QUASICONFORMAL MAPS WITH CONTROLLED LAPLACIAN

By

DAVID KALAJ[1] AND EERO SAKSMAN[2]

Abstract. We establish that every K-quasiconformal mapping w of the unit disk $\mathbb{D}$ onto a C^2-Jordan domain Ω is Lipschitz provided that $\Delta w \in L^p(\mathbb{D})$ for some $p > 2$. We also prove that if in this situation $K \to 1$ with $\|\Delta w\|_{L^p(\mathbb{D})} \to 0$, and $\Omega \to \mathbb{D}$ in $C^{1,\alpha}$-sense with $\alpha > 1/2$, then the bound for the Lipschitz constant tends to 1. In addition, we provide a quasiconformal analogue of the Smirnov theorem on absolute continuity over the boundary.

1 Introduction

Recall that the map $w : \mathbb{D} \to \mathbb{C}$ of the unit disk to the complex plane is quasiconformal if it is a sense-preserving homeomorphism that has locally L^2-integrable weak partial derivatives and it satisfies, for almost every $z \in \mathbb{D}$, the distortion inequality $|w_{\bar{z}}| \leq k|w_z|$, where $k < 1$. In this situation, we say that w is K-**quasiconformal**, with $K := (1 + k)/(1 - k)$. We refer to [2] and [4] for basic notions and results of the quasiconformal theory. Quasiconformal self-maps of the disc, even when locally C^2-smooth inside $\mathbb{D}$, need not to be Lipschitz. However, Pavlović [18] proved that in the situation where $w : \mathbb{D} \to \mathbb{D}$ is a quasiconformal homeomorphism that is also harmonic, f is bi-Lipschitz. Many generalisations of this result for harmonic maps heve been proved since; we refer, e.g., to [12] and [5] and the references therein.

Our paper addresses the following problem: how much one can relax the condition of harmonicity of the quasiconformal map w, while still being able to deduce the Lipschitz property of w? In this situation, it is less natural to require w to be bi-Lipschitz. Better insight into this kind of question ought to be useful also in applications to non-linear elasticity.

[1] A part of this project was finished during the visit of the first author to Helsinki University in April 2014.

[2] Supported by the Finnish CoE in Analysis and Dynamics Research, and by the Academy of Finland, projects 113826 and 118765.

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 137 (2019)
DOI 10.1007/s11854-018-0072-5

A natural measure for the deviation from harmonic functions to consider is $\|\Delta w\|_{L^p(\mathbb{D})}$ for some $p \geq 1$ and one can ask whether finiteness of this quantity enables one to make the desired conclusion. Our first result is the following theorem.

Theorem 1. *Assume that $g \in L^p(\mathbb{D})$ and $p > 2$. If w is a K-quasiconformal solution of $\Delta w = g$ that maps the unit disk onto a bounded Jordan domain $\Omega \subset \mathbb{C}$ with C^2-boundary, then w is Lipchitz continuous. The result is sharp since it fails in general if $p = 2$.*

The proof is given in Section 2.

Our second result shows that, in the setting of Theorem 1 (actually under weaker regularity condition on the boundary), the Lipschitz constant of a normalised map f becomes arbitrarily close to 1 as the image domain Ω approaches the unit disk in a suitably defined $C^{1,\alpha}$-sense and if the deviations both from conformality and harmonicity tend to 0. Below, we identify $[0, 2\pi)$ and the boundary of the unit disk $\mathbb{T}$ in the usual way.

Theorem 2. *Let $p > 2$, and assume that $w_n : \mathbb{D} \to \Omega_n$ is a K_n-quasiconformal normalised map, normalised by $w(0) = 0$, and with*

$$\lim_{n\to\infty} K_n = 1 \quad \textit{and} \quad \lim_{n\to\infty} \|\Delta w_n\|_{L^p(\mathbb{D})} = 0.$$

Moreover, assume that for each $n \geq 1$ the bounded Jordan domain Ω_n approaches the unit disk in the $C^{1,\alpha}$-bounded sense[1] *Then for large enough n, the function w_n is Lipschitz and its Lipschitz constant tends to 1 as $n \to \infty$, i.e.,*

$$\lim_{n\to\infty} \|\nabla w\|_{L^\infty(\mathbb{D})} = 1. \tag{1}$$

This result is a corollary of slightly more general results that we provide in Section 3 below. Together, Theorems 1 and 2 considerably improve the main result of the first author and Pavlović from [15], where it was instead assumed that $\Delta w \in C(\overline{\mathbb{D}})$. Other related results are contained in [13]; we refer to [6] and references therein for other types of connections between quasiconformal and Lipschitz maps.

Before stating our last theorem, we recall the result of V. I. Smirnov [9, Theorem 1, p. 409], stating that a conformal mapping of the unit disk onto a Jordan domain Ω with rectifiable boundary has a absolutely continuous extension to the boundary. This implies, in particular, that if $E \subset \mathbf{T}$ is a set of zero 1-dimensional

[1] More precisely, this means that there is a parametrisation $\partial\Omega_n = \{f_n(\theta) : \theta \in \mathbf{T}\}$, where f_n satisfies $\|f_n(\theta) - e^{i\theta}\|_{L^\infty(\mathbf{T})} \to 0$ as $n \to \infty$ and $\sup_{n\geq 1} \|f_n(\theta)\|_{C^{1,\alpha}(\mathbb{T})} < \infty$ for some $\alpha > 1/2$.

Hausdorff measure, its image $f(E)$ is a set of zero 1-dimensional Hausdorff measure in $\partial\Omega$. This result has been generalized for the class of q.c. harmonic mapping by several authors; see, e.g., [17, 14]. On the other hand, if f is merely quasiconformal, its boundary function need not, in general, be an absolutely continuous function. In Section 4, we prove the following generalization of Smirnov's theorem for quasiconformal mappings, subject again to a size condition on their Laplacian.

Theorem 3. *Let f be a quasiconformal mapping of the unit disk onto a Jordan domain with rectifiable boundary. Assume that Δf is locally integrable and satisfies $|\Delta f(z)| \leq C(1-|z|)^{-a}$ for some constants $a < 1$ and $C < \infty$. Then $f_{|\mathbf{T}}$ is an absolutely continuous function.*

The result is optimal: there is a quasiconformal self-map $f : \mathbb{D} \to \mathbb{D}$, with non-absolutely continuous boundary values and such that $f \in C^\infty(\mathbb{D})$ and with $|\Delta f(z)| \leq C(1-|z|)^{-1}$ in $\mathbb{D}$.

The following corollary is another variant of the previous theorem.

Corollary 1. *If f is a quasiconformal mapping of the unit disk onto a Jordan domain with rectifiable boundary such that $\Delta f \in L^p(\mathbb{D})$ for some $p > 1$, then $f_{|\mathbf{T}}$ is an absolutely continuous function.*

The conclusion of the corollary fails in general if $p < 1$. Further comments, generalizations, and open questions related to the above results are included in Sections 2–4.

2 Proof of Theorem 1: Lipschitz-property of q.c. solutions of $\Delta f = g$

In what follows, we say that a bounded Jordan domain $\Omega \subset \mathbb{C}$ has C^2-boundary if it is the image of the unit disk $\mathbb{D}$ under a C^2-diffeomorphism of the whole complex plane onto itself. For planar Jordan domains, this is well-known to be equivalent to the more standard definition that requires the boundary to be locally isometric to the graph of a C^2-function on $\mathbb{R}$. In what follows, Δ refers to the distributional Laplacian. We make use of the following well-known fact, whose proof we provide.

