# LIOUVILLE THEOREMS FOR NONLINEAR ELLIPTIC EQUATIONS IN HALF-SPACES

*By*

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**Abstract.** In this paper we study the existence of nonnegative supersolutions of the nonlinear elliptic problem $-\Delta u + |\nabla u|^q = \lambda u^p$ in the half-space $\mathbb{R}^N_+$, where $N \geq 2$, $q > 1$, $p > 0$ and $\lambda > 0$. We obtain Liouville theorems for positive, bounded supersolutions, depending on the exponents $q$ and $p$, the dimension $N$, and, in some critical cases, also on the parameter $\lambda > 0$.

## 1 Introduction

Liouville type theorems (LTT) play an important role in the theory of elliptic and parabolic PDE. Similarly to the original Liouville result on bounded harmonic functions, a LTT usually states that a PDE has no nontrivial solutions, and in most cases is restricted to signed solutions in some kind of an unbounded domain.

The discovery of various LTT for nonlinear PDE in the last forty years was instrumental in the development of the theory of such equations. Probably the most outstanding theorem of this type is attached to the equation

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $p > 0$, and was obtained in [26]. It was shown there that there do not exist positive solutions of (1.1) provided that $1 < p < \frac{N+2}{N-2}$ (see also [17], [13] or [38] for different proofs).

Another important and more general question concerns nonexistence theorems for positive supersolutions of equations such as (1.1). Then one actually asks what is the smallest power $p$ such that the "concavity" (more precisely, superharmonicity) induced by the negative Laplacian does not prevent supersolutions to exist globally. Specifically, for (1.1) it was shown in [25] that if

$$p \leq \frac{N}{N-2},$$



JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 139 (2019)
DOI 10.1007/s11854-019-0066-y

then no positive supersolutions may exist, and this range is optimal. It turns out that this property remains true when the equation is set only in an exterior domain, instead of $\mathbb{R}^N$ (see [12] and the references there).

Problem (1.1) admits many generalizations, which mainly consist in replacing the Laplacian with different elliptic operators in divergence or non-divergence form, the power nonlinearity with a more general positive function or both. We refer the reader to [32], [20], [6], [21], [35], [5], [34], [30], and the references therein.

A generalization which has a somewhat different flavor is obtained when the differential operator contains a gradient term, to obtain a problem like

$$-\Delta u + |\nabla u|^q = \lambda u^p \tag{1.2}$$

in $\mathbb{R}^N \setminus B_{R_0}$, where $q > 1$ and $\lambda > 0$. Here and in what follows, $B_{R_0}$ stands for the ball of radius $R_0$ centered at the origin.

It is outside the scope of this paper to list all the various gradient-dependent problems which have (1.2) as a model case, and which appear in practice. We will only note here that gradient-dependent problems are particularly abundant in PDE which arise from control theory and economic applications. Problems which generalize (1.2) also appear in the theory of mean-field games, which witnesses quick development in the last years. The interested reader may consult for instance [19], [16], and the many references given in these works. For classical results on gradient-dependent problems we refer to [28], [15].

Problems like (1.2) are also of theoretical importance, since they do not have many of the properties which have been heavily used in previous nonexistence results. In particular, the differential operator in (1.2) does not have any variational structure, nor is it homogeneous when $q > 1$. Also, supersolutions of (1.2) are not necessarily superharmonic. The introduction of the gradient term thus forces us to search for alternative proofs.

Problem (1.2) in an exterior domain has been analyzed recently by some of the authors in [3] (see also [1] for slightly more general nonlinearities and [4] for the case $q = 1$). There are also some previous results available, which however almost exclusively dealt with radially symmetric solutions. See [18], [39], [35], [41], [23], [24] and [40].

Another line of research on nonexistence results for solutions of problems related to (1.1) arises when the underlying domain is different from $\mathbb{R}^N$ or an exterior domain. Such results appeared for instance in [8], [9], [11], [10], [29], and [6], where cone-like domains (and in particular half-spaces) were studied. All these works do not allow the appearance of a gradient term as in (1.2). We remark that one of the most important technical difficulties when dealing with this type of domains arises because their boundary is not compact; this might explain the absence of nonexistence results for gradient-dependent problems set in domains with a noncompact boundary.

The present paper can be viewed as a first step in this direction. We analyze bounded supersolutions of the problem (1.2) in a half-space $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$ (we are adopting the usual convention to write $x = (x', x_N)$ for a point $x \in \mathbb{R}^N$):

$$-\Delta u + |\nabla u|^q \geq \lambda u^p, \ u > 0 \quad \text{in } \mathbb{R}^N_+, \quad u \in C^1(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+). \tag{1.3}$$

In our subsequent discussion, we will always assume that $N \geq 2$, $q > 1$, $p > 0$ and $\lambda > 0$ are parameters.

Supersolutions of (1.3) will be defined in the weak sense, that is,

$$\int_{\mathbb{R}^N_+} \nabla u \nabla \phi + \int_{\mathbb{R}^N_+} |\nabla u|^q \phi \geq \lambda \int_{\mathbb{R}^N_+} u^p \phi$$

for every nonnegative $\phi \in C_0^\infty(\mathbb{R}^N_+)$. By shifting the half-space, if necessary, we can always assume that $u \in C^1(\overline{\mathbb{R}^N_+})$ and that $u > 0$ in $\overline{\mathbb{R}^N_+}$.

The following two theorems are our main results. They fully describe the solvability of (1.3), in terms of the parameters of this problem.

**Theorem 1.** *Assume* $1 < q < \frac{N+1}{N}$. *Then if*

$$p < \frac{q}{2-q}, \tag{1.4}$$

*the problem* (1.3) *does not have a solution. If* $p = \frac{q}{2-q}$, *then there exists*

$$\lambda_0 = \lambda_0(q, N) > 0$$

*such that for* $\lambda > \lambda_0$ *the same conclusion holds.*

**Theorem 2.** *Assume* $q \geq \frac{N+1}{N}$. *Then if*

$$p < \frac{N+1}{N-1}, \tag{1.5}$$

*the problem* (1.3) *does not have a solution. In addition, if* $q > \frac{N+1}{N}$ *and* $p = \frac{N+1}{N-1}$, *the same conclusion holds.*

**Theorem 3.** *For all values of* $N \geq 2$, $q > 1$, $p, \lambda > 0$ *not covered by the previous two theorems, there exists a solution of* (1.3).

We now discuss the threshold values for $p$ and $q$ which appear in these theorems, and explain their appearance.

First, recall that the problem without gradient dependence

$$-\Delta u \geq \lambda u^p \tag{1.6}$$

does not have positive supersolutions in a half-space if and only if $p \leq \frac{N+1}{N-1}$, by the well-known result of [9]. That is, if $p$ is large enough, positive solutions which are small at infinity may exist, since the right-hand side of (1.6) is sufficiently small to accommodate the superharmonicity.

In the same line of thought, when a gradient term is added to the left-hand side of (1.6), one may expect that there will be a threshold value for the power of the gradient, which determines the solvability of (1.3). The theorems above uncover the value of this threshold, as well as describe the interplay between $p$ and $q$. When $q$ is large, the gradient term turns out to be "too small to make a difference," and the full problem is solvable just when (1.6) is solvable. On the other hand, for small $q$ it becomes "easier" to satisfy the inequality, and the nonexistence range for $p$ is reduced. More specifically, (1.3) is not solvable if and only if $(p, q)$ is above the hyperbola given by (1.4).

Observe the particular (and even surprising at first sight) phenomenon which appears in the "critical" case $p = q/(2 - q)$—the existence of a threshold value for the multiplicative parameter $\lambda > 0$. This critical case is certainly the most difficult one to study. In a sense, it is also the most important one, since it is precisely in this critical case that the equation is invariant with respect to a "blow-up", i.e., rescaling of the dependent and independent variables. Specifically, if we set $u(x) = t^{\alpha}\tilde{u}(y)$, $y = tx$, $t > 0$, the function $\tilde{u}$ satisfies

$$-t^{\alpha+2}\Delta\tilde{u} + t^{q(\alpha+1)}|\nabla\tilde{u}|^q \geq \lambda t^{\alpha p}\tilde{u}^p$$

and the three powers of $t$ in this expression can be equalized by a suitable choice of $\alpha$ if and only if $p = q/(2 - q)$.

