BOUNDARY GAUSS–LUCAS TYPE THEOREMS ON THE DISK

By

KONSTANTIN M. DYAKONOV*

Abstract. The classical Gauss–Lucas theorem describes the location of the critical points of a polynomial. There is also a hyperbolic version, due to Walsh, in which the role of polynomials is played by finite Blaschke products on the unit disk. We consider similar phenomena for generic inner functions, as well as for certain "locally inner" self-maps of the disk. More precisely, we look at a unit-norm function $f \in H^\infty$ that has an angular derivative on a set of positive measure (on the boundary) and we assume that its inner factor, I, is nontrivial. Under certain conditions to be discussed, it follows that f' must also have a nontrivial inner factor, say J, and we study the relationship between the boundary singularities of I and J. Examples are furnished to show that our sufficient conditions cannot be substantially relaxed.

1 Introduction and results

The functions considered in this paper are holomorphic self-maps of the unit disk. Our purpose is to find out when the presence of a nontrivial inner factor in the function's canonical factorization (i.e., the property of being non-outer) survives differentiation. This clearly happens if the original function, f, has multiple zeros, since these will also be zeros for f'. Somewhat less obvious is the fact that, under certain natural assumptions, the passage from f to f' preserves singular inner factors. (In a sense, these are responsible for the boundary zeros of infinite multiplicity.) Much subtler is the case of simple zeros inside, and this is what chiefly interests us here. The results that arise can be viewed as descendants of the classical Gauss–Lucas theorem, or rather of its disk version, and we begin by recalling those prototypical results.

If f is a holomorphic—or perhaps meromorphic—function living (at least) on a domain $\Omega \subset \mathbb{C}$, we write $\mathcal{Z}_\Omega(f)$ for its zero set there. Thus,

$$\mathcal{Z}_\Omega(f) := \{z \in \Omega : f(z) = 0\}.$$

*Supported in part by grant MTM2017-83499-P from El Ministerio de Economía y Competitividad (Spain) and grant 2017-SGR-358 from AGAUR (Generalitat de Catalunya).

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 138 (2019)
DOI 10.1007/s11854-019-0042-6

The Gauss–Lucas theorem tells us that, given a nonconstant polynomial P, the set of its critical points, $\mathcal{Z}_{\mathbb{C}}(P')$, is contained in the convex hull of $\mathcal{Z}_{\mathbb{C}}(P)$; see, e.g., [16, Chapter 2].

This fact admits a certain "hyperbolic" analogue, which was found by Walsh [19]. The plane $\mathbb{C}$ is now replaced by the disk

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\},$$

while the role of polynomials is played by finite Blaschke products. A (**finite**) **Blaschke product** B of degree n is, by definition, given by

$$B(z) = c \prod_{j=1}^{n} \frac{z - a_j}{1 - \bar{a}_j z}$$

(with some $a_1, \dots, a_n \in \mathbb{D}$ and a unimodular constant c), a formula known to provide the general form of an n-to-1 mapping from $\mathbb{D}$ onto itself. Now, Walsh's theorem says that, for such a B, the set $\mathcal{Z}_{\mathbb{D}}(B')$ is contained in the hyperbolic convex hull of $\mathcal{Z}_{\mathbb{D}}(B)$, defined appropriately; see [19] for a precise statement.

It should be mentioned that the set $\mathcal{Z}_{\mathbb{C}}(P')$ in the Gauss–Lucas theorem is automatically—and trivially—nonempty, provided that $\deg P \ge 2$. Similarly, in Walsh's theorem, we have $\mathcal{Z}_{\mathbb{D}}(B') \ne \emptyset$ whenever n, the degree of B, satisfies $2 \le n < \infty$. To see why, assume that B' does not vanish on the set $\{0, a_1, \dots, a_n\}$ (otherwise the statement is trivial) and note that $\mathcal{Z}_{\mathbb{C}}(B') = \mathcal{Z}_{\mathbb{C}}(B'/B)$. The formula

$$\frac{B'(z)}{B(z)} = \sum_{j=1}^{n} \frac{1 - |a_j|^2}{(z - a_j)(1 - \bar{a}_j z)}$$

shows then that the (nonempty) set $\mathcal{Z}_{\mathbb{C}}(B')$ is symmetric with respect to the circle $\mathbb{T} := \partial\mathbb{D}$, so precisely one half of its points must be in $\mathbb{D}$.

Thus, the Gauss–Lucas and Walsh theorems actually assert the existence of critical points in the appropriate region and also describe their location; this last part roughly amounts to saying that the zeros of P' or B' are to be found not too far from those of P or B, respectively.

Our purpose is to elaborate on Walsh's theorem by moving from finite Blaschke products to infinite ones, as well as to generic inner functions, and still further—namely, to fairly general analytic self-maps of the disk—and to study similar (Gauss–Lucas type) phenomena in these cases.

At this point, we pause to recall some basic terminology and notation. A function θ in H^∞ (i.e., a bounded holomorphic function on $\mathbb{D}$) is said to be **inner** if $\lim_{r\to 1^-} |\theta(r\zeta)| = 1$ for m-almost all $\zeta \in \mathbb{T}$. Here and throughout, m is the normalized Lebesgue measure on the unit circle $\mathbb{T}$, so that $dm(\zeta) = (2\pi)^{-1}|d\zeta|$. It is well known that every inner function θ can be factored canonically as $\theta = \lambda BS$, where $\lambda \in \mathbb{T}$ is a constant, B is a Blaschke product and S is a **singular** inner function; see [9, Chapter II]. More explicitly, the factors involved are of the form

$$B(z) = B_{\{a_j\}}(z) := \prod_j \frac{|a_j|}{a_j} \frac{a_j - z}{1 - \bar{a}_j z}, \quad z \in \mathbb{D}, \tag{1.1}$$

where $\{a_j\} \subset \mathbb{D}$ is a sequence, possibly finite or empty, with $\sum_j (1 - |a_j|) < \infty$ (if $a_j = 0$, one puts $|a_j|/a_j = -1$), and

$$S(z) = S_\mu(z) := \exp\left\{ -\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right\}, \quad z \in \mathbb{D}, \tag{1.2}$$

with μ a (nonnegative) singular measure on $\mathbb{T}$. The set $\mathbb{T} \cap \operatorname{clos}(\{a_j\} \cup \operatorname{supp} \mu)$ coincides with the **boundary spectrum** $\sigma(\theta)$ of θ, defined as the set of its boundary singularities (i.e., the smallest closed set $E \subset \mathbb{T}$ such that θ is analytic across $\mathbb{T} \setminus E$).

Further, a zero-free holomorphic function F on $\mathbb{D}$ is said to be **outer** if $\log |F|$ coincides with the harmonic extension (Poisson integral) of an integrable function on $\mathbb{T}$. When normalized by the condition $F(0) > 0$, an outer function F takes the form

$$F(z) = \mathcal{O}_h(z) := \exp\left\{ \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log h(\zeta)\, dm(\zeta) \right\}, \quad z \in \mathbb{D}, \tag{1.3}$$

where h is a nonnegative function on $\mathbb{T}$ with $\log h \in L^1(\mathbb{T}, m)$. This h actually agrees with the nontangential boundary values of $|F|$ almost everywhere on $\mathbb{T}$.

The functions f that admit a factorization of the form $f = \theta F$, with θ inner and F outer, are precisely those lying in the **Smirnov class** $\mathcal{N}^+$; see [9, Chapter II]. Alternatively, we can define (or characterize) $\mathcal{N}^+$ as the set of ratios u/v, where $u, v \in H^\infty$ and v is outer. When v is merely assumed to be zero-free on $\mathbb{D}$, such ratios range over the **Nevanlinna class** $\mathcal{N}$.

We write the canonical factorization of a function $f \in \mathcal{N}^+$, $f \not\equiv 0$, in the form

$$f = BSF, \tag{1.4}$$

the three factors on the right being (1.1), (1.2) and (1.3), respectively (and we take the liberty to ignore the unimodular constant factor involved). In particular, this canonical representation applies whenever f is in the Hardy space H^p with some $p \in (0, \infty]$ (see [9, Chapter II]); in fact, we have $H^p = \mathcal{N}^+ \cap L^p(\mathbb{T}, m)$.

