^{p^*+\epsilon} - \bar{U}^{p^*+\epsilon} - (p^*+\epsilon)\bar{U}^{p^*+\epsilon-1}\phi],$$
$$N_\epsilon^2 = -\beta\omega(x)e^{-(p^*-q)x}[(\bar{U}+\phi)^q - \bar{U}^q - q\bar{U}^{q-1}\phi],$$
$$R_\epsilon = \sum_{i=1}^k U_i^{p^*} + \beta e^{\epsilon x}\bar{U}^{p^*+\epsilon} - \beta\omega(x)e^{-(p^*-q)x}\bar{U}^q.$$

Next we prove that (21) has a solution for a certain choice of ξ_i. In order to do that we first analyze its linear part, i.e., given a function h, we consider the problem of finding ϕ such that

$$\begin{cases} \mathcal{L}_\epsilon\phi = h + \sum_{i=1}^k c_i Z_i, \\ \phi(x) \to 0, \qquad |x| \to \infty, \\ \int_{\mathbb{R}} Z_i\phi dx = 0, \quad i = 1, \cdots, k. \end{cases} \tag{22}$$

In order to analyze invertibility properties of $\mathcal{L}_\epsilon$ under the orthogonality conditions, we introduce the following norm for function $\psi : \mathbb{R} \to \mathbb{R}$:

$$\|\psi\|_* = \sup_{x\in\mathbb{R}} \left(\sum_{i=1}^k e^{-\sigma|x-\xi|}\right)^{-1} |\psi(x)|,$$

where $\sigma > 0$ is a small constant to be fixed later.

The following result holds.

Proposition 3.1. *There exist positive numbers ϵ_0, δ_0, R_0 such that if*

$$R_0 < \xi_i, \quad R_0 < \min_{1\le i<k-1}(\xi_{i+1}-\xi_i), \quad \xi_k < \frac{\delta_0}{\epsilon}, \tag{23}$$

then for all $0 < \epsilon < \epsilon_0$ and for all $h \in C(\mathbb{R})$ with $|h|_* < +\infty$, problem (22) has a unique solution $\psi =: T_\epsilon(h)$, such that

$$\|T_\epsilon(h)\|_* \le C\|h\|_*, \quad |c_i| \le \|h\|_*.$$

Lemma 3.1. *Assume there is a sequence $\epsilon_n \to 0$ and points ξ_i's satisfying $0 < \xi_1^n < \cdots < \xi_k^n$ with*

$$\xi_1^n \to \infty, \quad \min_{1\le i<k-1}(\xi_{i+1}^n - \xi_i^n) \to \infty, \quad \xi_k^n = o(\epsilon_n^{-1}) \tag{24}$$

such that for certain functions ϕ_n and h_n with $\|h_n\|_* \to 0$, and scalars c_i^n, one has in $\mathbb{R}$

$$\begin{cases} \mathcal{L}_{\epsilon_n}(\phi_n) = h_n + \sum_{i=1}^k c_i^n Z_i^n, \\ \phi_n(x) \to 0, \qquad |x| \to \infty, \\ \int_{\mathbb{R}} Z_i^n\phi_n dx = 0, \quad i = 1, \ldots, k. \end{cases} \tag{25}$$

with $Z_i^n(x) = U'(x-\xi_i^n)$. Then $\lim_{n\to\infty}\|\phi_n\|_* = 0$

Proof. We first establish the weaker assertion that

$$\lim_{n\to\infty} \|\phi_n\|_\infty = 0.$$

For a contradiction, we may assume that $\|\phi_n\|_\infty = 1$. Testing (25) against Z_l^n and integrating by parts we get

$$\sum_{i=1}^{k} c_i^n \int_{\mathbb{R}} Z_i^n Z_l^n dx = \int_{\mathbb{R}} \mathcal{L}_{\epsilon_n}(Z_l^n)\phi_n dx - \int_{\mathbb{R}} h_n Z_l^n dx.$$

