SPECTRAL MULTIPLIERS FOR SUB-LAPLACIANS ON SOLVABLE EXTENSIONS OF STRATIFIED GROUPS

By

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Abstract. Let $G = N \rtimes A$, where N is a stratified group and $A = \mathbb{R}$ acts on N via automorphic dilations. Homogeneous sub-Laplacians on N and A can be lifted to left-invariant operators on G, and their sum is a sub-Laplacian Δ on G. We prove a theorem of Mihlin–Hörmander type for spectral multipliers of Δ. The proof of the theorem hinges on a Calderón–Zygmund theory adapted to a sub-Riemannian structure of G and on L^1-estimates of the gradient of the heat kernel associated to the sub-Laplacian Δ.

1 Introduction

Let N be a stratified Lie group of homogeneous dimension $Q \geq 2$. Let G be the semidirect product $N \rtimes A$, where $A = \mathbb{R}$ acts on N via automorphic dilations. The group G is a solvable extension of N that is not unimodular and has exponential volume growth; see Section 2 for more details. For each $p \in [1, \infty]$, let $L^p(G)$ denote the L^p space with respect to a right Haar measure μ on G.

Consider a system $\check{X}_1, \dots, \check{X}_q$ of left-invariant vector fields on N that form a basis of the first layer of the Lie algebra of N, and let $\check{X}_0$ be the standard basis of the Lie algebra of A. The vector fields $\check{X}_0$ on A and $\check{X}_1, \dots, \check{X}_q$ on N can be lifted to left-invariant vector fields $X_0, X_1, \dots, X_q$ on G which generate the Lie algebra of G and define a sub-Riemannian structure on G with associated left-invariant Carnot–Carathéodory distance ϱ.

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Let Δ be the left-invariant sub-Laplacian on G defined by

$$\Delta = -\sum_{j=0}^{q} X_j^2. \tag{1.1}$$

The operator Δ extends uniquely to a positive self-adjoint operator on $L^2(G)$. For all bounded Borel functions $F : [0,\infty) \to \mathbb{C}$, the operator $F(\Delta)$ defined via the spectral theorem is left-invariant and bounded on $L^2(G)$ and, by the Schwartz kernel theorem,

$$F(\Delta)f = f * k_{F(\Delta)} \quad \text{for all } f \in L^2(G), \tag{1.2}$$

for some convolution kernel $k_{F(\Delta)}$, which in general is a distribution on G. The object of this paper is the multiplier problem for Δ, i.e., the study of conditions on F sufficient to imply the L^p-boundedness of $F(\Delta)$ for some $p \neq 2$.

Our main result provides a sufficient condition of Mihlin–Hörmander type for operators of the form $F(\Delta)$ to be bounded on $L^p(G)$ for $1 < p < \infty$; endpoint results are also obtained, both of weak type $(1,1)$ and in terms of the Hardy space $H^1(G)$ and bounded mean oscillation space $BMO(G)$ introduced in [51]; see Section 3.

Let ψ be a function in $C_c^\infty(\mathbb{R})$, supported in $[1/4, 4]$, such that

$$\sum_{j\in\mathbb{Z}} \psi(2^j \lambda) = 1 \quad \text{for all } \lambda \in (0,\infty). \tag{1.3}$$

For each $s \geq 0$, we define $\|F\|_{0,s}$ and $\|F\|_{\infty,s}$ as follows:

$$\|F\|_{0,s} = \sup_{t<1} \|F(t\cdot)\psi(\cdot)\|_{H^s(\mathbb{R})}, \qquad \|F\|_{\infty,s} = \sup_{t\geq 1} \|F(t\cdot)\psi(\cdot)\|_{H^s(\mathbb{R})},$$

where $H^s(\mathbb{R})$ denotes the L^2-Sobolev space of order s on $\mathbb{R}$. We say that a bounded Borel function $F : [0,\infty) \to \mathbb{C}$ satisfies a **mixed Mihlin–Hörmander condition of order** (s_0, s_∞) if $\|F\|_{0,s_0} < \infty$ and $\|F\|_{\infty,s_\infty} < \infty$.

Theorem 1.1. *Suppose that* $s_0 > 3/2$ *and* $s_\infty > (Q+1)/2$. *If* F *satisfies a mixed Mihlin–Hörmander condition of order* (s_0, s_∞), *then* $F(\Delta)$ *extends to an operator of weak type* $(1,1)$ *and bounded on* $L^p(G)$ *for all* $p \in (1,\infty)$, *bounded from* $H^1(G)$ *to* $L^1(G)$ *and from* $L^\infty(G)$ *to* $BMO(G)$.

Spectral multiplier theorems for Laplacians and sub-Laplacians have been obtained in many different contexts, so we do not attempt to give a complete account of the existing literature but instead restrict our discussion to the works that are more closely related to our result. The interested reader is referred to the cited works and references therein for more details.