Lemma 1. *Assume that $w \in C(\overline{\mathbb{D}})$ is such that $\|\Delta w\|_{L^p(\mathbb{D})} < \infty$ with $p > 1$.*

(i) *If $p > 2$, then $\|\nabla w\|_{L^\infty(B(0,r))} < \infty$ for all $r < 1$. Moreover, if $w_{|\partial\mathbb{D}} = 0$, then there exists $C_p < \infty$ such that $\|\nabla w\|_{L^\infty(\mathbb{D})} \leq C_p \|\Delta w\|_{L^p(\mathbb{D})}$.*

(ii) *If $1 < p < 2$ and $w_{|\partial\mathbb{D}} = 0$, then $\|\nabla w\|_{2p/(2-p)} < \infty$.*

Proof. By the classical representation, we have, for $|z| < 1$,

$$w(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\varphi})w(e^{i\varphi})d\varphi + \int G(z,\omega)\Delta w(\omega)\, dA(\omega), \tag{2}$$

where P stands for the Poisson kernel and $G(z,\omega) := \frac{1}{2\pi}\log\left|\frac{1-z\overline{\omega}}{z-\omega}\right|$ is the Green's function of $\mathbb{D}$. We observe first that since G is real-valued, $|\nabla G| = 2|\partial_z G|$, so

$$|\nabla G(z,\omega)| = \frac{1}{2\pi}\left|\frac{-\overline{\omega}}{1-z\overline{\omega}} - \frac{1}{z-\omega}\right| \le \frac{1}{\pi|z-\omega|}. \tag{3}$$

Hence, an application of Hölder's inequality shows that the second term in (2) has uniformly bounded gradient in $\mathbb{D}$. To conclude part (i), it suffices to observe that the first term vanishes if $w_{|\partial\mathbb{D}} = 0$; and, in the general case, it has uniformly bounded gradient in compact subsets of $\mathbb{D}$. Finally, part (ii) follows immediately from (3) by the standard mapping properties of the Riesz potential I_1 with the kernel $|z-\omega|^{-1}$; see [21]. □

Proof of Theorem 1. It would be natural to try to generalise the ideas in [13], where differential inequalities were applied while treating related problems. However, it turns out that the approach of [11], where the use of distance functions was initiated, is flexible enough for further development.

In the sequel, we say $a \approx b$ if there is a constant $C \ge 1$ such that $a/C \le b \le Ca$, and we say $a \lesssim b$ if there is a constant $C > 0$ such that $a \le Cb$.

By our assumption on the domain, we may fix a diffeomorphism $\psi : \overline{\Omega} \to \overline{\mathbb{D}}$ that is C^2 up to the boundary. Let $H := 1 - |\psi|^2$, which is C^2-smooth in $\overline{\Omega}$ and vanishes on $\partial\Omega$ with $|\nabla H| \approx 1$ in a neighborhood of $\partial\Omega$. We may then define $h : \mathbb{D} \to [0,1]$ by setting

$$h(z) := H \circ w(z) = 1 - |\psi(w(z))|^2 \quad \text{for } z \in \mathbb{D}.$$

The quasiconformality of f and the behavior of ∇H near $\partial\Omega$ imply the existence of $r_0 \in (0,1)$ such that the weak gradients satisfy

$$|\nabla h(x)| \approx |\nabla w(x)| \quad \text{for } r_0 \le |x| < 1. \tag{4}$$

Moreover, by Lemma 1(i), we have $|\nabla h(x)| \lesssim |\nabla w(x)| \le C$ for $|x| \le r_0$. It follows that for all $q \in (1,\infty]$,

$$\nabla h \in L^q(\mathbb{D}) \quad \text{if and only if } \nabla w \in L^q(\mathbb{D}). \tag{5}$$

A direct computation, simplified by the fact that H is real valued, yields

$$\Delta h = \Delta(H\circ w)) = (\Delta H)(w)(|w_z|^2 + |w_{\overline{w}}|^2) + 2\mathrm{Re}\Big(4H_{zz}(w)w_z w_{\overline{z}} + H_z(w)\Delta w\Big). \tag{6}$$

In particular, since $H \in C^2(\overline{D})$, we have

$$|\Delta h| \lesssim |\nabla w|^2 + |g|. \tag{7}$$

The higher integrability of quasiconformal self-maps of $\mathbb{D}$ makes sure that $\nabla(\psi \circ w) \in L^q(\mathbb{D})$ for some $q > 2$, which implies that $\nabla w \in L^q(\mathbb{D})$. By combining this with the fact that $g \in L^p(\mathbb{D})$ with $p > 2$, we deduce that $\Delta h \in L^r(\mathbb{D})$ with $r = \min(p, q/2) > 1$. This information is not enough in case $q \leq 4$, but we show that one may improve the situation to $q > 4$ via a bootstrapping argument based on the following observation: in our situation,

$$\text{if } \nabla w \in L^q(\mathbb{D}) \text{ with } 2 < q < 4, \quad \text{then } \nabla w \in L^{2q/(4-q)}(\mathbb{D}). \tag{8}$$

To prove (8), assume that $\nabla w \in L^q(\mathbb{D})$ for an exponent $q \in (2,4)$. Then (7) and our assumption on g verify that $\Delta h \in L^{q/2}(\mathbb{D})$. Since h vanishes continuously on the boundary $\partial\mathbb{D}$, we may apply Lemma 1(ii) to obtain that $\nabla h \in L^{2q/(4-q)}(\mathbb{D})$, which yields the claim, according to (5).

We then claim that, in our situation, $\nabla w \in L^q(\mathbb{D})$ with some exponent $q > 4$. To prove the claim, fix an exponent $q_0 > 2$ obtained from the higher integrability of the quasiconformal map w such that $\nabla w \in L^{q_0}(\mathbb{D})$. Diminishing q_0 if necessary, we may assume that $q_0 \in (2,4)$ and $q_0 \notin \{2^n/(2^{n-1}-1),\ n = 3, 4, \ldots\}$. Then we may iterate (8) and deduce inductively that $\nabla w \in L^{q_k}(\mathbb{D})$ for $k = 0, 1, 2 \ldots k_0$, where the indexes q_k satisfy the recursion $q_{k+1} = \frac{2q_k}{4-q_k}$ and k_0 is the first index such that $q_{k_0} > 4$. Such an index exists since, by induction, we have the relation $(1 - 2/q_{k+1}) = 2^k(1 - 2/q_0)$ for $k \geq 0$

Thus we may assume that $\nabla w \in L^q(\mathbb{D})$ with $q > 4$. At this stage, (7) shows that $\Delta h \in L^{p\wedge(q/2)}(\mathbb{D})$. As $p \wedge (q/2) > 2$, Lemma 1(ii) verifies that ∇h is bounded. Finally, by (5), we have the same conclusion for ∇w; hence w is Lipschitz, as claimed.