This defines a linear system in the c_i's which is "almost diagonal" as $n \to \infty$. Moreover, the assumptions made plus the fact that the Z_l^n solves

$$-Z'' + (1 - p^*\beta U_l^{p^*-1} Z) = 0$$

yield, after an application of dominated convergence, that $\lim_{n\to\infty} c_i^n = 0$. If we set $x_n \in \mathbb{R}^N$ such that $\phi_n(x_n) = 1$, we can assume that there exists $i \in \{1, \ldots, k\}$ such that for n large enough we have $|\xi_i^n - x_n| < R$ for some fixed $R > 0$. We set $\tilde{\phi}_n = \phi_n(x + \xi_i^n)$. From (25), we see that passing to a suitable subsequence, $\tilde{\phi}_n(x)$ converges uniformly over compacts to a nontrivial bounded solution $\tilde{\phi}$ of

$$-\tilde{\phi}'' + \tilde{\phi} - \beta p^* U^{p^*}\tilde{\phi} = 0, \quad \text{in } \mathbb{R}.$$

Hence for some $c \neq 0$, $\tilde{\phi} = cU'$. However the orthogonality condition passes to the limit as

$$0 = \int_{\mathbb{R}} Z_l^n \phi_n \to c \int_{\mathbb{R}} (U')^2$$

which is a contradiction. Then $\lim_{n\to\infty} \|\phi_n\|_\infty = 0$.

Now, we observe that this shows that (25) takes the form

$$-\phi_n'' + \phi_n = g_n \tag{26}$$

where

$$g_n = h_n + \sum_{i=1}^{k} c_i^n Z_i^n + \beta[(p^* + \epsilon_n)e^{\epsilon_n x}\bar{U}^{p^*+\epsilon_n-1} - q\omega(x)e^{-(p^*-q)x}\bar{U}^{q-1}]\phi_n.$$

We estimate g_n:

$$\begin{aligned}|g_n| \le \|h_n\|_*\left(\sum_{i=1}^{k} e^{-\sigma|x-\xi_i^n|}\right) + c_l^n \sum_{i=1}^{n} o(e^{-|x-\xi_i^n|}) \\ + \|\phi_n\|_\infty\left(\sum_{i=1}^{n} o(e^{-(p^*-1)|x-\xi_i^n|}) + \sum_{i=1}^{n} o(e^{-(2q-p^*-1)|x-\xi_i^n|})\right),\end{aligned}$$

since $V \in L^\infty(\mathbb{R})$. If $0 < \sigma < \min\{1, p^* - 1, 2q - p^* - 1\}$, we have

$$|g_n(x)| \leq \theta_n \sum_{i=1}^{k} e^{-\sigma|x-\xi_i|} =: \psi_n(x),$$

with $\theta_n \to 0$. We see that the function $C\psi_n$, for $C > 0$ sufficiently large, is a supersolution for (26), so that $\phi_n \leq C\psi_n$. Similarly, we have $\phi_n \geq -C\psi_n$. Thus, the proof is concluded. □

The proof of Proposition 3.1 then follows from Lemma 3.1 as in [12].

Next we study some differentiability properties of T_ϵ on ξ_i. We write $\xi = (\xi_1, \ldots, \xi_k)$. We let $\mathcal{C}_*$ be the Banach space of all continuous ψ defined in $\mathbb{R}$ satisfying $\|\psi\|_* < \infty$, endowed with the norm $\|\cdot\|_*$. Also, let $\mathcal{L}(\mathcal{C}_*)$ be the space of linear operators of $\mathcal{C}_*$.

The following result can be established.

