ON THE STABILITY FOR ALEXANDROV'S SOAP BUBBLE THEOREM

By

ROLANDO MAGNANINI AND GIORGIO POGGESI

To Prof. Shigeru Sakaguchi on the occasion of his 60-th birthday

Abstract. Alexandrov's Soap Bubble Theorem dates back to 1958 and states that a compact embedded hypersurface in $\mathbb{R}^N$ with constant mean curvature must be a sphere. For its proof, A. D. Alexandrov invented his reflection principle. In 1977, R. Reilly gave an alternative proof, based on integral identities and inequalities, connected with the torsional rigidity of a bar. In this article we study the stability of the spherical symmetry: the question is how near is a hypersurface to a sphere, when its mean curvature is near to a constant in some norm?
We present a stability estimate that states that a compact hypersurface $\Gamma \subset \mathbb{R}^N$ can be contained in a spherical annulus whose interior and exterior radii, say ρ_i and ρ_e, satisfy the inequality

$$\rho_e - \rho_i \le C \|H - H_0\|_{L^1(\Gamma)}^{\tau_N},$$

where $\tau_N = 1/2$ if $N = 2, 3$, and $\tau_N = 1/(N + 2)$ if $N \ge 4$. Here, H is the mean curvature of Γ, H_0 is some reference constant, and C is a constant that depends on some geometrical and spectral parameters associated with Γ. This estimate improves previous results in the literature under various aspects. We also present similar estimates for some related overdetermined problems.

1 Introduction

Alexandrov's **Soap Bubble Theorem** states that a compact hypersurface embedded in $\mathbb{R}^N$ that has constant mean curvature H must be a sphere. To prove that result, A. D. Alexandrov introduced his reflection principle (see [Al1],[Al2]), later adapted and refined by J. Serrin into the method of moving planes, which has turned out to be effective to prove radial symmetry of the solutions of certain overdetermined problems in potential theory (see [Se]).

We now know that the fact that essentially the same method works successfully for both problems is not accidental. To see that, we recall that in its simplest formulation the result obtained by Serrin states that the overdetermined boundary

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 139 (2019)
DOI 10.1007/s11854-019-0058-y

value problem,

$$\Delta u = N \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \tag{1.1}$$

$$u_\nu = R \quad \text{on } \Gamma, \tag{1.2}$$

admits a solution for some positive constant R if and only if Ω is a ball of radius R and $u(x) = (|x|^2 - R^2)/2$. Here, Ω denotes a bounded domain in $\mathbb{R}^N$, $N \ge 2$, with sufficiently smooth boundary Γ and u_ν is the outward normal derivative of u on Γ.

The connection between (1.1)–(1.2) and the Soap Bubble problem is hinted at by the simple differential identity

$$\Delta u = |\nabla u| \operatorname{div} \frac{\nabla u}{|\nabla u|} + \frac{\langle \nabla^2 u \nabla u, \nabla u\rangle}{|\nabla u|^2};$$

here, ∇u and $\nabla^2 u$ are the gradient and the hessian matrix of u, as standard. If we agree to still denote by ν the vector field $\nabla u/|\nabla u|$ (that on Γ coincides with the outward unit normal), the above identity and (1.1) inform us that

$$u_{\nu\nu} + (N-1) H u_\nu = N, \tag{1.3}$$

on every non-critical level surface of u, and hence on Γ, since a well-known formula states that the mean curvature H of a regular level surface of u equals

$$\frac{1}{N-1} \operatorname{div} \frac{\nabla u}{|\nabla u|}.$$

Based on (1.3), R. C. Reilly gave in [Re1] (see also [Re2]) an alternative proof of Alexandrov's theorem, that hinges on an assortment of integral inequalities and identities and culminates in showing that the inequality

$$(\Delta u)^2 \le N |\nabla^2 u|^2, \tag{1.4}$$

that is a simple consequence of Cauchy-Schwarz inequality, holds pointwise with the equality sign on the whole Ω. (In our notation, $|\nabla^2 u|^2$ is the sum of the squares of the entries of $\nabla^2 u$.) In fact, equality takes place in (1.4) if and only if u is a quadratic polynomial q of the form

$$q(x) = \frac{1}{2}(|x-z|^2 - a), \tag{1.5}$$

for some choice of $z \in \mathbb{R}^N$ and $a \in \mathbb{R}$, since (1.1) is in force. This fact clearly implies that Ω must be a ball.

The aim of this paper is to study the stability issue for the Soap Bubble Theorem. The question is to ascertain under which conditions the smallness (in some norm) of the deviation of H from being a constant implies the closeness of Ω to a ball. The key ingredient to accomplish that goal is the following integral identity for the solution of (1.1):

$$\frac{1}{N-1}\int_\Omega \left\{|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\right\}dx = N|\Omega| - \int_\Gamma H(u_\nu)^2 dS_x \tag{1.6}$$

(see Theorem 2.1 below). We shall refer to the first integrand in (1.6) as the **Cauchy-Schwarz deficit** for $\nabla^2 u$.

If H is constant on Γ, from **Minkowski's identity**,

$$\int_\Gamma H(x)\langle x - p, \nu(x)\rangle dS_x = |\Gamma|, \quad p \in \mathbb{R}^N, \tag{1.7}$$

we find that $H \equiv |\Gamma|/N|\Omega|$ and hence the Hölder inequality

$$\left(\int_\Gamma u_\nu dS_x\right)^2 \le |\Gamma| \int_\Gamma (u_\nu)^2 dS_x \tag{1.8}$$

yields the non-positivity of the right-hand side of (1.6), that gives the equality sign in (1.4), as desired.

If H is not constant, we can take the mean curvature of a ball as a reference under the form

$$H_0 = \frac{|\Gamma|}{N|\Omega|};$$

by applying (1.8) as before, from (1.6) we obtain that

$$\frac{1}{N-1}\int_\Omega \left\{|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\right\}dx \le \int_\Gamma (H_0 - H)(u_\nu)^2 dS_x. \tag{1.9}$$

It is interesting to note that (1.9) implies the spherical symmetry of u (or Ω) if its right-hand side is non-positive, with no need to use (1.7), and this certainly holds if $H \ge H_0$. (Of course, if H equals some constant on Γ, then (1.7) implies that $H \equiv H_0$ and hence $H \ge H_0$, too.)

Inequality (1.9) can also be rearranged as

$$\frac{1}{N-1}\int_\Omega \left\{|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\right\}dx + \int_\Gamma (H_0 - H)^-(u_\nu)^2 dS_x \le \int_\Gamma (H_0 - H)^+(u_\nu)^2 dS_x$$

(here, we use the positive and negative part functions $(t)^+ = \max(t, 0)$ and $(t)^- = \max(-t, 0)$). That inequality tells us that, if we have an a priori bound M for u_ν on Γ, then its left-hand side is small if the integral

$$\int_\Gamma (H_0 - H)^+ dS_x$$

is also small. In particular, if H is not too much smaller than H_0, then it cannot be too much larger than H_0 and the Cauchy-Schwarz deficit cannot be too large. Thus, to achieve our aim, it remains to quantitavely transform this smallness into closeness of Ω to a ball.

In Theorem 4.1 we shall prove that, for some point $z \in \Omega$, the radius of the largest ball centered at z and contained in Ω and that of the smallest concentric ball that contains Ω, that is

$$\rho_i = \min_{x \in \Gamma} |x - z| \quad \text{and} \quad \rho_e = \max_{x \in \Gamma} |x - z|, \tag{1.10}$$

satisfy the following stability estimate:

$$\rho_e - \rho_i \le C \left\{ \int_\Gamma (H_0 - H)^+ dS_x \right\}^{\tau_N}, \tag{1.11}$$

where $\tau_N = 1/2$ for $N = 2, 3$ and $\tau_N = 1/(N+2)$ for $N \geq 4$. Here, the constant C depends on N and some geomerical and spectral parameters associated with Ω (see Theorem 4.1 for details).

Inequality (1.11) improves similar estimates given in [CV] and [CM], under various aspects. In fact, it replaces the uniform measure $\|H - H_0\|_{\infty,\Gamma}$ of the deviation from H_0, considered in [CV] and [CM], by a weaker L^1-type norm; it is thus extended to a larger class of hypersurfaces, being not restricted (as it is in [CM]) to those with positive mean curvature. Also, it improves the exponent τ_N obtained in [CM]—even in this weaker setting and for all $N \geq 2$—to the extent that it obtains, for the cases $N = 2, 3$, what seems to be the optimal exponent attainable with this approach. Finally, even if it does not improve the exponent $\tau_N = 1$, obtained in [CV] by means of an adaptation of the reflection principle, it favors the computability of the constant C, as shown in [CM] and differently from [CV].

