SPARSE BOUNDS FOR SPHERICAL MAXIMAL FUNCTIONS

By

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Abstract. We consider the averages of a function f on $\mathbb{R}^n$ over spheres of radius $0 < r < \infty$ given by $A_r f(x) = \int_{\mathbb{S}^{n-1}} f(x - ry) d\sigma(y)$, where σ is the normalized rotation invariant measure on $\mathbb{S}^{n-1}$. We prove a sharp range of sparse bounds for two maximal functions, the first the lacunary spherical maximal function, and the second the full maximal function.

$$M_{\text{lac}} f = \sup_{j \in \mathbb{Z}} A_{2^j} f, \quad M_{\text{full}} f = \sup_{r>0} A_r f.$$

The sparse bounds are very precise variants of the known L^p bounds for these maximal functions. They are derived from known L^p-improving estimates for the localized versions of these maximal functions, and the indices in our sparse bound are sharp. We derive novel weighted inequalities for weights in the intersection of certain Muckenhoupt and reverse Hölder classes.

1 Introduction

For a smooth function f on $\mathbb{R}^n$, let $A_r f(x) = \int_{\mathbb{S}^{n-1}} f(x - ry) d\sigma(y)$ be the average of f over the sphere centered at x and of radius r. Here, σ is normalized measure on $\mathbb{S}^{n-1}$. We consider the two maximal functions

$$M_{\text{lac}} f = \sup_{j \in \mathbb{Z}} A_{2^j} f, \quad M_{\text{full}} f = \sup_{r>0} A_r f.$$

The first is the lacunary maximal function, and the second is the full maximal function, introduced by E. M. Stein [34]. For both of these, we prove **sparse bounds.** The latter are particular quantifications of the known L^p inequalities for these operators. In particular, these bounds quickly imply novel weighted inequalities, for weights in intersections of certain Muckenhoupt and reverse Hölder classes. These inequalities are the sharpest known for these operators.

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We set notation for the sparse bounds. Call a collection of cubes $\mathcal{S}$ in $\mathbb{R}^n$ **sparse** if there are sets $\{E_S : S \in \mathcal{S}\}$ which are pairwise disjoint, $E_S \subset S$ and satisfy $|E_S| > \frac{1}{4}|S|$ for all $S \in \mathcal{S}$. For any cube Q and $1 \leq r < \infty$, set $\langle f \rangle_{Q,r}^r = |Q|^{-1} \int_Q |f|^r \, dx$. Then the $(r, s)_m$-sparse form $\Lambda_{\mathcal{S},r,s,m} = \Lambda_{r,s}$, indexed by the sparse collection $\mathcal{S}$, is

$$\Lambda_{S,r,s,m}(f, g) = \sum_{S \in \mathcal{S}} |S| \langle f \rangle_{S,r} \langle g \mathbf{1}_{F_S} \rangle_{S,s}. \tag{1.1}$$

Here, the subscript $_m$ is a reminder that the form has a maximal function component: The sets $\{F_S : S \in \mathcal{S}\}$ are a collection of pairwise disjoint sets with $F_S \subset S$ for all $S \in \mathcal{S}$ (with no requirement on a lower bound on the measure of F_S). If there is no subscript $_m$, we mean the same bilinear form, but with $\mathbf{1}_{F_S} \equiv \mathbf{1}_S$ for all cubes S. The sparse collection $\mathcal{S}$ is also frequently suppressed in the notation.

Given a sublinear operator T, and $1 \leq r, s < \infty$, we set $\|T : (r, s)_m\|$ to be the infimum over constants C so that for all bounded compactly supported functions f, g,

$$|\langle Tf, g \rangle| \leq C \sup \Lambda_{r,s,m}(f, g),$$

where the supremum is over all sparse forms. It is essential that the sparse form be allowed to depend upon f and g. But the point is that the sparse form itself varies over a class of operators with very nice properties.

We include a discussion of the lacunary maximal operator for pedagogical reasons. The following L^p bounds are well known.

Theorem A ([6, 3])**.** *For all $1 < p < \infty$, and dimensions n, we have $\|M_{\mathrm{lac}} : L^p \mapsto L^p\| < \infty$.*

The proofs for the result above compare to the Hardy–Littlewood maximal function, and pass through a square function. For the sparse bound, we will argue directly. The bounds below contains the L^p bounds as a trivial corollary, and so it represents a new proof of this fact, one that is intrinsic, in that it only uses properties of spherical averages.

Theorem 1.2. *Let $\mathbf{L}_n$ be the triangle with vertexes $(0, 1)$, $(1, 0)$ and $(\frac{n}{n+1}, \frac{n}{n+1})$. (See Figure 2.) For $n \geq 2$, and all $(\frac{1}{r}, \frac{1}{s})$ in the interior of $\mathbf{L}_n$, we have the inequality*

$$\|M_{\mathrm{lac}} : (r, s)_m\| < \infty. \tag{1.3}$$

Moreover, for $\frac{1}{r} + \frac{1}{s} > 1$ not in the closed set $\mathbf{L}_n$, the inequality (1.3) fails.

The case of the full maximal operator is more delicate. The foundational work is due to E. M. Stein, in dimensions $n \geq 3$, and Bourgain in the delicate case of $n = 2$.

Theorem B ([6, 3]). *For dimensions $n \geq 2$, we have*

$$\|M_{\text{full}} : L^p \mapsto L^p\| < \infty, \qquad \frac{n}{n-1} < p < \infty.$$

The sparse bound below is again a very precise refinement of the well-known inequalities above.

Theorem 1.4. *For $n \geq 2$, let $\mathbf{F}_n$ be the trapezium with vertexes $P_1 = (0, 1)$, $P_2 = (\frac{n-1}{n}, \frac{1}{n})$, $P_3 = (\frac{n-1}{n}, \frac{n-1}{n})$, and $P_4 = (\frac{n^2-n}{n^2+1}, \frac{n^2-n+2}{n^2+1})$. (See Figure 2.) For all $(\frac{1}{r}, \frac{1}{s})$ in the interior of $\mathbf{F}_n$, we have*

$$\|M_{\text{full}} : (r, s)_m\| < \infty. \tag{1.5}$$

Moreover, for $\frac{1}{r} + \frac{1}{s} > 1$ not in the closed set $\mathbf{F}_n$, the inequality (1.5) *fails.*

One of the great advantages of sparse bounds is that one can easily derive weighted inequalities for sparse operators, indeed inequalities with sharp dependence upon the Muckenhoupt and reverse Hölder constants. We will discuss this in detail in §6. Weighted inequalities for the spherical maximal function in the category of Muckenhoupt and reverse Hölder classes has been studied in [8, 13]. We recover and extend their results using the sparse bound. See for instance Proposition 6.7

Sparse bounds for different operators is a recent topic of research. These arguments have delivered the most powerful known proof [19] of the A_2 conjecture. They quickly prove sharp weighted estimates for commutators [23]. In other settings, they establish weighted inequalities [9] for the bilinear Hilbert transform, as well as other objects in phase plane analysis [12]. Some of these arguments are rather short and elegant, using familiar TT^* style arguments [7] to provide remarkably sharp control of rough singular integrals. Also see [16, 25, 17] for further work in this direction. In the setting of Radon transforms, the paper [10] discusses a particular arithmetic example, showing that sparse bounds are possible in that setting. Random examples have been considered in [18, 21, 15]. This paper proves the first sparse bounds for a Radon transform in the continuous case.

Our sparse bounds are sharp in the scale of L^p averages. Sharper results can be obtained using local Lorentz–Orlicz averages at the endpoint cases. The latter is the focus of the article of Richard Oberlin [29]. Given the close association between sparse bounds and weighted inequalities in other settings, one then suspects that the weighted inequalities that follow are the best possible in the category of Muckenhoupt and reverse Hölder classes. In another direction, the core innovation is the identification of the central role of the L^p improving inequalities. The sharp range of improving inequalities are known for a wide range of Radon transforms. Many of these can now be extended to sparse bounds for allied maximal functions.