We recall that one of the most frequent and fundamental uses of Liouville type theorems is that they imply a priori bounds for positive solutions of elliptic PDE in bounded domains, through the "blow-up" method of Gidas and Spruck [26]. There have been works where this method was applied to equations with weak gradient dependence which disappears after the blow-up—see [22]. Our results here (and those in [3] which cover Liouville theorems in the whole space) now make it possible to apply the Gidas–Spruck method to problems with strong dependence in the gradient, in which the gradient stays after a blow-up change of scale.

Let us give a brief hint to our proofs of Theorems 1 and 2, which borrow tools from [7] and Section 5 of [6]. When $q < \frac{N+1}{N}$, we construct and use a subsolution $\Phi$ of problem (1.3) with $\lambda = 0$, vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$, which plays a similar role as the fundamental solution used in [6]. We assume a positive supersolution of (1.3) exists and we define the quantity

$$\rho(R) = \inf_{\substack{x\in\mathbb{R}^N_+ \\ |x|=R}} \frac{u(x)}{\Phi(x)}, \qquad R > 0.$$

A contradiction is reached by obtaining suitable upper and lower bounds for this quantity. The lower bounds follow because of the comparison principle, while the upper bounds are a consequence of arguments which involve the "doubling lemma" in [36]. Much more work is needed in the critical case $p = \frac{q}{2-q}$. In addition, in this case some arguments of [6] need to be extended and adapted to our situation. It turns out that the fundamental monotonicity property of $\rho$ is valid only for sufficiently large values of $\lambda$. It is also worth mentioning that when $q \geq \frac{N+1}{N}$, the function $\Phi$ which makes the procedure work is not exactly a subsolution of the homogeneous problem, but a fundamental solution of the Laplacian which vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$. In this case we need to show that a subsolution of a rescaled version of (1.3) with $\lambda = 0$ (see (5.3)) does not decay too quickly at infinity and then use a comparison argument in rescaled dyadic balls in order to reach a contradiction.

In the end we comment on the possibility of extending our results to more general cones in $\mathbb{R}^N$ than half-spaces. The only part of the proofs presented below which would require substantial modifications when dealing with more general cones is the construction of explicit subsolutions in Lemmas 4, 6, and 11. One would need to look for subsolutions as powers of the radial variable and functions of the angular variables of the cone. We will not deal with this rather technical question here, leaving it to a further study.

The rest of the paper is organized as follows: in Section 2 we obtain lower bounds for all positive supersolutions. Section 3 is dedicated to obtaining upper bounds for some special supersolutions, while Sections 4 and 5 deal with the proofs of Theorems 1, 2 and 3.

## 2 Lower bounds

The purpose of this section is to obtain lower bounds for all positive supersolutions of (1.3). These lower bounds will be obtained by comparing with appropriate subsolutions of the equation

$$-\Delta u + |\nabla u|^q = 0 \quad \text{in } \mathbb{R}^N_+ \tag{2.1}$$

which vanish on $\partial\mathbb{R}^N_+ \setminus \{0\}$. Note that when $q > 1$, in the context of weak solutions, the comparison principle is furnished for instance by Theorem 3.5.1 in [37] (see also Theorem 10.1 in [27] for classical solutions).

It turns out that the cases $1 < q < \frac{N+1}{N}$ and $q \geq \frac{N+1}{N}$ have to be analyzed separately. We begin with the first one and look for a subsolution involving the function

$$\Phi(x) := \frac{x_N^{\alpha}}{|x|^{\beta}}, \qquad x_N > 0, \tag{2.2}$$

where $x = (x', x_N)$ and $\alpha, \beta > 0$. Our first result is a simple calculation.

**Lemma 4.** *Assume* $1 < q < \frac{N+1}{N}$. *The function* $A\Phi$ *is a classical subsolution of* (2.1) *provided that* $0 < A \leq A_0$, *where*

$$\begin{cases} \alpha = \theta - N + 2, \\ \beta = 2\theta - N + 2, \\ A_0 = \left(\frac{\alpha(\alpha-1)}{\theta^q}\right)^{\frac{1}{q-1}}, \end{cases} \tag{2.3}$$

*and* $\theta = \frac{2-q}{q-1}$.

**Proof.** After straightforward calculations, we see that $\Phi$ being a classical subsolution of (2.1) is equivalent to

$$\begin{aligned} -\frac{x_N^{\alpha-2}}{|x|^{\beta}}&\left(\beta(\beta-2\alpha+2-N)\frac{x_N^2}{|x|^2} + \alpha(\alpha-1)\right) \\ &+ A^{q-1}\frac{x_N^{(\alpha-1)q}}{|x|^{\beta q}}\left|(\beta^2-2\alpha\beta)\frac{x_N^2}{|x|^2} + \alpha^2\right|^{\frac{q}{2}} \leq 0 \end{aligned} \tag{2.4}$$

if $x_N > 0$. Since $\beta > 0$ and $\beta - 2\alpha = N - 2 > 0$, we have

$$\left|(\beta^2-2\alpha\beta)\frac{x_N^2}{|x|^2} + \alpha^2\right| = (\beta^2-2\alpha\beta)\frac{x_N^2}{|x|^2} + \alpha^2 \leq (\beta-\alpha)^2 = \theta^2$$

so that (2.4) is implied by

$$-\alpha(\alpha-1) + \theta^q A^{q-1}\frac{x_N^{(\alpha-1)q-\alpha+2}}{|x|^{\beta(q-1)}} \leq 0.$$

However, this last inequality is easily seen to hold since $(\alpha-1)q-\alpha+2 = \beta(q-1)$ and $-\alpha(\alpha-1) + \theta^q A^{q-1} \leq 0$ (observe also that $\alpha > 1$ is a consequence of $q < \frac{N+1}{N}$). □

To proceed further, we introduce the notation $B_R^+ = \{x \in \mathbb{R}_+^N : |x| < R\}$, $S_R^+ = \{x \in \mathbb{R}_+^N : |x| = R\}$, for $R > 0$. For a given positive function $u$ which satisfies (1.3), we consider the function

$$\rho(R) = \inf_{x \in S_R^+} \frac{u(x)}{\Phi(x)}, \quad R > 0, \tag{2.5}$$

where $\Phi$ is given by (2.2) with the choice of $\alpha$, $\beta$ in (2.3). Observe that $u/\Phi$ is bounded from below by a positive constant on $S_R^+$, according to Hopf's principle (applied to $-\Delta u + c(x)|\nabla u| \geq 0$ with $c = |\nabla u|^{q-1}$) and to the fact that $\alpha > 1$. Therefore we always have $\rho(R) > 0$.

The first important property of the function $\rho$ when $u$ is a positive supersolution of (1.3) is the following:

**Lemma 5.** *Assume $u \in C^1(\mathbb{R}_+^N)$ is a positive weak supersolution of* (1.3) *and let $\rho$ be defined by* (2.5). *Then*

$$\liminf_{R \to +\infty} \rho(R) > 0.$$

*Moreover, the function $\rho(R)$ is increasing in any interval where it verifies $0 < \rho(R) < A_0$ ($A_0$ is the constant from the previous lemma), and if $\rho(R_1) = A_0$ for some $R_1 > 0$, then $\rho(R) \geq A_0$ for every $R > R_0$.*

**Proof.** Let us show that if $\rho(R_1) < A_0$ for some $R_1$, then $\rho(R) > \rho(R_1)$ for every $R > R_1$, where $A_0$ is given in (2.3).

Choose $\varepsilon > 0$. Observe that $\Phi \leq R_2^{-\theta} \leq \varepsilon/\rho(R_1)$ on $|x| = R_2$ if $R_2$ is large enough, thus $u + \varepsilon \geq \varepsilon \geq \rho(R_1)\Phi$ on $S_{R_2}^+$. By definition, we also have $u+\varepsilon \geq u \geq \rho(R_1)\Phi$ on $S_{R_1}^+$. Since $\rho(R_1) < A_0$, the function $\rho(R_1)\Phi$ is a subsolution of $-\Delta v + |\nabla v|^q \leq 0$ in $A^+(R_1, R_2) := \{x \in \mathbb{R}_+^N : R_1 < |x| < R_2\}$ by Lemma 4, and using comparison we get

$$u + \varepsilon \geq \rho(R_1)\Phi \quad \text{in } A^+(R_1, R_2).$$

Letting $R_2 \to +\infty$ and then $\varepsilon \to 0$, we arrive at $u \geq \rho(R_1)\Phi$ in $\mathbb{R}_+^N \cap \{|x| > R_1\}$, in other words,

$$\rho(R_1) = \inf_{x \in \mathbb{R}_+^N \setminus B_{R_1}} \frac{u(x)}{\Phi(x)}.$$

In particular, by (2.5), $\rho(R) \geq \rho(R_1)$ for every $R > R_1$, as was to be shown. As an immediate consequence, we have $\liminf_{R \to +\infty} \rho(R) > 0$, which concludes the proof. ∎

In the case $q \geq \frac{N+1}{N}$, we look for a subsolution of (2.1) similar to the one discovered before:

$$\Psi_\varepsilon(x) = \frac{x_N^{1+\varepsilon}}{|x|^{N+2\varepsilon}}, \tag{2.6}$$

where $\varepsilon > 0$ (observe that when $\varepsilon = 0$, this function reduces to one of the fundamental solutions of the Laplacian in $\mathbb{R}^N_+$ which vanish on $\partial\mathbb{R}^N_+ \setminus \{0\}$). The proof of the next result is similar to that of Lemma 4.