Now, going back to Walsh's theorem and trying to adapt it to an infinite Blaschke product B (to begin with), we already have to face the new phenomenon that the set $\mathcal{Z}_{\mathbb{D}}(B')$ may be empty. Moreover, this may well happen for a Blaschke product B with $B' \in \mathcal{N}^+$. An example can be furnished as follows: fix a number $\alpha \in \mathbb{D} \setminus \{0\}$ and put

$$B_\alpha(z) := \frac{S(z) - \alpha}{1 - \overline{\alpha} S(z)},$$

where S is the "atomic" singular inner function given by

$$S(z) := \exp\left(\frac{z+1}{z-1}\right).$$

It is well known (and easy to verify) that B_α is a Blaschke product. At the same time, differentiation yields

$$B'_\alpha(z) = \frac{1 - |\alpha|^2}{(1 - \overline{\alpha} S(z))^2} \cdot \frac{-2S(z)}{(z-1)^2},$$

and it is clear that the right-hand side is zero-free in $\mathbb{D}$. Furthermore, $B'_\alpha \in H^p$ for every $p \in (0, \frac{1}{2})$, and the inner factor of B'_α is S.

On the other hand, given a nonconstant inner function θ with $\theta' \in \mathcal{N}^+$, it turns out that θ' must have a nontrivial inner factor, unless θ is a Möbius transformation (see [6, Corollary 2.2] or [7]). These observations seem to suggest that we modify our viewpoint appropriately. Namely, as long as our variations on Walsh's theme involving an inner function θ are supposed to deal with something a priori existent, rather than pertain to the "theory of the empty set," we feel that we should look at $\operatorname{inn}(\theta')$, the inner factor of θ', rather than at the zero set $\mathcal{Z}_{\mathbb{D}}(\theta')$. (Here and below, we use the notation $\operatorname{inn}(f)$ for the inner factor of a function $f \in \mathcal{N}^+$.) We should then try to understand the relationship between the (suitably defined) **smallness set** of $\operatorname{inn}(\theta')$ and that of θ. More precisely, we shall be actually concerned with the boundary spectra $\sigma(\operatorname{inn}(\theta'))$ and $\sigma(\theta)$, i.e., with those parts of the unit circle $\mathbb{T}$ where the two smallness sets hit it. One consequence of our results is that

$$\sigma(\operatorname{inn}(\theta')) = \sigma(\theta),$$

and we regard this as a boundary version of the Gauss–Lucas–Walsh theorem for inner functions.

In fact, we are not going to restrict ourselves to inner functions, even though moving beyond this class makes things more complicated. This time, turning to a general function $f \in H^\infty$ with $f' \in \mathcal{N}^+$, we can no longer expect that f' will necessarily have an inner factor whenever f does. For instance, suppose that h is a holomorphic function on $\mathbb{D}$, with $\operatorname{Re} h$ bounded above, whose range $h(\mathbb{D})$ contains infinitely many points of the form $c + 2\pi ik$, where $c \in \mathbb{C}$ is fixed and k ranges over (a subset of) $\mathbb{Z}$. The function $f := e^h - e^c (\in H^\infty)$ then vanishes on the set $\bigcup_k h^{-1}(c + 2\pi ik)$; thus, f is divisible by an infinite Blaschke product, while $f' = h'e^h$ may well be outer. More sophisticated examples in this vein are given in Section 4 below.

At the same time, we single out a class of H^∞-functions that does obey the "Gauss–Lucas principle," or perhaps the "Walsh principle," meaning that the property of being non-outer is inherited by f' from f and that the boundary spectra of the two inner factors, $\mathrm{inn}(f')$ and $\mathrm{inn}(f)$, are related appropriately. The class in question appears to be (almost) optimal.

Before stating the results, let us recall that a unit-norm function $f \in H^\infty$ is said to possess an **angular derivative** (in the sense of Carathéodory) at a point $\zeta \in \mathbb{T}$ if both f and f' have nontangential limits at ζ and, once we agree to denote the two limits by $f(\zeta)$ and $f'(\zeta)$, the former of these satisfies $|f(\zeta)| = 1$. The classical Julia–Carathéodory theorem (see [2, Chapter VI], [3, Chapter I] or [15, Chapter VI]) asserts that this happens if and only if

$$\liminf_{z \to \zeta} \frac{1 - |f(z)|}{1 - |z|} < \infty.$$

Further, given a point $z \in \mathbb{D}$, we shall denote by ω_z the **harmonic measure** associated with it. Thus $d\omega_z = P_z dm$ on $\mathbb{T}$, where P_z stands for the corresponding **Poisson kernel**:

$$P_z(\zeta) := \frac{1 - |z|^2}{|\zeta - z|^2}, \qquad \zeta \in \mathbb{T}.$$

The quantity $\omega_z(E) = \int_E d\omega_z$, where E is a (Lebesgue) measurable subset of $\mathbb{T}$, can be roughly interpreted as the normalized angle at which E is seen from z.

Also, we need to recall that a Blaschke product b with zeros $\{z_j\}$ is said to be **thin** if

$$\lim_{k \to \infty} \prod_{j:\, j \neq k} \left| \frac{z_j - z_k}{1 - \bar{z}_j z_k} \right| = 1,$$

a condition that can be rewritten in the form

$$\lim_{k \to \infty} |b'(z_k)|(1 - |z_k|^2) = 1.$$

The sequence $\{z_j\}$ itself is then also called thin, whereas non-thin sequences (and the corresponding Blaschke products) will be termed **thick**. In the literature, one encounters thin (or thick) sequences in many places. In particular, they turn up in connection with maximal ideals in uniform algebras and with various interpolation problems. One of the first occurrences can be found in [20]; see also [8, 10, 18].

Now suppose that $\mathcal{E}$ is a (Lebesgue) measurable subset of $\mathbb{T}$, and

$$\widetilde{\mathcal{E}} := \mathbb{T} \setminus \mathcal{E}$$

(this notation will be used throughout), while f is an H^∞-function with $f' \in \mathcal{N}^+$. Further, let $\operatorname{inn}(f) = BS$, where B is a Blaschke product and S a singular inner function. We then write

$$\sigma^{\mathrm{i}}_{\mathcal{E}}(B) := \sigma(B) \cap \operatorname{ess\,int} \mathcal{E},$$

where $\operatorname{ess\,int} \mathcal{E}$ is the **essential interior** of $\mathcal{E}$. (By definition, a point $\zeta \in \mathbb{T}$ is in $\operatorname{ess\,int} \mathcal{E}$ if there exists a set $\Lambda \subset \mathbb{T}$ with $m(\Lambda) = 0$ such that ζ is an interior point of $\mathcal{E} \cup \Lambda$ with respect to $\mathbb{T}$.) Also, we shall denote by $\sigma^{\mathrm{b}}_{\mathcal{E},f}(B)$ the set of points $\zeta \in \mathbb{T} \setminus \operatorname{ess\,int} \mathcal{E}$ with the following property: there exists a thick sequence $\{z_n\} \subset \mathcal{Z}_{\mathbb{D}}(B)$ with $z_n \to \zeta$ satisfying

$$\omega_{z_n}(\widetilde{\mathcal{E}}) \log \frac{1}{1 - |z_n|} \to 0 \tag{1.5}$$

and

$$\int_{\widetilde{\mathcal{E}}} \log |f'| \, d\omega_{z_n} \to 0. \tag{1.6}$$

The superscripts "i" and "b" in $\sigma^{\mathrm{i}}_{\mathcal{E}}(B)$ and $\sigma^{\mathrm{b}}_{\mathcal{E},f}(B)$ stand for "interior" and "boundary," respectively. (It should be noted that the latter set is contained in $\operatorname{clos} \mathcal{E}$, so its elements are "essentially boundary" points for $\mathcal{E}$.) Finally, we put

$$\sigma_{\mathcal{E}}(f) := \sigma(S) \cup \sigma^{\mathrm{i}}_{\mathcal{E}}(B) \cup \sigma^{\mathrm{b}}_{\mathcal{E},f}(B).$$

Theorem 1.1. *Let $f \in H^\infty$ be a nonconstant function with $\|f\|_\infty = 1$, and let $\mathcal{E}$ be a measurable subset of $\mathbb{T}$ such that f has an angular derivative almost everywhere on $\mathcal{E}$. Suppose that each of the three factors in the canonical factorization* (1.4) *has its derivative in $\mathcal{N}^+$ (whence also $f' \in \mathcal{N}^+$). Assume, finally, that $\sigma_{\mathcal{E}}(f) \neq \emptyset$. Then f' has a nontrivial inner factor, say J, with $\sigma(J) \supset \sigma_{\mathcal{E}}(f)$.*

The last inclusion should be compared with the fact that $\sigma(J)$ is always contained in $\sigma(f)$, the set of boundary singularities for f. Indeed, if f is analytic in a neighborhood of a point $\zeta_0 \in \mathbb{T}$, then so is f', and hence also its inner factor, J (see [9, Chapter II] in connection with the latter implication).