We prove the sparse bounds for M_{lac} first, followed by that for M_{full}. Both use the same tool, the L^p improving mapping properties of the unit scale version of the maximal operators. In fact, we need a 'continuity' version of these inequalities. These appear to be new, and are proved in §4. Once the continuity inequalities are established, the remaining argument is a variant, but not a corollary, of the innovative paper of Conde, Culiuc, Di Plinio and Ou [7]. The argument is presented in detail. We then turn to the consequences for weighted inequalities in §6. A final section includes various complements.

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2 The Lacunary case

The argument has two components, one being a (small) improvement to the classical L^p-improving properties of the spherical averages due to Littman [26] and Strichartz [35]. We set $\mathbf{L}_n$ to be the triangle with vertexes $(0, 1)$, $(1, 0)$ and $(\frac{n}{n+1}, \frac{n}{n+1})$. Consider the dual to $\mathbf{L}_n$, defined by $\mathbf{L}_n' = \{(\frac{1}{p}, \frac{1}{q}) : (\frac{1}{p}, 1 - \frac{1}{q}) \in \mathbf{L}_n\}$. See Figure 2.

Theorem C ([26, 35])**.** *For any point $(\frac{1}{r}, \frac{1}{s})$ in the closed triangle $\mathbf{L}_n'$, there holds*

$$\|A_1 : L^r \mapsto L^s\| < \infty.$$

The inequality strengthens as s increases. In particular, the critical case is vertex $(\frac{1}{r}, \frac{1}{s}) = (\frac{n}{n+1}, \frac{1}{n+1})$. The improvement is a 'continuity' condition, namely the inequality is preserved, with a small gain, under small translations. Let $\tau_y f(x) = f(x - y)$ be the translation of f by y.

Theorem 2.1. *Let $\mathbf{L}_n'$ be the closed triangle with vertexes $(0, 0)$, $(1, 1)$ and $(\frac{n}{n+1}, \frac{1}{n+1})$. For $(\frac{1}{r}, \frac{1}{s})$ in the interior of $\mathbf{L}_n'$ we have the inequalities*

$$\|A_1 - \tau_y A_1 : L^r \mapsto L^s\| \lesssim |y|^\eta, \quad |y| \le 1, \tag{2.2}$$

for a choice of $\eta = \eta(n, r, s) > 0$.

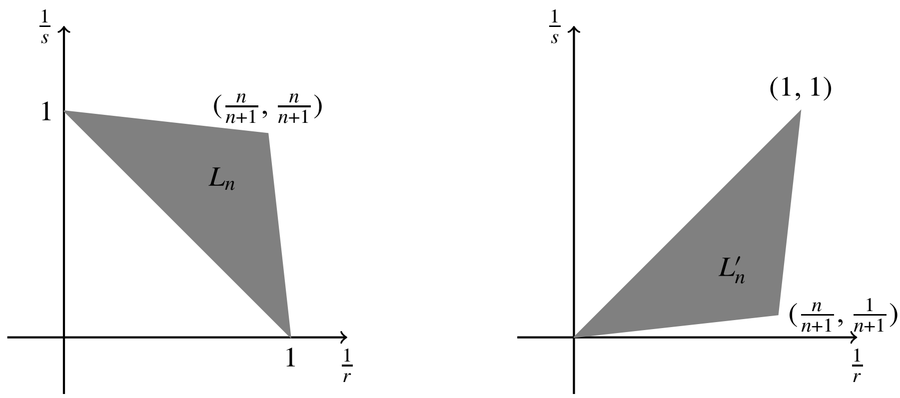

Figure 1. The triangle L_n on the left, and L'_n on the right.

A proof is presented in §4. We need a scale invariant version of the inequalities above, which is very easy to prove by a change of variables.

Lemma 2.3. *Let f_1, f_2 be supported on a cube Q, and let $t \simeq \ell Q$. For $(\frac{1}{r}, \frac{1}{s})$ as in the interior of $\mathbf{L}_n$, we have*

$$|\langle A_t f - A_t \tau_y f_1, f_2\rangle| \lesssim |y/\ell Q|^\eta |Q| \langle f_1\rangle_{Q,r} \langle f_2\rangle_{Q,s}, \quad |y| \leq \ell Q. \tag{2.4}$$

We set some notation for the statement of the main lemma. For a cube Q with side length 2^q, for $q \in \mathbb{Z}$, let

$$A_Q f = A_{2^{q-2}}(f \mathbf{1}_{\frac{1}{3}Q}).$$

It is important for the proof below that the support of $A_Q f$ is contained in Q. There are a choice of 3^n dyadic grids $\mathcal{D}_1, \ldots, \mathcal{D}_{3^n}$ so that

$$A_{2^{q-2}} f = \sum_{t=1}^{3^n} \sum_{Q \in \mathcal{D}_t : \ell Q = 2^q} A_Q f.$$

Therefore, it suffices to prove the sparse bound for each of the maximal operators

$$M_{\mathcal{D}_t} f := \sup_{Q \in \mathcal{D}_t} A_Q f, \quad 1 \leq t \leq 3^n. \tag{2.5}$$

The specific dyadic grid in question is immaterial, so we fix such a grid below, and write $\mathcal{D} = \mathcal{D}_t$. This is the kernel of the proof.

Lemma 2.6. *Let* $1 < r, s < \infty$ *be as in Theorem 1.2, and let* $C_0 > 1$ *be a constant. Let* $\mathcal{Q}$ *be a collection of sub cubes of* $Q_0 \in \mathcal{D}$ *for which*

$$\sup_{Q' \in \mathcal{Q}} \sup_{Q : Q' \subset Q \subset Q_0} \left\{ \frac{\langle f_1 \rangle_{Q,r}}{\langle f_1 \rangle_{Q_0,r}} + \frac{\langle f_2 \rangle_{Q,s}}{\langle f_2 \rangle_{Q_0,s}} \right\} < C_0.$$

Then,

$$\Big\langle \sup_{Q \in \mathcal{Q}} A_Q f_1, f_2 \Big\rangle \lesssim |Q_0| \langle f_1 \rangle_{Q_0,r} \langle f_2 \rangle_{Q_0,s}. \tag{2.7}$$

Proof. By homogeneity, we can assume $\langle f_1 \rangle_{Q_0,r} = \langle f_2 \rangle_{Q_0,s} = 1$. The supreumum is linearized. Thus, for pairwise disjoint sets $\{F_Q : Q \in \mathcal{Q}\}$ with $F_Q \subset Q$, set $f_Q = f_2 \mathbf{1}_{F_Q}$. We estimate

$$\sum_{Q \in \mathcal{Q}} \langle A_Q f_1, \mathbf{1}_{F_Q} f_2 \rangle. \tag{2.8}$$

We take $\mathcal{B}$ to be the maximal dyadic subcubes of Q_0 so that we have

$$\langle f_1 \rangle_{Q,r} + \langle f_Q \rangle_{Q,s} > 2C_0. \tag{2.9}$$

Perform a standard Calderón–Zygmund decomposition on f_1. Set $f_1 = g_1 + b_1$ where

$$b_1 = \sum_{P \in \mathcal{B}} (f_1 - \langle f_1 \rangle_P) \mathbf{1}_P = \sum_{k=-\infty}^{q_0 - 1} \sum_{P \in \mathcal{B}(k)} (f_1 - \langle f_1 \rangle_P) \mathbf{1}_P =: \sum_{k=-\infty}^{q_0-1} B_{1,k}, \tag{2.10}$$

where above we write $\ell Q_0 = 2^{q_0}$, and set $\mathcal{B}(k) = \{P \in \mathcal{B} : \ell P = 2^k\}$.

The bilinear expression in (2.8) is dominated by a sum of two terms. The first places the good function g_1 in the first place. It is a bounded function, so that

$$\sum_{Q \in \mathcal{Q}} |\langle A_Q g_1, f_Q \rangle| \lesssim \sum_{Q \in \mathcal{Q}} \|f_2 \mathbf{1}_{F_Q}\|_1 \lesssim |Q_0|.$$

This just depends upon the disjointness of the sets F_Q.