To verify the sharpness of the result, consider the map $w_0(z) = z \log^a\left(\frac{e}{|z|^2}\right)$, where $a \in (0, 1/2)$; w_0 is a self-homeomorphism of $\mathbb{D}$ that is quasiconformal with continuous Beltrami-coefficient. We can easily compute

$$(w_0)_z = \log^{a-1}\left(\frac{e}{|z|^2}\right) \log\left(\frac{e^{1-a}}{|z|^2}\right) \quad \text{and} \quad (w_0)_{\bar z} = -a\frac{z}{\bar z} \log^{a-1}\left(\frac{e}{|z|^2}\right),$$

so the complex dilatation of w_0 satisfies

$$|\mu_{w_0}(z)| = \Big| -a\frac{z}{\bar z}\Big(\log\Big(\frac{e^{1-a}}{|z|^2}\Big)\Big)^{-1}\Big| \leq \frac{a}{1-a} < 1.$$

In addition, we see that $\Delta w_0 \in L^2(\mathbb{D})$ since

$$|\Delta w_0(z)| = \left|4\frac{d}{d\overline{z}}(w_0)_z(z)\right| = \left|\frac{4a}{\overline{z}}\log^{a-2}\left(\frac{e}{|z|^2}\right)\left((a-1)-\log\left(\frac{e}{|z|^2}\right)\right)\right|$$
$$\lesssim |z|^{-1}\left(\log\left(\frac{e}{|z|^2}\right)\right)^{a-1}.$$

Finally, it remains to observe that w is not Lipschitz at the origin. □

Remark 2.1. Invoking the known sharp L^p-integrability results of q.c. maps (see Astaka [4, Theorem 13.2.3]), one sees that, in the above proof, no iteration is needed in case $K < 2$. One should also observe that the counterexample given above in the case $p = 2$ is already based on the behaviour of w near origin, not to any boundary effect. So, in this sense, Theorem 1 is quite sharp. We have not pursued seriously the optimality question related to C^2-regularity assumption on Ω.

Remark 2.2. Assume that $w : B(0,1) \to B(0,1)$ is quasiconformal, where $B(0,1) \subset \mathbb{R}^d$ is the d-dimensional unit ball, with $d \geq 3$, and is such that $\Delta w_k \in L^p(B(0,1))$ with $p > n$ for each component of w (here $k = 1,\ldots,d$). Then the above proof, with some modifications, applies and shows that w is Lipschitz. Actually, in a recent preprint [5] Astala and Manojlović proved that quasiconformal harmonic gradient mapping of the unit ball B^3 on to itself are bi-Lipschitz. They also provide a short new proof of the Lipschitz-property of quasiconformal harmonic maps of the unit ball onto a domain with C^2 boundary on $\mathbb{R}^d$ (cf. [13, Theorem C]), using bootstrapping argument. The results of [5] and of the present paper were obtained independently.

3 Proof of Theorem 2: quantitative bounds as $\Omega \to \mathbb{D}$

We start with an auxiliary lemma.

Lemma 2. *There exists a function $\psi : (1, 3/2) \to \mathbb{R}^+$ such that if $w : \mathbb{D} \to \mathbb{D}$ is a K-quasiconformal self-map normalised with $\psi(0) = 0$, then*

$$\||w_z|^2 + |w_{\overline{z}}|^2 - 1\|_{L^3(\mathbb{D})} \leq \psi(K).$$

Moreover, $\lim_{K\to 1^+} \psi(K) = 0.$

Proof. By the sharp area distortion, $\|\nabla w\|_{L^6(\mathbb{D})} < \infty$ for $K < 3/2$. Reflecting w over the boundary $\partial\mathbb{D}$, we may also assume that w extends to a K-quasiconformal map (still denoted by w) to the whole plane. Rotating the plane, if necessary, we may also impose the condition that $w(1) = 1$. Furthermore, we may even assume that $w_{\mathbb{C}\setminus B(0,e^{3\pi})}$ is the identity map, since we may use standard quasiconformal surgery (choose $k = (K-1)/(K+1)$ and $\alpha = 2k$ in [4, Theorem 12.7.1]) to produce $\frac{3K-1}{3-K}$-quasiconformal modification (still denoted by w) that equals the original function w in $\mathbb{D}$ and satisfies $w(z) = z$ for $|z| \geq e^{3\pi}$. Observe that it is a principal solution. Since $\frac{3K-1}{3-K} \to 1$ as $K \to 1$ and we are interested in only small values of K, it thus suffices to prove the corresponding claim only for principal solutions with complex dilatation supported in $B(0, e^{3\pi})$.

Denote by M the norm of the Beurling operator on $L^6(\mathbb{C})$. Fix $R_0 > 0$ and consider a principal solution w to the Beltrami equation $w_{\bar{z}} = \mu w_z$ with $|\mu| \leq k < 1/2M$. Then we have the standard Neumann-series representation

$$w_{\bar{z}} = \mu + \mu T\mu + \mu T\mu T\mu + \cdots \quad \text{and} \quad w_z - 1 = Tw_{\bar{z}}$$

and thus obtain

$$\|w_{\bar{z}}\|_{L^6(\mathbb{C})} \leq \|\mu\|_{L^6(\mathbb{C})} \left(1 + \frac{M}{2M} + \left(\frac{M}{2M}\right)^2 + \cdots\right) \leq 2\|\mu\|_{L^6(\mathbb{C})} \leq Ck^{1/6}$$

and, a fortiori,

$$\|w_z - 1\|_{L^6(\mathbb{C})} \leq MCk^{1/6} = C'k^{1/6}.$$

The desired L^3-estimate for $|f_{\bar{z}}|^2$ follows, since $k \to 0$ as $K \to 1$. The estimate for $|f_z|^2 - 1$ follows by noting that $\big||f_z|^2 - 1\big| \leq |f_z - 1|(|f_z - 1| + 2)$ and applying Hölder's inequality. □

Before proving the more general convergence result stated in the introduction it is useful to consider first the case where the image domain is fixed, and in fact equals $\mathbb{D}$.

Proposition 3.1. *For $p > 2$, there exist a function*

$$[1, \infty) \times [0, \infty) \ni (K, t) \to \widetilde{C}_p(K, t)$$

such that if $w : \mathbb{D} \to \mathbb{D}$ is a K-quasiconformal self map of the unit disc, normalised by $w(0) = 0$, and with $\Delta w \in L^p(\mathbb{D})$, then

$$\|\nabla w\|_{L^\infty(\mathbb{D})} \leq \widetilde{C}_p(K, \|\Delta w\|_p).$$

Moreover, the function $\widetilde{C}_p$ satisfies

$$\lim_{K\to 1^+,\, t\to 0^+} \widetilde{C}_p(K, t) = 1. \tag{9}$$

Proof. We follow the line of the proof of Theorem 1; in particular, we employ its notation, but this time we strive to make the conclusion quantitative. We may assume that $p \le 3$. Let us then assume that w is as in the assumption of the proposition with $K < 1 + 1/100$, say. In addition, we may assume that $w(1) = 1$. As the image domain is $\mathbb{D}$, the function h from the proof of Theorem 1 takes the form $h(z) = 1 - |w(z)|^2$. Let us write $h_0(z) = 1 - |z|^2$, which corresponds to h when w is the identity map. An application of (6) and Lemma 2 allows us to estimate

$$\begin{aligned} \|\Delta(h-h_0)\|_{L^p(\mathbb{D})} &= \big\|4(1-|w_z|^2) - 4|w_{\bar z}|^2 + 2\mathrm{Re}(\overline{w}g)\big\|_{L^p(\mathbb{D})} \\ &\le 4\|(|w_z|^2-1) + |w_{\bar z}|^2\|_{L^3(\mathbb{D})} + \|g\|_{L^p(\mathbb{D})} \\ &\le 4\psi(K) + \|g\|_{L^p(\mathbb{D})}. \end{aligned} \tag{10}$$

Lemma 1 implies that $\|\nabla h - \nabla h_0\| \le c_p(\psi(K) + \|g\|_{L^p(\mathbb{D})})$.