To prove (1.11), we consider the function $h = q - u$; h is harmonic in Ω, $h = q$ on Γ, and we can deduce from (1.9) that

$$\frac{1}{N-1} \int_\Omega |\nabla^2 h|^2 dx \le \int_\Gamma (H_0 - H)^+ (u_\nu)^2 dS_x.$$

Notice that this inequality holds regardless of the choice of the parameters $z \in \mathbb{R}^N$ and $a \in \mathbb{R}$ in (1.5). Thus, to ensure that z is in Ω, we choose it as a minimum (or any critical) point of u; then, since $\nabla h(z) = 0$, we show that the oscillation of h on Γ,

$$\max_\Gamma h - \min_\Gamma h,$$

can be bounded in terms of a power of the quantity

$$\int_\Omega |\nabla^2 h|^2 dx.$$

Since h attains its extrema when q does, then

$$\max_\Gamma h - \min_\Gamma h = \frac{1}{2}(\rho_e^2 - \rho_i^2), \tag{1.12}$$

and hence an a priori bound for u_ν on Γ and the observation that $\rho_i + \rho_e$ can be bounded from below by the volume of Ω give the desired estimate.

In Section 2, we collect all the relevant identities on which our result is based. To make the presentation self-contained, we also include a version of Reilly's proof of the Soap Bubble Theorem. We also discuss versions of those identities that give radial symmetry for some overdetermined problems associated with (1.1). In particular, we present a new proof of Serrin's symmetry result which, however, only works if Ω is strictly star-shaped with respect to some origin.

Section 3 contains the estimates on harmonic functions and the torsional creep function that are instrumental to derive (1.11). The key result is Lemma 3.4, in which we are able to bound the difference $\rho_e - \rho_i$ in terms of $\|\nabla^2 h\|_{2,\Omega}$. Theorem 3.10 provides a simple bound for the gradient of u on Γ in terms of the diameter of Ω and the radius of the exterior uniform touching ball. This bound is important, since it allows to treat the general case of $C^{2,\alpha}$-smooth hypersurfaces, and is obtained by elementary arguments and seems to be new, generalizing the classical work of Payne and Philippin [PP], that concerned the case of strictly mean convex domains, in which H is positive at each point in Γ.

Finally, in Section 4, we assemble the identities and inequalities proved in the previous sections and establish our stability results. As a corollary of our main inequality contained in Theorem 4.1, we obtain an estimate of closeness to an aggregate of balls, in the spirit of [CM]. With more or less the same techniques employed for Theorem 4.1, we also present stability bounds for some of the overdetermined problems considered in Section 2.

2 Alexandrov's Soap Bubble Theorem

In this section, we review the details of Reilly's proof, with some modifications, that will enable us to derive our stability results. The proof we present is based on the identity (1.6). We also show how to use (1.6) to obtain other symmetry results, old and new.

The identity (1.6) is a consequence of the differential identity for the solution u of (1.1),

$$|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} = \Delta P, \tag{2.1}$$

that associates the Cauchy-Schwarz deficit with the **P-function**

$$P = \frac{1}{2}|\nabla u|^2 - u, \tag{2.2}$$

and is easily obtained by direct computation. Notice that (2.1) also implies that P is subharmonic, since the left-hand side is non-negative by the Cauchy-Schwarz inequality.

In the next theorem, for the sake of completeness, we give the proof of (1.6), which can also be found in [Re1].

Theorem 2.1 (Fundamental Identity)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class $C^{2,\alpha}$ and let H be the mean curvature of Γ.*

If u is the solution of (1.1)*, then* (1.6) *holds:*

$$\frac{1}{N-1}\int_\Omega \left\{|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\right\}dx = N|\Omega| - \int_\Gamma H(u_\nu)^2 dS_x.$$

Proof. Let P be given by (2.2). By the divergence theorem we can write

$$\int_\Omega \Delta P dx = \int_\Gamma P_\nu dS_x. \tag{2.3}$$

To compute P_ν, we observe that ∇u is parallel to ν on Γ, that is $\nabla u = (u_\nu)\nu$ on Γ. Thus,

$$P_\nu = \langle D^2 u \nabla u, \nu\rangle - u_\nu = u_\nu \langle (D^2 u)\nu, \nu\rangle - u_\nu = u_{\nu\nu}u_\nu - u_\nu.$$

By Reilly's identity (1.3), we know that

$$u_{\nu\nu}u_\nu + (N-1)H(u_\nu)^2 = N u_\nu,$$

and hence

$$P_\nu = (N-1)u_\nu - (N-1)H(u_\nu)^2$$

on Γ.

Therefore, (1.6) follows from this identity, (2.1), (2.3) and the formula

$$\int_\Gamma u_\nu dS_x = N|\Omega|, \tag{2.4}$$

which is an easy consequence of the divergence theorem. □

The fundamental identity (1.6) can be re-arranged at least into two ways to yield the Soap Bubble Theorem. The former follows the lines of Reilly's proof. The latter gives Alexandrov's theorem via the Heintze–Karcher inequality (2.10) below and we will present it at the end of this section.

Theorem 2.2 (Soap Bubble Theorem)**.** *Let* $\Gamma \subset \mathbb{R}^N$ *be a surface of class* $C^{2,\alpha}$, *which is the boundary of a bounded domain* $\Omega \subset \mathbb{R}^N$, *and let* u *be the solution of* (1.1)*. Let two positive constants be defined by*

$$R = \frac{N|\Omega|}{|\Gamma|} \quad \text{and} \quad H_0 = \frac{1}{R} = \frac{|\Gamma|}{N|\Omega|}. \tag{2.5}$$

Then, the following identity holds:

$$\frac{1}{N-1}\int_\Omega \Big\{|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\Big\}dx + \frac{1}{R}\int_\Gamma (u_\nu - R)^2 dS_x = \int_\Gamma (H_0 - H)(u_\nu)^2 dS_x. \tag{2.6}$$

Therefore, if the mean curvature H *of* Γ *satisfies the inequality* $H \geq H_0$ *on* Γ, *then* Γ *must be a sphere (and hence* Ω *is a ball) of radius* R.

In particular, the same conclusion holds if H *equals some constant on* Γ.

Proof. Since, by (2.4), we have that

$$\frac{1}{R}\int_\Gamma u_\nu^2 dS_x = \frac{1}{R}\int_\Gamma (u_\nu - R)^2 dS_x + N|\Omega|,$$

then

$$\begin{aligned}\int_\Gamma H(u_\nu)^2 dS_x &= H_0\int_\Gamma (u_\nu)^2 dS_x + \int_\Gamma (H - H_0)(u_\nu)^2 dS_x \\ &= \frac{1}{R}\int_\Gamma (u_\nu - R)^2 dS_x + N|\Omega| + \int_\Gamma (H - H_0)(u_\nu)^2 dS_x.\end{aligned}$$

Thus, (2.6) follows from this identity and (1.6) at once.

If $H \geq H_0$ on Γ, then the right-hand side in (2.6) is non-positive and hence both summands at the left-hand side must be zero, being non-negative. (Note in passing that this fact implies that the second summand is zero and hence $u_\nu \equiv R$ on Γ, that is u satisfies (1.1)–(1.2).)

The fact that also the first summand is zero gives that the Cauchy-Schwarz deficit for the hessian matrix $\nabla^2 u$ must be identically zero and, since $\Delta u = N$, that occurs if and only if $\nabla^2 u$ equals the identity matrix I. Thus, u must be a quadratic polynomial q, as in (1.5), for some $z \in \mathbb{R}^N$ and $a \in \mathbb{R}$.

Since $u = 0$ on Γ, then $|x - z|^2 = a$ for $x \in \Gamma$, that is a must be positive and

$$\sqrt{a}|\Gamma| = \int_\Gamma |x - z| dS_x = \int_\Gamma (x - z) \cdot \nu(x) dS_x = N|\Omega|.$$

In conclusion, Γ must be a sphere centered at z with radius R.

If H equals some constant, instead, then (1.7) tells us that the constant must equal H_0, and hence we can apply the previous argument. □

Remark 2.3. (i) As pointed out in the previous proof, before showing that Ω is a ball, we have also proved that, if H is constant, then u satisfies (1.1)–(1.2). It would be interesting to show that also the converse is true. That would show that the two problems are equivalent.

(ii) We observe that the assumption that $H \geq H_0$ on Γ implies that $H \equiv H_0$, anyway, if Ω is strictly star-shaped with respect to some origin p. In fact, by Minkowski's identity (1.7), we obtain that

$$0 \leq \int_\Gamma [H(x) - H_0] \langle (x - p), \nu(x) \rangle dS_x = |\Gamma| - H_0 \int_\Gamma \langle (x - p), \nu(x) \rangle dS_x = 0,$$

and we know that $\langle (x - p), \nu(x) \rangle > 0$ for $x \in \Gamma$.

Before presenting the proof of Alexandrov's theorem based on the Heintze–Karcher inequality, we prove two symmetry results for overdetermined problems (one of which is Serrin's result under some restriction), that have their own interest.