It is already known in the literature that, unlike other sub-Laplacians on solvable groups (see, e.g., [8, 27]), the sub-Laplacian Δ on the group G has L^p-differentiable functional calculus. More precisely, Hebisch [25] proved that if F is compactly supported and $F \in H^s(\mathbb{R})$ for some $s > (Q + 5)/2$, then $F(\Delta)$ is bounded on $L^p(G)$ for all $p \in [1, \infty]$. Mustapha [43] proved the same result pushing down the smoothness condition on the multiplier F, i.e., requiring that $F \in H^s(\mathbb{R})$ for some $s > 2$. A further improvement with condition $s > 3/2$ is stated in [29, Theorem 6.1]. Subsequently, Gnewuch [18] obtained similar results for sub-Laplacians on compact extensions of a class of solvable groups, which strictly include the groups we consider here.

All these results differ from Theorem 1.1 because they treat only the case of compactly supported multipliers F belonging to a Sobolev space of suitable order and show that, in that case, the convolution kernel $k_{F(\Delta)}$ is integrable on G. Our result instead is a genuine Mihlin–Hörmander theorem for multipliers F which need not be compactly supported nor have bounded derivatives at 0. In this case, the convolution kernels $k_{F(\Delta)}$ need not be integrable; indeed, for the endpoint values $p = 1$ and $p = \infty$ we prove boundedness only in the weak type (1, 1) sense and in terms of Hardy and BMO spaces.

Other multiplier theorems on solvable extensions of stratified groups were previously obtained for distinguished full Laplacians. More precisely, Cowling, Giulini, Hulanicki and Mauceri [11] proved a multiplier theorem for a distinguished Laplacian L on NA groups coming from the Iwasawa decomposition of a semisimple Lie group of arbitrary rank: they showed that if $F \in H^{s_0}_{\mathrm{loc}}(\mathbb{R})$ and $\|F\|_{\infty,s_\infty} < \infty$ for suitable orders s_0, s_∞ depending on the topological dimension and the pseudodimension of the group, then $F(L)$ is of weak type (1, 1) and bounded on L^p for all $p \in (1, \infty)$. An analogous result was then proved by Astengo [2] for a distinguished Laplacian on Damek–Ricci spaces, i.e., groups of the form $H \rtimes \mathbb{R}$, where H is a Heisenberg-type group [15].

Hebisch and Steger [29, Theorem 2.4] improved the results in [11] by proving a genuine Mihlin–Hörmander theorem for spectral multipliers of a distinguished Laplacian L on the group $\mathbb{R}^Q \rtimes \mathbb{R}$, which corresponds to the case of real hyperbolic spaces (and coincides with our Theorem 1.1 in the case N is abelian). Their theorem was generalized in [52] to a distinguished Laplacian on Damek–Ricci spaces. The results in [29, 52] hinge on a new abstract Calderón–Zygmund theory developed by Hebisch and Steger and L^1-estimates of the gradient of the heat kernel associated to L.

All the aforementioned results for multipliers of a full Laplacian L make strong use of spherical analysis either on semisimple Lie groups or Damek–Ricci spaces.

In particular, on Damek–Ricci spaces, the convolution kernels $k_{F(L)}$ have the property that $m^{-1/2}k_{F(L)}$ is radial, where m is the modular function and, moreover, an explicit formula for the heat kernel associated to L is known. These tools are not available for the analysis of the sub-Laplacian Δ on G (unless N is abelian). So we need new techniques to obtain weighted estimates of the convolution kernels of multipliers of Δ and to study the horizontal gradient of the heat kernel associated to Δ. A brief illustration of these techniques and of our strategy of proof follows.

In Section 2, we obtain a precise description of the left-invariant Carnot–Carathéodory distance on G in terms of the analogous distance on N. This is done by relating solutions to the Hamilton–Jacobi equations on G and N. These equations are analogous to the geodesic equations on Riemannian manifolds. However on sub-Riemannian manifolds there may exist “strictly abnormal minimizers”, i.e., length-minimizing curves that do not correspond to solutions of the Hamilton–Jacobi equations. Nevertheless, a density result by Agrachev [3] allows us to transfer information from solutions of the Hamilton–Jacobi equations to the corresponding sub-Riemannian distances.

Based on our analysis of distances, in Section 3, we develop a Calderón–Zygmund theory adapted to the sub-Riemannian structure of G. More precisely, we show that the metric measure space (G, ϱ, μ) satisfies the axioms of the abstract Calderón–Zygmund theory introduced in [29] and further developed in [51]. The crucial step is the construction of a suitable family of “admissible sets” which play the role that in the classical Calderón–Zygmund theory on spaces of homogeneous type would be played by balls or “dyadic cubes”; cf. [6]. In this way, when we study spectral multipliers of the sub-Laplacian Δ, we can use the theorems for singular integral operators proved in [29] for the boundedness of type $(1, 1)$ and those contained in [51] for the boundedness on Hardy and BMO spaces.

In Section 4, we focus on the properties of Δ and its functional calculus. In particular, Section 4.2 is devoted to an L^1-estimate of the horizontal gradient of the heat kernel associated to Δ at any real time. This estimate is well known (in much greater generality) for small time, but appears to be new for large time (and nonabelian N). Our proof is based on a formula that relates the sub-Riemannian heat kernels on G and N; this relation was already used in [43, 19] to estimate the heat kernel on G at complex time $1 + i\tau$, $\tau \in \mathbb{R}$.