**Lemma 6.** *Assume $q \geq \frac{N+1}{N}$. For $\varepsilon > 0$, the function $A\Psi_\varepsilon$ is a classical subsolution of* (2.1) *in $\mathbb{R}^N_+ \setminus B_1^+$, provided that*

$$0 < A \leq \left(\frac{\varepsilon(1+\varepsilon)}{(N-1+\varepsilon)^q}\right)^{\frac{1}{q-1}}.$$

**Proof.** It suffices to have, for $x_N > 0$ and $|x| > 1$,

$$-\varepsilon(1+\varepsilon) + (N-1+\varepsilon)^q A^{q-1} \frac{x_N^{1+\varepsilon(q-1)}}{|x|^{(N+2\varepsilon)(q-1)}} \leq 0. \tag{2.7}$$

Since $q \geq \frac{N+1}{N}$, we see that $(N+2\varepsilon)(q-1) \geq 1 + \varepsilon(q-1)$, hence we deduce for $|x| > 1$

$$\frac{x_N^{1+\varepsilon(q-1)}}{|x|^{(N+2\varepsilon)(q-1)}} \leq \left(\frac{x_N}{|x|}\right)^{1+\varepsilon(q-1)} \leq 1.$$

Then inequality (2.7) is a consequence of our choice of $A$. The proof is concluded. □

In analogy with (2.5), for a given positive function $u$ defined in $\mathbb{R}^N_+$ we introduce the function of $R$:

$$\rho_\varepsilon(R) = \inf_{x \in S_R^+} \frac{u(x)}{\Psi_\varepsilon(x)}, \qquad R > 1. \tag{2.8}$$

The following property can be shown with an entirely similar proof to that of Lemma 5, and will thus be omitted.

**Lemma 7.** *Assume $u \in C^1(\mathbb{R}^N_+)$ is a positive weak supersolution of* (1.3) *and let $\rho_\varepsilon$ be defined by* (2.8)*. Then*

$$\liminf_{R \to +\infty} \rho_\varepsilon(R) > 0.$$

## 3 Upper bounds

Our next task is to obtain upper bounds for positive, bounded supersolutions of the problem

$$-\Delta u + |\nabla u|^q = \lambda u^p \quad \text{in } \mathbb{R}^N_+. \tag{3.1}$$

Our tool to achieve them is the doubling lemma proved in [36]. But in order to be able to use the results there, we first need to get supersolutions with some additional regularity properties.

Actually, in the case $1 < q \le 2$, the existence of a positive, bounded, weak supersolution suffices to guarantee the existence of a bounded solution with bounded derivatives.

**Lemma 8.** *Assume* $1 < q \le 2$ *and* $u \in C^1(\mathbb{R}^N_+)$ *is a positive, bounded weak supersolution of* (3.1)*. Then there exists a positive classical solution* $v \in C^2(\mathbb{R}^N_+)$ *of* (3.1) *verifying* $\|v\|_{C^2(\mathbb{R}^N_+)} < +\infty$.

**Proof.** Set $\Upsilon = \Phi$ when $1 < q < \frac{N+1}{N}$ while $\Upsilon = \Psi_\varepsilon$ for $q \ge \frac{N+1}{N}$, where $\Phi$ and $\Psi_\varepsilon$ are given by (2.2) and (2.6), respectively, and $\varepsilon$ is small enough. Fix $R_1 > 0$ and for $R_2 > R_1$ denote again $A^+(R_1, R_2) = \{x \in \mathbb{R}^N_+ : R_1 < |x| < R_2\}$. By Lemmas 5 and 7 we have $u \ge \delta\Upsilon$ for $x \in \mathbb{R}^N_+$, $|x| > R_1$ and some small positive $\delta$.

Next, consider the problem

$$\begin{cases} -\Delta v + |\nabla v|^q = \lambda v^p & \text{in } A^+(R_1, R_2), \\ v = \delta\Upsilon & \text{on } \partial A^+(R_1, R_2). \end{cases} \tag{3.2}$$

It is clear that $u$ is a weak supersolution of (3.2), while $\underline{v} = \delta\Upsilon$ is a weak subsolution, and they are ordered. Since $q \le 2$, we may apply the classical method of sub- and supersolutions to obtain a weak solution $v_{R_2}$ verifying $\delta\Upsilon \le v_{R_2} \le u$ in $A^+(R_1, R_2)$ (cf. [14]). Notice that $v_{R_2}$ is classical by standard regularity (see [31] and [27]).

In particular, the set $\{v_{R_2}\}_{R_2 > R_1}$ is uniformly bounded. Using, for instance, Theorem 7 in [2] we deduce that the set $\{|\nabla v_{R_2}|\}_{R_2 > R_1}$ is locally bounded. Thus we obtain also local bounds for $\{|\Delta v_{R_2}|\}_{R_2 > R_1}$, which by standard theory provide with $C^{1,\alpha}$ local bounds and therefore $C^{2,\alpha}$ local bounds. Thus, by compactness and using a diagonal argument, we may select a sequence $R_{2,n} \to +\infty$ such that $v_{R_{2,n}} \to v$ in $C^2_{\text{loc}}(\mathbb{R}^N_+ \setminus B_{R_1})$. Hence $v$ is a classical solution of

$$-\Delta v + |\nabla v|^q = \lambda v^p \quad \text{in } \mathbb{R}^N_+ \setminus B_{R_1},$$

verifying in addition $v \ge \delta\Upsilon$. Thus $v > 0$ in $\mathbb{R}^N_+ \setminus B_{R_1}$.

Finally, since $u \in L^\infty(\mathbb{R}^N_+)$, we also have $v \in L^\infty(\mathbb{R}^N_+ \setminus B_{R_1})$, and using Theorem 7 in [2] again this implies $|\nabla v| \in L^\infty(\mathbb{R}^N_+ \setminus B_{R_1}) \cap \{x_N > 2R_1\})$. By classical theory we deduce also $|D^2 v| \in L^\infty(\mathbb{R}^N_+ \setminus B_{R_1}) \cap \{x_N > 3R_1\})$. Now a convenient shift $\tilde{v}$ of $v$ gives a solution of problem (3.1) in $\mathbb{R}^N_+$, verifying $\tilde{v} \in C^2(\mathbb{R}^N_+)$ and $\|\tilde{v}\|_{C^2(\mathbb{R}^N_+)} < +\infty$. This concludes the proof. □

The case $q > 2$ is a bit more delicate, since standard theory is not available. Nevertheless, gradient estimates can still be used, and they allow us to obtain a slightly weaker result, however enough for our purposes below.

**Lemma 9.** *Assume $q \geq 2$ and $u \in C^1(\mathbb{R}^N_+)$ is a positive, bounded weak supersolution of* (3.1). *Then there exists a positive, weak supersolution $v \in C^{1,\alpha}(\mathbb{R}^N_+)$ of* (3.1) *with $\|v\|_{C^{1,\alpha}(\mathbb{R}^N_+)} < +\infty$ for every $\alpha \in (0, 1)$.*

**Proof.** Our intention is to apply the method of sub- and supersolutions again, but since now $q > 2$, we have to work with some more care.