Theorem 2.4 (Two overdetermined problems)**.** *Let $u \in C^{2,\alpha}(\overline{\Omega})$ be the solution of* (1.1).

Then, Ω is a ball if and only if u satisfies one of the following conditions:

(i) $u_\nu(x) = 1/H(x)$ *for every* $x \in \Gamma$*;*

(ii) (1.2) *holds and* $\langle (x - p), \nu(x) \rangle > 0$ *for every* $x \in \Gamma$ *and some* $p \in \Omega$.

Proof. It is clear that, if Ω is a ball, then (i) and (ii) hold. Conversely, we shall check that the right-hand side of (1.6) is zero when one of the items (i) or (ii) occurs.

(i) Notice that our assumption implies that H must be positive, since u_ν is positive and finite. Since (2.4) holds, then (1.6) can be written as

$$\frac{1}{N-1} \int_\Omega \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} dx = \int_\Gamma (1 - H u_\nu) u_\nu dS_x, \tag{2.7}$$

and the conclusion follows at once.

(ii) Let u_ν be constant on Γ; by (2.4) we know that that constant equals the value R given in (2.5). Also, notice that $1 - H u_\nu \geq 0$ on Γ. In fact, the function P

in (2.2) is subharmonic in Ω, since $\Delta P \ge 0$ by (2.1). Thus, it attains its maximum on Γ, where it is constant. We thus have that

$$0 \le P_\nu = u_\nu u_{\nu\nu} - u_\nu = (N-1)(1 - H u_\nu) u_\nu \quad \text{on } \Gamma.$$

Now,

$$0 \le \int_\Gamma [1 - H(x) u_\nu(x)] \langle (x-p), \nu(x) \rangle dS_x = \int_\Gamma [1 - H(x) R] \langle (x-p), \nu(x) \rangle dS_x = 0,$$

by (1.7). Thus, $1 - H u_\nu \equiv 0$ on Γ and hence (i) applies. □

Remark 2.5. The proof of (ii) seems to be new. Even if it is restricted to the case of strictly star-shaped domains, it might be used to obtain better stability estimates for Serrin's symmetry result.

We recall that, by following the tracks of Weinberger's proof ([We]) and its modification due to Payne and Schaefer ([PS]), one can write the identity

$$\int_\Omega (-u) \Big\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \Big\} dx = \frac{1}{2} \int_\Gamma (u_\nu^2 - R^2)(u_\nu - x \cdot \nu) dS_x, \tag{2.8}$$

that gives at once spherical symmetry if u_ν is constant on Γ, without major restrictions on Ω other than on the regularity of Γ. The presence of the factor $-u$ at the left-hand side, however, may cause additional difficulties in the study of the stability issue.

We conclude this section by showing that (1.6) can be rearranged into an identity that implies the Heintze–Karcher inequality (see [HK]). This proof is slightly different from that of A. Ros in [Ro] and relates the equality case for the Heintze–Karcher inequality to the overdetermined problem considered in (i) of Theorem 2.4.

Theorem 2.6 (Soap Bobble Theory and the Heintze–Karcher inequality)**.** *Let $\Gamma \subset \mathbb{R}^N$ be a surface of class $C^{2,\alpha}$, which is the boundary of a bounded domain $\Omega \subset \mathbb{R}^N$, and let $u \in C^{2,\alpha}(\overline{\Omega})$ be the solution of* (1.1).

If Γ is strictly mean-convex, then we have the following identity:

$$\frac{1}{N-1} \int_\Omega \Big\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \Big\} dx + \int_\Gamma \frac{(1 - H u_\nu)^2}{H} dS_x = \int_\Gamma \frac{dS_x}{H} - N|\Omega|. \tag{2.9}$$

In particular, the Heintze–Karcher inequality

$$\int_\Gamma \frac{dS_x}{H} \ge N|\Omega| \tag{2.10}$$

holds and the sign of equality is attained in (2.10) *if and only if Ω is a ball.*

Thus, if H is constant on Γ, then Γ is a sphere.

Proof. By integrating on Γ the identity

$$\frac{(1-Hu_\nu)^2}{H} = -(1-Hu_\nu)u_\nu + \frac{1}{H} - u_\nu,$$

summing the result up to (2.7) and taking into account (2.4), we get (2.9).

Both summands at the left-hand side of (2.9) are non-negative and hence (2.10) follows. If the right-hand side is zero, those summands must be zero. The vanishing of the first summand implies that Ω is a ball, as already noticed. Note in passing that the vanishing of the second summand gives that $u_\nu = 1/H$ on Γ, which also implies radial symmetry, by Theorem 2.4.

Finally, if H equals some constant on Γ, we know that such a constant must have the value H_0 in (2.5), that implies that the right-hand side of (2.9) is null and hence, once again, Ω must be a ball. □

3 Some estimates for harmonic functions

We begin by setting some relevant notation. By $\Gamma \subset \mathbb{R}^N$, $N \geq 2$, we shall always denote a hypersurface of class $C^{2,\alpha}$, $0 < \alpha < 1$, that is the boundary of a bounded domain Ω. By $|\Omega|$ and $|\Gamma|$, we will denote indifferently the N-dimensional Lebesgue measure of Ω and the surface measure of Γ. The **diameter** of Ω will be indicated by d_Ω.

Moreover, since Γ is bounded and of class $C^{2,\alpha}$, it has the properties of the uniform interior and exterior sphere condition, whose respective radii will be designated by r_i and r_e; namely, there exists $r_e > 0$ (resp., $r_i > 0$) such that for each $p \in \Gamma$ there exists a ball $B \subset \mathbb{R}^N \setminus \overline{\Omega}$ (resp. $B \subset \Omega$) of radius r_e (resp., r_i) such that $\overline{B} \cap \Gamma = \{p\}$.

The assumed regularity of Γ ensures that the unique solution of (1.1) is of class $C^{2,\alpha}$. Thus, we can define

$$M = \max_{\overline{\Omega}} |\nabla u| = \max_{\Gamma} u_\nu. \tag{3.1}$$

We finally recall that, for a point $z \in \Omega$, ρ_i and ρ_e denote the radius of the largest ball centered at z and contained in Ω and that of the smallest ball that contains Ω with the same center, as defined in (1.10).

As already mentioned in the Introduction, the first summand in (2.6) or in (2.9) can be suitably re-written, in terms of the harmonic function $h = q - u$, as

$$\frac{1}{N-1}\int_\Omega \Big\{|\nabla^2 u|^2 - \frac{(\Delta u)^2}{N}\Big\} dx = \frac{1}{N-1}\int_\Omega |\nabla^2 h|^2 dx, \tag{3.2}$$

where q is any quadratic polynomial of the form (1.5). Also, if we choose the center z of the paraboloid (1.5) in Ω, we have (1.12), that is

$$\max_\Gamma h - \min_\Gamma h = \max_\Gamma q - \min_\Gamma q = \frac{1}{2}(\rho_e^2 - \rho_i^2).$$

A stability estimate for the spherical symmetry of Γ will then be obtained, via identity (2.6) or (2.9), if we associate the oscillation of h on Γ with (3.2).

To realize this agenda, we start by proving some Poincaré-type inequalities for harmonic functions.

Lemma 3.1 (Poincaré-type inequalities)**.** *There exist two positive constants, $\overline{\mu}(\Omega)$ and $\mu_0(\Omega)$, such that*

$$\int_\Omega v^2 dx \le \overline{\mu}(\Omega)^{-1} \int_\Omega |\nabla v|^2 dx, \tag{3.3}$$

or

$$\int_\Omega v^2 dx \le \mu_0(\Omega)^{-1} \int_\Omega |\nabla v|^2 dx, \tag{3.4}$$

for every function $v \in W^{1,2}(\Omega)$ which is harmonic in Ω, and such that

$$\int_\Omega v\, dx = 0, \tag{3.5}$$

or, respectively,

$$v(x_0) = 0, \tag{3.6}$$

where x_0 is a given point in Ω.

Proof. We define

$$\overline{\mu}(\Omega) = \inf\left\{ \int_\Omega |\nabla v|^2 dx : \int_\Omega v^2 dx = 1,\ \Delta v = 0 \text{ in } \Omega,\ \int_\Omega v\, dx = 0 \right\} \tag{3.7}$$

and

$$\mu_0(\Omega) = \inf\left\{ \int_\Omega |\nabla v|^2 dx : \int_\Omega v^2 dx = 1,\ \Delta v = 0 \text{ in } \Omega,\ v(x_0) = 0 \right\}. \tag{3.8}$$

If we prove that the two infima are attained, we obtain that they are positive and hence (3.3) and (3.4) hold.