Another important consequence of the relation between heat kernels on G and N is discussed in Section 4.3. It turns out that, for all multipliers F, the L^2-norm of the convolution kernel $k_{F(\Delta)}$ on G coincides with the L^2-norm of the convolution kernel $k_{F(\tilde{\Delta})}$ on the real hyperbolic space $\tilde{G} = \mathbb{R}^Q \rtimes \mathbb{R}$, where $\tilde{\Delta}$ is a full Laplacian on $\tilde{G}$. In fact, it is even possible to estimate weighted L^2-norms of $k_{F(\Delta)}$ on G by weighted L^2-norms of $k_{F(\tilde{\Delta})}$ on $\tilde{G}$, where spherical analysis can be applied. This crucial observation is already contained, with a different proof, in [24].

Finally, in Section 5, we combine all these ingredients to prove Theorem 1.1.

A natural question is whether the smoothness condition $s_\infty > (Q+1)/2$ on the multiplier in Theorem 1.1 is sharp. In fact, via transplantation (cf. [33]), Theorem 1.1 implies a similar theorem for a homogeneous sub-Laplacian on the nilpotent contraction $N \times A$ of G, with a smoothness condition of order $(Q+1)/2$. This is just a particular case of the multiplier theorem of Christ [7] and Mauceri and Meda [39] on stratified groups, because $Q+1$ is the homogeneous dimension of $N \times A$. If N is abelian, then the transplanted result is sharp and, a fortiori, the condition $s_\infty > (Q+1)/2$ in Theorem 1.1 is sharp. However, for many nonabelian stratified groups N, the transplanted result is not sharp: in fact, in several cases, it is possible to push down the smoothness condition to half the topological dimension of the group [23, 44, 37, 38]. For this reason, it might be expected that the smoothness condition $s_\infty > (Q+1)/2$ in Theorem 1.1 can also be pushed down, at least for some nonabelian N.

Recently, the second and third named authors, extending a result in [28], have proved a multiplier theorem for some Laplacians with drift on Damek–Ricci spaces [46]; part of the proof of their result hinges on a Mihlin–Hörmander type theorem for a distinguished Laplacian without drift. Inspired by [46], we think that Theorem 1.1 could be an ingredient for proving a multiplier theorem for sub-Laplacians with drift on the solvable groups considered here. We recall that among these sub-Laplacians with drift there is the “intrinsic hypoelliptic Laplacian” associated with the sub-Riemannian structure on G; see [4].

Let us fix some notation that is used throughout. $\mathbb{R}^+$ and $\mathbb{R}^+_0$ denote the open and closed positive half-lines,respectively, in $\mathbb{R}$. $\bigcup \mathcal{R}$ denotes the union of a family of sets $\mathcal{R}$, i.e., $\bigcup \mathcal{R} = \bigcup_{R\in\mathcal{R}} R$. The letter C and variants, such as C_s, denote finite positive constants that may vary from place to place. Given two expressions A and B, $A \lesssim B$ means that there exists a finite positive constant C such that $A \leq CB$. Moreover, $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

2 Solvable extensions of stratified groups

In this section, we introduce the class of Lie groups that we study in the sequel and recall their main properties. In particular, we discuss their metric properties in Subsection 2.2 and some useful integral formulas in Subsection 2.3.

2.1 Stratified groups and their extensions. Let N be a **stratified group**. In other words, N is a simply connected Lie group whose Lie algebra $\mathfrak{n}$ is endowed with a derivation D such that the eigenspace of D corresponding to the eigenvalue 1 generates $\mathfrak{n}$ as a Lie algebra. In particular, the eigenvalues of D are positive integers $1, \dots, S$, and $\mathfrak{n}$ is the direct sum of the eigenspaces of D, which are called **layers**: the jth layer corresponds to the eigenvalue j. Moreover, $\mathfrak{n}$ is S-step nilpotent, where S is the maximum eigenvalue.

The exponential map $\exp_N : \mathfrak{n} \to N$ is a diffeomorphism and provides global coordinates for N that are used in the sequel without further mention. Any chosen Lebesgue measure on $\mathfrak{n}$ is then a left and right Haar measure on N. Let us fix such a measure and write $|E|$ for the measure of a measurable subset $E \subset N$.

The formula $\delta_t = \exp((\log t)D)$ defines a family of automorphic dilations $(\delta_t)_{t>0}$ on N. For all measurable sets $E \subset N$ and $t > 0$, $|\delta_t E| = t^Q |E|$, where $Q = \operatorname{tr} D$ is the homogeneous dimension of N. Note that $Q \geq d$, where $d = \dim \mathfrak{n}$ is the topological dimension of N; in fact, $Q = d$ if and only if $S = 1$, i.e., if and only if N is abelian. Note, moreover, that, if $Q = 1$, then $N \cong \mathbb{R}$. In the following, we assume that $Q \geq 2$, since the case $Q = 1$ has already been treated in [29].