The second has b_1 in the first position. We have this following easy, but essential, fact: For all $Q \in \mathcal{Q}$ and $P \in \mathcal{B}$, if $Q \cap P \neq \emptyset$, then $P \subsetneq Q$. Therefore, for any $Q \in \mathcal{Q}$, with $\ell Q = 2^q$, we have, using the notation of (2.10),

$$\langle A_Q b_1, f_Q \rangle = \sum_{k : k < q} \langle A_Q B_{1,k}, f_Q \rangle = \sum_{k=1}^{\infty} \langle A_Q B_{1,q-k}, f_Q \rangle.$$

Therefore,

$$\Big| \sum_{Q \in \mathcal{Q}} \langle A_Q b_1, f_Q \rangle \Big| \leq \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{Q}} |\langle A_Q B_{1,q-k}, f_Q \rangle|. \quad (\ell Q = 2^q)$$

We achieve the desired bound, with geometric decay in k, derived from our continuity inequalities. For $Q \in \mathcal{Q}$, with $\ell Q = 2^q$, we estimate as follows, using the mean zero properties of the bad functions.

$$
\begin{aligned}
&|\langle A_Q B_{1,q-k}, f_Q \rangle| \\
&\quad = |\langle B_{1,q-k}, A_Q^* f_Q \rangle| \\
&\quad = \sum_{P \in \mathcal{B}(q-k)} \frac{1}{|P|} \Big| \int_P \int_P [A_Q^* f_Q(x) - A_Q^* f_Q(x')] \cdot B_{1,q-k}(x) dx dx' \Big| \\
&\quad \lesssim \frac{1}{|P_0|} \Big| \int [A_Q^* f_Q(x) - \tau_y A_Q^* f_Q(x)] \cdot B_{1,q-k}(x) dx \Big| dy \\
&\quad \lesssim 2^{-\eta k} |Q| \langle B_{1,q-k} \mathbf{1}_Q \rangle_{Q,r} \langle f_Q \rangle_{Q,s}.
\end{aligned}
\tag{2.11}
$$

Above, P_0 is the cube of side length 2^{q-k+1} centered at the origin, and we use our continuity inequality (2.2).

It remains to argue that uniformly in $k \geq 1$,

$$
\sum_{Q \in \mathcal{Q}} |Q| \langle B_{1,q-k} \mathbf{1}_Q \rangle_{Q,r} \langle f_2 \mathbf{1}_{F_Q} \rangle_{Q,s} \lesssim |Q_0|.
$$

This follows from (a) the disjointness of the sets F_Q, (b) the disjointness of the supports of $B_{1,k}\mathbf{1}_Q$, for $k \geq 1$ fixed, and (c) $1/r + 1/s \geq 1$. In particular, the inequality is clear in the case of $1/r + 1/s = 1$, and also clear in the case of $\min\{r, s\} = 1$, so that the remaining cases follow by interpolation. □

Proof of Theorem 1.2. We deduce the m-sparse bound for the operator $M_{\mathcal{D}}$ in (2.5). From this it follows that M_{lac} is bounded by the sum of a finite number of sparse forms. But, the principle described in (5.2) shows that there is a constant C, so that given f, g, there is a fixed sparse form $\Lambda_{\mathcal{S}_0,r,s}$, so that

$$
\sup_{\mathcal{S}} \Lambda_{\mathcal{S},r,s}(f, g) \leq C \Lambda_{\mathcal{S}_0,r,s}(f, g).
$$

Thus, the sparse bound as claimed will follow.

We can assume that f_1, f_2 are bounded functions supported on a dyadic cube $Q_0 \in \mathcal{D}$. Indeed, we can even assume that for any cube $Q \supsetneq Q_0$, we have $A_Q f \equiv 0$. Namely, for the construction of the sparse bound, we need only consider cubes $Q \subset Q_0$.

We then add the cube Q_0 to $\mathcal{S}$. We take the $\mathcal{S}$-children of Q_0 to be the collection $\mathcal{E}$ of maximal children $P \subsetneq Q_0$ for which $\langle f_1 \rangle_{P,r} > C_n \langle f_1 \rangle_{Q_0,r}$, or $\langle f_2 \rangle_{P,s} > C_n \langle f_2 \rangle_{Q_0,s}$. Let E be the union of these maximal children. For a choice of constant $C_n > 1$, we have $|E| < \frac{1}{2}|Q_0|$. Set $\mathcal{Q} = \{P \subset Q_0 : P \not\subset E\}$. Associated to the set Q_0 we need the set

$$F_{Q_0} = \{x \in Q_0 : M_{\mathcal{D}} f(x) = \sup_{Q \in \mathcal{Q}} A_Q f(x)\}.$$

Then apply Lemma 2.6 to the collection $\mathcal{Q}$, with the second function being $f_2 \mathbf{1}_{F_{Q_0}}$. We see from (2.7), and the support condition on $A_Q f$, that it remains to recurse inside the cubes $\mathcal{E}$. The proof is complete. □

3 The full supremum

The analog of the L^p-improving properties of A_1 in Theorem C concern the 'unit scale' maximal function $\tilde{M} f = \sup_{1 \le t \le 2} A_t f$. This is due to Schlag [30]; also see Schlag and Sogge [31].

Theorem D. *Let $\mathbf{F}'_n$ be the closed convex hull of the four points $P'_1 = (0, 0)$, $P'_2 = (\frac{n-1}{n}, \frac{n-1}{n})$, $P'_3 = (\frac{n-1}{n}, \frac{1}{n})$, and $P'_4 = (\frac{n^2-n}{n^2+1}, \frac{n-1}{n^2+1})$. For all $(\frac{1}{r}, \frac{1}{s})$ in $\mathbf{F}'_n$, we have*

$$\|\tilde{M} : L^r \mapsto L^s\| < \infty. \tag{3.1}$$

This 'continuity property' is a corollary.

Theorem 3.2. *For all $(\frac{1}{r}, \frac{1}{s})$ in the interior of $\mathbf{F}'_n$, for some $\eta = \eta(n, r, s) > 0$, we have*

$$\| \sup_{1 \le t \le 2} |A_t f - \tau_y A_t f| \|_s \lesssim |y|^{\eta} \|f\|_r, \quad |y| < 1. \tag{3.3}$$

We will delay the proof of this theorem to the next section. See Figure 2 for a picture of the trapeziums $\mathbf{F}_n$ and $\mathbf{F}'_n$.

We again make a dyadic reduction. For a cube Q with side length 2^q, for $q \in \mathbb{Z}$, let

$$\tilde{M}_Q f = \sup_{2^{q-3} \le t < 2^{q-2}} A_t(f \mathbf{1}_{\frac{1}{3}Q}), \quad \ell Q = 2^q.$$

There are a choice of 3^n dyadic grids $\mathcal{D}_1, \ldots, \mathcal{D}_{3^n}$ so that

$$\sup_{2^{q-3} \le t < 2^{q-2}} A_t(f \mathbf{1}_{\frac{1}{3}Q}) \le \sum_{s=1}^{3^n} \sum_{Q \in \mathcal{D}_s : \ell Q = 2^q} \tilde{M}_Q f.$$

Therefore, it suffices to prove the sparse bound for each of the maximal operators

$$M_{\mathcal{D}_s} f := \sup_{Q \in \mathcal{D}_t} \tilde{M}_Q f, \quad 1 \le s \le 3^n.$$

We fix such a grid below, and write $\mathcal{D} = \mathcal{D}_s$. The main Lemma is as before. We will prove it, and leave the details of the derivation of Theorem 1.4 to the reader.