We know that we can find a minimizing sequence $\{v_n\}_{n\in\mathbb{N}}$ of (3.7) or (3.8) that converges in $L^2(\Omega)$ and weakly in $W^{1,2}(\Omega)$ to a function v in $W^{1,2}(\Omega)$. Also, by the mean value property for harmonic functions, this sequence converges uniformly on compact subsets of Ω, that implies that v is harmonic in Ω. Thus, we easily infer that

$$\int_\Omega v^2 dx = 1,$$

and $\int_\Omega v dx = 0$, if $\{v_n\}_{n\in\mathbb{N}}$ is minimizing the problem (3.7), or $v(x_0) = 0$, if it is minimizing (3.8).

Finally, we have that

$$\overline{\mu}(\Omega) = \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2 dx \geq \int_\Omega |\nabla v|^2 dx \geq \overline{\mu}(\Omega),$$

by the weak convergence in $W^{1,2}(\Omega)$. This same conclusion holds for problem (3.8). □

Remark 3.2. It is clear that $\overline{\mu}(\Omega) > \mu_2(\Omega)$, where $\mu_2(\Omega)$ is the **second Neumann eigenvalue**. Moreover,

$$\mu_0(\Omega) \leq \overline{\mu}(\Omega).$$

In fact, let

$$v_0 = \frac{v - v(x_0)}{\sqrt{1 + |\Omega| v(x_0)^2}},$$

where v is a minimizer for (3.7); v_0 is harmonic in Ω, $v_0(x_0) = 0$, and $\int_\Omega v_0^2 dx = 1$. Therefore,

$$\mu_0(\Omega) \leq \int_\Omega |\nabla v_0|^2 dx = \frac{\int_\Omega |\nabla v|^2 dx}{1 + |\Omega| v(x_0)^2} = \frac{\overline{\mu}(\Omega)}{1 + |\Omega| v(x_0)^2} \leq \overline{\mu}(\Omega).$$

The crucial result is Theorem 3.4, in which we associate the oscillation of the already defined harmonic function $h = q - u$, and hence the difference $\rho_e - \rho_i$, with the L^2-norm $\|\nabla^2 h\|_{2,\Omega}$ of its Hessian matrix. In the following lemma, we start by linking that oscillation with the L^2-norm of $h - h(z)$. With this aim, we define the **parallel set** as

$$\Omega_\sigma = \{y \in \Omega : \text{dist}(y, \Gamma) > \sigma\} \quad \text{for } 0 < \sigma < r_i.$$

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary of class $C^{2,\alpha}$. Set $h = q - u$, where u is the solution of* (1.1) *and q is any quadratic polynomial as in* (1.5) *with $z \in \Omega$.*

Then, if

$$\|h - h(z)\|_{2,\Omega} < \frac{\sqrt{|B|}}{N 2^{N+1}} M r_i^{\frac{N+2}{2}} \tag{3.9}$$

holds, we have that

$$\rho_e - \rho_i \le a_N \frac{M^{\frac{N}{N+2}}}{|\Omega|^{\frac{1}{N}}} \|h - h(z)\|_{2,\Omega}^{2/(N+2)}, \tag{3.10}$$

where

$$a_N = \frac{2^{2+\frac{N}{N+2}}(N+2)}{N^{\frac{N}{N+2}}} |B|^{\frac{1}{N}-\frac{1}{N+2}}. \tag{3.11}$$

Proof. Let x_i and x_e be points in Γ that minimize (resp., maximize) q on Γ and, for

$$0 < \sigma < r_i,$$

define the two points in $y_i, y_e \in \partial\Omega_\sigma$ by $y_j = x_j - \sigma\nu(x_j)$, $j = i, e$.

We have that

$$h(y_j) - h(x_j) = -\int_0^\sigma \langle \nabla h(x_j - t\nu(x_j)), \nu(x_j)\rangle dt$$

and hence, by recalling that $\nabla h(x) = x - z - \nabla u(x)$ and that $x_j - z$ is parallel to $\nu(x_j)$, we obtain that

$$h(y_j) - h(x_j) = -|x_j - z|\sigma + \frac{1}{2}\sigma^2 + \int_0^\sigma \langle \nabla u(x_j - t\nu(x_j)), \nu(x_j)\rangle dt.$$

Thus, the fact that $2h(x_j) = \rho_j^2$ and $|x_j - z| = \rho_j$ yields that

$$\frac{1}{2}\rho_j^2 - \rho_j\sigma + \frac{1}{2}\sigma^2 = h(y_j) - \int_0^\sigma \langle \nabla u(x_j - t\nu(x_j)), \nu(x_j)\rangle dt, \quad j = i, e,$$

and hence

$$\frac{1}{2}(\rho_e - \rho_i)(\rho_e + \rho_i - 2\sigma) \le h(y_e) - h(y_i) + 2M\sigma, \tag{3.12}$$

for every $0 < \sigma < \min\{\frac{\rho_e+\rho_i}{2}, r_i\}$.

Since h is harmonic and $y_j \in \overline{\Omega}_\sigma$, $j = i, e$, we can use the mean value property for the balls with radius σ centered at y_j and obtain

$$\begin{aligned} |h(y_j) - h(z)| &\le \frac{1}{|B|\sigma^N} \int_{B_\sigma(y_j)} |h - h(z)| dy \le \frac{1}{\sqrt{|B|\sigma^N}} \left[\int_{B_\sigma(y_j)} |h - h(z)|^2 dy\right]^{1/2} \\ &\le \frac{1}{\sqrt{|B|\sigma^N}} \left[\int_\Omega |h - h(z)|^2 dy\right]^{1/2} \end{aligned}$$

by Hölder's inequality. This and inequality (3.12) then yield that

$$\frac{1}{2}(\rho_e + \rho_i - 2\sigma)(\rho_e - \rho_i) \le 2\Big[\frac{\|h - h(z)\|_{2,\Omega}}{\sqrt{|B|}\sigma^{N/2}} + M\sigma\Big],$$

for every $0 < \sigma < \min\{\frac{\rho_e+\rho_i}{2}, r_i\}$.

Now, observe that, for $0 < \sigma < \sigma_0$ with

$$\sigma_0 = \frac{1}{4}\Big(\frac{|\Omega|}{|B|}\Big)^{1/N},$$

then $\rho_e + \rho_i - 2\sigma > 2\sigma_0$, and hence

$$\rho_e - \rho_i \le \frac{2}{\sigma_0}\Big[\frac{\|h - h(z)\|_{2,\Omega}}{\sqrt{|B|}\sigma^{N/2}} + M\sigma\Big] \quad \text{for every } 0 < \sigma < \min\{\sigma_0, r_i\}. \tag{3.13}$$

Therefore, by minimizing the right-hand side of (3.13), we can conveniently choose

$$\sigma = \Big(\frac{N\|h - h(z)\|_{2,\Omega}}{2|B|^{1/2}M}\Big)^{2/(N+2)}$$

in (3.13) and obtain (3.10), if $\sigma < r_i/4 < \min\{\sigma_0, r_i\}$; (3.9) will then follow. □

To simplify formulas, in the remainder of this section and in Section 4 we shall always denote the constants only depending on the dimension by k_N and α_N. Their computation will be clear from the relevant proofs.

A way to conveniently choose z inside Ω is to let z be any (local) minimum point of u in $\overline{\Omega}$; we are thus sure that $z \in \Omega$ and also obtain that $\nabla h(z) = 0$. This remark and Lemmas 3.1 and 3.3 give the following result.

Theorem 3.4. *Let z be any (local) minimum point of the solution u of* (1.1) *in $\overline{\Omega}$. Set $h = q - u$, where q is given by* (1.5).

If $N = 2$ or 3, then

$$\rho_e - \rho_i \le C\|\nabla^2 h\|_{2,\Omega}, \tag{3.14}$$

where

$$C = k_N c \frac{d_\Omega^\gamma}{|\Omega|^{\frac{1}{N}}} \frac{1 + \mu_0(\Omega)}{\mu_0(\Omega)},$$

γ is any number in (0, 1) *for $N = 2$, $\gamma = 1/2$ for $N = 3$, and c is the Sobolev immersion constant of $W^{2,2}(\Omega)$ in $C^{0,\gamma}(\overline{\Omega})$.*

If $N \geq 4$, then

$$\rho_e - \rho_i \le C\|\nabla^2 h\|_{2,\Omega}^{2/(N+2)} \tag{3.15}$$

for

$$\|\nabla^2 h\|_{2.\Omega} < \varepsilon, \tag{3.16}$$

where

$$C = k_N \frac{M^{\frac{N}{N+2}}}{\mu_0(\Omega)^{\frac{2}{N+2}}|\Omega|^{\frac{1}{N}}} \quad \text{and} \quad \varepsilon = \alpha_N M \mu_0(\Omega) r_i^{\frac{N+2}{2}}.$$

Proof. (i) Let $N = 2$ or 3. By the Sobolev immersion theorem (see, for instance, [Gi, Theorem 3.12] or [Ad, Chapter 5]), we have that there is a constant c such that, for any $v \in W^{2,2}(\Omega)$, we have that

$$\frac{|v(x) - v(y)|}{|x - y|^\gamma} \le c\|v\|_{W^{2,2}(\Omega)} \quad \text{for any } x, y \in \overline{\Omega} \text{ with } x \neq y,$$

where γ is any number in $(0, 1)$ for $N = 2$ and $\gamma = 1/2$ for $N = 3$.