Fix $R_2 > R_1 > 0$. By Lemma 7, we obtain $u \geq \delta\Psi_\varepsilon$ in $\mathbb{R}^N_+ \setminus B_{R_1}$ for some small enough $\delta$. Consider the unique solution $z$ of the problem

$$\begin{cases} -\Delta z = \lambda u^p & \text{in } A^+(R_1, R_2), \\ z = \delta\Psi_\varepsilon & \text{on } \partial A^+(R_1, R_2). \end{cases}$$

By standard regularity, $z \in W^{2,s}(A^+(R_1, R_2)) \cap C^{1,\alpha}(\overline{A^+(R_1, R_2)})$ for every $s > 1$ and every $\alpha \in (0, 1)$ (cf. [27]). Thus it is clear that $z$ is a (strong) supersolution of

$$\begin{cases} -\Delta v + |\nabla v|^q = \lambda u^p & \text{in } A^+(R_1, R_2), \\ v = \delta\Psi_\varepsilon & \text{on } \partial A^+(R_1, R_2), \end{cases} \tag{3.3}$$

while $\delta\Psi_\varepsilon$ is a strong subsolution of the same problem, and they coincide on the boundary. By the comparison principle we have $z \geq \delta\Psi_\varepsilon$ in $A^+(R_1, R_2)$.

We may then use Theorem III.1 in [33] (we remark that the proof there can be adapted to deal with nonhomogeneous Dirichlet problems, provided only that the sub- and supersolution coincide on the boundary) to ensure the existence of a strong solution $v_{R_2} \in W^{2,s}(A^+(R_1, R_2))$ of (3.3). Again by comparison, $u \geq v_{R_2}$ in $A^+(R_1, R_2)$, so that the family $\{v_{R_2}\}_{R_2 > R_1}$ is locally bounded in $\mathbb{R}^N_+ \setminus B_{R_1}$.

We can pass to the limit in a similar way as in Lemma 8 to obtain that, for some sequence $R_{2,n} \to +\infty$, $v_{R_{2,n}} \to v$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N_+ \setminus B_{R_1})$. Of course, this function verifies

$$-\Delta v + |\nabla v|^q = \lambda u^p \geq \lambda v^p \quad \text{in } \mathbb{R}^N_+ \setminus B_{R_1}$$

in the weak sense. Observe that both $v$ and $u$ are bounded in $\mathbb{R}^N_+ \setminus B_{R_1}$. Thus, we may use again Theorem 7 in [2] to conclude that $|\nabla v|$ is also bounded in $x_N > 2R_1$. By classical regularity, $\|v\|_{C^{1,\alpha}(\mathbb{R}^N_+ \setminus B_{R_1})}$ is finite for every $\alpha \in (0, 1)$ and we may take as before a shift of $v$ to obtain the sought supersolution in $\mathbb{R}^N_+$. □

Once we have appropriate positive supersolutions of (3.1), we can obtain adequate information on their decay at infinity with the "doubling method", introduced in [36].

**Lemma 10.** *Assume* $1 < q \leq 2$ *and* $v \in C^2(\mathbb{R}^N_+)$ *is a positive classical solution of* (3.1)*, or* $q > 2$ *and* $v \in C^1(\mathbb{R}^N_+)$ *is a positive weak supersolution of* (3.1)*. Assume in addition that* $\|v\|_{C^{1,\alpha}(\mathbb{R}^N_+)} < +\infty$ *for some* $\alpha \in (0,1)$ *and one of the following conditions hold:*

(a) $1 < q \leq \frac{N+1}{N}$ *and* $0 < p < \frac{q}{2-q}$*;*
(b) $1 < q \leq \frac{N+1}{N}$, $p = \frac{q}{2-q}$ *and* $\lambda$ *is large enough;*
(c) $q > \frac{N+1}{N}$ *and* $p \leq \frac{N+1}{N-1}$.

*Then there exists a constant* $C > 0$ *such that*

$$v(x) \leq C x_N^{-\frac{2}{p-1}}, \quad |\nabla v(x)| \leq C x_N^{-\frac{p+1}{p-1}}, \quad \textit{for } x \in \mathbb{R}^N_+. \tag{3.4}$$

**Proof.** Let us show first that

$$\lim_{x_N \to +\infty} v(x) = \lim_{x_N \to +\infty} |\nabla v(x)| = 0. \tag{3.5}$$

For this, choose an arbitrary sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $x_{n,N} \to +\infty$, where $x_{n,N}$ stands for the $N$-th component of $x_n$. Let $w_n(x) = v(x + x_n)$, which solves (3.1) in $x_N > -x_{n,N}$. By hypothesis $\{w_n\}_{n \in \mathbb{N}}$ is bounded in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$, so we obtain, passing to a subsequence, that $w_n \to w$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$, where $w$ is a nonnegative, bounded weak solution of

$$-\Delta w + |\nabla w|^q \geq \lambda w^p \quad \text{in } \mathbb{R}^N.$$

Now we observe that our assumptions (a), (b), (c) imply that one of the following cases occurs:

(i) $1 < q < \frac{N}{N-1}$ and $0 < p < \frac{q}{2-q}$;
(ii) $1 < q < \frac{N}{N-1}$, $p = \frac{q}{2-q}$ and $\lambda$ is large enough;
(iii) $q \geq \frac{N}{N-1}$ and $p < \frac{N}{N-2}$.

Therefore we can use the results in [3] (specifically Theorems 1, 2 and 3 therein) to obtain that $w \equiv 0$. Thus $v(x_n) \to 0$, $|\nabla v(x_n)| \to 0$, and this shows (3.5).

Next assume (3.4) does not hold and introduce the function

$$M(x) := v(x)^{\frac{p-1}{2}} + |\nabla v(x)|^{\frac{p-1}{p+1}}, \quad x \in \mathbb{R}^N_+.$$

Then there exist points $y_k \in \mathbb{R}^N_+$ such that

$$M(y_k) > 2k y_{k,N}^{-1} \quad \text{for every } k \in \mathbb{N}.$$

We may use Lemma 5.1 in [36] (see also Remark 5.2 (b) there) to obtain points $x_k \in \mathbb{R}^N_+$ such that $M(x_k) \geq M(y_k)$, $M(x_k) > 2k x_{k,N}^{-1}$ and

$$M(x) \leq 2M(x_k) \quad \text{if } |x - x_k| \leq kM(x_k)^{-1}.$$

Define the functions

$$z_k(y) = \mu_k^{\frac{2}{p-1}} v(x_k + \mu_k y), \quad y \in B_k$$

($B_k$ stands for the ball with radius $k$ centered at the origin), where $\mu_k = M(x_k)^{-1}$. Observe that $M$ is a bounded function by hypothesis, therefore $x_{k,N} \geq 2k/C$ for some positive constant $C$, hence $x_{k,N} \to +\infty$ as $k \to +\infty$. Thus

$$M(x_k) = v(x_k)^{\frac{p-1}{2}} + |\nabla v(x_k)|^{\frac{p-1}{p+1}} \to 0$$

by (3.5), so that $\mu_k \to +\infty$.

From this moment on, we need to distinguish the cases $1 < q \leq 2$ and $q > 2$. In the first case, $v$ is a solution of (3.1), so it is not hard to check that the functions $z_k$ verify

$$-\Delta z_k + \mu_k^{\frac{2p}{p-1} - \frac{p+1}{p-1}q} |\nabla z_k|^q = \lambda z_k^p \quad \text{in } B_k$$

together with

$$z_k(0)^{\frac{p-1}{2}} + |\nabla z_k(0)|^{\frac{p-1}{p+1}} = \mu_k M(x_k) = 1$$

and

$$z_k(y)^{\frac{p-1}{2}} + |\nabla z_k(y)|^{\frac{p-1}{p+1}} = \mu_k M(x_k + \mu_k y) \leq 2\mu_k M(x_k) = 2 \quad \text{in } B_k.$$

Using that both $z_k$ and its gradient are bounded in $B_k$, we deduce that $\Delta z_k$ is bounded in $B_k$, so that by standard regularity we obtain $C^{1,\alpha}$ local bounds for $z_k$. This entails that actually $\Delta z_k$ is locally bounded in $C^\alpha$, which in turn gives $C^{2,\alpha}$ local bounds. Passing to a subsequence through a diagonal procedure, we may assume $z_k \to z$ in $C^2_{\text{loc}}(\mathbb{R}^N)$, where $z$ verifies $z(0)^{\frac{p-1}{2}} + |\nabla z(0)|^{\frac{p-1}{p+1}} = 1$ and is a nonnegative classical solution of

$$-\Delta z = \lambda z^p \quad \text{in } \mathbb{R}^N$$

when $p < \frac{q}{2-q}$ or

$$-\Delta z + |\nabla z|^q = \lambda z^p \quad \text{in } \mathbb{R}^N$$

when $p = \frac{q}{2-q}$. By the strong maximum principle we see that $z > 0$ in $\mathbb{R}^N$. In the first case we obtain a contradiction with Theorem 1.1 in [26], while in the second a contradiction is also reached with Theorem 1 in [3] if $\lambda$ is large enough. This shows (3.4) in cases (a), (b) and (c) when $q \leq 2$.