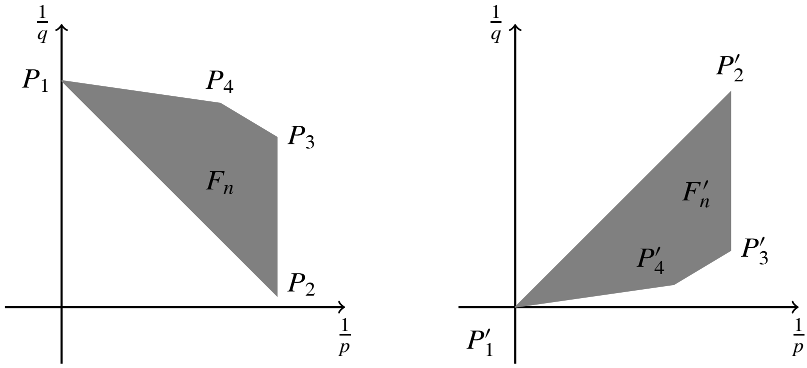

Figure 2. The trapezium F_n on the left, and F'_n on the right. (When $n = 2$, they are in fact triangles.)

Lemma 3.4. *Let* $(\frac{1}{r}, \frac{1}{s})$ *be in the interior of* $\mathbf{F}_n$. *Let* $\mathcal{Q} \subset \mathcal{D}$ *be a collection of sub cubes of* Q_0 *so that*

$$\sup_{Q' \in \mathcal{Q}} \sup_{Q : Q' \subset Q \subset Q_0} \left\{ \frac{\langle f_1 \rangle_{Q,r}}{\langle f_1 \rangle_{Q_0,r}} + \frac{\langle f_2 \rangle_{Q,s}}{\langle f_2 \rangle_{Q_0,s}} \right\} < C_0.$$

Then, there holds

$$\left\langle \sup_{Q \in \mathcal{Q}} \tilde{M}_Q f_1, f_2 \right\rangle \lesssim |Q_0| \langle f_1 \rangle_{Q_0,r} \langle f_2 \rangle_{Q_0,s}.$$

Proof. The proof closely follows the lines of the proof of Lemma 2.6. Assume $\langle f_1 \rangle_{Q_0,r} = \langle f_2 \rangle_{Q_0,s} = 1$. Define the collection of 'bad' cubes $\mathcal{B}$ as in (2.9). We bound the bilinear form

$$\sum_{Q \in \mathcal{Q}} \langle \tilde{M}_Q f_1, f_Q \rangle, \tag{3.5}$$

where $\{F_Q : Q \in \mathcal{Q}\}$ is a family of disjoint sets with $F_Q \subset Q$, and $f_Q = \mathbf{1}_{F_Q} f_2$.

Use the Calderón–Zygmund decomposition, just like in (2.10). The bilinear form in (3.5) is divided into two terms, of which the first has the good function g_1 in the first place:

$$\sum_{Q \in \mathcal{Q}} |\langle \tilde{M}_Q g_1, \mathbf{1}_{F_Q} f_2 \rangle| \lesssim \sum_{Q \in \mathcal{Q}} \int_{F_Q} |f_2| dx \lesssim |Q_0|.$$

The second term has b_1 in the first place, and f_Q in the second. Namely, we have to bound

$$\sum_{Q\in\mathcal{Q}} |\langle \tilde{M}_Q b_1, f_Q\rangle| \le \sum_{k=1}^{\infty}\sum_{Q\in\mathcal{Q}} |\langle \tilde{M}_Q B_{1,q-k}, f_Q\rangle| \quad (\ell Q = 2^q).$$

Above, we have used the expansion in (2.10). We will use the continuity inequality (3.3) to establish the desired bound with geometric decay in k. Let us argue by duality. For each $Q \in \mathcal{Q}$ we can replace $\tilde{M}_Q$ by $L_Q\phi(x) = A_{t_Q(x)}\phi(x)$, where $t_Q : \frac{1}{3}Q \mapsto [2^{q-2}, 2^{q-1}]$ is measurable. Then, estimate

$$\begin{aligned}
|\langle L_Q B_{1,q-k_1}, f_Q\rangle| &= |\langle B_{1,q-k}, L_Q^* f_Q\rangle| \\
&\le \sum_{\substack{P\in\mathcal{B}(q-k)\\ P\subset Q}} \frac{1}{|P|}\int_P \left| \int_P B_{1,q-k_1}(x)\cdot (L_Q^* f_Q(x) - L_Q^* f_Q(x'))dx \right| dx' \\
&\lesssim 2^{-\eta k}|Q|\langle B_{1,q-k}\rangle_{Q,r}\langle f_Q\rangle_{Q,s}.
\end{aligned}$$

Here, the notation is similar to (2.11), and we appeal to the scale-invariant and dual form of (3.3). The remainder of the argument is exactly as in the proof of Lemma 2.6. □

4 Proof of the continuity inequalities

4.1 Proof of Theorem 2.1. From Plancherel's theorem, we have

$$\|A_1 f - A_1\tau_y : L^2 \mapsto L^2\| = \|(1-e^{iy\cdot\xi})\widehat{d\sigma}(\xi)\|_\infty \lesssim |y|^{\eta_0}, \qquad \eta_0 = \eta_o(n) > 0.$$

To see this last inequality, we need only appeal to the well known decay estimate for $|\widehat{d\sigma}(\xi)|$ which we recall below.

In interpolation between this L^2 estimate and the L^r improving estimates of Theorem C, it is clear that the conclusion (2.2) holds for $(\frac{1}{r}, \frac{1}{s})$ in the interior of the triangle $\mathbf{L}'_n$.

4.2 Proof of Theorem 3.2. We recall that the Fourier transform of σ, the uniform measure on the sphere $\mathbb{S}^{n-1}$, is

$$\widehat{d\sigma}(\xi) = e^{-i|\xi|}a_-(\xi) + e^{i|\xi|}a_+(\xi), \tag{4.1}$$

where $|\partial^\alpha a_\pm(\xi)| \lesssim (1+|\xi|)^{-(n-1)/2-|\alpha|}$.

The trapezium $\mathbf{F}'_n$ is contained in the triangle $\mathbf{L}'_n$. Thus, if $\mathcal{T} \subset [1,2]$ is a finite set, it follows from Theorem 2.1 that we have

$$\|\sup_{t\in\mathcal{T}}|A_t f - \tau_y A_t f|\|_{p_2} \lesssim \sharp(\mathcal{T})^{1/p_2}\cdot |y|^\eta \|f\|_{p_1}, \qquad \left(\frac{1}{p_1}, \frac{1}{p_2}\right) \in \mathbf{F}'_n \setminus \{(0,1),(1,0)\}.$$

Taking $\mathcal{T}$ be a $|y|^\eta$-net in $[1,2]$, it clearly suffices to show this modulus of continuity result.

Proposition 4.2. *Subject to $(\frac{1}{r}, \frac{1}{s})$ satisfying the hypotheses of Theorem 3.2, there is a $\eta > 0$ so that for all $0 < \delta < \frac{1}{2}$, we have*

$$\Big\| \sup_{\substack{s,t\in[1,2]\\ |s-t|<\delta}} |A_t f - A_s f| \Big\|_s \lesssim \delta^\eta \|f\|_r. \tag{4.3}$$

The proof in dimensions $n \geq 3$. It suffices to prove a version of (4.3) at the point $(\frac{1}{2}, \frac{1}{2})$, and then interpolate to the other points in the interior of $\mathbf{F}'_n$. Using (4.1) and Plancherel, we see that there is a full derivative in t:

$$\|\partial_t A_t f\|_{L^2(\mathbb{R}^n\times[1,2))} \lesssim \|f\|_2.$$

It follows that for each $x \in \mathbb{R}^n$, $A_t f(x)$ continuously embeds as a function of t into the class $\mathrm{Lip}(\frac{1}{4})$, so that (4.3) follows.

The proof in dimension $n = 2$. We rely upon the detailed analysis of Sanghyuk Lee [22], which refines the work of Schlag [30] and Schlag–Sogge [31] in the convolution setting. Again, we prove the estimate (4.3) at a single point in the triangle $\mathbf{F}'_2$, and obtain the result as stated by interpolation.