Also, in the theorem above, we may replace the hypothesis that f has an angular derivative a.e. on $\mathcal{E}$ by the seemingly weaker condition that $|f| = 1$ a.e. on $\mathcal{E}$. (The reason is that the other assumptions imply the existence of nontangential limits for f' a.e. on $\mathbb{T}$.) The functions f that arise can thus be viewed as "locally inner." Because of the role that angular derivatives play in Theorem 1.4 below, we have chosen to state Theorem 1.1 in similar terms; the relation between the two results might in this way become clearer.

We now make a remark concerning the meaning of conditions (1.5) and (1.6) that were used to define the set $\sigma^{\mathrm{b}}_{\mathcal{E},f}(B)$. Given an (essentially) boundary point ζ of $\mathcal{E}$ and a sequence $\{z_n\} \subset \mathcal{Z}_{\mathbb{D}}(B)$ with $z_n \to \zeta$, the two conditions basically mean that the z_n's tend to ζ tangentially enough "on the $\mathcal{E}$ side" (i.e., they lie much closer to $\mathcal{E}$ than to $\widetilde{\mathcal{E}}$). The examples constructed at the end of the paper will show that nontangential convergence would not do, and moreover, that the qualitative tangency conditions (1.5) and (1.6) cannot be substantially relaxed.

One may find it unfortunate that condition (1.6) involves f', instead of being stated in terms of f alone. We note, however, that it only depends on the boundary values of $|f'|$ (or, equivalently, on the outer factor of f'), whereas the conclusion of Theorem 1.1 concerns the inner factor of f'. Besides, under further hypotheses, we shall come up with simpler sufficient conditions replacing (1.5) and (1.6) that will lead to more transparent formulations.

The following corollary deals with the situation where $\sigma(B)$ is contained in $\operatorname{ess\,int}\mathcal{E}$, in which case we have $\sigma^{\mathrm{i}}_{\mathcal{E}}(B) = \sigma(B)$, $\sigma^{\mathrm{b}}_{\mathcal{E},f}(B) = \emptyset$ and

$$\sigma_{\mathcal{E}}(f) := \sigma(B) \cup \sigma(S) = \sigma(BS).$$

Corollary 1.2. *Let $f \in H^\infty$ be a nonconstant function with $\|f\|_\infty = 1$, and let $\mathcal{E}$ be a measurable subset of $\mathbb{T}$ such that f has an angular derivative almost everywhere on $\mathcal{E}$. Suppose that each of the three factors in the canonical factorization* (1.4) *has its derivative in $\mathcal{N}^+$. Assume, finally, that $\sigma(BS) \neq \emptyset$, while $\sigma(B) \subset \operatorname{ess\,int}\mathcal{E}$. Then $J := \operatorname{inn}(f')$ is a nontrivial inner function and $\sigma(BS) \subset \sigma(J)$.*

In the special case where $\mathcal{E} = \mathbb{T}$, this reduces to the following result.

Corollary 1.3. *Let θ be a nonconstant inner function, other than a Möbius transformation, with $\theta' \in \mathcal{N}^+$. Then $\mathcal{J} := \operatorname{inn}(\theta')$ is a nontrivial inner function and $\sigma(\theta) = \sigma(\mathcal{J})$.*

In view of the discussion following Theorem 1.1, we have the (trivial) inclusion $\sigma(\mathcal{J}) \subset \sigma(\theta)$. Now, if $\sigma(\theta) \neq \emptyset$, Corollary 1.3 is a special case of the preceding result (just take $\mathcal{E} = \mathbb{T}$ and $f = \theta = BS$). Otherwise, we are only concerned with the nontriviality of $\operatorname{inn}(\theta')$, and this is guaranteed by the above-mentioned result from [6, 7].

Regarding the hypothesis $\theta' \in \mathcal{N}^+$ (for θ inner), we recall that this is actually equivalent to $\theta' \in \mathcal{N}$. Furthermore, each of these holds if and only if $\log^+|\theta'| \in L^1(\mathbb{T}, m)$, where θ' is understood as the angular derivative. Also, for $\theta = BS$ to satisfy $\theta' \in \mathcal{N}$ (or $\theta' \in \mathcal{N}^+$), it is necessary and sufficient that both B' and S' be in $\mathcal{N}$ (or $\mathcal{N}^+$). These results are due to Ahern and Clark; see [1, Corollary 4].

The next result, also a consequence of Theorem 1.1, contains a simple sufficient condition for $\operatorname{inn}(f')$ to be nontrivial when $\mathcal{E}$ is taken to be an arc. In what follows, we write $\mathcal{A}$ for the **disk algebra** $H^\infty \cap C(\mathbb{T})$.

Theorem 1.4. *Let $\mathcal{E} = \{e^{it} : 0 \le t \le t_0\}$, where $0 < t_0 \le \pi$, and let $F \in H^\infty$ be an outer function such that $\|F\|_\infty = 1$, $F' \in \mathcal{A}$, and $|F| = 1$ on $\mathcal{E}$. Further, suppose $\{z_n\} \subset \mathbb{D}$ is a thick sequence with the properties that $\operatorname{Im} z_n > 0$ ($n \in \mathbb{N}$), $\lim_{n\to\infty} z_n = 1$ and*

$$\sum_n \frac{1-|z_n|^2}{|1-z_n|^2} < \infty. \tag{1.7}$$

Finally, assume that the Blaschke product $B = B_{\{z_n\}}$ satisfies $B' \in \mathcal{N}^+$ and put $f := BF$. Then f' lies in $\mathcal{N}^+$ and has a nontrivial inner factor, J, with $1 \in \sigma(J)$.

A few remarks are in order. First, it is easy to construct an outer function F satisfying the hypotheses of Theorem 1.4 by defining its modulus $|F|\big|_{\mathbb{T}} =: h$ appropriately. Namely, it suffices to assume that $h \in C^{2+\varepsilon}(\mathbb{T})$ for some $\varepsilon > 0$ (i.e., that h'' is Lipschitz continuous of order ε), in addition to the obvious conditions that $\log h \in L^1(\mathbb{T}, m)$, $0 \le h \le 1$ on $\mathbb{T}$, and $h\big|_{\mathcal{E}} = 1$. Now, for $F = \mathcal{O}_h$, the fact that $F' \in \mathcal{A}$—and actually the stronger conclusion that $F \in C^{1+\varepsilon/2}(\mathbb{T})$—is guaranteed by the Havin–Shamoyan–Carleson–Jacobs theorem, or rather by its higher order version, as given in [17, Chapter 2]. (See also [11, 12] for the original statement, as well as [4, 5] for alternative proofs and approaches.) In the case where h is strictly positive, the regularity assumption can be relaxed to $h \in C^{1+\varepsilon}(\mathbb{T})$, since $F = \mathcal{O}_h$ will then be in the same class. This follows from standard properties of the Hilbert transform; see [9, Chapter III].

Secondly, condition (1.7) means precisely that B has an angular derivative at the point 1. Roughly speaking, it says that the zero sequence $\{z_n\}$ approaches its limit point 1 in a suitably tangential manner. The (sufficient) tangency condition expressed by (1.7) should be compared with the weaker condition (1.5), which, alone, does not suffice to conclude that $\operatorname{inn}(f')$ is nontrivial; see Example 2 in Section 4 below.