Thus only the case $q > 2$ in (c) remains to be proved. In this situation the functions $z_k$ verify

$$-\Delta z_k + \mu_k^{\frac{2p}{p-1} - \frac{p+1}{p-1}q} |\nabla z_k|^q \geq \lambda z_k^p \quad \text{in } B_k$$

and the exponent of $\mu_k$ is negative. Since $|\nabla z_k|$ is bounded we have, by passing to a subsequence, that $z_k \to z$ uniformly in compact sets of $\mathbb{R}^N$ and $z_k \rightharpoonup z$ in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$. As $\mu_k^{\frac{2p}{p-1}-\frac{p+1}{p-1}q} \to 0$, we deduce $-\Delta z \geq \lambda z^p$ in $\mathbb{R}^N$ in the $H^1$-weak sense, and since $z$ is continuous also in the viscosity sense. This contradicts for instance Theorem 2.1 in [6], since $p \leq \frac{N+1}{N-1} < \frac{N}{N-2}$. This contradiction proves (3.4) in case (c) and the proof is concluded. □

## 4 The case $1 < q < \frac{N+1}{N}$

We devote this section to the proof of Theorem 1. Our first step is to obtain a slightly more accurate version of Lemma 5 which will be used when we analyze the critical case $p = \frac{q}{2-q}$. For this purpose we consider the problem

$$-\Delta u + |\nabla u|^q = B\Phi^p \quad \text{in } \mathbb{R}^N_+ \tag{4.1}$$

for $B > 0$, where $\Phi$ is given by (2.2). The key point is to obtain a positive subsolution of (4.1) vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$. The next lemma is similar to Lemma 4.

**Lemma 11.** *Assume $1 < q < \frac{N+1}{N}$ and $p = \frac{q}{2-q}$. With $\alpha$ and $\theta$ as given in Lemma 4 set*

$$\lambda_0 = \frac{\theta^q(q-1)}{p-1}\left(\frac{\theta^q(p-q)}{\alpha(\alpha-1)(p-1)}\right)^{\frac{p-q}{q-1}}. \tag{4.2}$$

*Then the function $A\Phi$ is a classical subsolution of* (4.1) *provided that*

$$A \leq \left(\frac{B}{\lambda_0}\right)^{\frac{1}{p}}.$$

**Proof.** Similarly as in the proof of Lemma 4, we see that being a classical subsolution of (4.1) is implied by

$$-A\alpha(\alpha-1) + A^q\theta^q \frac{x_N^{(\alpha-1)q-\alpha+2}}{|x|^{\beta(q-1)}} \leq B\frac{x_N^{\alpha p-\alpha+2}}{|x|^{\beta(p-1)}} \quad \text{in } x_N > 0.$$

Taking into account that $(\alpha-1)q-\alpha+2 = \beta(q-1)$ and $\alpha(p-1)+2 = (p-1)\beta$, this reduces to

$$-A\alpha(\alpha-1) + A^q\theta^q\left(\frac{x_N}{|x|}\right)^{\beta(q-1)} \leq B\left(\frac{x_N}{|x|}\right)^{\beta(p-1)} \quad \text{for } x_N > 0. \tag{4.3}$$

Denote temporarily $a = A\alpha(\alpha-1)$, $b = A^q\theta^q$, $c = B$, $s = \beta(q-1)$, $t = \beta(p-1)$, and consider the function $G(\xi) = -a + b\xi^s - c\xi^t$ for $\xi \in [0,1]$. Then (4.3) will hold in $\mathbb{R}^N_+$ when $G(\xi) \leq 0$ in $[0,1]$.

Observe that with our hypotheses $p > q$, so that $t > s$. Then it is not hard to see that

$$G(\xi) \leq -a + \frac{b}{t}\left(\frac{bs}{ct}\right)^{\frac{s}{t-s}}(t-s) \leq 0$$

provided we take $a$ to verify $a \geq \frac{b}{t}(\frac{bs}{ct})^{\frac{s}{t-s}}(t-s)$. After some calculations, we see that this inequality is equivalent to $A^p\lambda_0 \leq B$. This concludes the proof. □

With a similar procedure as in the proof of Lemma 5, the subsolution given by Lemma 11 can be used to obtain the following property:

**Lemma 12.** *Let $\lambda_0$ be given by* (4.2). *Then, if $u \in C^1(\mathbb{R}^N)$ is a positive weak supersolution of* (1.3) *with $p = \frac{q}{2-q}$ and $\lambda > \lambda_0$, the function $\rho$ given by* (2.5) *is increasing in $R > 0$.*

**Proof.** Observe first that Lemma 5 shows that $\rho$ is increasing as long as $\rho < A_0$. Also, if there exists $R_1 > 0$ such that $\rho(R_1) = A_0$, then $\rho(R) \geq A_0$ for every $R > R_1$.

Thus assume the existence of such a $R_1$. Then

$$-\Delta u + |\nabla u|^q \geq \lambda A_0^p \Phi^p \quad \text{in } |x| \geq R_1. \tag{4.4}$$

Set $A_1 = A_0(\lambda/\lambda_0)^{\frac{1}{p}} > A_0$. If $\rho(R_2) < A_1$ for some $R_2 > R_1$, then $\rho(R_2)\Phi$ is a subsolution of (4.4) by Lemma 11, while $u + \varepsilon$ is a supersolution for small positive $\varepsilon$. Thus an argument like the one used in the proof of Lemma 5 shows that $\rho(R) \geq \rho(R_2)$ for $R \geq R_2$. Thus $\rho(R)$ is nondecreasing as long as $A_0 \leq \rho(R) < A_1$ and if $\rho$ reaches the value $A_1$ it always stays above this value. Defining recursively $A_k = A_{k-1}(\lambda/\lambda_0)^{\frac{1}{p}}$ we deduce that $\rho$ is increasing as long as $A_{k-1} \leq \rho(R) < A_k$, and since $\lim_{k\to+\infty} A_k = +\infty$ we obtain that $\rho$ is increasing in $R > 0$. □

Our proof in the critical case $p = \frac{q}{2-q}$ follows arguments introduced in [6]. For this, several ideas have to be generalized. The next property is essentially an adaptation of the so-called 'quantitative maximum principle' there (Lemma 2.2 in [6]).

**Lemma 13.** *Let $\Omega$ be a bounded Lipschitz domain of $\mathbb{R}^N$ and $v, z \in C^2(\Omega)$ verifying*

$$\begin{cases} -\Delta v + |\nabla v|^q \geq -\Delta z + 2^{q-1}|\nabla z|^q + f & \text{in } \Omega, \\ v \geq z & \text{on } \partial\Omega, \end{cases}$$

*where $f \in L^\infty(\Omega)$, $f \geq 0$, $f \not\equiv 0$. Then for every $\Omega' \subset\subset \Omega$ there exists $c = c(f, \Omega') > 0$ such that*

$$v \geq z + c \quad \text{in } \Omega'.$$

**Proof.** Let $\phi$ be the unique (positive) solution of

$$\begin{cases} -\Delta\phi + 2^{q-1}|\nabla\phi|^q = f & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

(this problem is uniquely solvable since $q \le 2$). Since

$$\begin{aligned} -\Delta(z+\phi) + |\nabla(z+\phi)|^q &\le -\Delta z - \Delta\phi + 2^{q-1}|\nabla z|^q + 2^{q-1}|\nabla\phi|^q \\ &= -\Delta z + 2^{q-1}|\nabla z|^q + f \\ &\le -\Delta v + |\nabla v|^q \end{aligned}$$

in $\Omega$, with $z+\phi = z \le v$ on $\partial\Omega$, we deduce by comparison that $z+\phi \le v$ in $\Omega$. The conclusion of the lemma follows with $c = \inf_{\Omega'} \phi$. □

For our last preliminary result we introduce some notation: let $\mathcal{C}$ be an open cone with vertex at zero contained in $\mathbb{R}^N_+$ with $\partial\mathcal{C} \cap \partial\mathbb{R}^N_+ = \{0\}$. Denote $D = B_4^+ \setminus \overline{B_{1/2}^+}$, $E = (B_2^+ \setminus \overline{B_1^+}) \cap \mathcal{C}$. The following is a version of Lemma 5.5 in [6].