We now set $v = h - h(z)$, $x_0 = z$, and apply (3.4) twice: to v and to each first derivative of v (since $\nabla v(z) = \nabla h(z) = 0$). We obtain that

$$\|h - h(z)\|_{W^{2,2}(\Omega)} \le \sqrt{1 + \mu_0(\Omega)^{-1} + \mu_0(\Omega)^{-2}}\,\|\nabla^2 h\|_{2,\Omega}.$$

Since $h - h(z)$ is harmonic, it attains its extrema on Γ and hence we have that

$$\frac{1}{2}(\rho_e^2 - \rho_i^2) = \max_\Gamma h - \min_\Gamma h \le c d_\Omega^\gamma \sqrt{1 + \mu_0(\Omega)^{-1} + \mu_0(\Omega)^{-2}}\,\|\nabla^2 h\|_{2,\Omega}.$$

Thus, (3.14) follows by observing that $\rho_e + \rho_i \ge \rho_e \ge (|\Omega|/|B|)^{1/N}$ and that the square root can be bounded by $1 + \mu_0(\Omega)^{-1}$.

(ii) Let $N \ge 4$. By the same choice of v and x_0 as in (i), we obtain that

$$\|h - h(z)\|_{2,\Omega} \le \mu_0(\Omega)^{-1}\|\nabla^2 h\|_{2,\Omega}.$$

The conclusion then follows from Lemma 3.3. □

Remark 3.5. We recall that if Ω has the strong local Lipschitz property (for the definition see [Ad, Section 4.5]), the immersion constant c depends only on N and the two Lipschitz parameters of the definition (see [Ad, Chapter 5]). In our case Ω is of class $C^{2,\alpha}$, hence obviously it has the strong local Lipschitz property and the two Lipschitz parameters can be easily estimated in terms of $\min\{r_i, r_e\}$.

If we use (3.3) instead of (3.4), we obtain a similar result, but we must suppose that Ω contains its center of mass.

Theorem 3.6. *Let z be the center of mass of Ω and suppose that $z \in \Omega$. Set $h = q - u$, where q is given by* (1.5)*, and set the constant a in q such that*

$$\int_\Omega [h(x) - h(z)]dx = 0.$$

If $N = 2$ or 3, then (3.14) *holds with*

$$C = k_N c \frac{d_\Omega^\gamma}{|\Omega|^{\frac{1}{N}}} \frac{1 + \overline{\mu}(\Omega)}{\overline{\mu}(\Omega)},$$

where γ and c are the constants introduced in Theorem 3.4.

If $N \geq 4$, *then* (3.15) *holds if* (3.16) *is in force, where*

$$C = k_N \frac{M^{\frac{N}{N+2}}}{\overline{\mu}(\Omega)^{\frac{2}{N+2}} |\Omega|^{\frac{1}{N}}} \quad \text{and} \quad \varepsilon = \alpha_N M \overline{\mu}(\Omega) r_i^{\frac{N+2}{2}}.$$

Proof. The proof is similar to that of Theorem 3.4. Since z is the center of mass of Ω, we have that

$$\int_\Omega \nabla h(x) dx = \int_\Omega [x - z - \nabla u(x)] dx = \int_\Omega x dx - |\Omega| z - \int_\Gamma u(x) \nu(x) dS_x = 0.$$

We can thus apply (3.3) to the first derivatives of h and obtain that

$$\|\nabla h\|_{2,\Omega} \leq \overline{\mu}(\Omega)^{-1/2} \|\nabla^2 h\|_{2,\Omega}.$$

Since we chose the constant a in q such that

$$\int_\Omega [h(x) - h(z)] dx = 0,$$

we can apply (3.3) again to obtain

$$\|h - h(z)\|_{2,\Omega} \leq \overline{\mu}(\Omega)^{-1/2} \|\nabla h\|_{2,\Omega}.$$

Thus, we can write, as in Theorem 3.4, that

$$\|h - h(z)\|_{W^{2,2}(\Omega)} \leq \sqrt{1 + \overline{\mu}(\Omega)^{-1} + \overline{\mu}(\Omega)^{-2}} \|\nabla^2 h\|_{2,\Omega},$$

and

$$\|h - h(z)\|_{2,\Omega} \leq \overline{\mu}(\Omega)^{-1} \|\nabla^2 h\|_{2,\Omega}.$$

The rest of the proof runs similarly to that of Theorem 3.4. □

If Ω is convex, the presence of the spectral quantity $\mu_0(\Omega)$ in (3.15) and (3.16) can be removed and replaced by a purely geometric quantity. This can be done by modifying Lemma 3.3. In this case, we know that the solution of (1.1) has only one minimum point (see [Ko], for instance) and this can be joined to any boundary point by a segment.

Lemma 3.7. *Let* $\Omega \subset \mathbb{R}^N$ *be a convex domain. Let* z *be the minimum point of the solution* u *of* (1.1). *Set* $h = q - u$, *where* q *is given by* (1.5).

Then (3.15) *holds, if* (3.16) *is in force, with*

$$C = k_N \frac{d_\Omega^{\frac{4}{N+2}} M^{\frac{N}{N+2}}}{|\Omega|^{\frac{1}{N}}} \quad \text{and} \quad \varepsilon = \alpha_N \frac{M r_i^{\frac{N+2}{2}}}{d_\Omega^2}.$$

Proof. We begin by proceeding as in the proof of Lemma 3.3. We let x_i and x_e be points in Γ that minimize (resp., maximize) q on Γ and, for

$$0 < \sigma < r_i,$$

define the two points in $y_i, y_e \in \partial\Omega_\sigma$ by $y_j = x_j - \sigma\nu(x_j)$, $j = i, e$. As already done, we obtain the inequality

$$\frac{1}{2}(\rho_e - \rho_i)(\rho_e + \rho_i - 2\sigma) \le h(y_e) - h(y_i) + 2M\sigma$$

for $0 < \sigma < \min\{\frac{\rho_e+\rho_i}{2}, r_i\}$.

Since Ω_σ is convex, we can join each y_j to z by a segment and, since $\nabla h(z) = 0$, we can write the identity

$$h(y_j) - h(z) = \int_0^1 (1-t)\frac{d^2 h}{dt^2}(z + t(y_j - z))dt.$$

Thus,

$$|h(y_j) - h(z)| \le |y_j - z|^2 |\nabla^2 h(z_j)| \le \rho_j^2 |\nabla^2 h(z_j)|,$$

where z_j is some point in Ω_σ.

Then, we apply the mean value property to $|\nabla^2 h|$ (in fact this is subharmonic) in the ball $B_\sigma(z_j)$ and obtain as before that

$$|h(y_j) - h(z)| \le \frac{\rho_j^2}{\sqrt{|B|}\sigma^{N/2}} \|\nabla^2 h\|_{2,\Omega}.$$

Therefore, we find the inequality

$$\frac{1}{2}(\rho_e - \rho_i)(\rho_e + \rho_i - 2\sigma) \le \frac{\rho_i^2 + \rho_e^2}{\sqrt{|B|}\sigma^{N/2}} \|\nabla^2 h\|_{2,\Omega} + 2M\sigma$$

for $0 < \sigma < \min\{\frac{\rho_e+\rho_i}{2}, r_i\}$.

Observing that $\rho_i^2 + \rho_e^2 \le \frac{5}{4} d_\Omega^2$ we get that

$$\rho_e - \rho_i \le \frac{1}{\sigma_0}\Big[\frac{\frac{5}{4} d_\Omega^2}{\sqrt{|B|}\sigma^{N/2}} \|\nabla^2 h\|_{2,\Omega} + 2M\sigma\Big]$$

and we finally conclude as in the proof of Lemma 3.3. □

Remark 3.8. (i) It is clear that Lemma 3.7 still holds in a domain for which we can claim that y_e can be joined to z by a segment. We stress the fact that, instead, the point y_i can always be joined to z by a segment.

(ii) As observed in Remark 3.2, the value $\overline{\mu}(\Omega)$ can be bounded below by the second Neumann eigenvalue $\mu_2(\Omega)$ that, in turn, can be estimated by geometrical parameters or isoperimetric constants. In the case that Ω is convex, estimates involving the diameter d_Ω alone can be found in [PW] and [ENT]. In the case of a general Lipschitz bounded domain, a lower bound for $\mu_2(\Omega)$ involving the best isoperimetric constant relative to Ω can be found in [BCT].