Thirdly, we could have stated Theorem 1.4 in a more general form, where a nontrivial singular factor S (with $S' \in \mathcal{N}^+$) is present in the canonical factorization (1.4). The conclusion would have been that $\sigma(S) \cup \{1\} \subset \sigma(J)$. However, the main issue being the location of zeros, we have chosen to restrict ourselves to the current version.

The proofs of Theorems 1.1 and 1.4 are given in Sections 2 and 3, respectively, while the last section contains a couple of examples to the effect that the hypotheses of Theorem 1.1 are close to being sharp.

Acknowledgment. I thank Pascal Thomas for a helpful remark concerning the formulation of Theorem 1.1.

2 Proof of Theorem 1.1

First of all, since

$$f' = B'SF + BS'F + BSF', \tag{2.1}$$

our hypotheses on the three factors guarantee that $f' \in \mathcal{N}^+$.

Furthermore, because $\sigma_{\mathcal{E}}(f) \neq \emptyset$, we know that either $\sigma(S) \neq \emptyset$ or

$$\sigma^{\mathrm{i}}_{\mathcal{E}}(B) \cup \sigma^{\mathrm{b}}_{\mathcal{E},f}(B) =: \sigma_{\mathcal{E},f}(B) \neq \emptyset.$$

Assuming that $\sigma(S) \neq \emptyset$ (so that the singular factor S is nontrivial), we now rewrite (2.1) as

$$\frac{f'}{S} = B'F + BF' + BF\frac{S'}{S}$$

and claim that each of the three terms on the right is in $\mathcal{N}^+$. Indeed, for the last term, this is ensured by Ahern and Clark's result (see [1, Corollary 4]) which says that $S'/S \in \mathcal{N}^+$ whenever $S' \in \mathcal{N}^+$; the preceding terms present no difficulty. It follows that $f'/S \in \mathcal{N}^+$, and so $J := \operatorname{inn}(f')$ is divisible by S. In particular, we have then $J \not\equiv \mathrm{const}$ and $\sigma(S) \subset \sigma(J)$.

To deal with the case where $\sigma_{\mathcal{E},f}(B) \neq \emptyset$, more work is needed. Let G stand for the outer factor of f', so that

$$G(z) = \exp\left\{\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|f'(\zeta)|\, dm(\zeta)\right\}, \quad z \in \mathbb{D},$$

and let $G_{\mathcal{E}}$ (resp., $G_{\widetilde{\mathcal{E}}}$) be defined by a similar formula, where the integral is taken over $\mathcal{E}$ (resp., over $\widetilde{\mathcal{E}} := \mathbb{T} \setminus \mathcal{E}$). Thus, in particular, $G_{\mathcal{E}}$ is the outer function with modulus $|f'|\chi_{\mathcal{E}} + \chi_{\widetilde{\mathcal{E}}}$ and $G_{\widetilde{\mathcal{E}}} = G/G_{\mathcal{E}}$.

Our plan is to deduce the nontriviality of the inner function $J := f'/G$, plus the fact that $\sigma(J)$ contains $\sigma_{\mathcal{E},f}(B)$, from the inequality

$$|J(z)||G_{\widetilde{\mathcal{E}}}(z)| \le \frac{|f'(z)|\,(1-|z|^2)}{1-|f(z)|^2} \cdot \gamma_{\mathcal{E}}(z), \quad z \in \mathbb{D}, \tag{2.2}$$

where

$$\gamma_{\mathcal{E}}(z) := \left\{ \frac{2}{(1-|z|)\,\omega_z(\widetilde{\mathcal{E}})} \right\}^{\omega_z(\widetilde{\mathcal{E}})}.$$

This crucial estimate will be established later on. Right now, we take it for granted and complete the proof.

Suppose $\zeta \in \sigma^{\mathrm{i}}_{\mathcal{E}}(B)$. By adding a suitable null-set to $\mathcal{E}$ if necessary, we may assume that ζ is an interior point of $\mathcal{E}$. To show that $\zeta \in \sigma(J)$, we argue by contradiction. Let Γ be an open subarc of $\mathbb{T}$ with $\zeta \in \Gamma \subset \mathcal{E}$ such that J is analytic across Γ, and fix a point $\xi \in \Gamma$. Then

$$|J(z)| \to 1 \quad \text{as } z \to \xi \tag{2.3}$$

(it is always understood that z is restricted to $\mathbb{D}$). Furthermore, since ξ lies at a positive distance from $\widetilde{\mathcal{E}}$, the Poisson kernels P_z satisfy

$$\sup\{P_z(\eta) : \eta \in \widetilde{\mathcal{E}}\} \le C \cdot (1-|z|)$$

whenever z is close enough to ξ; here $C = C(\xi, \mathcal{E})$ is a positive constant. This implies that the quantities

$$\log \gamma_{\mathcal{E}}(z) = \omega_z(\widetilde{\mathcal{E}}) \cdot \log \left\{ \frac{2}{(1-|z|)\,\omega_z(\widetilde{\mathcal{E}})} \right\}$$

and

$$\log |G_{\widetilde{\mathcal{E}}}(z)| = \int_{\widetilde{\mathcal{E}}} \log |f'(\eta)|\, d\omega_z(\eta)$$

both tend to 0 as $z \to \xi$, whence

$$\gamma_{\mathcal{E}}(z) \to 1 \quad \text{and} \quad |G_{\widetilde{\mathcal{E}}}(z)| \to 1, \quad \text{as } z \to \xi. \tag{2.4}$$

Combining (2.2) with (2.3) and (2.4), in conjunction with the Schwarz–Pick estimate

$$\frac{|f'(z)|\,(1-|z|^2)}{1-|f(z)|^2} \le 1,$$

we see that

$$\frac{|f'(z)|\,(1-|z|^2)}{1-|f(z)|^2} \to 1 \quad \text{as } z \to \xi. \tag{2.5}$$

This being true for every $\xi \in \Gamma$, we invoke a result of Kraus, Roth and Ruscheweyh (see [13, Theorem 1.1]) to conclude from (2.5) that f is analytic across Γ. However, this conclusion is incompatible with the fact that the zeros of f cluster at $\zeta (\in \Gamma)$. The contradiction proves that $\zeta \in \sigma(J)$.

We have thus established the inclusion $\sigma^{\mathrm{i}}_{\mathcal{E}}(B) \subset \sigma(J)$. In particular, it follows that J is nonconstant whenever $\sigma^{\mathrm{i}}_{\mathcal{E}}(B) \neq \emptyset$.

Now suppose that $\zeta \in \sigma^{\mathrm{b}}_{\mathcal{E},f}(B)$. By definition, this means that we can find a thick sequence $\{z_n\} \subset \mathcal{Z}_{\mathbb{D}}(B)$ with $z_n \to \zeta$ satisfying (1.5) and (1.6). Let b be the Blaschke product with zeros $\{z_n\}$, so that b is a subproduct of B and

$$\liminf_{n\to\infty} |b'(z_n)|(1 - |z_n|^2) < 1. \tag{2.6}$$

Also, we have $f = gb$ for some $g \in H^\infty$ with $\|g\|_\infty = 1$; this in turn implies that

$$|f'(z_n)| = |g(z_n)| \cdot |b'(z_n)| \leq |b'(z_n)|$$

for each n. Consequently, applying (2.2) with $z = z_n$ yields

$$|J(z_n)||G_{\widetilde{\mathcal{E}}}(z_n)| \leq \gamma_{\mathcal{E}}(z_n) \cdot |b'(z_n)| \cdot (1 - |z_n|^2). \tag{2.7}$$

Now, (1.5) shows that the quantity

$$\log \gamma_{\mathcal{E}}(z_n) = \omega_{z_n}(\widetilde{\mathcal{E}}) \cdot \log \left\{ \frac{2}{(1 - |z_n|)\, \omega_{z_n}(\widetilde{\mathcal{E}})} \right\}$$

tends to 0 as $n \to \infty$, while (1.6) leads us to a similar conclusion about the quantity

$$\log |G_{\widetilde{\mathcal{E}}}(z_n)| = \int_{\widetilde{\mathcal{E}}} \log |f'(\eta)|\, d\omega_{z_n}(\eta).$$

Therefore,

$$\lim_{n\to\infty} \gamma_{\mathcal{E}}(z_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} |G_{\widetilde{\mathcal{E}}}(z_n)| = 1. \tag{2.8}$$

Finally, taking (2.6) and (2.8) into account, we deduce from (2.7) that

$$\liminf_{n\to\infty} |J(z_n)| < 1.$$

Recalling that $z_n \to \zeta (\in \mathbb{T})$, we readily conclude that $\zeta \in \sigma(J)$.