**Lemma 14.** *For every $b > 0$, there exists $\varepsilon = \varepsilon(b) \in (0, \frac{1}{2})$ such that, for every $v, z \in C^2(D) \cap C(\overline{D})$ verifying*

$$\begin{cases} -\Delta v + |\nabla v|^q \ge -\Delta z + 2^{q-1}|\nabla z|^q & \text{in } D \\ v \ge z & \text{on } \partial\mathbb{R}^N_+ \cap (B_4 \setminus \overline{B_{1/2}}) \\ v \ge z + b & \text{on } E \end{cases}$$

*and $v \ge z - \varepsilon$ in $D$, we have that $v \ge z$ in $\overline{B_2^+} \setminus B_1^+$.*

**Proof.** Let $\varepsilon > 0$ be small and $\phi_\varepsilon$ the solution of the problem

$$\begin{cases} -\Delta\phi + |\nabla\phi|^q = 0 & \text{in } D \setminus \overline{E}, \\ \phi = 0 & \text{on } \partial\mathbb{R}^N_+ \cap (B_4 \setminus \overline{B_{1/2}}), \\ \phi = b & \text{on } \partial E, \\ \phi = -\varepsilon & \text{on the rest of } \partial(D \setminus E). \end{cases} \tag{4.5}$$

We claim that there exists $\varepsilon > 0$ such that $\phi_\varepsilon > 0$ on $B_2^+ \setminus \overline{B_1^+}$.

If this were not true, there would exist sequences $\varepsilon_n \to 0$ and points $x_n \in B_2^+ \setminus \overline{B_1^+}$ with $\phi_{\varepsilon_n}(x_n) \le 0$. Passing to subsequences we may assume $x_n \to x_0 \in \overline{B_2^+} \setminus B_1^+$, $\phi_n \to \phi$ uniformly in $\overline{D} \setminus E$, and also in $C^1_{\text{loc}}((D \setminus \overline{E}) \cup (\partial\mathbb{R}^N_+ \cap (B_4 \setminus \overline{B_{1/2}})))$, where $\phi$ is the solution of (4.5) with $\varepsilon = 0$.

Since $\phi > 0$ in $D\setminus\overline{E}$ and $\phi(x_0) \le 0$, we deduce $x_0 \in \partial\mathbb{R}^N_+ \cap (B_2\setminus\overline{B_1})$. According to Hopf's principle, $\frac{\partial\phi}{\partial x_N}(x_0) > 0$, so that there exists a neighborhood of $x_0$ with $\frac{\partial\phi_{\varepsilon_n}}{\partial x_N} > 0$. Hence $\phi_{\varepsilon_n} > 0$ in this neighborhood, contradicting $\phi_{\varepsilon_n}(x_n) \le 0$. This shows the claim.

We next argue as in the proof of Lemma 13: we have

$$-\Delta(z+\phi_\varepsilon) + |\nabla(z+\phi_\varepsilon)|^q \le -\Delta v + |\nabla v|^q \quad \text{in } D \setminus E,$$

with $z+\phi_\varepsilon \le v$ on $\partial(D\setminus E)$. By comparison $z+\phi_\varepsilon \le v$ in $D\setminus E$, and since $\phi_\varepsilon > 0$ in $B_2^+\setminus\overline{B_1^+}$ for small $\varepsilon$, we deduce that $z \le v$ in $B_2^+\setminus\overline{B_1^+}$ if $\varepsilon$ is small enough (depending only on $b$). This concludes the proof. □

We can finally proceed to the proof of Theorem 1.

**Proof of Theorem 1.** Assume there exists a positive, bounded, weak supersolution of (1.3). Thanks to Lemma 8, there exists a classical solution $v$ of (1.3) verifying $\|v\|_{C^2(\mathbb{R}^N_+)} < +\infty$. Consider the function $\rho$ defined by (2.5), with $u$ replaced by $v$. According to Lemma 5, we have $\rho(R) \ge c > 0$ for large $R$. On the other hand, by Lemma 10

$$\rho(R) \le \frac{v(Re_N)}{\Phi(Re_N)} = R^\theta v(Re_N) \le CR^{\theta-\frac{2}{p-1}}. \tag{4.6}$$

In the case $p < \frac{q}{2-q}$, since $\theta < \frac{2}{p-1}$, we obtain an immediate contradiction by letting $R$ go to infinity.

Thus, we focus on the critical case $p = \frac{q}{2-q}$. With a minor modification of the proof of Lemmas 11 and 12, we see that there exists an increasing sequence $\{B_k\}_{k=1}^\infty$ of nonnegative numbers with $\lim_{k\to+\infty} B_k = +\infty$, such that, if $B_{k-1} \le A < B_k$ then

$$-\Delta(A\Phi) + 2^{q-1}|\nabla(A\Phi)|^q \le \frac{\lambda}{2}B_{k-1}^p\Phi^p \quad \text{in } \mathbb{R}^N_+,$$

provided that $\lambda$ is larger than some value $\lambda_0 = \lambda_0(q)$.

By Lemma 12, $\rho$ is increasing and it is bounded by (4.6) (since $\theta = \frac{2}{p-1}$), hence we may set $\ell = \lim_{R\to+\infty}\rho(R)$. We may choose $k$ so that

$$B_{k-1} \le \rho\left(\frac{R}{2}\right) < \ell < B_k$$

for sufficiently large $R$ (if $\ell = B_k$ for some $k$ we can modify the choice of $B_k$ by replacing $\lambda/2$ by something slightly larger).

For large $R$, let $w(x) = R^\theta v(Rx)$ (recall that $\theta = \frac{2-q}{q-1}$). Since $\rho(R)$ is increasing

$$w(x) \ge R^\theta\rho\left(\frac{R}{2}\right)\Phi(Rx) = \rho\left(\frac{R}{2}\right)\Phi(x)$$

when $|x| \geq \frac{1}{2}$. Moreover

$$-\Delta w + |\nabla w|^q \geq \lambda \rho\Big(\frac{R}{2}\Big)^p \Phi^p \geq \lambda B_{k-1}^p \Phi^p$$

while

$$-\Delta\Big(\rho\Big(\frac{R}{2}\Big)\Phi\Big) + 2^{q-1}\Big|\nabla\Big(\rho\Big(\frac{R}{2}\Big)\Phi\Big)\Big|^q \leq \frac{\lambda}{2} B_{k-1}^p \Phi^p$$

in $D$. Using Lemma 13 we obtain

$$w \geq \rho\Big(\frac{R}{2}\Big)\Phi + c \quad \text{in } E$$

for some $c > 0$, independent of $R$. Since $\Phi$ is bounded from above in $E$ we get indeed

$$w \geq \Big(\rho\Big(\frac{R}{2}\Big) + c\Big)\Phi \quad \text{in } E$$

(the constant $c$ need not be the same but the only important point is that it is independent of $R$). Diminishing $c$ if necessary we may also assume $0 < c < 1$ and

$$B_{k-1} \leq \rho\Big(\frac{R}{2}\Big) + c < \ell + c < B_k$$

for large $R$. Take $\varepsilon \in (0, \frac{1}{2})$. We have

$$-\Delta\Big(\Big(\rho\Big(\frac{R}{2}\Big) + c\varepsilon\Big)\Phi\Big) + 2^{q-1}\Big|\nabla\Big(\rho\Big(\frac{R}{2}\Big) + c\varepsilon\Big)\Phi\Big|^q \leq \lambda B_k^p \Phi^p \quad \text{in } D.$$

Observe that

$$\begin{aligned}
&w \geq \Big(\rho\Big(\frac{R}{2}\Big) + c\varepsilon\Big)\Phi && \text{in } \partial\mathbb{R}_+^N \setminus (B_4 \setminus \overline{B_{1/2}}),\\
&w - \Big(\rho\Big(\frac{R}{2}\Big) + c\varepsilon\Big)\Phi \geq c(1-\varepsilon)\Phi \geq \frac{c}{2}\Phi \geq b > 0 && \text{in } E,\\
&w - \Big(\rho\Big(\frac{R}{2}\Big) + c\varepsilon\Big)\Phi \geq -c\varepsilon\Phi \geq -K\varepsilon && \text{in } D,
\end{aligned}$$

for some $K, b > 0$. If we now choose $\varepsilon = \varepsilon(b)$ as in Lemma 14 we deduce

$$w \geq \Big(\rho\Big(\frac{R}{2}\Big) + c\varepsilon\Big)\Phi \quad \text{in } B_2^+ \setminus \overline{B_1^+}.$$