Proposition 3.2. *Under the assumptions of Proposition 3.1, consider the map $T_\epsilon(\xi)$ with values in $\mathcal{L}(\mathcal{C}_*)$. Then T_ϵ is C^1 and*

$$\|D_\xi T_\xi\|_{\mathcal{L}(\mathcal{C}_*)} \leq C$$

uniformly on ξ satisfying (23)*, for some constant C.*

Proof. Fix $h \in \mathcal{C}_*$ and let $\phi = T_\epsilon(h)$ for $\epsilon < \epsilon_0$. Notice that ϕ satisfies (22) and the orthogonality conditions, for some uniquely determined constants c_i. For $l \in \{1, \ldots, k\}$, if we define the constant b_l as follows:

$$b_l \int_{\mathbb{R}} |Z_l|^2 = \int_{\mathbb{R}} \phi \partial_{\xi_l} Z_l,$$

then by differentiating with respect to ξ_l we obtain that

$$\partial_{\xi_l}\phi = T_\epsilon(f) + b_l Z_l$$

where

$$f = -b_l \mathcal{L}_\epsilon Z_l + c_l \partial_{\xi_l} Z_l + \beta[(p^* + \epsilon)e^{\epsilon x}(\partial_{\xi_l} \bar{U}^{p^*+\epsilon-1}) - q\omega(x)e^{-(p^*-q)x}(\partial_{\xi_l} \bar{U}^{q-1})]\phi.$$

Moreover $\|f\|_* \leq C\|h\|_*$, $|b_l| \leq C\|\phi\|_*$ so that $\|\partial_{\xi_l}\phi\| \leq C\|h\|_*$. Besides, $\partial_{\xi_l}\phi$ depends continuously on ξ for this norm. Thus, the result follows. □

We are now ready to prove that (21) is uniquely solvable with respect to $\|\phi\|_*$. In order to do that we restrict the range of the parameters ξ_i's in a convenient way. We assume that, for a fixed $M > 0$ large, the following conditions hold:

$$\log(M\epsilon)^{-1} < \min_{1 \leq i < k-1}(\xi_{i+1} - \xi_i), \quad \xi_k < k \log(M\epsilon)^{-1}. \tag{27}$$

Then we can estimate R_ϵ, $N_\epsilon^1 + N_\epsilon^2$, and their derivatives, by direct calculation, as follows.

Lemma 3.2. *If* $\|\phi\|_1 \leq \frac{1}{2}\|\bar{U}\|_1$ *then*

$$\|N_\epsilon(\phi)\|_* \leq C(\|\phi\|_*^{\min\{p^*,2\}} + \|\phi\|_*^{\min\{2q-p^*,2\}}),$$
$$\|D_\phi N_\epsilon(\phi)\|_* \leq C(\|\phi\|_*^{\min\{p^*-1,2\}} + \|\phi\|_*^{\min\{2q-p^*-1,2\}}),$$

where $\|\phi\|_1 := \sup_{x\in\mathbb{R}}(\sum_{i=1}^k e^{|x-\xi_i|})^{-1}|\phi(x)|$ *and* $N_\epsilon(\phi) = N_\epsilon^1(\phi)+N_\epsilon^2(\phi)$. *In addition, if* (27) *holds then*

$$\|R_\epsilon\|_* \leq C\epsilon^{\frac{1+\tau}{2}}, \quad \|\partial_\xi R_\epsilon\|_* \leq C\epsilon^{\frac{1+\tau}{2}},$$

where $\tau > 0$ *is small.*

The next result allows for the reduction to a finite-dimensional problem, as we will see in the next section. The proof is very similar to [12, Proposition 3] and we omit it here.

Proposition 3.3. *Assume* (27) *holds. Then, for all* ϵ *small enough, there exists a unique solution* $\phi = \phi(\xi)$ *to problem* (20) *which satisfies*

$$\|\phi\|_* \leq C\epsilon^{\frac{1+\tau}{2}}.$$

Moreover, the map $\xi \mapsto \phi(\xi)$ *is of class* C^1 *for the norm* $\|\cdot\|_*$ *and*

$$\|D_\xi\phi\|_* \leq C\epsilon^{\frac{1+\tau}{2}}.$$

4 The finite-dimensional variational problem

In this section we fix a large constant $M > 0$ and assume the conditions (27) for ξ. Our problem is equivalent to that of finding ξ_i's satisfying $c_i(\xi) = 0$, for all $i = 1, 2, \ldots, k$. In this case, $v = \bar{U} + \phi$ is a solution for (10) satisfying the desired formula.