Now we know that $\sigma^{\mathrm{b}}_{\mathcal{E},f}(B) \subset \sigma(J)$; and this clearly implies that $J \not\equiv \mathrm{const}$ whenever $\sigma^{\mathrm{b}}_{\mathcal{E},f}(B) \neq \emptyset$.

It remains to verify (2.2). Let $z \in \mathbb{D}$ be fixed. Then, for almost all $\zeta \in \mathcal{E}$, Julia's lemma (see [9, p. 41]) gives

$$\frac{|f(\zeta) - f(z)|^2}{1 - |f(z)|^2} \leq |f'(\zeta)| \cdot \frac{|\zeta - z|^2}{1 - |z|^2}, \tag{2.9}$$

or equivalently,

$$\frac{1-|z|^2}{1-|f(z)|^2}\cdot\left|\frac{1-\overline{f(z)}f(\zeta)}{1-\bar{z}\zeta}\right|^2\le|f'(\zeta)| \tag{2.10}$$

(recall that $|f(\zeta)|=1$ whenever f has an angular derivative at ζ). Next, we consider the H^∞-function

$$\Phi_z(w):=\frac{1-|z|^2}{1-|f(z)|^2}\cdot\left(\frac{1-\overline{f(z)}f(w)}{1-\bar{z}w}\right)^2 \tag{2.11}$$

and rewrite (2.10) in the form

$$|\Phi_z(\zeta)|\le|f'(\zeta)|,\quad \zeta\in\mathcal{E}. \tag{2.12}$$

Further, we define Ψ_z to be the outer function with modulus

$$|\Psi_z(\zeta)|=|f'(\zeta)|\cdot\chi_{\mathcal{E}}(\zeta)+|\Phi_z(\zeta)|\cdot\chi_{\widetilde{\mathcal{E}}}(\zeta),\quad \zeta\in\mathbb{T},$$

and observe that

$$|\Phi_z(\zeta)|\le|\Psi_z(\zeta)|,\quad \zeta\in\mathbb{T}. \tag{2.13}$$

In fact, for $\zeta\in\mathcal{E}$, this inequality coincides with (2.12), while for $\zeta\in\widetilde{\mathcal{E}}$ the two sides are obviously equal.

Since Ψ_z is outer, the estimate (2.13) extends into $\mathbb{D}$, so that

$$|\Phi_z(w)|\le|\Psi_z(w)|,\quad w\in\mathbb{D}.$$

In particular, this holds for $w=z$, whence

$$|\Phi_z(z)|\le|\Psi_z(z)|. \tag{2.14}$$

A glance at (2.11) reveals that

$$|\Phi_z(z)|=\Phi_z(z)=\frac{1-|f(z)|^2}{1-|z|^2}, \tag{2.15}$$

and we take further steps to estimate $|\Psi_z(z)|$.

We have

$$\log|\Psi_z(z)|=\int_{\mathbb{T}}\log|\Psi_z(\zeta)|\,d\omega_z(\zeta)=I_1(z)+I_2(z), \tag{2.16}$$

where

$$I_1(z):=\int_{\mathcal{E}}\log|f'(\zeta)|\,d\omega_z(\zeta)=\log|G_{\mathcal{E}}(z)| \tag{2.17}$$

and

$$I_2(z) := \int_{\widetilde{\mathcal{E}}} \log |\Phi_z(\zeta)|\, d\omega_z(\zeta). \tag{2.18}$$

The arithmetic/geometric mean inequality yields

$$\begin{aligned} I_2(z) &= \omega_z(\widetilde{\mathcal{E}}) \cdot \int_{\widetilde{\mathcal{E}}} \log |\Phi_z(\zeta)| \frac{d\omega_z(\zeta)}{\omega_z(\widetilde{\mathcal{E}})} \\ &\le \omega_z(\widetilde{\mathcal{E}}) \cdot \log \Big\{ \frac{1}{\omega_z(\widetilde{\mathcal{E}})} \int_{\widetilde{\mathcal{E}}} |\Phi_z(\zeta)|\, d\omega_z(\zeta) \Big\} \\ &\le \omega_z(\widetilde{\mathcal{E}}) \cdot \log \Big\{ \frac{1}{\omega_z(\widetilde{\mathcal{E}})} \int_{\mathbb{T}} |\Phi_z(\zeta)|\, d\omega_z(\zeta) \Big\}. \end{aligned} \tag{2.19}$$

We proceed by noticing that, for almost all $\zeta \in \mathbb{T}$,

$$\begin{aligned} |\Phi_z(\zeta)| &\le \frac{2}{1-|z|} \cdot \frac{|1-\overline{f(z)}f(\zeta)|^2}{1-|f(z)|^2} \\ &= c_{f,z}\{1 - 2\,\mathrm{Re}(\overline{f(z)}f(\zeta)) + |f(z)|^2 |f(\zeta)|^2\}, \end{aligned} \tag{2.20}$$

where

$$c_{f,z} := \frac{2}{(1-|z|)(1-|f(z)|^2)}.$$

Also, we introduce the (harmonic) function

$$u_z(\zeta) := 1 - 2\,\mathrm{Re}(\overline{f(z)}f(\zeta)) + |f(z)|^2$$

and go on to observe that

$$|\Phi_z(\zeta)| \le c_{f,z} \cdot u_z(\zeta), \qquad \zeta \in \mathbb{T}.$$

(This follows from (2.20) and the fact that $|f(\zeta)| \le 1$ on $\mathbb{T}$.) Consequently,

$$\int_{\mathbb{T}} |\Phi_z(\zeta)|\, d\omega_z(\zeta) \le c_{f,z} \int_{\mathbb{T}} u_z(\zeta)\, d\omega_z(\zeta) = c_{f,z} \cdot u_z(z) = \frac{2}{1-|z|}. \tag{2.21}$$

Plugging the resulting inequality from (2.21) into (2.19), we now get

$$I_2(z) \le \omega_z(\widetilde{\mathcal{E}}) \log \frac{2}{(1-|z|) \cdot \omega_z(\widetilde{\mathcal{E}})}. \tag{2.22}$$

This done, we combine (2.16) with (2.17) and (2.22) to infer that

$$\log |\Psi_z(z)| \le \log |G_{\mathcal{E}}(z)| + \omega_z(\widetilde{\mathcal{E}}) \log \frac{2}{(1-|z|) \cdot \omega_z(\widetilde{\mathcal{E}})}$$

and hence

$$|\Psi_z(z)| \le |G_{\mathcal{E}}(z)| \left\{ \frac{2}{(1-|z|)\cdot \omega_z(\widetilde{\mathcal{E}})} \right\}^{\omega_z(\widetilde{\mathcal{E}})} = |G_{\mathcal{E}}(z)| \cdot \gamma_{\mathcal{E}}(z). \tag{2.23}$$

Since

$$G_{\mathcal{E}}(z) = \frac{G(z)}{G_{\widetilde{\mathcal{E}}}(z)} = \frac{f'(z)}{G_{\widetilde{\mathcal{E}}}(z)J(z)},$$

we may further rewrite (2.23) as

$$|\Psi_z(z)| \le \left| \frac{f'(z)}{G_{\widetilde{\mathcal{E}}}(z)J(z)} \right| \cdot \gamma_{\mathcal{E}}(z). \tag{2.24}$$

On the other hand, recalling (2.14) and (2.15), we see that

$$|\Psi_z(z)| \ge \frac{1-|f(z)|^2}{1-|z|^2}. \tag{2.25}$$

Finally, a juxtaposition of (2.24) and (2.25) yields

$$\frac{1-|f(z)|^2}{1-|z|^2} \le \left| \frac{f'(z)}{G_{\widetilde{\mathcal{E}}}(z)J(z)} \right| \cdot \gamma_{\mathcal{E}}(z),$$

which is precisely (2.2). The proof is therefore complete.