Let $A = \mathbb{R}$, considered as an abelian Lie group. Again we identify A with its Lie algebra $\mathfrak{a}$. Then A acts on N by dilations, that is, we have a homomorphism $A \ni u \mapsto \delta_{e^u} \in \operatorname{Aut}(N)$ and we can define the corresponding semidirect product $G = N \rtimes A$, with operations

$$(z, u) \cdot (z', u') = (z \cdot e^{uD} z', u + u'), \quad (z, u)^{-1} = (-e^{-uD} z, -u)$$

and identity element $0_G = (0_N, 0)$. The Lie algebra $\mathfrak{g}$ of G is then canonically identified [54, §3.14-3.15] with the semidirect product of Lie algebras $\mathfrak{n} \rtimes \mathfrak{a}$, where

$$[(z, u), (z', u')] = ([z, z'] + uDz' - u'Dz, 0).$$

The group G is not nilpotent, but is a solvable Lie group of topological dimension $d+1$. The left and right Haar measures μ_ℓ and μ on G are given, respectively, by

$$\mathrm{d}\mu_\ell(z, u) = e^{-Qu}\, \mathrm{d}z\, \mathrm{d}u, \quad \mathrm{d}\mu(z, u) = \mathrm{d}z\, \mathrm{d}u$$

[30, Section (15.29)], and the modular function m is given by $m(z, u) = e^{-Qu}$. In particular, G is not unimodular and has exponential volume growth [21, Lemme I.3]. In the following, unless otherwise specified, the right Haar measure μ is used to define Lebesgue spaces $L^p(G) = L^p(G, \mathrm{d}\mu)$ on G, and $\|f\|_p$ denotes the $L^p(G)$-norm of a function f on G.

2.2 Metric structure and geodesics. Consider a system $\check{X}_1, \dots, \check{X}_q$ of left-invariant vector fields on N that form a basis of the first layer of $\mathfrak{n}$. These vector fields provide a global frame for a subbundle HN of the tangent bundle TN of N, called the **horizontal distribution**. Since N is stratified, the first layer generates $\mathfrak{n}$ as a Lie algebra; consequently, the horizontal distribution is bracket-generating.

Let g^N be the left-invariant sub-Riemannian metric on the horizontal distribution of N which makes $\check{X}_1, \dots, \check{X}_q$ an orthonormal basis. Using the metric g^N, by integrating the g^N-norm of the tangent vector, we can define the length of horizontal curves on N (i.e., absolutely continuous curves $\gamma : [a, b] \to N$ whose tangent vector $\dot{\gamma}(t)$ lies in the horizontal distribution for almost all $t \in [a, b]$). The **Carnot–Carathéodory distance** ϱ^N on N associated to g^N is then defined by

$$\varrho^N(z, z') = \inf\{\text{lengths of horizontal curves joining } z \text{ to } z'\}$$

for all $z, z' \in N$. Since the horizontal distribution is bracket-generating, the distance ϱ^N is finite and induces on N the usual topology, by the Chow–Rashevskii theorem. Moreover, since $\check{X}_1, \dots, \check{X}_q$ are left-invariant and belong to the first layer, the distance ϱ^N is left-invariant and homogeneous with respect to the automorphic dilations δ_t. For each $z_0 \in N$ and $r > 0$, we denote by $B_N(z_0, r)$ the ball in N centered at z_0 of radius r, i.e., $B_N(z_0, r) = \{z \in N : \varrho^N(z, z_0) < r\}$. Then $|B_N(z_0, r)| = r^Q |B_N(0_N, 1)|$ for all $z_0 \in N$, for all $r > 0$.

Let $\check{X}_0 = \partial_u$ be the canonical basis of $\mathfrak{a}$. The vector fields $\check{X}_0$ on A and $\check{X}_1, \dots, \check{X}_q$ on N can be lifted to left-invariant vector fields $X_0|_{(z,u)} = \check{X}_0|_u = \partial_u$, $X_j|_{(z,u)} = e^u \check{X}_j|_z$, for $j = 1, \dots, q$ on G. Analogously, as above, the system $X_0, \dots, X_q$ generates the Lie algebra $\mathfrak{g}$ and defines a sub-Riemannian structure on G with associated left-invariant Carnot–Carathéodory distance ϱ. For each $(z_0, u_0) \in G$ and $r > 0$, we denote by $B_\varrho\big((z_0, u_0), r\big)$ the ball in G centered at (z_0, u_0) with radius r with respect to the distance ϱ.

We give a more precise description of the distance ϱ and precise asymptotics for the volume of balls by means of geodesics. Note that the characterization of length-minimizing curves in sub-Riemannian geometry is more complicated than in the Riemannian case, because a length-minimizing curve need not correspond to a solution of the Hamilton–Jacobi equations associated to the metric (see, e.g., [41] for an insight). However, by means of a density result of Agrachev [3], we are able to characterize the distance ϱ by studying the solutions of the Hamilton–Jacobi equations on N and G.