We consider the functional

$$\mathcal{J}_\epsilon(\xi) = E_\epsilon(\bar{U} + \phi),$$

where $\phi = \phi(\xi)$ is that of Proposition 3.3 and E_ϵ is the energy functional defined in (11). It is known that finding the desired c_i's is equivalent to finding a critical point of $\mathcal{J}_\epsilon(\xi)$; see, for instance, [12]. That is, we need to find a point ξ satisfying

$$\nabla\mathcal{J}_\epsilon(\xi) = 0. \tag{28}$$

In order to do that, the following expansion result will be crucial.

Lemma 4.1. *The following expansion holds:*

$$\mathcal{J}_\epsilon(\xi) = E_\epsilon(\bar{U}) + o(\epsilon),$$

where $o(\epsilon)$ is uniform in the C^1-sense over all points ξ satisfying (27).

Proof. First, notice that $DE_\epsilon(\bar{U}+\phi)[\phi] = 0$. It then follows from a Taylor expansion that

$$\begin{aligned} E_\epsilon(\bar{U}+\phi) - E_\epsilon(\bar{U}) &= \int_0^1 D^2 E_\epsilon(\bar{U}+t\phi)[\phi^2] t dt \\ &= \int_0^1 \int_{\mathbb{R}} [N_\epsilon(\phi) + R_\epsilon]\phi t dt \\ &\quad + \int_0^1 \int_{\mathbb{R}} \beta(p^*+\epsilon) e^{\epsilon x}[\bar{U}^{p^*+\epsilon-1} - (\bar{U}+t\phi)^{p^*+\epsilon-1}]\phi^2 t dt \\ &\quad - \int_0^1 \beta q \int_{\mathbb{R}} \omega(x) e^{-(p^*-q)x}[\bar{U}^{q-1} - (\bar{U}+t\phi)^{q-1}]\phi^2 t dt. \end{aligned} \tag{29}$$

Since $\|\phi\|_* \leq C\epsilon^{\frac{1+\tau}{2}}$, from Lemma 3.2 we get

$$\mathcal{J}_\epsilon(\xi) - E_\epsilon(\bar{U}) = o(\epsilon^{1+\tau})$$

uniformly on points satisfying (27). Next we differentiate with respect to ξ and get, from (29) that

$$\begin{aligned} D_\xi[\mathcal{J}_\epsilon(\xi) - E_\epsilon(\bar{U})] &= \int_0^1 \int_{\mathbb{R}} D_\xi[N_\epsilon(\phi) + R_\epsilon]\phi t dt \\ &\quad + \int_0^1 \int_{\mathbb{R}} \beta(p^*+\epsilon) e^{\epsilon x} D_\xi[\bar{U}^{p^*+\epsilon-1} - (\bar{U}+t\phi)^{p^*+\epsilon-1}]\phi^2 t dt \\ &\quad - \int_0^1 \beta q \int_{\mathbb{R}} \omega(x) e^{-(p^*-q)x} D_\xi[\bar{U}^{q-1} - (\bar{U}+t\phi)^{q-1}]\phi^2 t dt. \end{aligned}$$

Using similar arguments as in Proposition 3.2, we find that

$$D_\xi[\mathcal{J}_\epsilon(\xi) - E_\epsilon(\bar{U})] = o(\epsilon^{1+\tau}).$$

Thus the result follows. □

In what follows we prove Theorem 1.1.

Proof of Theorem 1.1. Recall that

$$\xi_1 = -\frac{1}{p^*-q}\log\epsilon - \log\Lambda_1,$$

$$\xi_{i+1} - \xi_i = -\log\epsilon - \log\Lambda_{i+1}, \quad i = 1, \ldots, k-1,$$

where Λ_i's are positive parameters. Thus, it is sufficient to find a critical point of

$$\Phi_\epsilon(\Lambda) = \epsilon^{-1}\mathcal{J}_\epsilon(\xi(\Lambda)).$$

Now, from Lemma 2.1, we get

$$\nabla\Phi_\epsilon(\Lambda) = \nabla\Psi_k + o(1),$$

where $o(1)$ is uniform with respect to parameters Λ with $M^{-1} < \Lambda_i < M$, for fixed large M.