3 Proof of Theorem 1.4

It suffices to check that the hypotheses of the current theorem imply those of Theorem 1.1, with $S \equiv 1$, and that $\sigma_{\mathcal{E}}(f) = \sigma^{\mathrm{b}}_{\mathcal{E},f}(B) = \{1\}$. An application of Theorem 1.1 will then do the job.

First of all, the assumptions on B and F guarantee that $f = BF$ possesses an angular derivative everywhere on $\mathcal{E} = \{e^{it} : 0 \le t \le t_0\}$. Indeed, each of the two factors enjoys a similar property there; in particular, (1.7) tells us that B has an angular derivative at the endpoint 1. This said, it remains to verify conditions (1.5) and (1.6), where $\widetilde{\mathcal{E}}$ is the arc complementary to $\mathcal{E}$.

The verification of (1.5) is straightforward. In fact, from (1.7) it clearly follows that

$$\frac{(1-|z_n|)^{1/2}}{|1-z_n|} \to 0.$$

It is also obvious that

$$(1-|z_n|)^{1/2} \log \frac{1}{1-|z_n|} \to 0.$$

Consequently, the product of the two quantities, which is

$$\frac{1-|z_n|}{|1-z_n|}\log\frac{1}{1-|z_n|},$$

also tends to 0 as $n \to \infty$. Together with the elementary estimate

$$\omega_{z_n}(\widetilde{\mathcal{E}}) \asymp \frac{1-|z_n|}{|1-z_n|},$$

this yields (1.5).

Here and below, the sign $\asymp$ is used to mean that the quantities involved are comparable (i.e., their ratio lies between two positive constants).

To check (1.6), we are going to prove the following claim: *There exist numbers* $C > 1$ *and* $\delta \in (0, \frac{\pi}{2})$ *such that*

$$C^{-1} \le |f'(e^{it})| \le C \quad \textit{whenever } -\delta < t < 0. \tag{3.1}$$

Once this is established, (1.6) comes out easily. Indeed, on the arc

$$\gamma_\delta := \{e^{it} : -\delta < t < 0\} \tag{3.2}$$

we have $-M \le \log|f'| \le M$ with $M := \log C$, whence

$$\left|\int_{\gamma_\delta} \log|f'|\, d\omega_{z_n}\right| \le M\omega_{z_n}(\gamma_\delta) \le M\omega_{z_n}(\widetilde{\mathcal{E}}) \to 0. \tag{3.3}$$

Now, for $\zeta \in \widetilde{\mathcal{E}} \setminus \gamma_\delta$, we have

$$P_{z_n}(\zeta) \le \text{const} \cdot (1-|z_n|)$$

(because the limit point 1 of the z_n's lies at a positive distance from $\widetilde{\mathcal{E}} \setminus \gamma_\delta$), and so

$$\left|\int_{\widetilde{\mathcal{E}}\setminus\gamma_\delta} \log|f'|\, d\omega_{z_n}\right| \le \text{const} \cdot (1-|z_n|)\int_{\mathbb{T}} |\log|f'||dm \to 0. \tag{3.4}$$

Combining (3.3) and (3.4), we arrive at (1.6).

We now turn to proving the claim above; see (3.1) and the italicized text preceding it. Since $f' = B'F + BF'$, the right-hand inequality in (3.1) will be established as soon as we show that $|B'(e^{it})|$ is bounded for $-\frac{\pi}{2} \le t < 0$. To this end, we write $z_n = r_n e^{i\varphi_n}$ (with $r_n > 0$ and $0 < \varphi_n < \pi$) and estimate the quantity

$$|B'(\zeta)| = \sum_n \frac{1-r_n^2}{|\zeta - z_n|^2}$$

at a point $\zeta = e^{it}$ with $-\frac{\pi}{2} \le t < 0$. There is no loss of generality in assuming that $r_n \ge \frac{1}{2}$ and $0 < \varphi_n \le \frac{\pi}{2}$, since this is true for all but finitely many z_n's. We now combine the elementary inequalities

$$\frac{2}{\pi^2}[(1-r_n)^2 + (\varphi_n - t)^2] \le |\zeta - z_n|^2 \le (1-r_n)^2 + (\varphi_n - t)^2, \tag{3.5}$$

valid in this case, with the fact that

$$|\varphi_n - t| = \varphi_n + |t| \ge \varphi_n \tag{3.6}$$

to infer that

$$\begin{aligned} |B'(\zeta)| &\le \frac{\pi^2}{2}\sum_n \frac{1-r_n^2}{(1-r_n)^2 + (\varphi_n - t)^2} \le \frac{\pi^2}{2}\sum_n \frac{1-r_n^2}{(1-r_n)^2 + \varphi_n^2} \\ &\le \frac{\pi^2}{2}\sum_n \frac{1-r_n^2}{|1-z_n|^2}. \end{aligned} \tag{3.7}$$

The last quantity being finite by (1.7), it follows that $|B'(e^{it})|$ is bounded for $-\frac{\pi}{2} \le t < 0$. Consequently,

$$\sup\left\{|f'(e^{it})| : -\frac{\pi}{2} \le t < 0\right\} < \infty, \tag{3.8}$$

which proves "half" of (3.1).

Moving on to the left-hand inequality in (3.1), we first note that the modulus of the (angular) derivative $F'(\zeta)$ at a point $\zeta \in \mathcal{E}$ coincides with the nontangential limit of

$$Q_F(z) := \frac{1 - |F(z)|^2}{1 - |z|^2}$$

as $z \to \zeta$; this forms part of the Julia–Carathéodory theorem. Secondly, we recall that

$$Q_F(z) \ge \frac{1 - |F(0)|}{1 + |F(0)|} =: \eta(= \eta_F) > 0, \quad z \in \mathbb{D},$$

a well-known consequence of Schwarz's lemma (see, e.g., [14]). Therefore,

$$|F'(\zeta)| \ge \eta, \quad \zeta \in \mathcal{E}. \tag{3.9}$$

Now let $N \in \mathbb{N}$ be a number such that

$$\sum_{n=N+1}^{\infty} \frac{1 - |z_n|^2}{|1 - z_n|^2} < \frac{\eta}{2\pi^2}, \tag{3.10}$$

and let B_0 and B_1 be the Blaschke products with zero sets $\{z_n : 1 \le n \le N\}$ and $\{z_n : n > N\}$, respectively. Then put $G := FB_0$, so that $f = GB_1$.

Since F and B_0 both have an angular derivative on $\mathcal{E}$, the same is true for G. Moreover, it follows (see [1, Corollary 1]) that

$$|G'(\zeta)| = |F'(\zeta)| + |B_0'(\zeta)|, \quad \zeta \in \mathcal{E}.$$

Recalling (3.9), we see that $|G'| \geq |F'| \geq \eta$ on $\mathcal{E}$; and since $G'(= F'B_0 + FB_0')$ is continuous on $\mathbb{T}$, we can find a number $\delta \in (0, \frac{\pi}{2})$ such that

$$|G'(\zeta)| \geq \frac{\eta}{2}, \quad \zeta \in \gamma_\delta \tag{3.11}$$

(here γ_δ is the arc defined by (3.2)). Furthermore, because $|G| \leq 1 = |B_1|$ on $\mathbb{T}$, we have

$$|f'| \geq |G'B_1| - |GB_1'| \geq |G'| - |B_1'| \tag{3.12}$$

there; in particular, this holds on γ_δ.