The sub-Riemannian metric g^N determines a dual metric $(g^N)^*$ on the cotangent bundle T^*N of N. When $S > 1$, $(g^N)^*$ is degenerate: its kernel at each point of

N is the annihilator of the horizontal distribution. If N is identified as a manifold with the vector space $\mathfrak{n}$ via the exponential map (see Section 2.1), then, for all $z \in N$, the tangent space $T_z N$ at z is identified with $\mathfrak{n}$ and the cotangent space $T_z^* N$ is identified with $\mathfrak{n}^*$. Let us in turn identify $\mathfrak{n}^*$ with $\mathfrak{n}$ by choosing an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$, and let us fix orthonormal coordinates on $\mathfrak{n}$. Then

$$(g^N)_z^*(\zeta, \zeta') = \langle M_z \zeta, \zeta' \rangle,$$

where $M_z : \mathfrak{n} \to \mathfrak{n}$ is a symmetric linear map depending smoothly on $z \in N$. Moreover, $H_z N$ is the range of M_z, the restriction $M_z|_{H_z N} : H_z N \to H_z N$ is invertible, and

$$g_z^N(Z, Z') = \langle (M_z|_{H_z N})^{-1} Z, Z' \rangle.$$

In the chosen coordinates, the Hamilton–Jacobi equations associated to g^N read

$$\dot{z}_j = \frac{\partial H^N}{\partial \zeta_j}, \qquad \dot{\zeta}_j = -\frac{\partial H^N}{\partial z_j} \tag{2.1}$$

($j = 1, \dots, d$), where the Hamiltonian $H^N : T^* N \to \mathbb{R}$ is given by

$$H^N(z, \zeta) = \frac{1}{2} (g^N)_z^*(\zeta, \zeta) = \frac{1}{2} \langle M_z \zeta, \zeta \rangle.$$

A solution $(z, \zeta) : I \to T^* N$ of the Hamilton–Jacobi equations (2.1), where $I \subset \mathbb{R}$ is an interval, is called an **HJ-curve on** N. It is known that the projection to N of such a curve, namely, $z : I \to N$, is horizontal and locally length-minimizing. Moreover, z has constant speed, since $g_z^N(\dot{z}, \dot{z}) = 2H^N(z, \zeta)$ is constant along the HJ-curve (z, ζ). We define the **length** of an HJ-curve as the length of its projection. Analogously, we say that an HJ-curve **joins** two points on N if its projection does.

Note that, if $S \leq 2$, all length-minimizing horizontal curves on N are "normal minimizers", i.e., projections of HJ-curves (see, e.g., the argument after [42, Theorem 4]). However, on higher-step groups N, there may exist "strictly abnormal length-minimizers" [20], i.e., length-minimizers that are not projections of HJ-curves.

An analogous discussion can be conducted on G. If G is identified as a manifold with the vector space $\mathfrak{n} \times \mathfrak{a}$ via the map $\mathfrak{n} \times \mathfrak{a} \ni (z, u) \mapsto (\exp_N(z), u) \in G$ (as in Section 2.1), the left-invariant sub-Riemannian metric g on the horizontal distribution of TG is given by

$$g_{(z,u)}((Z, U), (Z', U')) = e^{-2u} g_z^N(Z, Z') + UU'.$$

Hence the dual metric g^* on the cotangent bundle $T^* G$ of G is

$$g_{(z,u)}^*((\zeta, \nu), (\zeta', \nu')) = e^{2u} (g^N)_z^*(\zeta, \zeta') + \nu\nu',$$

and the Hamilton–Jacobi equations on G read

$$\begin{aligned} \dot{z}_j &= \frac{\partial H}{\partial \zeta_j}, & \dot{\zeta}_j &= -\frac{\partial H}{\partial z_j}, \\ \dot{u} &= \frac{\partial H}{\partial \nu}, & \dot{\nu} &= -\frac{\partial H}{\partial u} \end{aligned} \tag{2.2}$$

($j = 1, \dots, d$), where the Hamiltonian $H : T^*G \to \mathbb{R}$ is given by

$$H(z, u, \zeta, \nu) = \frac{1}{2} g^*_{(z,u)}((\zeta, \nu), (\zeta, \nu)) = \frac{1}{2}\left(e^{2u}\langle M_z \zeta, \zeta\rangle + \nu^2\right).$$

A solution $(z, u, \zeta, \nu) : I \to T^*G$ of (2.2) is called an **HJ-curve on** G.

We now look for HJ-curves on G of the form

$$(z, u, \zeta, \nu) = (z^N \circ \upsilon, u, \zeta^N \circ \upsilon, \nu),$$

where (z^N, ζ^N) is an HJ-curve on N and υ is a suitable change of variables. Plugging these expressions in the Hamilton–Jacobi equations for G and using the fact that (z^N, ζ^N) satisfies the Hamilton–Jacobi equations for N, we obtain the following result.

Lemma 2.1. *Let (z^N, ζ^N) be an HJ-curve on N. Then $(z^N \circ \upsilon, u, \zeta^N \circ \upsilon, \nu)$ is an HJ-curve on G, provided the functions υ, u, ν satisfy*

$$\dot{\upsilon} = e^{2u}, \quad \dot{u} = \nu, \quad \dot{\nu} = -2H_0^N e^{2u}, \tag{2.3}$$

where H_0^N is the constant value of H^N along (z^N, ζ^N). Moreover, H_0^N is related to the constant value H_0 of H along $(z^N \circ \upsilon, u, \zeta^N \circ \upsilon, \nu)$ by

$$H_0 = e^{2u} H_0^N + \nu^2/2.$$

This leads us to the following definition.