A Littlewood–Paley decomposition is needed. Let $\mathbf{1}_{[1,2]} \leq \zeta \leq \mathbf{1}_{[\frac{1}{2},\frac{5}{2}]}$ be a smooth function on $\mathbb{R}$ so that $\sum_{j\geq 1} \zeta(y/2^j) = 1$, if $|y| \geq 4$. Then set $\zeta_0 = 1 - \sum_{j\geq 1}\zeta(y/2^j)$. For $f \in L^2(\mathbb{R}^n)$, set $\widehat{f_j}(\xi) = \zeta(|\xi|/2^j)\widehat{f}(\xi)$, for $j \geq 1$, and $\widehat{f_0} = \zeta_0 \widehat{f}$.

Let $\mathcal{M}_\delta$ be the maximal function in (4.3), and let $\mathcal{M}_{\delta,j} f = \mathcal{M}_\delta f_j$. We have

$$\mathcal{M}_\delta f \leq \sum_{j\geq 0} \mathcal{M}_{\delta,j} f.$$

Now, it follows from [22, just above eqn. (1.5)] that

$$\|M_{\delta,j} : L^p \mapsto L^q\| \lesssim 2^{j(1-\frac{5}{q})}, \qquad \frac{1}{p} + \frac{3}{q} = 1, \; q > \frac{14}{3}. \tag{4.4}$$

The exponent on j above is negative for $\frac{14}{3} < q < 5$. At $q = 5$, we have $(p,q) = (\frac{5}{2}, 5)$, which corresponds to the crucial vertex $(\frac{2}{5}, \frac{1}{5})$ of the triangle $\mathbf{F}'_2$. See Figure 3.

It again follows from (4.1) that

$$\|\partial_t A_t f_j\|_{L^2(\mathbb{R}^2\times[1,2))} \lesssim 2^{\frac{j}{2}} \|f\|_2.$$

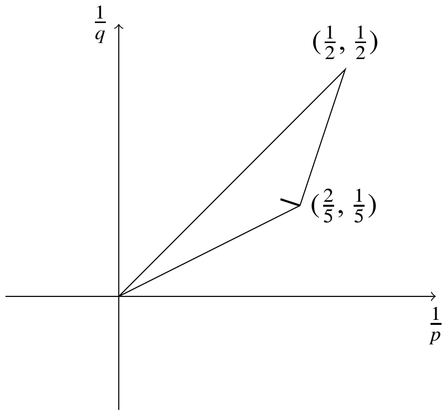

Figure 3. The triangle $\mathbf{F}_2'$, and the elements of the proof of Theorem 3.2 in the case of dimension 2. The thick line inside the triangle goes from $(\frac{5}{14}, \frac{3}{14})$ to $(\frac{2}{5}, \frac{1}{5})$. The point $(\frac{1}{3}, \frac{2}{9})$ is on the thick line, and we will interpolate between an estimate at that point, and an estimate at $(\frac{1}{2}, \frac{1}{2})$.

As a consequence, $A_t f_j$ continuously embeds into $\mathrm{Lip}(\frac{1}{4})$ with norm at most $2^{j/2}$. That is, we have the bound

$$\|M_{\delta,j} : L^2 \mapsto L^2\| \lesssim \delta^{\frac{1}{4}} 2^{\frac{j}{2}}.$$

Interpolation with (4.4), say with $p = 3, q = \frac{9}{2}$, shows that with $(\frac{1}{p}, \frac{1}{q})$ sufficiently close to $(\frac{1}{3}, \frac{2}{9})$, we have for a positive choice of $\eta > 0$,

$$\|M_{\delta,j} : L^p \mapsto L^q\| \lesssim \delta^{\eta} 2^{-\eta j}.$$

This is summable in $j \geq 0$, so completes our proof.

5 Sharpness of the sparse bounds

Sharpness of the sparse bounds is not immediate from the sharpness of the L^p improving estimates, as the sparse bound is defined as the largest possible sparse bound. Nevertheless, sharpness will follow from the examples that show that the L^p improving estimates are sharp.

Proposition 5.1. *Suppose that* $1 \leq r, s < \infty$ *satisfy* $\frac{1}{r} + \frac{1}{s} \geq 1$.

(1) *If the sparse bound* $\|M_{\mathrm{lac}} : (r, s)_m\| < \infty$ *holds, then* $(\frac{1}{r}, \frac{1}{s}) \in \mathbf{L}_n$, *where the last set is the triangle defined in Theorem 1.2.*

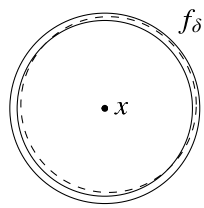

Figure 4. The example showing sharpness of the bounds in Theorem 2.1. The function f_δ is the indicator of the thin annulus, of width δ. For a point x within say $\delta/2$ of the center of the annulus, one has $A_1 f_\delta(x) \geq c$. The dashed circle is centered at x, and has radius 1. At least $\frac{1}{4}$ of the dashed circle is inside the support of f_δ. This leads to the inequality (5.3).

(2) *If the sparse bound* $\|M_{\text{full}} : (r, s)_m\| < \infty$ *holds, then* $(\frac{1}{r}, \frac{1}{s}) \in \mathbf{F}_n$*, where the latter set is the trapezium defined in Theorem 1.4.*

We recall this elementary fact, [20, Lemma 4.7]. For all $1 \leq r, s < \infty$, there is a constant C so that for all f and g, there is a sparse form Λ_0 so that

$$\sup_{\mathcal{S}} \Lambda_{\mathcal{S},r,s}(f, g) \leq C\Lambda_0(f, g). \tag{5.2}$$

For the pairs f, g that we describe below, it will be very easy to verify this principle. The largest sparse form Λ_0 will consist of a single cube, namely one that contains the support of the functions defined below, and is of minimal side length.

Proof of Proposition 5.1(1). We begin with the lacunary maximal operator, M_{lac}, and the L^p-improving bounds of Littman [26] and Strichartz [35]. For $0 < \delta < \frac{1}{4}$, let $f_\delta = \mathbf{1}_{||x|-1|<\delta}$ be the indicator of a thin annulus around the unit circle. Note that for small absolute constant c, we have

$$A_1 f_\delta(x) \geq c g_\delta(x) = c\mathbf{1}_{|x|<c\delta}.$$

This example is illustrated in Figure 4. It establishes the sharpness of exponents r and s in Theorem 2.1. Suppose that M_{lac} satisfies an (r, s)-bound, where $1/r + 1/s > 1$. We then have

$$\delta^n \lesssim \langle A_1 f_\delta, g_\delta\rangle = \langle f_\delta, A_1 g_\delta\rangle \lesssim \min\{\Lambda_{\mathcal{S},r,s}(f_\delta, g_\delta), \Lambda_{\mathcal{S}',s,r}(f_\delta, g_\delta)).$$

for some choice of sparse collections $\mathcal{S}$ and $\mathcal{S}'$. Note that we have two bounds on the right, due to the convolution structure of the question.

But each cube in the collections $\mathcal{S}$ and $\mathcal{S}'$ should intersect the support of f and of g. That is, we can assume that $\{x : |x| < 2\} \subset Q$, for each $Q \in \mathcal{S}$. But then, the

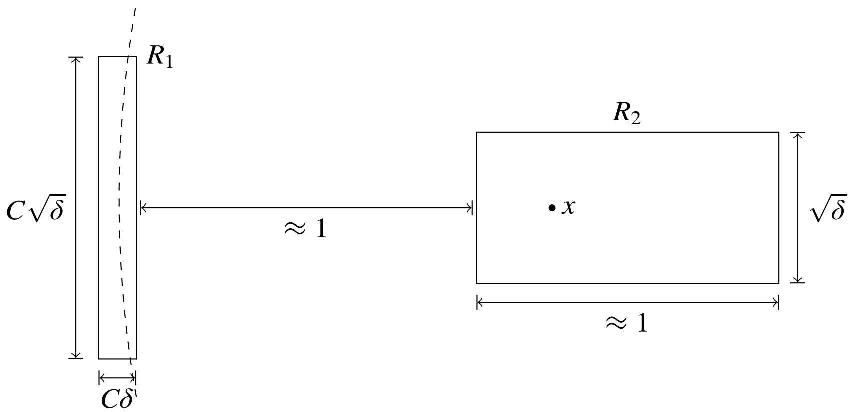

Figure 5. An example for the operator $\tilde{M}$. The rectangle R_1 is on the left, and at each point $x \in R_2$, there is a circle of radius $1 \le r \le 2$ which intersects a substantial portion of the rectangle R_1, as indicated by the dashed arc of a circle. We have $\tilde{M}\mathbf{1}_{R_1}(x) \gtrsim \delta^{\frac{n-1}{2}}$. The assumed (r, s) bound leads to the restriction (5.4).