(iii) A lower bound for $\mu_0(\Omega)$ can be obtained as follows. For any $B_r(x_0) \subset \Omega$, by the mean value property we have that

$$\mu_0(\Omega) = \inf\left\{ \int_\Omega |\nabla v|^2 dx : \int_\Omega v^2 dx = 1,\ \Delta v = 0 \text{ in } \Omega,\ \int_{B_r(x_0)} v dx = 0 \right\}$$

and clearly,

$$\mu_0(\Omega) \geq \inf\left\{ \int_\Omega |\nabla v|^2 dx : v \in W^{1,2}(\Omega),\ \int_\Omega v^2 dx = 1,\ \int_{B_r(x_0)} v dx = 0 \right\}.$$

Here, the right-hand side is the reciprocal of the optimal constant in the following Poincaré-type inequality considered in [Me, Theorem 1]

$$\int_\Omega v^2 dx \leq C \int_\Omega |\nabla v|^2 dx,$$

which holds with

$$C = (1 + r^{-N/2} \sqrt{|\Omega|/|B|})^2 (1 + \mu_2(\Omega)^{-1}) - 1,$$

for every $v \in W^{1,2}(\Omega)$ that has null mean value on $B_r(x_0)$ (C has been computed by using [Me, Theorem 1] and [AMR, Theorem 3.3 and Example 3.5]). It is clear that $\mu_0(\Omega) \geq 1/C$.

We conclude this section by presenting a simple method to estimate the number M in a quite general domain. The following lemma results from a simple inspection and by the uniqueness for the Dirichlet problem.

Lemma 3.9 (Torsional creep in an annulus)**.** *Let $A = A_{r,R} \subset \mathbb{R}^N$ be the annulus centered at the origin and radii $0 < r < R$, and set $\kappa = r/R$.*

Then, the solution w of the Dirichlet problem

$$\Delta w = N \text{ in } A, \quad w = 0 \text{ on } \partial A,$$

is defined for $r \leq |x| \leq R$ by

$$w(x) = \begin{cases} \frac{1}{2}|x|^2 + \frac{R^2}{2}(1-\kappa^2)\frac{\log(|x|/r)}{\log \kappa} - \frac{r^2}{2} & \text{for } N = 2, \\ \frac{1}{2}|x|^2 + \frac{1}{2}\frac{R^2}{1-\kappa^{N-2}}\{(1-\kappa^2)(|x|/r)^{2-N} + \kappa^N - 1\} & \text{for } N \geq 3. \end{cases}$$

Theorem 3.10 (A bound for the gradient on Γ)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain that satisfies the uniform interior and exterior conditions with radii r_i and r_e and let $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of* (1.1) *in Ω.*

Then, we have that

$$r_i \le |\nabla u| \le c_N \frac{d_\Omega(d_\Omega + r_e)}{r_e} \quad \text{on } \Gamma, \tag{3.17}$$

where d_Ω is the diameter of Ω and $c_N = 3/2$ for $N = 2$ and $c_N = N/2$ for $N \ge 3$.

Proof. We first prove the first inequality in (3.17). Fix any $p \in \Gamma$. Let $B = B_{r_i}$ be the interior ball touching Γ at p and place the origin of cartesian axes at the center of B.

If w is the solution of (1.1) in B, that is $w(x) = (|x|^2 - r_i^2)/2$, by comparison we have that $w \ge u$ on $\overline{\Omega}$ and hence, since $u(p) = w(p) = 0$, we obtain

$$u_\nu(p) \ge w_\nu(p) = r_i.$$

To prove the second inequality, we place the origin of axes at the center of the exterior ball $B = B_{r_e}$ touching Γ at p. Denote by A the smallest annulus containing Ω, concentric with B and having ∂B as internal boundary, and let R be the radius of its external boundary.

If w is the solution of (1.1) in A, by comparison we have that $w \le u$ on $\overline{\Omega}$. Moreover, since $u(p) = w(p) = 0$, we have that

$$u_\nu(p) \le w_\nu(p).$$

By Lemma 3.9 we then compute that

$$w_\nu(p) = \frac{R(R - r_e)}{r_e} f(\kappa)$$

where, for $0 < \kappa < 1$,

$$f(\kappa) = \begin{cases} \frac{2\kappa^2 \log(1/\kappa) + \kappa^2 - 1}{2(1-\kappa)\log(1/\kappa)} & \text{for } N = 2, \\ \frac{2\kappa^N - N\kappa^2 + N - 2}{2(1-\kappa)(1-\kappa^{N-2})} & \text{for } N \ge 3. \end{cases} \tag{3.18}$$

Notice that f is bounded since it can be extended to a continuous function on $[0, 1]$. Tedious calculations yield that

$$\sup_{0<\kappa<1} f(\kappa) = \begin{cases} \frac{3}{2} & \text{for } N = 2, \\ \frac{N}{2} & \text{for } N \ge 3. \end{cases}$$

Finally, observe that $R \le d_\Omega + r_e$. □

Remark 3.11. To the best of our knowledge, inequality (3.17) is not present in the literature for general smooth domains and is not sharp. Other estimates are given in [PP] for planar strictly convex domains (but the same argument can be generalized to general dimension for strictly mean convex domains) and in [CM] for strictly mean convex domains in general dimension. In particular, in [CM, Lemma 2.2] the authors prove that there exists a universal constant c_0 such that

$$|\nabla u| \le c_0 |\Omega|^{1/N} \quad \text{in } \overline{\Omega}. \tag{3.19}$$

Since the focus of this paper is not on the sharpness of constants, we chose to present the elementary proof of Theorem 3.10.

4 Stability for the Soap Bubble Theorem and some overdetermined problems

In this section, we collect our results on the stability of the spherical configuration by putting together the identities derived in Section 2 and the estimates obtained in Section 3.

It is clear that we may replace $\|H_0 - H\|_{1,\Gamma}$ by the weaker deviation

$$\int_\Gamma (H_0 - H)^+ dS_x$$

in all the relevant formulas in the sequel.

We begin with our main result.

Theorem 4.1 (General stability for the Soap Bubble Theorem)**.** *Let Γ be the connected boundary of class $C^{2,\alpha}$, $0 < \alpha < 1$, of a bounded domain $\Omega \subset \mathbb{R}^N$, $N \ge 2$. Denote by H its mean curvature function and let H_0 be the constant defined in* (2.5)*.*

There is a point $z \in \Omega$ such that

(i) *if $N = 2$ or $N = 3$, there exists a positive constant C such that*

$$\rho_e - \rho_i \le C \|H_0 - H\|_{1,\Gamma}^{1/2}; \tag{4.1}$$

(ii) *if $N \ge 4$, there exist two positive constants C and ε such that*

$$\rho_e - \rho_i \le C \|H_0 - H\|_{1,\Gamma}^{1/(N+2)} \quad \text{if } \|H_0 - H\|_{1,\Gamma} < \varepsilon. \tag{4.2}$$

The constants C and ε depend on the dimension N, the geometrical quantities $|\Omega|$, d_Ω, r_e, r_i, the spectral parameter $\mu_0(\Omega)$ defined in (3.8)*, and in the case (i) also on the immersion constant c introduced in Theorem 3.4. Their explicit expressions are given in* (4.4) *and* (4.5)*.*

Proof. Let u be the solution of (1.1) and let $z \in \Omega$ be any local minimum point of u in Ω. Set $h = q - u$, where q is given by (1.5). From (2.6) and (3.2), we infer that

$$\|\nabla^2 h\|_{2,\Omega} \le M\sqrt{N-1}\,\|H - H_0\|_{1,\Gamma}^{1/2}. \tag{4.3}$$

If $N = 2$ or $N = 3$, by (3.14) and (3.17) we obtain (4.1) at once with

$$C = k_N c \frac{d_\Omega^\gamma}{|\Omega|^{\frac{1}{N}}} \frac{1 + \mu_0(\Omega)}{\mu_0(\Omega)} \frac{d_\Omega(d_\Omega + r_e)}{r_e}. \tag{4.4}$$

When $N \ge 4$, if (3.16) holds, then (3.15) informs us that

$$\begin{aligned} \rho_e - \rho_i &\le \frac{a_N M^{N/(N+2)}}{\mu_0(\Omega)^{2/(N+2)}|\Omega|^{1/N}} \|\nabla^2 h\|_{2,\Omega}^{2/(N+2)} \\ &\le \frac{a_N (N-1)^{1/(N+2)}}{\mu_0(\Omega)^{2/(N+2)}|\Omega|^{1/N}} M \|H - H_0\|_{1,\Gamma}^{1/(N+2)}, \end{aligned}$$

where a_N is the constant defined in (3.11). Thus, there are constants k_N and α_N such that (4.2) holds with

$$C = k_N \frac{d_\Omega(d_\Omega + r_e)}{\mu_0(\Omega)^{\frac{2}{N+2}} |\Omega|^{\frac{1}{N}} r_e} \quad \text{and} \quad \varepsilon = \alpha_N \mu_0(\Omega)^2 r_i^{N+2}, \tag{4.5}$$

by (3.17). □

Remark 4.2. (i) The distance of a minimum point of u from Γ may be estimated from below, in terms of geometrical and spectral parameters, by following the arguments contained in [BMS].