Definition 2.2. We say that an HJ-curve $(z^N, \zeta^N) : J \to T^*N$ on N and an HJ-curve $(z, u, \zeta, \nu) : I \to T^*G$ on G are **associated** if there exists a diffeomorphism $\upsilon : I \to J$ such that $z = z^N \circ \upsilon$, $\zeta = \zeta^N \circ \upsilon$, and $(\upsilon, u, \nu) : I \to \mathbb{R}^3$ solves (2.3).

The Cauchy problem for the autonomous system of equations (2.3) is solved as follows.

Lemma 2.3. *Suppose that $u_0, \nu_0, H_0^N \in \mathbb{R}$ and $H_0^N \geq 0$. In the case $H_0^N > 0$, the maximal solution (υ, u, ν) to (2.3) with initial data*

$$\upsilon(0) = 0, \quad u(0) = u_0, \quad \nu(0) = \nu_0 \tag{2.4}$$

is given by

$$\begin{aligned} \upsilon(t) &= \frac{1}{2H_0^N}(\omega \tanh(\omega(t-t_*)) + \nu_0), \\ u(t) &= u_* - \log\cosh(\omega(t-t_*)), \\ \nu(t) &= -\omega\tanh(\omega(t-t_*)), \end{aligned}$$

where

$$\omega = \sqrt{\nu_0^2 + 2H_0^N e^{2u_0}}, \quad u_* = \log\frac{\omega}{\sqrt{2H_0^N}}, \quad t_* = \frac{1}{\omega}\operatorname{arctanh}\frac{\nu_0}{\omega}.$$

In the case $H_0^N = 0$*, the solution with initial data* (2.4) *is given by*

$$\upsilon(t) = \begin{cases} e^{2u_0}\frac{e^{2\nu_0 t}-1}{2\nu_0} & \text{if } \nu_0 \neq 0, \\ e^{2u_0} t & \text{if } \nu_0 = 0, \end{cases} \quad u(t) = u_0 + \nu_0 t, \quad \nu(t) = \nu_0.$$

All these solutions (υ, u, ν) *are defined globally in time, and* υ *is an increasing diffeomorphism onto its image. Moreover, for all* $u_1 \in \mathbb{R}$ *and* $T > 0$*, the following conditions are equivalent:*

(i) T *is in the range of* υ *and* $u(\upsilon^{-1}(T)) = u_1$*,*

(ii) $\nu_0 = (2T)^{-1}(e^{2u_1} - e^{2u_0}) + H_0^N T$*.*

Proof. It is not difficult to check that the above formulas give solutions of (2.3) with initial data (2.4). Since these solutions are defined globally in time, they must be the maximal solutions, and υ is an increasing diffeomorphism onto its image because $\dot{\upsilon} = e^{2u} > 0$. It remains to show the equivalence of the conditions (i) and (ii); we consider only the case $H_0^N > 0$, the other case being similar and easier.

Simple manipulations of the above formulas for u and υ yield

$$2u(t) = \log\left(\frac{\omega^2}{2H_0^N}\left(1 - \frac{(2H_0^N\upsilon(t) - \nu_0)^2}{\omega^2}\right)\right),$$

i.e.,

$$e^{2u(t)} = e^{2u_0} + 2\upsilon(t)(\nu_0 - H_0^N\upsilon(t)). \tag{2.5}$$

In particular, if there exists $t \in \mathbb{R}$ with $u(t) = u_1$ and $\upsilon(t) = T$, then, by solving (2.5) for ν_0, we obtain (ii).

Conversely, if (ii) holds, then $2H_0^N T^2 - 2T\nu_0 = e^{2u_0} - e^{2u_1} < e^{2u_0}$; hence

$$(2H_0^N T - \nu_0)^2 < 2H_0^N e^{2u_0} + \nu_0^2 = \omega^2.$$

Because of the explicit formula for υ, this means that T belongs to the range of υ, so $\upsilon(t) = T$ for some $t \in \mathbb{R}$; and (2.5), together with (ii), yields $u(t) = u_1$. □

From the above explicit solution, we derive several consequences. First, we can construct HJ-curves on G starting with HJ-curves on N.

Proposition 2.4. *Suppose that $T > 0$, $(z^N, \zeta^N) : [0, T] \to T^*N$ is an HJ-curve on N, and $u_0, u_1 \in \mathbb{R}$. Then there exists an HJ-curve on G associated to (z^N, ζ^N) that joins $(z^N(0), u_0)$ to $(z^N(T), u_1)$.*

Proof. Set $\nu_0 = (2T)^{-1}(e^{2u_1} - e^{2u_0}) + H_0^N T$. If (υ, u, ν) is the maximal solution of (2.3) with initial data (2.4), then, by Lemma 2.3, T is in the range of υ and $u(\upsilon^{-1}(T)) = u_1$. Therefore, by Lemma 2.1,

$$(z^N \circ \upsilon, u, \zeta^N \circ \upsilon, \nu) : [0, \upsilon^{-1}(T)] \to T^*G$$

is an HJ-curve on G associated to (z^N, ζ^N), which clearly joins $(z^N(0), u_0)$ to $(z^N(T), u_1)$. □

Conversely, HJ-curves on G determine HJ-curves on N.