The quasiconformality of w implies that for almost every z,

$$|\nabla h(z)| \ge K^{-1} |(\nabla h_0)(w(z))| |\nabla w(z)|.$$

Since $|\nabla h_0(z)| = 2|z|$, we obtain, by considering the annulus $1 - \varepsilon \le |z| < 1$ with arbitrarily small $\varepsilon > 0$,

$$\begin{aligned} \limsup_{|z|\to 1^-} |\nabla w(z)| &\le \frac{K}{2} \limsup_{|z|\to 1^-} \big(|\nabla h - \nabla h_0| + |\nabla h_0|\big) \\ &\le \frac{c_p K}{2}\big(\psi(K) + \|g\|_{L^p(\mathbb{D})}\big) + K. \end{aligned}$$

We now write w in terms of the standard Poisson decomposition $w = u + f$, where u is harmonic with $u_{|\partial\mathbb{D}} = w_{|\partial\mathbb{D}}$ and f has vanishing boundary values and satisfies $\Delta f = \Delta w = g$ in $\mathbb{D}$. The maximum principle applied to the subharmonic function $|\nabla u| = |u_z| + |u_{\bar z}| = |a'| + |b'|$, where a and b are analytic functions such that $u = a + \overline{b}$, together with Lemma 1, shows that $|\nabla w|$ is bounded by $c\|g\|_{L^p(\mathbb{D})}$. Combining these observations with (11), we deduce that

$$\begin{aligned} \sup_{|z|<1} |\nabla w(z)| &\le \limsup_{|z|\to 1^-} |\nabla u| + \sup_{|z|<1} |\nabla f(z)| \\ &\le \limsup_{|z|\to 1^-} |\nabla w| + 2 \sup_{|z|<1} |\nabla f(z)| \\ &\le \frac{c_p K}{2}\big(\psi(K) + \|g\|_{L^p(\mathbb{D})}\big) + K + 2c_p\|g\|_{L^p(\mathbb{D})}. \end{aligned} \tag{11}$$

We may thus choose, for small enough K,

$$\widetilde{C}_p(K,t) = K + \frac{c_p K}{2}\psi(K) + \frac{c_p(K+4)}{2} t;$$

the obtained bound has the desired behavior as $K \to 1$ and $t \to 0$. $\square$

Below, Id stands for the identity matrix. We refer to [21] for the standard definition of Sobolev spaces $W^{2,p}$ and for the Hölder(Zygmund)-classes C^{α} and $C^{1,\alpha}$.

Definition 3.1. Let $p > 2$. We say that the sequence of bounded Jordan domains $\Omega_n \subset \mathbb{C}$, with $0 \in \Omega_n$ for each $n \geq 1$, **converges in a $W^{2,p}$-controlled sense** to the unit disk $\mathbb{D}$ if there exist sense-preserving diffeomorphisms $\Psi_n : \mathbb{D} \to \Omega_n$, normalized by $\Psi_n(0) = 0$, such that

$$\lim_{n\to\infty} \|D\Psi_n - \mathrm{Id}\|_{L^\infty(\mathbb{D})} = 0, \quad \text{and} \quad \|\Psi_n\|_{W^{2,p}(\mathbb{D})} \leq M_0 \quad \text{for all } n \geq 1, \tag{12}$$

where $M_0 < \infty$ and

$$\|\Delta\Psi_n\|_{L^p(\mathbb{D})} \to 0 \quad \text{as } n \to \infty. \tag{13}$$

One should observe that since $\Psi_n \in W^{2,p}(D)$ with $p > 2$ in the above definition, it follows automatically that $\nabla\Psi_n \in C(\overline{\Omega})$. Hence requiring Ψ_n to be a diffeomorphism makes perfect sense in terms; and, in particular, by (12), the map Ψ_n is a bi-Lipschitz for large enough n. Also, each Ω_n is a bounded C^1- Jordan domain in the plane. As the following lemma shows, the above condition is, in a sense, symmetric with respect to the domains $\mathbb{D}$ and Ω.

Lemma 3. *Assume that Ω_n converges to $\mathbb{D}$ in a $W^{2,p}$-controlled sense, and let (Ψ_n) be the associated sequence of diffeomorphisms satisfying the conditions of definition 3.1. Then the inverse maps $\Phi_n := \Psi_n^{-1} : \Omega_n \to \mathbb{D}$ satisfy*

$$\lim_{n\to\infty} \|D\Phi_n - \mathrm{Id}\|_{L^\infty(\Omega_n)} = 0, \quad \textit{and} \quad \|\Phi_n\|_{W^{2,p}(\Omega_n)} \leq M_0' \quad \textit{for all } n \geq 1, \tag{14}$$

and

$$\|\Delta\Phi_n\|_{L^p(\Omega_n)} \to 0 \quad \textit{as } n \to \infty. \tag{15}$$

Proof. Conditions (14) follow easily from the formulas for the derivatives of the implicit function having first been approximated by smooth functions. As regards condition (15), we note that, in general, the inverse of a harmonic diffeomorphism need not to be harmonic, so (15) is not a direct consequence of (13). However, the first condition in (12) tells us that the maximal complex dilatation k_n of Ψ_n tends to 0 as $n \to \infty$, so Ψ_n is asymptotically conformal, and this makes (15) more plausible. Indeed, direct computations show that for C^2-diffeomorphism $\Psi : \mathbb{D} \to \Omega$ with maximal dilatation k and controlled derivative $|D\psi|, |(D\psi)^{-1}| \leq C$, $\Delta\Phi = A \circ \Phi$, where (recall that the Jacobian can be expressed as $J_\Psi = |\Psi_z|^2 - |\Psi_{\overline{z}}|^2$)

$$A = \frac{4}{(J_\Psi)^3}\Big[-\Psi_{\overline{z}}\Big(\overline{\Psi_{z\overline{z}}}J_\Psi - \overline{\Psi_z}\Big(\overline{\Psi_z}\Psi_{zz} + \Psi_z\overline{\Psi_{z\overline{z}}} - \overline{\Psi_{\overline{z}}}\Psi_{z\overline{z}} - \Psi_{\overline{z}}\overline{\Psi_{\overline{z}\overline{z}}}\Big)\Big) \tag{16}$$
$$+ \Psi_z\Big(\overline{\Psi_{zz}}J_\Psi - \overline{\Psi_z}\Big(\overline{\Psi_z}\Psi_{z\overline{z}} + \Psi_z\overline{\Psi_{zz}} - \overline{\Psi_{\overline{z}}}\Psi_{\overline{z}\overline{z}} - \Psi_z\overline{\Psi_{z\overline{z}}}\Big)\Big)\Big].$$

This formula is obtained using, as a starting point, the identity $\Delta\Phi = 4(d/d\overline{z})\Phi_z$ and expressing Φ_z in a standard manner in terms of the derivatives of Ψ. We next recall that Ψ_z is bounded, and $|\Psi_{\overline{z}}| \le k|\Psi_z|$, and then observe that on the right-hand side of (16), the terms that do not directly contain either $\Psi_{z\overline{z}}$ or $\Psi_{\overline{z}}$ as a factor sum to $\overline{\Psi_{zz}}(J_\Psi - |\Psi_z|^2) = -\overline{\Psi_{zz}}|\Psi_{\overline{z}}|^2$. We obtain $|A| \lesssim k|D^2\Psi| + |\Delta\Psi|$, and (15) follows from this applied to Ψ_n. □

We can now generalize Proposition 3.1 to include variable image domains that converge to the unit disk in a $W^{2,p}$-controlled sense.