Next we analyze the critical points of $\Psi_k(\Lambda)$, by writing

$$\Psi_k(\Lambda) = \varphi_1(\Lambda_1) + \sum_{i=2}^{k} \varphi_i(\Lambda_i),$$

where

$$\varphi_1(s) = a_5 V(0) s^{p^*-q} + a_3 k \log s,$$
$$\varphi_i(s) = (k-i+1)a_3 \log s - a_2 s, \quad i = 2, \dots, k.$$

Notice that φ_i has a unique maximum point $\Lambda_i^* = (k-i+1)\frac{a_3}{a_2}$, for $i = 2, \dots, k$. If we further assume that $V(0) < 0$, then $\varphi_1(s)$ also has a unique maximum point

$$\Lambda_1^* = \Big[-\frac{a_3 k}{a_5 V(0)(p^*-q)}\Big]^{\frac{1}{p^*-q}}.$$

Since the critical point

$$\Lambda^* = \bigg(\Big[-\frac{a_3 k}{a_5 V(0)(p^*-q)}\Big]^{\frac{1}{p^*-q}}, \frac{(k-1)a_3}{a_2}, \dots, \frac{a_3}{a_2}\bigg)$$

of Ψ_k is nondegenerate, it follows that the local degree $\deg(\nabla\Psi_k, \mathcal{V}, 0)$ is well defined and nonzero. Here $\mathcal{V}$ denotes a small neighborhood of Λ^* in $\mathbb{R}^k$. Hence $\deg(\nabla\Phi_\epsilon, \mathcal{V}, 0) \neq 0$, if ϵ is small enough. We conclude that there exists a critical point Λ_ϵ^* of Φ_ϵ satisfying

$$\Lambda_\epsilon^* = \Lambda^* + o(1).$$

For $\xi_\epsilon = \xi(\Lambda_\epsilon^*)$, the functions

$$\upsilon \;= \bar{U} + \phi(\xi_\epsilon)$$

are solutions of (10). From equation (20) and Proposition 3.3, we derive that $\upsilon \;= \bar{U}(1+o(1))$. If we set $\xi^* = \xi(\Lambda^*)$, then it is also true that

$$\upsilon\;(x) = \sum_{i=1}^{k} U(x-\xi_i^*)(1+o(1)).$$

Now, changing the variables back, we have that

$$u_\epsilon^*(r) = \gamma_N \sum_{i=1}^{k} \Big(\frac{1}{1 + e^{(p^*-1)\xi_i^*} r^2} \Big)^{\frac{N-2}{2}} e^{\xi_i^*}(1 + o(1)),$$

where $e^{\xi_i^*} = e^{-(i-1)-\frac{1}{p^*-q}} \Pi_{j=1}^{i} (\Lambda_j^*)^{-1}$ is a solution of (8). We conclude that the ansatz given for v provides a spike-tower solution for (2). □

5 Proof of Theorem 1.2

In this section we prove Theorem 1.2. The proof is very similar to that of Theorem 1.1, so we just highlight below the most critical changes. Since we are seeking for radial solutions of (1), we consider again the following slightly supercritical equation:

$$\begin{cases} u''(r) + \frac{N-1}{r} u'(r) + u^{p^*+\epsilon}(r) - V(r) u^q(r) = 0, \\ u(r) \to 0, \ \text{as } r \to \infty, \end{cases} \tag{30}$$

with $\epsilon > 0$, but this time we take $q > p^*$. We consider the transformation

$$v(x) = r^{\frac{2}{p^*-1}} u(r), \quad \text{with } r = e^{\frac{p^*-1}{2} x}, \tag{31}$$

for $x \in \mathbb{R}$. Then the problem (30) becomes

$$\begin{cases} v''(x) - v(x) + \beta[e^{-\epsilon x} v^{p^*+\epsilon}(x) - V(e^{\frac{p^*-1}{2}x}) e^{(p^*-q)x} v^q(x)] = 0, \\ 0 < v(x) \to 0, \ \text{as } |x| \to \infty. \end{cases} \tag{32}$$

We recall that $\beta = (\frac{2}{N-2})^2$. Again we denote $\omega(x) = V(e^{\frac{p^*-1}{2}x})$.