contribution of such cubes decreases as the side length of the cube increases. So, it suffices to have $\mathcal{S}$ to consist of just a single cube Q of side length, 2 say. Our assumption leads to the conclusion

$$\delta^n \lesssim \langle A_1 f_\delta, g_\delta\rangle \lesssim \min\{ \|f_\delta\|_r \|g_\delta\|_s, \|f_\delta\|_s \|g_\delta\|_r\} \lesssim \delta^{\max\{\frac{1}{r}+\frac{n}{s}, \frac{n}{r}+\frac{1}{s}\}}.$$

We conclude that we need to have the inequality below, which tells us that $(\frac{1}{r}, \frac{1}{s}) \in \mathbf{L}_n$:

$$\max\left\{\frac{1}{r}+\frac{n}{s}, \frac{n}{r}+\frac{1}{s}\right\} \le n. \tag{5.3}$$

And so, we cannot do better than the L^p-improving bounds of Littman and Strichartz for the lacunary maximal function. □

Proof of Proposition 5.1(2). We turn to the case of the full spherical maximal function. The sharpness of the trapezium in Theorem 3.2 is given by three examples. One of these is the thin annulus example just used, and this demonstrates the sharpness along the line from $P_1 = (0, 1)$ to $P_4 = (\frac{n^2-n}{n^2+1}, \frac{n^2-n+2}{n^2+1})$. Here, we are referring to the trapezium $\mathbf{F}_n$ in Figure 2.

The second example is a Knapp type example illustrated in Figure 5. Define two rectangles by

$$R_1 = [-C\sqrt{\delta}, C\sqrt{\delta}]^{n-1} \times [-C\delta, C\delta], \quad R_2 = [-\sqrt{\delta}, \sqrt{\delta}]^{n-1} \times \left[\frac{4}{3}, \frac{5}{3}\right].$$

Then, note that the localized maximal function $\tilde{M}$ applied to $\mathbf{1}_{R_1}$ satisfies $\tilde{M}\mathbf{1}_{R_1} \gtrsim \delta^{\frac{n-1}{2}}\mathbf{1}_{R_2}$. Then, assuming the (r, s) sparse bound for the full maximal function, we have

$$\delta^{n-1} \lesssim \langle \tilde{M} f, g\rangle \lesssim \Lambda_{\mathcal{S},r,s}(\mathbf{1}_{R_1}, \mathbf{1}_{R_2}).$$

The sparse form on the right is largest, up to a constant, taking $\mathcal{S}$ to consist of a single cube of bounded side length, which contains the two rectangles R_1 and R_2. We deduce that

$$\delta^{n-1} \lesssim |R_1|^{1/r}|R_2|^{1/s} \lesssim \delta^{\frac{n+1}{2r}+\frac{n-1}{2s}}.$$

From this, we see that we necessarily must have

$$\frac{n+1}{r} + \frac{n-1}{s} \leq 2(n-1). \tag{5.4}$$

This gives the restriction on the line from the point P_4 to $P_3 = (\frac{n-1}{n}, \frac{n-1}{n})$.

A third example of Stein is the function $h(x) = \mathbf{1}_{|x|<1}|x|^{1-n}(\log|x|)^{-1}$; we have $M_{\text{full}}h(x)$ is infinite on a set of positive measure. Hence, M_{full} is unbounded on L^p, for $1 < p \leq \frac{n}{n-1}$. Now, if M_{full} satisfies an (r, s) bound for any $1 < r \leq \frac{n}{n-1}$ and any finite s, it would follow that M_{full} is of weak-type L^r, which is impossible. This shows the sharpness of the line from P_2 toP_3. □

These examples also show that the 'continuity' condition cannot hold at the critical indexes for the L^p improving inequalities.

Proposition 5.5. *Suppose that* $1 \leq r, s < \infty$ *satisfy* $\frac{1}{r} + \frac{1}{s} > 1$.

(1) *If the inequality* (2.4) *holds, then* $(\frac{1}{r}, \frac{1}{s})$ *is in the interior of* $\mathbf{L}_n$*, the triangle defined in Theorem 1.2.*

(2) *If the inequality* (3.3) *holds, then,* $(\frac{1}{r}, \frac{1}{s})$ *is in the interior of* $\mathbf{F}_n$*, where the latter set is the trapezium defined in Theorem 1.4.*

Proof. This is a corollary to the fact that the relevant examples in the L^p improving estimates are supported on small sets.

(1) Suppose that $(\frac{1}{r}, \frac{1}{s})$ is on the boundary of $\mathbf{L}_n$, which is to say that it satisfies equality (5.3). We have the assumed inequality (2.2) with $|y|$ much smaller than one. Apply it to the function f_δ, where δ is much smaller than $|y|$. It follows that there is no cancellation after translation by y, so that

$$\|A_1 f_\delta - \tau_y A_1 f_\delta\|_s \simeq \|f_\delta\|_r \lesssim |y|^\eta \|f_\delta\|_r.$$

This is a contradiction.

(2) Suppose that $(\frac{1}{r}, \frac{1}{s})$ is on the boundary of $\mathbf{F}_n$, and that we have the assumed inequality (3.3). It follows from the first part of the argument that $(\frac{1}{r}, \frac{1}{s})$ cannot lie

on the line from P_1 to P_4, where we are referring to the points in Figure 2. By the example of Stein described above, it cannot lie on the line from P_2 to P_3. And, by a similar argument to the one above, but using the example from Figure 5, it also follows that $(\frac{1}{r}, \frac{1}{s})$ cannot lie on the line from P_3 to P_4. This is a contradiction, so the argument is complete. □

6 Weighted inequalities

The maximal function M_{lac} applied to the indicator of a ball B of radius 1 centered at the origin is dominated by

$$M_{\mathrm{lac}}\mathbf{1}_B(x) \lesssim \mathbf{1}_{2B}(x) + \sum_{k=1}^{\infty} 2^{-k(n-1)}\mathbf{1}_{||x|-2^k|\le 2}.$$

Thus, there is no reason to think that Muckenhoupt weights are the correct tool to understand the behavior of this (or the full) spherical maximal function in weighted spaces. (See Figure 5 for an example showing that the full supremum is poorly adapted to Muckenhoupt weights.)

Nevertheless, the question of weighted inequalities for weights of Muckenhoupt type has attracted interest [13, 8]. And the sparse bounds are especially efficient for such weights. We detail here some of the implications of our main theorems in this direction. We will see that our sparse bound contains the best known prior bound for M_{full}, and yields new information. The full implications would be a little technical, and so we do not develop them here.

We indicate here how easy it is to prove L^p bounds for sparse forms, and leave the details of the weighted case to the references. The familiar L^p bounds for the spherical maximal functions are seen to trivially follow from our sparse bounds.