Theorem 4. *Let $p > 2$, and assume that the planar Jordan domains Ω_n converge to $\mathbb{D}$ in a $W^{2,p}$-controlled sense. Assume, furthermore, that $w_n : \mathbb{D} \to \Omega_n$ is a K_n-quasiconformal normalised map normalised by $w(0) = 0$, such that*

$$\lim_{n\to\infty} K_n = 1 \quad \textit{and} \quad \lim_{n\to\infty} \|\Delta w_n\|_{L^p(\mathbb{D})} = 0.$$

Then, for large enough n, the function w_n is Lipschitz and its Lipschitz constant tends to 1 as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \|\nabla w_n\|_{L^\infty(\mathbb{D})} = 1. \tag{17}$$

Proof. Let $\Psi_n : \mathbb{D} \to \Omega_n$ be the maps as in Definition 3.1. By renumbering, if necessary, we may assume that that $|\Psi_n'(z) - 1| < 1/2$ for all n and $z \in \mathbb{D}$. Write $\Phi_n = \Psi_n^{-1}$ and define $\widetilde{w}_n := \Psi^{-1} \circ w_n = \Phi_n \circ w_n : \mathbb{D} \to \mathbb{D}$; $\widetilde{w}_n$ is $\widetilde{K}_n$-quasiconformal, with $\widetilde{K}_n \to 1$ as $n \to \infty$, by the first condition in (14). Fix an index $q \in (2, p)$. By conditions (12), (14), and Proposition 3.1, to prove (17) we need only verify that

$$\lim_{n\to\infty} \|\Delta\widetilde{w}_n\|_{L^q(\mathbb{D})} = 0. \tag{18}$$

A computation yields

$$\begin{aligned}\Delta\widetilde{w}_n &= (\Delta\Phi_n)(w_n)\Big(|(w_n)_z|^2 + |(w_n)_{\overline{z}}|^2\Big)\\ &\quad + 4\Big((\Phi_n)_{zz}(w)(w_n)_z(w_n)_{\overline{z}} + (\Phi_n)_{\overline{z}\overline{z}}(w_n)\overline{(w_n)_z(w_n)_{\overline{z}}}\Big)\\ &\quad + \Big((\Phi_n)_z(w_n)\Delta w_n + (\Phi_n)_{\overline{z}}(w_n)\overline{\Delta w_n}\Big)\\ &=: S_1 + S_2 + S_3.\end{aligned} \tag{19}$$

Since $|D\Phi_n|$ remains uniformly bounded and $\|\Delta w_n\|_{L^p(\mathbb{D})} \to 0$, we see that $\|S_3\|_{L^p(\mathbb{D})} \to 0$ as $n \to \infty$, hence the same is true for the L^q-norm. Set $\widetilde{q} := \sqrt{qp}$, so that $q < \widetilde{q} < p$. Since $\widetilde{w}_n$ is a normalized K_n-quasiconformal self-map of the unit disk $\mathbb{D}$, and $K_n \to 1$, we may assume (again by discarding small values of n and relabeling, if necessary), by the higher integrability of quasiconformal maps, that $\int_{\mathbb{D}} |\nabla w_n|^{2(\widetilde{q}/q)'} < C$ and $\int_\Omega (J_{w_n^{-1}})^{(p/\widetilde{q})'} dA(z) < C$ for all n. Here, e.g., $(\widetilde{q}/q)'$ stands for the dual exponent. Writing $k_n = (K_n - 1)/(K_n + 1)$, we thus obtain, for any measurable function F on Ω,

$$\begin{aligned}
\int_{\mathbb{D}} \left|F \circ w_n |(w_n)_z|^2\right|^q dA(z) &\le \Big(\int_{\mathbb{D}} \left|F \circ w_n\right|^{\widetilde{q}} dA(z)\Big)^{q/\widetilde{q}} \Big(\int_{\mathbb{D}} |\nabla w_n|^{2(\widetilde{q}/q)'} dA(z)\Big)^{1/(\widetilde{q}/q)'} \\
&\lesssim \Big(\int_{\mathbb{D}} \left|F \circ w_n\right|^{\widetilde{q}} dA(z)\Big)^{q/\widetilde{q}} \le \Big(\int_{\Omega} \left|F\right|^{\widetilde{q}} J_{w_n^{-1}} dA(z)\Big)^{q/\widetilde{q}} \\
&\lesssim \Big(\int_{\Omega} \left|F\right|^{p} dA(z)\Big)^{q/p} \Big(\int_{\Omega} (J_{w_n^{-1}})^{(p/\widetilde{q})'} dA(z)\Big)^{q/(\widetilde{q}(p/\widetilde{q})')} \\
&\le \Big(\int_{\Omega} \left|F\right|^{p} dA(z)\Big)^{q/p}.
\end{aligned}$$

Employing this formula and Lemma 3, we obtain immediately that

$$\|S_1\|_{L^q(\mathbb{D})} \lesssim \|\Delta \Phi_n\|_{L^p(\Omega)} \to 0 \quad \text{as} \quad n \to \infty.$$

In a similar vein,

$$\|S_2\|_{L^q(\mathbb{D})} \lesssim k_n \to 0 \quad \text{as} \quad n \to \infty. \qquad \square$$

We next examine what kind of convergence of the boundaries $\partial\Omega_n \to \partial\mathbb{D}$ implies $W^{2,p}$-controlled convergence of the domains itself. First of all, given $\psi_n : \mathbb{D} \to \Omega$ as in Definition 3.1 we have $\Psi_n \in W^{2,p}(\mathbb{D})$, so by the standard trace theorem for Sobolev spaces, the induced map $\Psi_{n|\partial\mathbb{D}}$ on the boundary satisfies $\Psi_{n|\partial\mathbb{D}} \in B^{2-1/p}_{p,p}(\mathbb{D})$. On the other hand, for $p > 2$, we may pick $\alpha, \alpha' \in (1/2, 1)$ such that

$$C^{1,\alpha'}(\partial\mathbb{D}) \subset B^{2-1/p}_{p,p}(\mathbb{D}) \subset C^{1,\alpha}(\partial\mathbb{D});$$

see [23]. Hence about the best one can hope is to have a theorem where the boundary converges in $C^{1,\alpha}$ for some $\alpha > 1/2$. In fact, this can be realized.