Proposition 2.5. *Every HJ-curve on G is associated to an HJ-curve on N.*

Proof. Let $(z, u, \zeta, \nu) : I \to T^*G$ be an HJ-curve on G. Without loss of generality, we may assume that $0 \in I$. Let $(z^N, \zeta^N) : J \to \mathbb{R}$ be the maximal solution of the Hamilton–Jacobi equations (2.1) on N with initial data $z^N(0) = z(0)$, $\zeta^N(0) = \zeta(0)$. Let H_0^N be the constant value of H^N along (z^N, ζ^N), and set $u_0 = u(0)$, $\nu_0 = \nu(0)$. Let $(\upsilon, \tilde{u}, \tilde{\nu})$ be the solution of (2.3) with initial data (2.4) given by Lemma 2.3. Then (z, u, ζ, ν) and $(z^N \circ \upsilon, \tilde{u}, \zeta^N \circ \upsilon, \tilde{\nu})$ are both solutions of (2.2) with the same initial condition; in particular (by uniqueness of solutions to ODEs), they coincide on the intersections of their intervals of definition.

To finish the proof, it suffices to show that I is contained in the domain $\tilde{I}$ of $(z^N \circ \upsilon, \tilde{u}, \zeta^N \circ \upsilon, \tilde{\nu})$. Note that the solution $(\upsilon, \tilde{u}, \tilde{\nu})$ of (2.3) given by Lemma 2.3 is defined globally in time, $\upsilon : \mathbb{R} \to \upsilon(\mathbb{R})$ is an increasing diffeomorphism, and $\tilde{I} = \upsilon^{-1}(J)$ is open. Therefore, if I is not contained in $\tilde{I}$, there is a (nonzero) element $t_0 \in I$ of minimum modulus that does not belong to $\tilde{I}$. Assume, without loss of generality, that $t_0 > 0$. Then $\upsilon(t_0)$ does not belong to the domain J of (z^N, ζ^N), but $[0, \upsilon(t_0)) \subset J$. The equation $(z^N(\upsilon(t)), \zeta^N(\upsilon(t))) = (z(t), \zeta(t))$, which is valid for all $t \in [0, t_0)$, and the fact that υ is a diffeomorphism show that

$$\lim_{\tau \to \upsilon(t_0)} (z^N(\tau), \zeta^N(\tau)) = (z(t_0), \zeta(t_0)).$$

This contradicts the fact that (z^N, ζ^N) is a maximal solution to (2.1). □

Finally, there is a relation between lengths of associated HJ-curves.

Proposition 2.6. *Let $I \subset \mathbb{R}$ be a compact interval. Let $(z, u, \zeta, \nu) : I \to T^*G$ be an HJ-curve on G of length L, which is associated to an HJ-curve on N of length L^N. Let u_0 and u_1 be the values of u at the endpoints of I. Then*

$$\cosh L = \frac{1 + e^{2(u_1 - u_0)} + (e^{-u_0} L^N)^2}{2e^{u_1 - u_0}}. \tag{2.6}$$

Proof. Let $(z^N, \zeta^N) : J \to T^*N$ be the associated HJ-curve on N, and let $\upsilon : I \to J$ be the diffeomorphism as in Definition 2.2. Without loss of generality we may assume that $I = [0, \tau]$ with $\tau > 0$ and that $\upsilon(0) = 0$. Set $T = \upsilon(\tau)$, $u_0 = u(0)$, $u_1 = u(\tau)$, $\nu_0 = \nu(0)$, and let H_0^N be the constant value of H^N along (z^N, ζ^N). Then, by Lemma 2.3, $\nu_0 = (2T)^{-1}(e^{2u_1} - e^{2u_0}) + H_0^N T$. Moreover, in the case $H_0^N \neq 0$,

$$\tau = \upsilon^{-1}(T) = \frac{1}{\omega}\left(\operatorname{arctanh}\frac{\nu_0}{\omega} + \operatorname{arctanh}\frac{2H_0^N T - \nu_0}{\omega}\right), \tag{2.7}$$

where $\omega = \sqrt{2e^{u_0} H_0^N + \nu_0^2}$, whereas, in the case $H_0^N = 0$,

$$\tau = \begin{cases} u^{-1}(u_1) = \frac{u_1 - u_0}{\nu_0} = \frac{2u_1 - 2u_0}{e^{2u_1} - e^{2u_0}} T & \text{if } \nu_0 \neq 0, \\ \upsilon^{-1}(T) = e^{-2u_0} T & \text{if } \nu_0 = 0. \end{cases}$$

Note that $L^N = T\sqrt{2H_0^N}$, whereas $L = \tau\sqrt{2e^{u_0} H_0^N + \nu_0^2}$. Easy manipulations of the above expressions then yield (2.6). For example, in the case $H_0^N > 0$, the equality $L = \tau\omega$ holds and (2.6) can be obtained by multiplying by ω both sides of (2.7), taking the cosh of both sides, and applying the addition formula for cosh. ☐

We can now turn the relation (2.6) between lengths into a relation between sub-Riemannian distances. We mention that formula (2.8) below was already given without proof in [24, p. 9]. The argument given here can be thought of as a precise proof of it.