(ii) Another version of Theorem 4.1 can be stated if we assume that Ω contains its center of mass. The proof runs similarly. In fact, it suffices to use Theorem 3.6 instead of Theorem 3.4. In this way, we simply obtain the constants given in (4.4) and (4.5), with $\mu_0(\Omega)$ replaced by $\overline{\mu}(\Omega)$. Remark 3.2 then informs us that such constants are slightly better.

(iii) In (4.2), the assumption that $\|H_0 - H\|_{1,\Gamma} < \varepsilon$ may leave the impression that (ii) of Theorem 4.1 is not a global stability result. However, if $\|H_0 - H\|_{1,\Gamma} \ge \varepsilon$, it is a trivial matter to obtain an upper bound for $\rho_e - \rho_i$ in terms of $\|H_0 - H\|_{1,\Gamma}$.

Since the estimate in Theorem 4.1 does not depend on the particular minimum point chosen, as a corollary, we obtain a result of closeness to a union of balls.

Corollary 4.3 (Closeness to an aggregate of balls)**.** *Let Γ, H, and H_0 be as in Theorem 4.1.*

Then, there exist points $z_1, \dots, z_n$ in Ω, $n \ge 1$, and corresponding numbers

$$\rho_i^j = \min_{x\in\Gamma} |x - z_j| \quad and \quad \rho_e^j = \min_{x\in\Gamma} |x - z_j|, \quad j = 1, \dots, n, \tag{4.6}$$

such that

$$\bigcup_{j=1}^{n} B_{\rho_i^j}(z_j) \subset \Omega \subset \bigcap_{j=1}^{n} B_{\rho_e^j}(z_j) \tag{4.7}$$

and

$$\max_{1\le j\le n} (\rho_e^j - \rho_i^j) \le C\|H_0 - H\|_{1,\Gamma}^{1/2},$$

if $N = 2$ *or* $N = 3$, *and*

$$\max_{1\le j\le n} (\rho_e^j - \rho_i^j) \le C\|H_0 - H\|_{1,\Gamma}^{1/(N+2)} \quad if\ \|H_0 - H\|_{1,\Gamma} < \varepsilon,$$

if $N \ge 4$. *Here, the relevant constants are those in* (4.4) *and* (4.5).

The number n can be chosen as the number of connected components of the set $\mathcal{M}$ *of all the local minimum points of the solution u of* (1.1).

Proof. Let $\mathcal{M}_j$, $j = 1, \dots, n$, be the connected components of $\mathcal{M}$ and pick one point z_j from each $\mathcal{M}_j$. By applying Theorem 4.1 to each z_j, the conclusion is then evident. □

Remark 4.4. The estimates presented in Theorem 4.1 and sketched in (ii) of Remark 4.2 may be interpreted as stability estimates, once some a priori information is available: here, we just illustrate the case (ii) of Theorem 4.1. Given four positive constants d, r, V, and μ, let $\mathcal{S} = \mathcal{S}(d, r, V, \mu)$ be the class of connected surfaces $\Gamma \subset \mathbb{R}^N$ of class $C^{2,\alpha}$, where Γ is the boundary of a bounded domain Ω, such that

$$d_\Omega \le d, \quad r_i(\Omega), r_e(\Omega) \ge r, \quad |\Omega| \ge V, \quad \mu_0(\Omega) \ge \mu.$$

Then, for every $\Gamma \in \mathcal{S}$ with $\|H_0 - H\|_{1,\Gamma} < \varepsilon$, we have that

$$\rho_e - \rho_i \le C\|H_0 - H\|_{1,\Gamma}^{1/(N+2)},$$

where C and ε are the constants in (4.5), with the relevant parameters replaced by the constants d, r, V, μ.

If we relax the a priori assumption that $\Gamma \in \mathcal{S}$, it may happen that, as the deviation $\|H_0 - H\|_{1,\Gamma}$ tends to 0, Ω tends to the ideal configuration of two or more mutually tangent balls, while ε tends to 0 and C diverges since r tends to 0. This behavior can be avoided by considering strictly mean convex surfaces, as done in [CM] by using the uniform deviation $\|H_0 - H\|_{\infty,\Gamma}$.

If we suppose that Γ is strictly mean convex, then we can use Theorem 2.6 to to obtain a stability result for the Heintze–Karcher inequality and we can improve the constants C and ε in (4.2).

Theorem 4.5 (Stability for the Heintze–Karcher inequality)**.** *Let Γ be the connected boundary of class $C^{2,\alpha}$, $0 < \alpha < 1$, of a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. Denote by H its mean curvature function and suppose that $H > 0$ on Γ.*

There is a point $z \in \Omega$ such that

(i) *if $N = 2$ or $N = 3$, there exists a positive constant C such that*

$$\rho_e - \rho_i \leq C\left(\int_\Gamma \frac{dS_x}{H} - N|\Omega| \right)^{1/2}; \tag{4.8}$$

(ii) *if $N \geq 4$, there exist two positive constants C and ε such that*

$$\rho_e - \rho_i \leq C\left(\int_\Gamma \frac{dS_x}{H} - N|\Omega| \right)^{1/(N+2)} \tag{4.9}$$

if

$$\left(\int_\Gamma \frac{dS_x}{H} - N|\Omega| \right) < \varepsilon.$$

The relevant constants will be given in (4.10) *and* (4.11).

Proof. We chose the point z in Ω as in the proof of Theorem 4.1. Moreover, by (2.9) and (2.10), we have that

$$\frac{1}{N-1} \int_\Omega |\nabla^2 h|^2 dx \leq \int_\Gamma \frac{dS_x}{H} - N|\Omega|.$$

We then proceed as in the proof of Theorem 4.1 and obtain (4.8) with

$$C = k_N c \frac{d_\Omega^\gamma}{|\Omega|^{\frac{1}{N}}} \frac{1 + \mu_0(\Omega)}{\mu_0(\Omega)}, \tag{4.10}$$

if $N = 2$ or $N = 3$, with the help of (3.14), and (4.9) with

$$C = k_N \frac{M^{\frac{N}{N+2}}}{\mu_0(\Omega)^{\frac{2}{N+2}} |\Omega|^{\frac{1}{N}}} \quad \text{and} \quad \varepsilon = \alpha_N \mu_0(\Omega)^2 M^2 r_i^{N+2}, \tag{4.11}$$

with the help of (3.15), (3.16).

To avoid the presence of M in the constants C and ε, we can use respectively (3.19) (obviously we could also use again the second inequality in (3.17) as before) and the first inequality in (3.17) and choose

$$C = k_N \mu_0(\Omega)^{-\frac{2}{N+2}} |\Omega|^{-\frac{2}{N(N+2)}} \quad \text{and} \quad \varepsilon = \alpha_N \mu_0(\Omega)^2 r_i^{N+4}.$$

□

The following theorem is in the spirit of the main result contained in [CV] (see also [CM]).

Theorem 4.6 (Stability for strictly mean convex hypersurfaces)**.** *Let* Γ *be the connected boundary of class* $C^{2,\alpha}$, $0<\alpha<1$, *of a bounded domain* $\Omega\subset\mathbb{R}^N$, $N\ge 2$. *Denote by* H *its mean curvature function, suppose that there exists a constant* $\underline{H}>0$ *such that* $H\ge\underline{H}$ *on* Γ, *and let* H_0 *be the constant defined in* (2.5).

There is a point $z\in\Omega$ *such that*

(i) *if* $N=2$ *or* $N=3$, *there exists a positive constant* C *such that*

$$\rho_e-\rho_i\le C\|H_0-H\|_{\infty,\Gamma}^{1/2}; \tag{4.12}$$

(ii) *if* $N\ge 4$, *there exist two positive constants* C *and* ε *such that*

$$\rho_e-\rho_i\le C\|H_0-H\|_{\infty,\Gamma}^{1/(N+2)} \tag{4.13}$$

if

$$\|H_0-H\|_{\infty,\Gamma}<\varepsilon.$$

The relevant constants will be given in (4.14) *and* (4.15).

Proof. We simply observe that

$$\int_\Gamma\frac{dS_x}{H}-N|\Omega|=\int_\Gamma\left[\frac{1}{H}-\frac{1}{H_0}\right]dS_x\le\frac{N|\Omega|}{|\Gamma|}\|H_0-H\|_{\infty,\Gamma}\int_\Gamma\frac{dS_x}{H},$$

and hence from (2.9) and the fact that $H\ge\underline{H}$ on Γ it follows that

$$\frac{1}{N-1}\int_\Omega|\nabla^2h|^2dx\le\frac{N|\Omega|}{\underline{H}}\|H_0-H\|_{\infty,\Gamma}.$$

The rest of the proof runs similarly to those of Theorems 4.1 and 4.5.