Proposition 6.1. *Let $1 \le r < p < s' < \infty$. We have the inequality*

$$\Lambda_{r,s}(f, g) \lesssim \|f\|_p \|g\|_{p'}.$$

Proof. The notation for the sparse form is in (2.8). Recall that to each cube Q in the sparse collection $\mathcal{S}$, there is a set $E_Q \subset Q$, with $|E_Q| \ge \frac{1}{2}|Q|$, so that the sets $\{E_Q : Q \in \mathcal{S}\}$ are pairwise disjoint. Thus

$$\begin{aligned}\Lambda_{r,s}(f, g) &< 2\int \sum_{Q\in\mathcal{S}} \mathbf{1}_{E_Q}\langle f\rangle_{Q,r}\langle g\rangle_{Q,s}dx \\ &\le \int M_r f \cdot M_s g dx \lesssim \|M_r f\|_p \|M_s g\|_{p'} \lesssim \|f\|_p\|g\|_{p'}.\end{aligned}$$

Above $M_r f = \sup_Q \langle f\rangle_{Q,r}\mathbf{1}_Q$ is the maximal function with rth powers. □

A **weight** is a function $w(x) > 0$ a.e., which is the density of a measure on $\mathbb{R}^n$, also written as $w(E) = \int_E w dx$. For $1 < p < \infty$, the dual space to $L^p(w)$ (with respect to Lebesgue measure) is $L^{p'}(\sigma)$, where $p' = \frac{p}{p-1}$ and $\sigma = w^{1-p'}$. Note that $w \cdot \sigma^{p-1} \equiv 1$. A weight $w \in A_p$ if this equality holds in an average sense, uniformly over all locations and scales. Namely, define

$$[w]_{A_p} = \sup_Q \langle w \rangle_Q \langle \sigma \rangle_Q^{p-1} < \infty, \quad \sigma = w^{1-p'}.$$

Above, the supremum is over all cubes Q. At $p = 1$, we define

$$[w]_{A_1} = \sup_Q \sup_{x \in Q} \frac{\langle w \rangle_Q}{w(x)}.$$

A weight w is in the **reverse Hölder** class RH_r, $1 \leq r < \infty$, if

$$[w]_{RH_r} = \sup_Q \frac{\langle w \rangle_{Q,r}}{\langle w \rangle_Q} < \infty.$$

Qualitatively, the conditions of a weight w being in the intersection of A_p and reverse Hölder spaces is the same as w having a factorization $w \in A_1^\alpha A_1^\beta = \{u_1^\alpha u_2^\beta : u_1, u_2 \in A_1\}$. This is made precise in this proposition.

Proposition 6.2. *Let $u_1, u_2 \in A_1$, and let $\rho > 0$, and $1 < r < p < \infty$. We have*

$$A_1^{\frac{1}{\rho}} A_1^{-\frac{p}{r}+1} = A_{\frac{p}{r}} \cap RH_\rho.$$

Proof. These two facts are well-known.

(1) A weight in A_p can be factored into the product of A_1 weights

$$w \in A_p \iff w = u_1 u_2^{1-p}, \quad u_1 u_2 \in A_1.$$

(2) The condition $w \in A_{p/r} \cap RH_\rho$ is equivalent to $w^\rho \in A_{\rho(p/r-1)+1}$. Combining these two facts proves the proposition.

□

We focus on qualitative aspects of weighted inequalities for the sparse maximal functions. While quantitative estimates are available, and not too hard to prove, we think that what we can prove right now is improvable. (See §7.2.) Set $\mathcal{L}_p$ to be those weights w for which M_{lac} maps $L^p(w)$ to $L^p(w)$, for $1 < p < \infty$. Use the same type of notation $\mathcal{F}_p$ for M_{full}.

We have these two corollaries to our sparse bounds for the lacunary and full spherical maximal operators. These are obtained by combining our main theorems

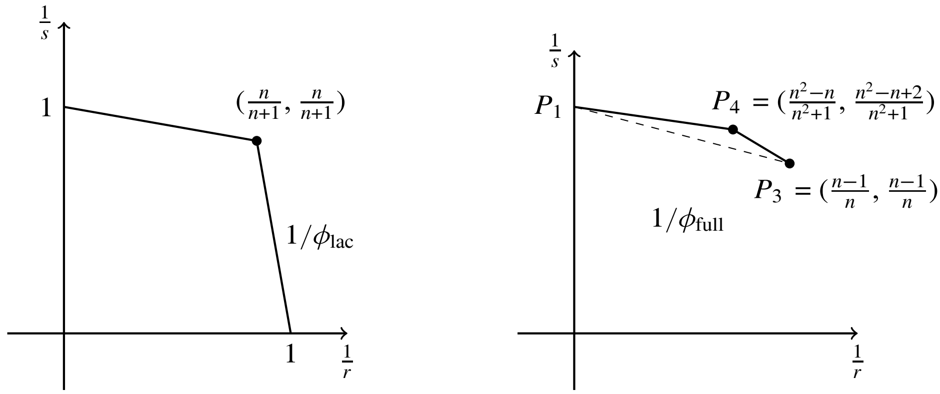

Figure 6. The two functions $1/\phi_{\rm lac}$ and $1/\phi_{\rm full}$ of Corollary 6.3. The dashed line is the function $1/\psi$, the function in (6.6).

with the bounds in Theorem G. As we only seek qualitative results, and the conditions of A_p and RH_r are open, we are free to work on the boundary of the figures $\mathbf{L}_n$ and $\mathbf{F}_n$. See Figure 6 for graphs of the two functions introduced below.

Corollary 6.3. *For the lacunary and full spherical maximal function, we have these two sets of weighted inequalities.*

(1) *Define* $\frac{1}{\phi_{\rm lac}(1/r)}$ *to be a piecewise linear function on* $[0, 1]$ *whose graph connects the points* $Q_1 = (0, 1)$*,* $Q_2 = (\frac{n-1}{n}, \frac{n-1}{n})$*, and* $Q_3 = (1, 0)$*. That is,*

$$\frac{1}{\phi_{\rm lac}(1/r)} = \begin{cases} 1 - \frac{1}{rn} & 0 < \frac{1}{r} \le \frac{n}{n+1}, \\ n(1 - \frac{1}{r}) & \frac{n}{n+1} < \frac{1}{r} < 1. \end{cases}$$

Assuming $1 < r < p < \phi(r)'$*, we have*

$$A_{p/r} \cap RH_{(\phi_{\rm lac}(r)'/p)'} \subset \mathcal{L}_p.$$

(2) *Define* $\frac{1}{\phi_{\rm full}(1/r)}$ *to be the piecewise linear function on* $[0, \frac{n-1}{n}]$ *whose graph connects the points* $P_1 = (0, 1)$*,* $P_4 = (\frac{n^2-n}{n^2+1}, \frac{n^2-n+2}{n^2+1})$ *and* $P_3 = (\frac{n-1}{n}, \frac{n-1}{n})$*. Assuming* $\frac{n}{n-1} < r < p < \phi_{\rm full}(r)'$*, we have*

$$A_{p/r} \cap RH_{(\phi_{\rm full}(r)'/p)'} \subset \mathcal{F}_p. \tag{6.4}$$

The case of radial weights has been completely analyzed by Duoandikoetxea and Vega [13]. Here, we recall this result, which records the possible inequalities for radial weights. These are sharp, except possibly the $a = 1 - n$ endpoint case in (6.5). (In particular, this shows that the class $\mathcal{L}_p$ does not satisfy the classical duality $\mathcal{L}_{p'} = \mathcal{L}_p^{1-p'}$. See [13] for more details.)

Theorem E ([13]). *Let $w_a(x) = |x|^a$ be a radial weight on $\mathbb{R}^n$, for $a \in \mathbb{R}$. We have the inequalities below, for $1 < p < \infty$:*

$$\begin{aligned} & w_a \in \mathcal{L}_p, \quad 1-n \le a < (n-1)(p-1), \\ & w_a \in \mathcal{F}_p, \quad 1-n < a < (n-1)(p-1)-n. \end{aligned} \tag{6.5}$$

In (6.5)*, the restriction on a implies that $\frac{n}{n-1} < p < \infty$.*

We cannot recover the full strength of this theorem. But this is to be expected: the category of A_p weights is not the correct one to characterize the weights for the spherical maximal function, and our sparse results are sharp. This suggests that the sparse bounds are proving the sharpest possible results in the category of Muckenhoupt type weights. We can improve upon the result below of Cowling, Garcia-Cuerva and Gunawan [8]. It gives sufficient conditions for M_{full} to satisfy a weighted inequality in terms of a factorization of the weight.