Theorem 5. *Let (Ω_n) be a sequence of bounded Jordan domains in $\mathbb{C}$ such that there is the parametrisation $\partial\Omega_n = \{f_n(\theta) : \theta \in (0, 2\pi)\}$ for each n, where f_n satisfies, for some $\alpha > 1/2$,*

$$\|f_n(\theta) - e^{i\theta}\|_{L^\infty(\mathbf{T})} \to 0 \quad \textit{as} \quad n \to \infty \quad \textit{and} \quad \sup_{n\ge1} \|f_n(\theta)\|_{C^{1,\alpha}(\mathbb{T})} < \infty. \tag{20}$$

Then the sequence (Ω_n) *converges to* $\mathbb{D}$ *in a* $W^{2,p}$*-controlled manner. In particular, the conclusion of Theorem 4 holds for the sequence* (Ω_n)*.*

Proof. Let us first observe that instead of imposing (20), we can fix $\alpha' \in (1/2, \alpha)$ and assume that $\|f_n(\theta) - e^{i\theta}\|_{C^{1,\alpha'}} \to 0$ as $n \to \infty$; this follows from an interpolation on (20). Write $g_n(\theta) = f_n(\theta) - e^{i\theta}$. Relabeling, if necessary, we may assume that $\|g_n\|_{C^{1,\alpha}(\mathbb{T})} \leq 1/10$, say, for all $n \geq 1$. Since $\mathrm{Id} : \mathbb{T} \to \mathbb{C}$ is 1-bi-Lipschitz and the Lipschitz norm of g_n is small, we obtain that $f_n : \mathbb{T} \to \partial\Omega_n$ is a diffeomorphism. We simply define Ψ_n to be the harmonic extension

$$\begin{aligned}\Psi_n(z) &= \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{it}) f_n(e^{it})dt = z + \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{it}) g_n(e^{it})dt\\ &= z + G_n(z), \quad z \in \mathbb{D}.\end{aligned}$$

Since $\|g_n'\|_\infty \to 0$ and $\|Hg_n'\|_\infty \to 0$ (recall that the Hilbert transform H is continuous in $C^\alpha(\mathbb{T})$), we may also assume that $|DG_n(z)| \leq 1/2$ for all n, and we have $\lim_{n\to\infty} \|DG_n\|_{L^\infty(D)} = 0$. In particular, $\Psi_n : \overline{\mathbb{D}} \to \overline{\Omega_n}$ is C^1 and bi-Lipschitz, hence a diffeomorphism. The first condition in (12) follows immediately, and condition (13) is immediate since Ψ_n is harmonic. It remains to verify the second condition in (12). To that end, we observe that by [21], the fact that $\|g_n\|_{C^{1,\alpha}(\mathbf{T})} \leq C$ for all n implies that the Poisson extension satisfies

$$\|D^2 G_n(z)\| \leq \frac{C'}{(1 - |z|)^{1-\alpha}}.$$

This obviously yields the desired uniform bound for $\|D^2 G_n\|_{L^p(\mathbb{D})}$; we take $p < (1 - \alpha)^{-1}$. □

Another condition is obtained by specializing to Riemann maps – the proof of the the preceding theorem can also be based on certain results of Smirnov concerning the regularity of conformal extensions and the following lemma.

Lemma 4. *Let* $p > 2$*. The sequence* $\Omega_n \subset \mathbb{C}$ *of bounded Jordan domains converges in a* $W^{1,p}$*-controlled sense to the unit disk* $\mathbb{D}$ *if the Riemann maps* $F_n : \mathbb{D} \to \Omega_n$ *(normalized by* $F_n(0) = 0$ *and* $\arg F_n'(0) > 0$*) satisfy*

$$\lim_{n\to\infty} \|F_n' - 1\|_{L^\infty(\mathbb{D})} = 0, \quad \textit{and} \quad \|F_n''\|_{L^p(D)} \leq M_0 \quad \textit{for all } n \geq 1, \tag{21}$$

with some $M_0 < \infty$*.*

This is obvious from the definition of controlled convergence.

Remark 3.1. It is an open question whether the conclusion of condition Theorem 2 holds with the condition $\alpha > 1/2$ weakened.

4 Proof of Theorem 3 and Corollary 1 : A Smirnov theorem for qc-maps

Proof of Theorem 3. We first assume that f is as in the theorem, so that $\Delta f(z) \leq (1-|z|)^{-a}$ with $a \in (0,1)$. Then we are to show that the boundary map induced by w is absolutely continuous. To that end, we need two simple lemmas.

Lemma 5. *Assume that $u \in C(\overline{D})$ is a harmonic mapping of the unit disk into $\mathbf{C}$ such that $U := u_{|\mathbf{T}}$ is a homeomorphism and $U(\mathbf{T}) = \Gamma$ is a rectifiable Jordan curve. Then $|\Gamma_r| := \int_{\mathbf{T}} |\partial_\theta u(re^{i\theta})|d\theta$ is increasing in r, so $|\Gamma_r| \leq |\Gamma|$. In particular, the angular derivative of u satisfies $\partial_\theta u(z) \in h^1$.*

Proof. Differentiating the Fourier-series representation

$$u(re^{i\theta}) = \sum_{n=-\infty}^{\infty} \widehat{g}_n r^{|n|} e^{in\theta},$$

we see immediately that $\partial_\theta u(z)$ is the harmonic extension to U of the distributional derivative $\partial_\theta g$. By assumption, g is of bounded variation; hence $\partial_\theta g$ is a finite (signed) Radon measure, which implies that $\partial_\theta u \in h^1$. It is well known (see [20, 11.17]) that since $\partial_\theta u \in h^1$, the integral average $\int_{\mathbf{T}} |\partial_\theta u(re^{i\theta})|d\theta$ is increasing in r. □

Lemma 6. *Let $g \in L^p(\ \ )$ with $p > 1$. Then there is a unique solution v of the Poisson equation $\triangle v = g$ satisfying $v \in C(\overline{U})$ and $v_{|\mathbf{T}} = 0$. Moreover, the weak derivative Dv can be modified on a set of measure zero so that*

$$\int_0^{2\pi} |Du(re^{i\theta})|d\theta \leq C(g) < \infty \quad \textit{for } r \in (1/2, 1).$$

Proof. The classical regularity theory for elliptic equations (see [1],[8]) yields a quick approach, as it guarantees that the Poisson equation has a unique solution v in the Sobolev space $W^{2,p}(U)$ (which, of course, is given by the Green potential; see (2)), and we have continuity up to the boundary. The derivatives $\partial_z, \partial_{\overline{z}}$ are in $W^{1,p}(U)$. We then have $\|Dv\|_{W^{1,p}}(B(0,r)) \leq C'$ for all $r \in (1/2, 1)$. At this stage, the trace theorem (see, e.g., [23]) for the space $W^{1,p}(U)$ and a simple scaling argument show that, for a suitable representative of Dv,

$$\|(Dv)_r\|_{B^{1-1/p}_{p,p}(\mathbb{D})} \leq C' \quad \text{for } \ r \in (1/2, 1).$$

Here, $(Dv)_r$ stands for the function $\mathbf{T} \ni \theta \mapsto v(re^{i\theta})$. The claim follows from the continuous embeddings $B^{1-1/p}_{p,p}(\mathbb{D}) \subset L^p(\mathbb{D}) \subset L^1(\mathbb{D})$. □