In particular, if $|x| = 1$ we obtain

$$\frac{v(Rx)}{\Phi(Rx)} = \frac{w(x)}{\Phi(x)} \geq \rho\Big(\frac{R}{2}\Big) + c\varepsilon,$$

so that $\rho(R) \geq \rho(\frac{R}{2}) + c\varepsilon$. Since $c$ and $\varepsilon$ do not depend on $R$, we may let $R$ go to infinity to arrive at $\ell \geq \ell + c\varepsilon$, a contradiction. This contradiction shows that no positive supersolutions of (1.3) exist. $\square$

## 5 The case $q \geq \frac{N+1}{N}$

We finally consider the case where the gradient term is 'negligible' when compared with the Laplacian. In this case, the relevant function to compare with is the fundamental solution of the Laplacian in $\mathbb{R}^N_+$ which vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and goes to zero at infinity, namely

$$\Psi(x) = \frac{x_N}{|x|^N}, \quad x_N > 0.$$

We introduce again the quantity

$$\widetilde{\rho}(R) = \inf_{x \in S_R^+} \frac{v(x)}{\Psi(x)}, \quad R > 0, \tag{5.1}$$

which will be relevant when dealing with the borderline situation $p = \frac{N+1}{N-1}$. Indeed, we have the following important property:

**Lemma 15.** *Assume* $q > \frac{N+1}{N}$ *and* $p = \frac{N+1}{N-1}$. *Let* $v \in C^{1,\alpha}(\mathbb{R}^N_+)$ *with* $\|v\|_{C^{1,\alpha}(\mathbb{R}^N_+)} < +\infty$ *be a positive weak supersolution of* (3.1). *If* $\widetilde{\rho}$ *is given by* (5.1)*, then*

$$\lim_{R \to +\infty} \widetilde{\rho}(R) = 0.$$

**Proof.** Let $R_k \to +\infty$ be an arbitrary sequence. Define

$$w_k(y) = R_k^{N-1} v(R_k e_N + R_k y), \quad y_N > 0.$$

Using Lemma 10, we see that for some positive constant $C$

$$w_k(y) \leq C(1 + y_N)^{1-N}, \quad |\nabla w_k(y)| \leq C(1 + y_N)^{-N}, \quad y_N > 0.$$

Thus, we may choose a subsequence, still denoted by $w_k$, such that $w_k \to w$ uniformly in compact sets of $\mathbb{R}^N_+$ and $w_k \rightharpoonup w$ in $H^1_{\text{loc}}(\mathbb{R}^N_+)$.

On the other hand, it is not hard to see that

$$-\Delta w_k + R_k^{N+1-qN} |\nabla w_k|^q \geq \lambda w_k^p \quad \text{in } \mathbb{R}^N_+, \tag{5.2}$$

so that we may pass to the limit in the weak formulation of (5.2) to obtain that $-\Delta w \geq \lambda w^p$ in $\mathbb{R}^N_+$ in the weak sense (therefore in the viscosity sense). Taking into account that $p = \frac{N+1}{N-1} < \frac{N}{N-2}$ and using Theorem 2.1 in [6] we have $w \equiv 0$, hence

$$\widetilde{\rho}(R_k) \leq \frac{v(R_k e_N)}{\Psi(R_k e_N)} = R_k^{N-1} v(R_k e_N) = w_k(0) \to 0.$$

Since $\{R_k\}$ is an arbitrary sequence we deduce $\lim_{R \to +\infty} \widetilde{\rho}(R) = 0$, as was to be shown. The proof is concluded. $\square$

Before giving the proof of Theorem 2 we still need to introduce an auxiliary function, which somehow connects the fundamental solution of the Laplacian $\Psi$ with the problem including a gradient term.

**Lemma 16.** *Assume $q > \frac{N+1}{N}$ and let $\gamma = qN - (N+1)$. Fix $\gamma' \in (0, \gamma)$. Then there is a value $R_0 = R_0(\gamma, \gamma') > 0$ with the following property: for every $R > R_0$, there exists a positive, bounded function $\psi_R \in C^2(\mathbb{R}^N_+ \setminus \overline{B_1^+})$ which verifies*

$$\begin{cases} -\Delta \psi_R + R^{-\gamma}|\nabla \psi_R|^q \le 0 & \text{in } \mathbb{R}^N_+ \setminus \overline{B_1^+} \\ \psi_R = \Psi & \text{on } \partial(\mathbb{R}^N_+ \setminus \overline{B_1^+}), \end{cases} \tag{5.3}$$

*together with*

$$\lim_{|x|\to+\infty} \psi_R(x) = 0.$$

*In addition, there exists a positive constant $C$ such that*

$$\psi_R(x) \ge (1 - CR^{-\gamma'})\Psi(x), \quad x \in S_2^+. \tag{5.4}$$

**Proof.** Set $q' = \min\{q, 2\}$, and for $R > 1$ consider the problem

$$\begin{cases} -\Delta \psi + R^{-\gamma'}|\nabla \psi|^{q'} = 0 & \text{in } \mathbb{R}^N_+ \setminus \overline{B_1^+}, \\ \psi = \Psi & \text{on } \partial(\mathbb{R}^N_+ \setminus \overline{B_1^+}). \end{cases}$$

It is clear that $\Psi$ is a supersolution, while 0 is a subsolution of this problem. Therefore, with an argument similar to that in Lemma 8, and since $q' \le 2$, we obtain the existence of a classical solution $\psi_R \in C^2(\mathbb{R}^N_+ \setminus \overline{B_1^+})$ verifying $0 < \psi_R < \Psi$. By standard regularity, $|\nabla \psi_R| \le C$ in $\mathbb{R}^N_+ \setminus \overline{B_1^+}$ for some positive constant $C$ which does not depend on $R$.

Observe that $\psi_R$ trivially verifies (5.3) for $R > 1$ when $q \le 2$, because $\gamma' < \gamma$. When $q > 2$, (5.3) will be verified provided that

$$R^{-\gamma}|\nabla \psi_R|^q \le R^{-\gamma'}|\nabla \psi_R|^2$$

in $\mathbb{R}^N_+ \setminus \overline{B_1^+}$. This is certainly true for large $R$, since $|\nabla \psi_R|$ is bounded independently of $R$. This shows the existence of $R_0$ as in the statement of the lemma.

We now turn to prove (5.4). Let $\phi \in C^2(\mathbb{R}^N_+ \setminus \overline{B_1^+})$ be the unique, bounded positive solution of

$$\begin{cases} -\Delta\phi = 1 & \text{in } \mathbb{R}^N_+ \setminus \overline{B_1^+} \\ \phi = 0 & \text{on } \partial(\mathbb{R}^N_+ \setminus \overline{B_1^+}) \end{cases}$$

and observe that, according to standard regularity, there exists a constant $C > 0$ with $\phi \le Cx_N$ in $\overline{B_3^+} \setminus B_{1/2}^+$, say. Moreover, since

$$\begin{cases} -\Delta(\Psi - \psi_R) = R^{-\gamma'} |\nabla \psi_R|^{q'} \le CR^{-\gamma'} & \text{in } \mathbb{R}^N_+ \setminus \overline{B_1^+}, \\ \Psi - \psi_R = 0 & \text{on } \partial(\mathbb{R}^N_+ \setminus \overline{B_1^+}), \end{cases}$$

we obtain by the Phragmèn–Lindelöf maximum principle that $\Psi - \psi_R \le CR^{-\gamma'}\phi$ in $\mathbb{R}^N_+ \setminus \overline{B_1^+}$. In particular,

$$\Psi - \psi_R \le CR^{-\gamma'} x_N \le CR^{-\gamma'}\Psi \quad \text{in } \overline{B_3^+} \setminus B_{1/2}^+.$$

Then inequality (5.4) follows, concluding the proof of the lemma. □

**Proof of Theorem 2.** Assume there exists a positive, bounded, weak supersolution of (3.1). By Lemmas 8 and 9, there exists a positive function $v$ which is a classical solution of (3.1) with $\|v\|_{C^2(\mathbb{R}^N_+)} < +\infty$ when $\frac{N+1}{N} \le q \le 2$ or a weak supersolution of (3.1) with $\|v\|_{C^{1,\alpha}(\mathbb{R}^N_+)} < +\infty$ for every $\alpha \in (0,1)$ when $q > 2$.