Theorem F ([8, Thm 3.1]). *Let $\frac{n}{n-1} < p < \infty$, and $\max\{0, 1-\frac{p}{n}\} \le \delta < \frac{n-2}{n-1}$. Then $A_1^\delta A_1^{\delta(d-1)-(d-2)} \subset \mathcal{F}_p$.*

We will deduce this as a special case of (6.4).

Proof of Theorem F. Rather than use the exact form of ϕ_{full} in (6.4), we use the restricted form

$$\psi(r)^{-1} = 1 - \frac{1}{r(n-1)}, \qquad \frac{n}{n-1} < r < \infty. \tag{6.6}$$

It follows that we have a sparse form bound $(r, \psi(r))$. This function corresponds to the dashed line in Figure 6. Provided $r < p < \psi(r)' = (r(n-1))' =: s'$, we have a weighted inequality, for $w \in A_{p/r} \cap RH_{(s'/p)'}$. Now, $(s'/p)' = \frac{r(n-1)}{r(n-1)-p} = 1 - \frac{p}{r(n-1)}$. By Proposition 6.2, we have $A_1^{1-\frac{p}{r(n-1)}} A_1^{1-\frac{p}{r}} \subset \mathcal{F}_p$. Setting $\delta = 1 - \frac{p}{r(n-1)}$, we have $1 - \frac{p}{r} = \delta(n-1) - (n-2)$. This matches the conclusion of the Theorem, so the proof is complete. □

As the proof above indicates, stronger results than those of Theorem F hold. The authors of [8] raised the possibility that $A_1^{1-\frac{1}{n}} \subset \mathcal{F}_p$. Here, we show that this is indeed the case, provided p is sufficiently large. It will be clear that more is true, but we do not pursue the details here.

Proposition 6.7. *For $n \ge 2$, we have $A_1^{1-\frac{1}{n}} \subset \mathcal{F}_p$, for $\frac{n^2+1}{n^2-n} < p < \infty$.*

Proof. We use the proof strategy for Theorem F, but use the sparse bound provided to us by the point $P_4 = (\frac{n^2-n}{n^2+1}, \frac{n^2-n+2}{n^2+1})$.

Indeed, assuming a sparse bound of the form (r_0, s_0), we have the inequality

$$\|M_{\text{full}} : L^p(w) \mapsto L^p(w)\| < \infty, \qquad w = u^{1/\rho},\ u \in A_1,$$

provided $r_0 < p < s_0'$, and $\rho = (s_0'/p)'$.

Setting $(1/r_0, 1/s_0) = P_4$, we have

$$\frac{1}{s_0} = \frac{n^2-n+2}{n^2+1}, \qquad \frac{1}{s_0'} = \frac{n-1}{n^2+1},$$
$$\frac{1}{r_0} = \frac{n^2-n}{n^2+1}, \qquad \frac{s_0'}{r_0} = n.$$

It follows that $\rho = (s_0'/p)' = \frac{n}{n-1}$. For $p > r_0$, we are allowed to take $w = u^{\frac{1}{\rho}} = u^{1-\frac{1}{n}}$, as claimed, provided $p > r_0$. □

7 Further remarks

7.1 Endpoint issues. Richard Oberlin [29] has investigated the endpoint issues. Namely, for a class of Radon transforms, a sparse bound is proved at the boundary of the sparse region. The 'local L^r norm' is adjusted with a logarithmic factor. It would be interesting to further develop the endpoint estimates.

7.2 Weighted estimates for m-sparse forms. For $1 < p < \infty$, the dual space to $L^p(w)$ (with respect to Lebesgue measure) is $L^{p'}(\sigma)$, where $p' = \frac{p}{p-1}$ and $\sigma = w^{1-p'}$. This is referenced in the statement of the Theorem below, which gives weighted inequalities for sparse forms. These estimates are sharp in the Muckenhoupt and reverse Hölder indices.

Theorem G ([1, §6]). *Let $1 \le r < s' < \infty$. Then,*

$$\Lambda_{r,s}(f,g) \le \{[w]_{A_{p/r}} \cdot [w]_{RH_{(s'/p)'}}\}^{\alpha} \|f\|_{L^p(w)} \|g\|_{L^{p'}(\sigma)}, \qquad r < p < s',$$
$$\text{where} \alpha = \max\left\{\frac{1}{p-1}, \frac{s'-1}{s'-p}\right\}.$$

For sparse forms of type $(1, 1)$, we recall that we have these estimates.

$$\Lambda_{1,1}(f,g) \lesssim [w]_{A_p}^{\max\{1,\frac{1}{p-1}\}} \|f\|_{L^p(w)} \|g\|_{L^{p'}(\sigma)},$$
$$\Lambda_{1,1,m}(f,g) \lesssim [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)} \|g\|_{L^{p'}(\sigma)}.$$

Both estimates are well-known. A very nice proof of the first bound can be found in [28]. The second follows from a comparison to the maximal function, namely Buckley's inequality [2]. Thus, the sparse forms and the m-sparse forms can obey different weighted estimates.

The papers [1, 24] supply explicit and sharp estimates for (r, s)-sparse forms. But, they do so only for the form (1.1), with $F_Q \equiv Q$. As this paper indicates, obtaining the sharp estimates for the m-sparse forms is also interesting.

7.3 Sharpness of the weighted estimates. We conjecture that the bounds in Corollary 6.3 are sharp in the category of weights allowed. For the sake of clarity, let us state a conjecture for the lacunary maximal function.

Conjecture 7.1. *Using the notation of Corollary 6.3, this holds. Let $1 < r < p < \phi_{\rm lac}(r)'$, and set $\rho = (\phi_{\rm lac}(r)'/p)'$. If $1/\rho < \alpha$, then there is a weight $w = u_1^{\alpha} u_2^{-\frac{p}{r}+1}$, for weights $u_1, u_2 \in A_1$, so that $M_{\rm lac}$ is not bounded on $L^p(w)$.*

7.4 The endpoint estimate. A result of Seeger, Tao and Wright addresses an endpoint estimate for the lacunary spherical maximal function, showing this.

Theorem H ([33])**.** *The lacunary maximal function $M_{\rm lac}$ is bounded as a map from $L \log\log L$ into weak L^1.*

Also see the recent significant improvement by Cladek and Krause [4]. The proof is based upon TT^* methods, and so it is tempting to think that a reading of the paper might prove a sparse bound for $M_{\rm lac}$ of the form $(r, 2)$, for all $1 < r < 2$. But such a sparse bound cannot hold. It is however interesting to speculate about what sparse bound the argument of [33] would imply.

7.5 Other themes. **(1)** As was pointed out by Duoandikoetxea and Vega [13], it is interesting to establish inequalities of Fefferman–Stein type, namely

$$\|M_{\rm lac} : L^p(w) \mapsto L^p(Nw)\|,$$

for some auxiliary maximal operator N. This has been addressed in [27]. It would be interesting to extend the results of this paper.

(2) The paper [8] studies weighted inequalities from L^p to L^q spaces for the maximal operator

$$\sup_{t>0} t^{\alpha} A_t f, \qquad \alpha = n(\frac{1}{p} - \frac{1}{q}).$$

Sparse bounds should be possible for such an operator.

(3) Variants of the maximal operator, formed over restricted ranges of radii of spheres, have been considered, namely,

$$\sup_{t\in E} A_t f, \quad E \subset (0,\infty).$$

See [32]. Subject to a dimensionality condition on E, a range of L^p inequalities can be proved. Again, sparse bounds should be available in this setting.

(4) The paper of Jones, Seeger and Wright [14, Thm 1.4] prove variational results for the full spherical maximal function. It would be interesting to extend this bound to a sparse bound. Also see [11] for some sparse variational results.

(5) Sparse bounds should hold for other Radon transforms. Key components would be (a) an appropriate dilation structure, and (b) variants of the continuity results Theorem 2.1 and Theorem 3.2. Note that these will become more involved in the cases in the variable curve case, as in [31].

(6) Cladek and Y. Ou [5] have studied sparse bounds for Hilbert transforms and averages along a general class of curves.

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