Recall also that any analytic (or anti-analytic) function in h^1 can be represented as the Poisson integral of an L^1-function; see [20, Theorem 17.11] or [9]. To proceed towards the absolute continuity of boundary values of f, we write $f = a + \overline{b} + v$, where v solves $\Delta v = g := \Delta f$ with $v_{|\mathbf{T}} = 0$ and a and b are analytic in the unit disk. Since $u := a + \overline{b} = \mathcal{P}[f_{|\mathbf{T}}]$, where $f_{|\mathbf{T}}$ is a homeomorphism, it follows from Lemma 5 that $\partial_\theta u = i(za' - \overline{zb'}) \in h^1(\quad)$, because $f(\mathbf{T})$ is a rectifiable curve. Furthermore, the weak derivatives satisfy $f_z = a' + v_z$ and $f_{\bar{z}} = \overline{b'} + v_{\bar{z}}$. Now we use the fact that $|f_{\bar{z}}| \le k|f_z|$, $k = (K-1)/(K+1)$, which implies that $|a' + v_z| \le k|b' + \overline{v_{\bar{z}}}|$. Since

$$b' = \frac{\bar{z}}{z}\overline{a'} - \frac{i}{z}\overline{u_\theta},$$

we obtain, for $z \neq 0$,

$$|a'| \le k\left|\frac{\bar{z}}{z}\overline{a}' - \frac{i}{z}\overline{u_\theta} + \overline{v_{\bar{z}}}\right| + |v_z|.$$

This yields for $|z| \ge 1/2$ the inequality, valid almost everywhere,

$$|a'| \le \frac{1}{1-k}(2|\overline{u_\theta}| + |\overline{v_{\bar{z}}}| + |v_z|).$$

Our assumption on the size of the Laplacian of f yields $\Delta f \in L^p(\mathbb{D})$ for some $p > 1$. Combining this with above inequality and noting that $\overline{u_\theta} \in h^1$ by Lemma 5, we infer (using a simple argument based on Fubini's theorem–the above inequality holds only for a.e. z) that $a' \in H^1$. Then the relation $b' = \frac{\bar{z}}{z}\overline{a'} - \frac{i}{z}\overline{u_\theta}$ verifies that also $b \in H^1$. Thus $\partial_\theta u$ is the Poisson integral of an L^1 function, and we conclude that $f_{|\mathbf{T}} = u_{|\mathbf{T}}$ is absolutely continuous.

To prove the optimality of Theorem 3, we construct quasiconformal maps with non-absolutely continuous boundary values but with not too large Laplacian. To that end, it is easier to work in the upper half space $\mathbb{C}^+ := \{z : \operatorname{Im} z > 0\}$. We construct the desired functions with the help of Zygmund measures.

Recall first that a bounded and continuous function $g : \mathbb{R} \to \mathbb{R}$ is **Zygmund** if

$$\bigl|g(x+t) + g(x-t) - 2g(x)\bigr| \le C|t| \quad \text{for all } x, t \in \mathbb{R}.$$

The **Zygmund norm** of g is the smallest possible C for which this inequality holds. If g is increasing, its derivative g' is a positive finite Borel measure on $\mathbb{R}$, i.e., $g' = \mu$. If, in addition, μ is singular, we call g a **singular Zygmund function**. It is well-known (see [19] or [10]) that there exist singular Zygmund measures. We refer the reader to the interesting article [3] for further information on this type of measure.

We next recall a modified version of the Beurling-Ahlfors extension of Fefferman, Kenig, and Pipher [7]. We denote by $\psi(x) := (2\pi)^{-1/2}e^{-x^2/2}$ the Gaussian density and notice that $-\psi'(x) = -x\psi(x)$. As is usual, for $t > 0$, we define the dilation $\psi_t(x) := t^{-1}\psi(x/t)$; ψ'_t is defined analogously. Then the extension u of an (at most polynomially) increasing homeomorphism $g : \mathbb{R} \to \mathbb{R}$ is defined by setting

$$u(x+it) := (\psi_t * g)(x) + i(-\psi'_t * g)(x), \quad \text{for all } x+it \in \mathbb{C}^+. \tag{22}$$

Obviously, u is smooth in $\mathbb{C}^+$ and has the correct boundary values.

Lemma 7 ([7, Lemma 4.4.]). *If $g : \mathbb{R} \to \mathbb{R}$ is quasisymmetric, the extension u defined by* (22) *defines a quasiconformal homeomorphism of $\mathbb{C}^+$ whose boundary map coincides with g.*

We need one more auxiliary result.

Lemma 8. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is Zygmund. Then the extension* (22) *of g satisfies*

$$|\Delta u(x+it)| \le Ct^{-1} \quad \text{and} \quad |\nabla u(x+it)| \le C \max\left(1, \log(t^{-1})\right),$$

for all $x \in \mathbb{R}$ and $t > 0$, where $C > 0$ is a constant.

Proof. Let us first observe that if g is Zygmund, then for all $\varphi \in W^{2,1}(\mathbb{R})$ (i.e., $\varphi, \varphi'' \in L^1(\mathbb{R})$),

$$\left\| \frac{d^2}{dx^2}\varphi_t * g \right\|_{L^\infty(\mathbb{R})} = O(t^{-1}) \quad \text{for all } t > 0. \tag{23}$$

In the case φ is even, this follows easily from the definition of Zygmund functions; but, for general φ, we use the fact that g can be decomposed as the sum $g = \sum_{j=0}^\infty g_j$, where $\|g_j\|_{L^\infty(\mathbb{R})} = O(2^{-j})$ and $\|g''_j\|_{L^\infty(\mathbb{R})} = O(2^j)$ for all $j \ge 0$,; see [21, Corollary 1, p. 256]. We can compute in two ways:

$$\frac{d^2}{dx^2}(\varphi_t * g(x)) = \int_{-\infty}^\infty \varphi_t(x-y)g''(y)dy = t^{-2}\int_{-\infty}^\infty \varphi''_t(x-y)g(y)dy.$$

Assuming first that $t \le 1$ with $t \sim 2^{-k}$, we apply the first formula above to the sum $g = \sum_{j=0}^k g_j$ and the second formula to the remainder $g = \sum_{j=k+1}^\infty g_j$. Noting that $\int_{-\infty}^\infty |\varphi_t(y)|dy = O(1)$ and $\int_{-\infty}^\infty |\varphi''_t(y)|dy = O(1)$, we obtain

$$\left|\frac{d^2}{dx^2}(\varphi_t * g(x))\right| = O\Big(\sum_{j=1}^k 2^j + t^{-2}\sum_{j=k+1}^\infty 2^{-j}\Big) = O(t^{-1}),$$

which proves (23) for $t \in (0,1]$. For $t > 1$, we simply apply the second formula directly to the bound $\|g\|_{L^\infty(\mathbb{R})} < \infty$ and obtain

$$\left\|\frac{d^2}{dx^2}(\varphi_t * g)\right\|_{L^\infty(\mathbb{R})} \le O(t^{-2}) = O(t^{-1})$$

for $t > 1$.

We then consider the Laplacian of the extension u of g. Since $\psi, \psi' \in W^{2,1}(\mathbb{R})$, we obtain immediately from (23) that $|\frac{d^2}{dx^2}u(x+it)| = O(t^{-1})$ uniformly in $x \in \mathbb{R}$. In turn, to consider differentiation with respect to t, we assume that $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and that $(1+|t|^2)\phi(t)$ is integrable. Then

$$\begin{aligned}\frac{d}{dt}\varphi_t * g(x) &= \int_{-\infty}^{\infty} \Big(- t^{-2}\varphi_t(x-y) - t^{-3}(x-y)\varphi_t'(x-y)\Big) g(y)dy \\ &= \int_{-\infty}^{\infty} g(y)\frac{d}{dy}\Big(t^{-2}(x-y)\varphi_t(x-y))\Big)dy \\ &= -t^{-1}\int_{-\infty}^{\infty} \frac{(x-y)}{t}\varphi\Big(\frac{x-y}{t}\Big)g'(y)dy \\ &= (\varphi_1)_t * g'(x),\end{aligned}$$

where $\varphi_1(y) := -y\varphi(y)$. An iteration gives (where $\varphi_2(y) := y^2\varphi(y)$)

$$\frac{d^2}{dt^2}\big(\varphi_t * g(x)\big) = (\varphi_2)_t * g''(x) = \frac{d^2}{dx^2}\big((\varphi_2)_t * g(x)\big). \tag{24}$$

Since all the functions $t\psi(t), t^2\psi(t), t\psi(t), t^2\psi(t)$ and their second derivatives are integral, we may apply (24) and obtain as before the desired estimate for $\frac{d^2}{dt^2}u(x+it)$.