Finally, we estimate the quantity

$$|B_1'(\zeta)| = \sum_{n=N+1}^{\infty} \frac{1 - |z_n|^2}{|\zeta - z_n|^2} \tag{3.13}$$

at a point $\zeta = e^{it}$ with $-\frac{\pi}{2} \leq t < 0$. As before, we write $z_n = r_n e^{i\varphi_n}$, assuming that $r_n \geq \frac{1}{2}$ and $0 < \varphi_n \leq \frac{\pi}{2}$ (this is certainly true for $n > N$, with N large enough), and we employ the elementary inequalities (3.5) and (3.6) to estimate the sum in (3.13). The estimate, which mimics (3.7), reads

$$\begin{aligned} |B_1'(\zeta)| &\leq \frac{\pi^2}{2} \sum_{n=N+1}^{\infty} \frac{1 - r_n^2}{(1 - r_n)^2 + (\varphi_n - t)^2} \leq \frac{\pi^2}{2} \sum_{n=N+1}^{\infty} \frac{1 - r_n^2}{(1 - r_n)^2 + \varphi_n^2} \\ &\leq \frac{\pi^2}{2} \sum_{n=N+1}^{\infty} \frac{1 - r_n^2}{|1 - z_n|^2} < \frac{\eta}{4}, \end{aligned} \tag{3.14}$$

where the last step relies on (3.10). Eventually, we obtain

$$|B_1'(\zeta)| < \frac{\eta}{4}, \quad \zeta \in \gamma_\delta \tag{3.15}$$

(we have actually checked this for the bigger arc $\{\zeta \in \mathbb{T} : -\frac{\pi}{2} < \arg \zeta < 0\}$, not just for γ_δ). Finally, we combine (3.12) with (3.11) and (3.15) to conclude that

$$|f'(\zeta)| \geq \frac{\eta}{4}, \quad \zeta \in \gamma_\delta.$$

This yields the left-hand side inequality in (3.1), with the appropriate C, and completes the proof.

4 Two examples

The purpose of this section is to show that conditions (1.5) and (1.6) appearing in Theorem 1.1, via the definition of $\sigma_{\mathcal{E}}(f)$, are indispensable and close to being sharp.

The two examples below follow the same pattern (and many more relevant examples can be furnished along these lines). Let $h : \mathbb{D} \to \Omega$ be a conformal mapping of the disk onto a domain Ω that is contained in the left half-plane

$$\mathcal{H} := \{w \in \mathbb{C} : \operatorname{Re} w < 0\}$$

and contains, for some fixed number c, infinitely many points of the form $c + 2\pi ik$ with $k \in \mathbb{Z}$. More precisely, we are assuming that there is a $c \in \mathcal{H}$ and an infinite subset Λ of $\mathbb{Z}$ such that

$$\{c + 2\pi ik : k \in \Lambda\} \subset \Omega.$$

(This is certainly the case if Ω contains a vertical line or half-line.) Further, put $a := e^c$ and note that $|a| < 1$; then define

$$g := e^h \quad \text{and} \quad f := \frac{g - a}{1 - \bar{a}g}. \tag{4.1}$$

Since $h(\mathbb{D}) = \Omega \subset \mathcal{H}$, it follows that g (and hence also f) is an H^∞-function of norm at most 1. Now, suppose that h maps a certain arc $\Gamma \subset \mathbb{T}$ continuously onto an interval—possibly infinite—of the imaginary axis $i\mathbb{R}$. We have then $|g| = |f| = 1$ on Γ (whence $\|g\|_\infty = \|f\|_\infty = 1$), and moreover, g and f will each have an angular derivative on Γ. In addition, f vanishes at the points $z_k := h^{-1}(c + 2\pi ik)$ with $k \in \Lambda$, because $g(z_k) = a$. Consequently, letting B denote the Blaschke product with zeros z_k, $k \in \Lambda$, we see that the inner part of f is divisible by B.

On the other hand,

$$f' = \frac{1 - |a|^2}{(1 - \bar{a}g)^2} gh'. \tag{4.2}$$

The function $(1 - \bar{a}g)^{-2}$ is outer (and even invertible in H^∞); therefore, if g and h' also happen to be outer, the same will be true for f'. The situation then stands in sharp contrast to the conclusion of Theorem 1.1: indeed, f has a nonconstant inner factor, while f' has none. This means that the current function f violates the hypotheses of the theorem. Specifically, if $\mathcal{E}$ is taken to be Γ (so that $\widetilde{\mathcal{E}} = \mathbb{T} \setminus \Gamma$) and if the zero sequence $\{z_k\}$ is thick, then either (1.5) or (1.6) must break down. Thus, a glance at a concrete example of the above type might reveal whether the two sufficient conditions are reasonably close to being necessary.

Example 1. Let

$$\Omega = \{ w \in \mathbb{C} : -\pi < \operatorname{Re} w < 0 \}.$$

The function

$$h(z) = i \log\left(\frac{1+z}{1-z}\right) - \frac{\pi}{2}$$

(where the principal branch of the logarithm is used) maps $\mathbb{D}$ conformally onto $\boldsymbol{\Omega}$. We now fix a number c with $-\pi < c < 0$, then put $a = e^c$ and define the functions g and f by (4.1), with the current h plugged in. Letting $\mathcal{E}$ stand for the arc $\{e^{it} : -\pi < t < 0\}$, we have $h(\mathcal{E}) = i\mathbb{R}$, and so f has an angular derivative on $\mathcal{E}$. Since $e^{-\pi} \le |g| \le 1$ on $\mathbb{D}$, it follows that g is outer. The function $h'(z) = 2i(1-z^2)^{-1}$ being outer as well, we may invoke (4.2) to deduce that f' is outer.

Now, the zeros z_k of f, given by

$$z_k = h^{-1}(c + 2\pi i k), \quad k \in \mathbb{Z}, \tag{4.3}$$

have the property that $\omega_{z_k}(\widetilde{\mathcal{E}})$ takes the constant value $|c|/\pi$. (This is best seen by looking at the images ζ_k of the z_k's under the transformation

$$z \mapsto \frac{1+z}{1-z}, \tag{4.4}$$

which maps $\mathbb{D}$ onto the right half-plane and $\mathcal{E}$ onto the half-line $i\mathbb{R}_- := \{ i\eta : \eta < 0 \}$. The points $\zeta_k = (1+z_k)/(1-z_k)$ are then determined by the formula

$$\zeta_k = \exp\left(2\pi k - ic - \frac{i\pi}{2}\right), \quad k \in \mathbb{Z}, \tag{4.5}$$

whence

$$\arg \zeta_k = -\frac{\pi}{2} - c = -\frac{\pi}{2} + |c|. \tag{4.6}$$

Recalling the well known interpretation of the harmonic measure in terms of angles, one readily arrives at the required fact.)

Finally, we observe that the sequence $\{z_k\}$ clusters at the points ± 1 and is thick. The latter claim can be verified with the help of a lemma by Sundberg and Wolff from [18]. (Precisely speaking, the version we need is obtained by combining Lemma 7.1 on p. 578 of [18] with the concluding paragraph on p. 580 after the lemma's proof. In fact, [18] treats a more general situation involving a Douglas algebra B, which we take to be $H^\infty + C$. See also [8, p. 4455] for the special case in question.) To state the thinness criterion given there, let $\{a_j\}$ be a sequence of distinct points in $\mathbb{D}$. Also, consider the arcs

$$I_{N,j} := \{ \zeta \in \mathbb{T} : |\zeta - a_j| \le N(1 - |a_j|) \}$$

with $N > 1$, and write $\mathcal{K}(N, j)$ for the set of those indices k, $k \neq j$, which satisfy $a_k/|a_k| \in I_{N,j}$ and $1 - |a_k| \le m(I_{N,j})$. This done, the Sundberg–Wolff result tells us that $\{a_j\}$ is thin if and only if, for every $N > 1$,

$$\lim_{j\to\infty} (1 - |a_j|)^{-1} \sum_{k\in\mathcal{K}(N,j)} (1 - |a_k|) = 0. \tag{4.7}$$

Now, a computation shows that setting $a_j = z_j$, where the z_j's are given by (4.3), makes (4.7) false (provided that N is large enough). Again, the easiest way to check this is to rephrase (4.7) for the right half-plane and then look at the images (4.5) of the z_k's under the conformal mapping (4.4). Thus, $\{z_k\}$ is indeed a thick sequence.