If $N=2$ or $N=3$ we obtain (4.12) with

$$C=k_Nc\frac{d_\Omega^\gamma|\Omega|^{\frac12-\frac1N}}{\underline{H}^{\frac12}}\frac{1+\mu_0(\Omega)}{\mu_0(\Omega)}. \tag{4.14}$$

If $N\ge 4$ we obtain (4.13) with

$$C=k_N\frac{M^{\frac{N}{N+2}}}{\mu_0(\Omega)^{\frac{2}{N+2}}|\Omega|^{\frac1N-\frac{1}{N+2}}\underline{H}^{\frac{1}{N+2}}}\quad\text{and}\quad\varepsilon=\alpha_N\frac{\underline{H}}{|\Omega|}\mu_0(\Omega)^2M^2r_i^{N+2}. \tag{4.15}$$

As before, the presence of M in C and ε can be avoided by means of (3.19) and the first inequality in (3.17). ☐

Remark 4.7. (i) In Theorem 4.6, if the deviation $\|H_0 - H\|_{\infty,\Gamma}$ is small enough, $\underline{H}$ can be replaced by a fraction of H_0. Also, from the proof of that theorem, it is evident that the norm $\|H_0 - H\|_{\infty,\Gamma}$ can be replaced by the weaker one $\|H_0 - H\|_{1,\Gamma}$.

(ii) When Ω is convex, by using Lemma 3.7 instead of Theorem 3.4, we can avoid the use of the spectral parameter $\mu_0(\Omega)$ in the constants of Theorems 4.1, 4.5, 4.6 and Corollary 4.3.

The inequalities of Section 3 can also be used to obtain stability estimates for one of the two overdetermined boundary value problems mentioned in Section 2.

Theorem 4.8 (Stability for an overdetermined problem)**.** *Let Γ and Ω be as in Theorem 4.1 and suppose that $H > 0$ on Γ.*

There is a point $z \in \Omega$ such that

(i) *if $N = 2$ or $N = 3$, there exists a positive constant C such that*

$$\rho_e - \rho_i \le C \|u_\nu - 1/H\|_{1,\Gamma}^{1/2}; \tag{4.16}$$

(ii) *if $N \ge 4$, there exist two positive constants C and ε such that*

$$\rho_e - \rho_i \le C \|u_\nu - 1/H\|_{1,\Gamma}^{\frac{1}{N+2}} \tag{4.17}$$

if

$$\|u_\nu - 1/H\|_{1,\Gamma} < \varepsilon.$$

The relevant constants will be given in (4.18) *and* (4.19).

Proof. We observe that

$$\int_\Gamma (1 - H u_\nu) u_\nu dS_x \le \int_\Gamma |u_\nu - 1/H| |H u_\nu| dS_x \le \frac{M}{r_i} \|u_\nu - 1/H\|_{1,\Gamma},$$

since $H \le 1/r_i$. Thus, by (2.7) and (3.2) we have that

$$\|\nabla^2 h\|_{2,\Omega}^2 \le (N-1) \frac{M}{r_i} \|u_\nu - 1/H\|_{1,\Gamma}.$$

By proceeding as before, we get (4.16) with

$$C = k_N c \frac{d_\Omega^\gamma}{|\Omega|^{\frac{1}{N}}} \frac{1 + \mu_0(\Omega)}{\mu_0(\Omega)} \sqrt{\frac{M}{r_i}} \tag{4.18}$$

if $N = 2$ or $N = 3$, and (4.17) with

$$C = k_N \frac{M^{\frac{N+1}{N+2}}}{\mu_0(\Omega)^{\frac{2}{N+2}} |\Omega|^{\frac{1}{N}} r_i^{\frac{1}{N+2}}} \quad \text{and} \quad \varepsilon = \alpha_N \mu_0(\Omega)^2 M r_i^{N+3} \tag{4.19}$$

if $N \ge 4$. As before, by (3.19) and the first inequality in (3.17), we can replace M in C and ε in (4.18) and (4.19). □

Remark 4.9. It is clear that estimates in the spirit of Corollary 4.3 can also be given for the situations treated in Theorems 4.5, 4.6 and 4.8.

Acknowledgements. The authors wish to thank Prof. S. Sakaguchi (Tohoku University) for bringing to their attention reference [Re2] and for many fruitful discussions.

Remarks 3.8 (iii) and 4.2 (iii) were suggested by the anonymous referee. The authors warmly thank him/her for the nice improvements to this paper.

The paper was partially supported by a grant iFUND-Azione 2 of the Università di Firenze, under a scientific and cultural agreement with Tohoku University, and by the GNAMPA (first author) and GNSAGA (second author) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[Ad] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[AMR] G. Alessandrini, A. Morassi and E. Rosset, *The linear constraints in Poincaré and Korn type inequalities*, Forum Math. **20** (2008), 557–569.

[Al1] A. D. Aleksandrov, *Uniqueness theorems for surfaces in the large. V*, Vestnik Leningrad Univ. **13** (1958), no. 19, 5–8. English transl.: Amer. Math. Soc. Transl. (Ser. 2) **21** (1962), 412–416.

[Al2] A. D. Alexandrov, *A characteristic property of spheres*, Ann. Math. Pura Appl. (4) **58** (1962), 303–315.

[BCT] B. Brandolini, F. Chiacchio and C. Trombetti, *Optimal lower bounds for eigenvalues of linear and nonlinear Neumann problems*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), 31–45.

[BMS] L. Brasco, R. Magnanini and P. Salani, *The location of the hot spot in a grounded convex conductor*, Indiana Univ. Math. J. **60** (2011), 633–659.

[CM] G. Ciraolo and F. Maggi, *On the shape of compact hypersurfaces with almost constant mean curvature*, Comm. Pure Appl. Math. **70** (2017), 665–716.

[CMV] G. Ciraolo, R. Magnanini and V. Vespri, *Hölder stability for Serrin's overdetermined problem*, Ann. Mat. Pura Appl. (4) **195** (2016), 1333–1345.

[CV] G. Ciraolo and L. Vezzoni, *A sharp quantitative version of Alexandrov's theorem via the method of moving planes*, J. Eur. Math. Soc. (JEMS) **20** (2018), 261–299.

[ENT] L. Esposito, C. Nitsch and C. Trombetti, *Best constants in Poincaré inequalities for convex domains*, J. Convex Anal. **20** (2013), 253–264.

[Gi] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific, River Edge, NJ, 2003.

[HK] E. Heintze and H. Karcher, *A general comparison theorem with applications to volume estimates for submanifolds*, Ann. Sci. École Norm. Sup. (4) **11** (1978), 451–470.

[Ko] N. Korevaar, *Convex solutions to nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J. **32** (1983), 603–614.

[Me] N. G. Meyers, *Integral inequalities of Poincaré and Wirtinger type*, Arch. Ration. Mech. Anal. **68** (1978), 113–120.

[PS] L. Payne and P. W. Schaefer, *Duality theorems in some overdetermined boundary value problems*, Math. Methods Appl. Sci. **11** (1989), 805–819.

[PP] L. E. Payne and G. A. Philippin, *Some applications of the maximum principle in the problem of torsional creep*, SIAM J. Appl. Math. **33** (1977), 446–455.

[PW] L. E. Payne and H. F. Weinberger, *An optimal Poincaré inequality for convex domains*, Arch. Ration. Mech. Anal. **5** (1960), 286–292.

[Re1] R. C. Reilly, *Applications of the Hessian operator in a Riemannian manifold*, Indiana Univ. Math. J. **26** (1977), 459–472.

[Re2] R. C. Reilly, *Mean curvature, the Laplacian, and soap bubbles*, Amer. Math. Monthly **89** (1982), 180–188, 197–198.

[Ro] A. Ros, *Compact hypersurfaces with constant higher order mean curvatures*, Rev. Mat. Iberoamericana **3** (1987), 447–453.

[Se] J. Serrin, *A symmetry problem in potential theory*, Arch. Ration. Mech. Anal. **43** (1971), 304–318.

[We] H. F. Weinberger, *Remark on the preceding paper of Serrin*, Arch. Ration. Mech. Anal. **43** (1971), 319–320.

Rolando Magnanini
Dipartimento di Matematica ed Informatica "U. Dini"
Università di Firenze
viale Morgagni 67/A
50134 Firenze, Italy
email: magnanini@unifi.it
http://web.math.unifi.it/users/magnanin

Giorgio Poggesi
Department of Mathematics and Statistics
The University of Western Australia
35 Stirling Highway
Crawley, Perth
WA 6009, Australia
email: giorgio.poggesi@uwa.edu.au

and
Dipartimento di Matematica ed Informatica "U. Dini"
Università di Firenze
viale Morgagni 67/A
50134 Firenze, Italy
email: giorgio.poggesi@unifi.it

(Received December 16, 2016)