The stated estimate for ∇u is proved in a similar way. We use the fact that in the decomposition $g = \sum_{j=0}^{\infty} g_j$, one may demand that also $\|g_j'\|_\infty \le C$ for all $j \ge 1$ (see [22, Formula (53), p. 254]), which yields, as before,

$$\left|\frac{d}{dx}(\varphi_t * g(x))\right| = O\Big(\sum_{j=1}^{k} 1 + t^{-1}\cdot \sum_{j=k+1}^{\infty} 2^{-j}\Big) = O\big(\log(t^{-1})\big)$$

for $t \sim 2^{-k} < 1$. The case $t \ge 1$ is trivial, and the case of the t-derivative is reduced to estimating the x-derivative as before. □

Given these preparations it is now a simple matter to produce the desired example. Let g_0 be a singular Zygmund function which is constant outside $[-1, 1]$. Set $g(x) = x + g_0(x)$ for $x \in \mathbb{R}$. Since g_0 is Zygmund, g is quasi symmetric. Its Fefferman-Kenig-Pipher extension $u : \mathbb{C}^+ \to \mathbb{C}^+$ is quasiconformal with non-absolutely continuous boundary values over $[-1, 1]$. Since the extension of the linear function $x \mapsto x$ is linear, we see that the Laplacian of u equals that of the extension of g_0; and, by the Lemma 8, we obtain the estimate

$$|\Delta u(x+it)| \le Ct^{-1} \quad \text{for all } x+it \in \mathbb{C}^+.$$

Next, let $h : \mathbb{D} \to \Omega'$ be conformal, where Ω' is a bounded and smooth Jordan domain that is contained in the upper half space $\mathbb{C}^+$ and contains $[-2, 2]$ as a boundary segment. Set $\Omega = u(\Omega')$, so that Ω is smooth, by construction. Finally, pick a conformal map $\widetilde{h} : \Omega \to \mathbb{D}$ and define $f := u \circ h$. Function f satisfies all the requirements: the main terms in the formula for the Laplacian of f (cf. (19)) are $|\Delta u|$ and $|\nabla u|^2$, and the Lemma 8 a yields suitable bounds for the gradient term.□

Proof of Corollary 1. Obviously, the example for optimality constructed above works also for the corollary. In a similar vein, the proof of the positive direction of Theorem 3 also applies as such for the corollary since in the proof we used as a starting point the fact that $\Delta u \in L^p(\mathbb{D})$ for some $p > 1$. □

Remark 4.1. There exists singular Zygmund functions g on the real line such that $g(x+t) + g(x-t) - 2g(x) = o(t)$ with quantitative little o on the right-hand side – the derivatives of such functions are sometimes called Kahane measures. A possible decay of the right-hand side is $o(t \log^{-1/2}(1/t))$ for small t, but one cannot decrease the power of log here. Use of this kind of measure in our construction gives examples of functions with Laplacian growth $o(t^{-1})$, where the little o can be made explicit.

However, it is an open problem whether the conclusion of Corollary 1 is valid for the exponent $p = 1$, as merely implementing the Kahane measures described above appears not to give enough extra decay for the Laplacian.

References

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for elliptic partial differential equations satisfying general boundary conditions. I,* Comm. Pure Appl. Math. **12** (1959), 623–727.

[2] L. Ahlfors, *Lectures on Quasiconformal Mappings*, D. Van Nostrand, Princeton, 1966.

[3] A. B. Aleksandrov, J. M. Anderson, and A Nicolau, *Inner functions, Bloch spaces and symmetric measures*, Proc. London Math. Soc. **79** (1999), 318–352.

[4] K. Astala, T. Iwaniec, and G. J. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton University Press, Princeton, 2009.

[5] K. Astala and V. Manojlović,*On Pavlović theorem in space*, Potential Analysis **43** (2015), 361–370.

[6] K. Astala, T. Iwaniec, I. Prause, and E. Saksman, *Bilipschitz and quasiconformal rotation, stretching and multifractal spectra*, Publ. Math. Inst. Hautes Études Sci. **121** (2015), 113–154.

[7] R. A. Fefferman, C. E. Kenig and J. Pipher, *The theory of weights and the Dirichlet problem for elliptic equations*, Ann. of Math. (2) **134** (1991), 65–124.

[8] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd edition, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[9] G. L. Goluzin, *Geometric Theory of Functions of a Complex Variable*, American Mathematical Society, Providence, RI, 1969.

[10] J. P. Kahane, *Trois notes sur les ensembles parfait linéares*, Enseign. Math. **15** (1969), 185–192.

[11] D. Kalaj, *Harmonic mappings and distance function*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **10** (2011), 669–681.

[12] D. Kalaj, *On boundary correspondences under quasiconformal harmonic mappings between smooth Jordan domains*, Math. Nachr. **285** (2012), 283–294.

[13] D. Kalaj, *A priori estimate of gradient of a solution to certain differential inequality and quasiregular mappings*, J. Anal. Math. **119** (2013), 63–88.

[14] D. Kalaj, M. Marković, and M. Mateljević, *Carathéodory and Smirnov type theorems for harmonic mappings of the unit disk onto surfaces*, Ann. Acad. Sci. Fenn. Math. **38** (2013), 565–580.

[15] D. Kalaj and M. Pavlović, *On quasiconformal self-mappings of the unit disk satisfying the Poisson equation*, Trans. Amer. Math. Soc. **363** (2011) 4043–4061.

[16] C. Pommerenke, *Boundary Behaviour of Conformal Maps*, , Springer-Verlag, Berlin-Heidelberg, 1992.

[17] D. Partyka and K. Sakan, *On bi-Lipschitz type inequalities for quasiconformal harmonic mappings*, Ann. Acad. Sci. Fenn. Math. **32** (2007), 579–594.

[18] M. Pavlović, *Boundary correspondence under harmonic quasiconformal homeomorphisms of the unit disc*, Ann. Acad. Sci. Fenn. Math. **27** (2002) 365–372.

[19] G. Piranian, *Two monotonic, singular, uniformly almost smooth functions*, Duke Math. J. **33** (1966), 255–262.

[20] W. Rudin, *Real and Complex Analysis*, third edition, McGraw-Hill, New York, 1987.

[21] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

[22] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.

[23] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, 2. Auflage. Barth, Heidelberg, 1995.

David Kalaj
UNIVERSITY OF MONTENEGRO
FACULTY OF NATURAL SCIENCES AND MATHEMATICS
CETINJSKI PUT B.B. 81000 PODGORICA, MONTENEGRO
email: davidkalaj@gmail.com

Eero Saksman
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF HELSINKI
PO BOX 68, FI-00014 HELSINKI, FINLAND
email: eero.saksman@helsinki.fi

(Received June 4, 2015 and in revised form September 11, 2015)