Consider first the case $q \ge \frac{N+1}{N}$ with $p < \frac{N+1}{N-1}$. By Lemmas 7 and 10 we see that

$$c\Psi_\varepsilon(x) \le v(x) \le Cx_N^{-\frac{2}{p-1}}, \quad x \in \mathbb{R}^N_+,$$

where $\Psi_\varepsilon$ is given by (2.6) and $c$, $C$ are positive constants. Setting $x = Re_N$ where $R > 0$, this implies

$$cR^{-(N-1+\varepsilon)} \le v(Re_N) \le CR^{-\frac{2}{p-1}}, \quad R > 0.$$

Choosing $\varepsilon$ small enough, we get a contradiction for large $R$.

Therefore to conclude the proof we may restrict ourselves to the case $q > \frac{N+1}{N}$ with $p = \frac{N+1}{N-1}$. Observe that we may apply Lemma 15 to have $\lim_{R\to+\infty} \widetilde{\rho}(R) = 0$, where $\widetilde{\rho}$ is given by (5.1).

Consider the scaled function $v_R(x) = R^{N-1}v(Rx)$, which verifies

$$-\Delta v_R + R^{-\gamma}|\nabla v_R|^q \ge 0 \quad \text{in } \mathbb{R}^N_+,$$

with $\gamma = qN - (N+1)$, and let $\psi_R$ be the function given by Lemma 16 for some $\gamma' \in (0,\gamma)$. Since $\widetilde{\rho}(R) < 1$ for large $R$, we obtain for sufficiently small $\varepsilon$:

$$-\Delta(v_R + \varepsilon) + R^{-\gamma}|\nabla(v_R + \varepsilon)|^q \ge -\Delta(\widetilde{\rho}(R)\psi_R) + R^{-\gamma}|\nabla(\widetilde{\rho}(R)\psi_R)|^q$$

in $\mathbb{R}^N_+ \setminus \overline{B_1^+}$. Also, by the definition of $\widetilde{\rho}(R)$,

$$v_R + \varepsilon \ge \widetilde{\rho}(R)\Psi = \widetilde{\rho}(R)\psi_R$$

on $S_1^+$, while the same inequality is true at infinity since $\psi_R(x) \to 0$ as $|x| \to +\infty$. By comparison we get the same inequality in $\mathbb{R}^N_+ \setminus \overline{B_1^+}$, and letting $\varepsilon \to 0$ we obtain in particular

$$v_R \geq \widetilde{\rho}(R)\psi_R \quad \text{on } S_2^+.$$

Taking into account (5.4), we have $v_R \geq \widetilde{\rho}(R)(1-CR^{-\gamma'})\Psi$ on $S_2^+$ for some positive constant $C$, which is equivalent to

$$\widetilde{\rho}(2R) \geq (1 - CR^{-\gamma'})\widetilde{\rho}(R). \tag{5.5}$$

Next, fix a value $R > 0$ so that $1 - CR^{-\gamma'} > 0$. An iteration of inequality (5.5) yields

$$\widetilde{\rho}(2^k R) \geq \prod_{i=0}^{k-1}(1 - C(2^i R)^{-\gamma'})\widetilde{\rho}(R),$$

for every $k \in \mathbb{N}$. Letting $k \to +\infty$ and observing that the infinite product

$$\prod_{i=0}^{\infty}(1 - C(2^i R)^{-\gamma'})$$

is convergent to a positive constant, we have the contradiction $0 \geq C\widetilde{\rho}(R)$. This contradiction proves the nonexistence of positive supersolutions of (3.1) in this case. $\square$

**Proof of Theorem 3.** Let us check that, in all cases not covered by our theorems, positive, bounded (classical) supersolutions can indeed be constructed. There are several cases to consider:

(a) $0 < q < \frac{N+1}{N}$ and $p > \frac{q}{2-q}$;
(b) $0 < q < \frac{N+1}{N}$ and $p = \frac{q}{2-q}$, but $\lambda$ is small enough;
(c) $q \geq \frac{N+1}{N}$ and $p > \frac{N+1}{N-1}$;
(d) $q = \frac{N+1}{N}$ and $p = \frac{N+1}{N-1}$.

Case (c) is straightforward: we can take a (bounded) supersolution of the problem

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^N_+$$

(cf. [30]), which will clearly be also a supersolution of (1.3). Thus we may assume in what follows that $q \leq \frac{N+1}{N}$. Moreover, observe that it is enough to construct a supersolution $u$ of (1.3) in $\mathbb{R}^N_+ \setminus B_R^+$ for some $R > 0$, since then the function $v(x) = u(x + 2Re_N)$ will be a supersolution of (1.3) defined in all $\mathbb{R}^N_+$.

We look for a supersolution of the form

$$u(x) = A\,\frac{x_N}{|x|^\beta}, \quad x_N > 0,$$

for $A > 0$ and $\beta > 1$ to be chosen. This will imply that $u$ is bounded in $\mathbb{R}^N_+ \setminus B^+_R$ for every $R > 0$. It is sufficient to have the inequality

$$-A\beta(\beta - N)\frac{x_N}{|x|^{\beta+2}} + \frac{A^q}{|x|^{\beta q}}\left|(\beta^2 - 2\beta)\frac{x_N^2}{|x|^2} + 1\right|^{\frac{q}{2}} \geq \lambda A^p \frac{x_N^p}{|x|^{\beta p}}$$

when $x_N > 0$ and $|x| \geq R \geq 1$. Notice that, when $\beta < 2$,

$$(\beta^2 - 2\beta)\frac{x_N^2}{|x|^2} + 1 \geq \beta^2 - 2\beta + 1 = (\beta - 1)^2,$$

while for $\beta \geq 2$ this quantity is simply greater than or equal to 1. Thus, $u$ will be a supersolution of (1.3) provided that

$$-A\beta|\beta - N|\frac{x_N}{|x|^{\beta+2}} + \frac{A^q\gamma^q}{|x|^{\beta q}} \geq \lambda A^p \frac{x_N^p}{|x|^{\beta p}} \tag{5.6}$$

for $x_N > 0$, $|x| \geq R$, where $\gamma = \min\{\beta - 1, 1\}$. We now choose $\beta = \frac{p}{p-q} > 1$ and observe that, since $x_N \leq |x|$, inequality (5.6) will be implied by

$$-A\beta|\beta - N|\frac{x_N}{|x|^{\beta(1-q)+2}} + A^q\gamma^q \geq \lambda A^p. \tag{5.7}$$

Next, consider case (a). Choose a small $A$ so that $A^q\gamma^q > \lambda A^p$ (notice that $p > q$ with the present assumptions), and let $\varepsilon$ be small so that $A^q\gamma^q - \varepsilon > \lambda A^p$. It is not hard to check that $\beta(1 - q) + 2 > 1$, and we deduce

$$A\beta|\beta - N|\frac{x_N}{|x|^{\beta(1-q)+2}} \leq \varepsilon\frac{x_N}{|x|} \leq \varepsilon$$

when $|x| \geq R$ and $R$ is large enough. Therefore (5.7) is a consequence of our choice of $A$ and $\varepsilon$.

As for case (b), we see that $\beta(1 - q) + 2 = 1$, so that (5.7) reduces to showing that a value of $A$ can be chosen to have

$$-A\beta|\beta - N| + A^q\gamma^q \geq \lambda A^p. \tag{5.8}$$

It is always possible to achieve (5.8) for a suitable value of $A$, provided that $\lambda$ verifies

$$0 < \lambda \leq \sup_{t>0} \frac{\gamma^q t^q - \beta|\beta - N|t}{t^p}. \tag{5.9}$$

Finally, case (d) falls under the assumption $p = \frac{q}{2-q}$, so that the discussion just made applies, with the only important difference that $\beta = N$ in this case. Thus the supremum in (5.9) is infinite and no restriction on the size of $\lambda$ is indeed needed. $\square$

**Acknowledgements.** J. G.-M. and A. Q. were partially supported by Ministerio de Economía y Competitividad under grants MTM2011-27998 and MTM2014-52822-P (Spain). A. Q. was also partially supported by Fondecyt Grant No. 1151180 Programa Basal, CMM. U. de Chile and Millennium Nucleus Center for Analysis of PDE NC130017.

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(Received March 18, 2017)