In summary, while a suitably tangential convergence (in the sense of (1.5) and (1.6)) of the zero sequence $\{z_k\}$ to the endpoints ± 1 of $\mathcal{E}$ would imply that f' has an inner factor, no kind of nontangential convergence would suffice. In fact, (4.6) shows that, letting c be appropriately small in modulus, we can arrange it for the ζ_k's to lie on a half-line that forms an arbitrarily small angle with the lower imaginary semiaxis (or, equivalently, for z_k to lie on a circular arc that forms an arbitrarily small angle with $\mathcal{E}$ at ± 1).

Example 2. Now let

$$\Omega = \{w \in \mathbb{C} : \operatorname{Re} w < 0, \operatorname{Im} w < 0\}.$$

This time, we take the conformal map $h : \mathbb{D} \to \Omega$ to be

$$h(z) = -e^{i\pi/4}\sqrt{\frac{1+z}{1-z}},$$

where the square root is supposed to satisfy $\sqrt{x} > 0$ for $x > 0$. We then fix a number $c \in (-\infty, 0)$ and define the functions g and f by (4.1), with $a = e^c$.

This done, we claim that g is an outer function. Indeed, since g is zero-free and has radial limit 0 only at the point 1, it follows that g has no inner factor, except possibly for the "atomic" singular function

$$S_\gamma(z) := \exp\left(\gamma \frac{z+1}{z-1}\right)$$

with some $\gamma > 0$. However, if g were divisible by S_γ, then we would have

$$|g(x)| \le \exp\left(\gamma \frac{x+1}{x-1}\right), \quad 0 < x < 1,$$

whereas g actually has a milder decay rate as $x \to 1^-$; in fact,

$$|g(x)| = \exp\left(-\frac{1}{\sqrt{2}} \cdot \sqrt{\frac{1+x}{1-x}}\right), \quad 0 < x < 1.$$

Thus, g is outer. So is the function

$$h'(z) = -e^{i\pi/4}(1+z)^{-1/2}(1-z)^{-3/2},$$

and we eventually conclude, by virtue of (4.2), that f' is outer also.

The semicircle $\{e^{it} : 0 < t < \pi\} =: \mathcal{E}$ is mapped by h onto the half-line $i\mathbb{R}_-$, so f has an angular derivative on $\mathcal{E}$. (Note that the current $\mathcal{E}$ is different from its namesake in Example 1.) Finally, the zeros z_k of f are now given by

$$z_k = h^{-1}(c - 2\pi ik), \quad k \in \mathbb{N}.$$

Equivalently, the points $\zeta_k = (1+z_k)/(1-z_k)$ (i.e., the images of the z_k's in the right half-plane under the transformation (4.4)) are determined by

$$\zeta_k = -i(c - 2\pi ik)^2, \quad k \in \mathbb{N}. \tag{4.8}$$

We have $z_k \to 1$ and

$$\omega_{z_k}(\widetilde{\mathcal{E}}) \asymp \frac{1}{k}, \quad k \in \mathbb{N}. \tag{4.9}$$

To verify (4.9), one may first rewrite (4.8) in the form

$$\zeta_k = \xi_k + i\eta_k,$$

where

$$\xi_k = 4\pi|c|k \quad \text{and} \quad \eta_k = 4\pi^2k^2 - c^2.$$

Now, the image of $\widetilde{\mathcal{E}}$ under the map (4.4) is $i\mathbb{R}_-$, and the angle at which this half-line is seen from ζ_k is comparable to its tangent, ξ_k/η_k (or equivalently, to $1/k$). Moving back to the disk, one arrives at (4.9). Furthermore, a calculation shows that

$$1 - |z_k| \asymp \frac{1}{k^3}, \quad k \in \mathbb{N}.$$

Together with (4.9), this ensures that

$$\omega_{z_k}(\widetilde{\mathcal{E}}) \cdot \log \frac{1}{1-|z_k|} \to 0,$$

making (1.5) true. By contrast, (1.6) breaks down, the reason being that $|f'|$ becomes too small near the endpoint 1 of $\widetilde{\mathcal{E}}$.

The conclusion is that condition (1.5) alone, or even its stronger version

$$\omega_{z_k}(\widetilde{\mathcal{E}}) = O((1-|z_k|)^{1/3}),$$

is not enough to guarantee the validity of our Gauss–Lucas type phenomenon (i.e., to ensure that f' has a nontrivial inner factor whenever f does). A different type of tangency condition, stated in terms of $|f'|$, should be added to make things work.

Finally, we remark that the sequence $\{z_k\}$ in this last example was thick. This, again, can be verified by means of the Sundberg–Wolff criterion (4.7), possibly transplanting everything to the right half-plane (for the sake of convenience) and working with the ζ_k's instead.

REFERENCES

[1] P. R. Ahern and D. N. Clark, *On inner functions with H^p-derivative*, Michigan Math. J. **21** (1974), 115–127.

[2] R. B. Burckel, *An Introduction to Classical Complex Analysis, Vol. I*, Academic Press, New York, 1979.

[3] C. Carathéodory, *Theory of Functions of a Complex Variable, Vol. II*, Chelsea Publ. Co., New York, 1954.

[4] K. M. Dyakonov, *Equivalent norms on Lipschitz-type spaces of holomorphic functions*, Acta Math. **178** (1997), 143–167.

[5] K. M. Dyakonov, *Approximate identities and geometric means of smooth positive functions*, J. Anal. Math. **78** (1999), 307–327.

[6] K. M. Dyakonov, *A reverse Schwarz–Pick inequality*, Comput. Methods Funct. Theory **13** (2013), 449–457.

[7] K. M. Dyakonov, *A characterization of Möbius transformations*, C. R. Math. Acad. Sci. Paris **352** (2014), 593–595.

[8] K. M. Dyakonov and A. Nicolau, *Free interpolation by nonvanishing analytic functions*, Trans. Amer. Math. Soc. **359** (2007), 4449–4465.

[9] J. B. Garnett, *Bounded Analytic Functions*, Springer, New York, 2007.

[10] P. Gorkin and R. Mortini, *Asymptotic interpolating sequences in uniform algebras*, J. London Math. Soc. (2) **6**7 (2003), 481–498.

[11] V. P. Havin, *A generalization of the Privalov–Zygmund theorem on the modulus of continuity of the conjugate function*, Izv. Akad. Nauk Armjan. SSR Ser. Mat. **6** (1971), 252–258 and 265–287.

[12] V. P. Havin and F. A. Shamoyan, *Analytic functions with Lipschitz moduli of the boundary values*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **19** (1970), 237–239.

[13] D. Kraus, O. Roth and S. Ruscheweyh, *A boundary version of Ahlfors' lemma, locally complete conformal metrics and conformally invariant reflection principles for analytic maps*, J. Anal. Math. **101** (2007), 219–256.

[14] D. Sarason, *Angular derivatives via Hilbert space*, Complex Variables Theory Appl. **10** (1988), 1–10.

[15] D. Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk*, Wiley, New York, 1994.

[16] T. Sheil-Small, *Complex Polynomials*, Cambridge University Press, Cambridge, 2002.

[17] N. A. Shirokov, *Analytic Functions Smooth Up to the Boundary*, Springer-Verlag, Berlin, 1988.

[18] C. Sundberg and T. H. Wolff, *Interpolating sequences for QA_B*, Trans. Amer. Math. Soc. **276** (1983), 551–581.

[19] J. L. Walsh, *Note on the location of zeros of the derivative of a rational function whose zeros and poles are symmetric in a circle*, Bull. Amer. Math. Soc. **45** (1939), 462–470.

[20] T. H. Wolff, *Two algebras of bounded functions*, Duke Math. J. **49** (1982), 321–328.

Konstantin M. Dyakonov
BGSMath, IMUB and Departament de Matemàtiques i Informàtica
Universitat de Barcelona
Gran Via 585
E-08007 Barcelona, Spain

and

ICREA, Pg. Lluís Companys 23
E-08010 Barcelona, Spain
email: konstantin.dyakonov@icrea.cat

(Received July 26, 2016)