The energy functional related to (32) is

$$\hat{E}_\epsilon(\psi) = \hat{I}_\epsilon(\psi) + \frac{\beta}{q+1} \int_{\mathbb{R}} \omega(x) e^{(p^*-q)x} |\psi|^{q+1} dx \tag{33}$$

where

$$\hat{I}_\epsilon(\psi) = \frac{1}{2} \int_{\mathbb{R}} (|\psi'|^2 + |\psi|^2) dx - \frac{\beta}{p^* + \epsilon + 1} \int_{\mathbb{R}} e^{-\epsilon x} |\psi|^{p^*+\epsilon+1} dx.$$

We choose, for small $\epsilon > 0$, the points ξ_i as follows:

$$\begin{aligned} \hat{\xi}_1 &= -\frac{1}{q - p^*} \log \epsilon - \log \hat{\Lambda}_1, \\ \hat{\xi}_{i+1} - \hat{\xi}_i &= -\log \epsilon - \log \hat{\Lambda}_{i+1}, \quad i = 1, \ldots, k-1 \end{aligned} \tag{34}$$

where the $\hat{\Lambda}_i$'s are positive parameters. We seek a solution of (32) of the form

$$\hat{v}(x) = \sum_{i=1}^{k} U(x - \hat{\xi}_i) + \phi(x),$$

where U is defined by (13) and ϕ is small. We set $\hat{\Lambda} = (\hat{\Lambda}_1, \hat{\Lambda}_2, \ldots, \hat{\Lambda}_k)$.

In this setting, Lemma 2.1 takes the following form.

Lemma 5.1. *Let $N \geq 3$, $\delta > 0$ fixed, $k \in \mathbb{N}$. Assume that*

$$\delta < \hat{\Lambda}_i < \delta^{-1}, \quad i = 1, 2, \ldots, k. \tag{35}$$

Then there exist positive numbers a_1, $i = 1, \ldots, 4$ and $\hat{a}_5$, depending on N, p and q, such that

$$\hat{E}_\epsilon(\hat{U}_S) = ka_1 + \epsilon\hat{\Psi}_k(\hat{\Lambda}) + k\epsilon\beta a_4 + \epsilon\hat{\theta}_\epsilon(\hat{\Lambda}) - \frac{a_3 k}{2(q-p^*)}((1-k)(q-p^*) - 2)\epsilon \log \epsilon$$

where

$$\hat{\Psi}_k(\hat{\Lambda}) = a_3 k \log \hat{\Lambda}_1 + \hat{a}_5 V_\infty \Lambda_1^{(q-p^*)} + \sum_{i=1}^{k} [(k-i+1)a_3 \log \hat{\Lambda}_i - a_2 \hat{\Lambda}_i] \tag{36}$$

and $\hat{\theta}_\epsilon(\hat{\Lambda}) \to 0$ as $\epsilon \to 0$ uniformly in C^1-sense on the set of $\hat{\Lambda}_i$'s satisfying (35). *Moreover, the constants a_i, $i = 1, 2, \ldots, 4$, are given as in* (19) *and $\hat{a}_5$ is defined by*

$$\hat{a}_5 = \frac{\beta}{q+1} \int_{\mathbb{R}} e^{-(q-p^*)x} U^{q+1}(x) dx. \tag{37}$$

If we assume that $V_\infty < 0$, then $\hat{\Psi}_k$ has a unique nondegenerate critical point given by

$$\hat{\Lambda}^* = \Big(\Big[\frac{a_3 k}{(p^* - q)\hat{a}_5 V_\infty} \Big]^{\frac{1}{q-p^*}}, \frac{(k-1)a_3}{a_2}, \frac{(k-2)a_3}{a_2}, \ldots, \frac{a_3}{a_2} \Big).$$

The finite-dimensional reduction and the conclusion of the theorem follows in a similar way to the proof of Theorem 1.1.

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