Proposition 2.7. *For all $(z_0, u_0), (z_1, u_1) \in G$,*

$$\begin{aligned} \varrho((z_0, u_0), (z_1, u_1)) &= \operatorname{arccosh} \frac{1 + e^{2(u_1 - u_0)} + (e^{-u_0} \varrho^N(z_0, z_1))^2}{2e^{u_1 - u_0}} \\ &= \operatorname{arccosh}\left(\cosh(u_0 - u_1) + e^{-(u_0 + u_1)} \varrho^N(z_0, z_1)^2/2\right). \end{aligned} \tag{2.8}$$

Proof. By left-invariance of ϱ and ϱ^N, it is sufficient to check the above formula in the case $(z_0, u_0) = 0_G$.

By the results in [3], there exists an open dense subset Ω of G made up of points which are joined to the origin 0_G by a unique length-minimizing curve, and this curve is a projection of an HJ-curve. Analogously, there exists an open dense subset Ω^N of N made up of points which are joined to the origin 0_N by a unique length-minimizing curve, and this curve is the projection of an HJ-curve.

Let $\tilde{\Omega} = \Omega \cap (\Omega^N \times A)$. Then $\tilde{\Omega}$ is a dense open subset of G. Moreover, for all $(z_1, u_1) \in \Omega$, the length L of the length-minimizing HJ-curve (z, u, ζ, ν) on G joining 0_G to (z_1, u_1) equals $\varrho(0_G, (z_1, u_1))$. Moreover, by Proposition 2.5, (z, u, ζ, ν) is of the form $(\zeta^N \circ \upsilon, u, \zeta^N \circ \upsilon, \nu)$ for some HJ-curve (z^N, ζ^N) on N, whose length L^N is related to L by (2.6).

We now claim that $L^N = \varrho^N(0_N, z_1)$. If not, the length-minimizing HJ-curve on N joining 0_N to z_1 (which exists, because $z_1 \in \Omega^N$) would have length less than L^N. So, via Proposition 2.4, we could construct an HJ-curve on G joining 0_G to (z_1, u_1) with length less than L, which leads to a contradiction.

The relation (2.6) between lengths yields (2.8) for all $(z_1, u_1) \in \tilde{\Omega}$. Since $\tilde{\Omega}$ is dense and ϱ, ϱ^N are continuous, (2.8) holds for all $(z_1, u_1) \in G$. □

2.3 Volume asymptotics and integral formulas for radial functions.

The expression (2.8) for the sub-Riemannian distance ϱ allows us to give precise formulas and asymptotics for the volume of the corresponding balls. It should be noted that detailed information on the local behavior of ϱ could be deduced by the Ball-Box Theorem (see [45] or [41]). For the global behavior, however, sufficiently precise general results seem not to be available, and formula (2.8) becomes crucial.

We obtain the volume formulas as corollaries of integral formulas for radial functions. By a radial function on G, we mean a function of the form $x \mapsto f(|x|_\varrho)$, where $f : \mathbb{R}_0^+ \to \mathbb{C}$ and $|x|_\varrho = \varrho(x, 0_G)$ is the distance of $x \in G$ from the origin. Analogously by a radial function on N, we mean a function of the form $z \mapsto f(|z|_N)$, where $|z|_N = \varrho^N(z, 0_N)$ is the distance of $z \in N$ from the origin.

The homogeneity of ϱ^N yields immediately the following integral formula for radial functions on N: for all Borel functions $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$,

$$\int_N f(|z|_N)\, \mathrm{d}z = V_N Q \int_0^\infty f(s) s^{Q-1}\, \mathrm{d}s, \tag{2.9}$$

where $V_N = |B_N(0_N, 1)|$. Clearly such a formula can be extended to complex-valued functions f, as soon as the integrals make sense. We now obtain a similar formula on G.

Proposition 2.8. *For all Borel functions* $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$,

$$\int_G f(|x|_\varrho)\,\mathrm{d}\mu(x) = \int_G f(|x|_\varrho) m(x)\,\mathrm{d}\mu(x) = c_N Q \int_0^\infty f(r)\sinh^Q r\,\mathrm{d}r, \tag{2.10}$$

where $c_N = V_N 2^{Q-1}\Gamma(Q/2)^2/\Gamma(Q)$. *In particular,*

$$\mu\big(B_\varrho(0,r)\big) = c_N Q \int_0^r \sinh^Q s\,\mathrm{d}s \sim \begin{cases} r^{Q+1} & \text{if } 0 < r \le 1, \\ e^{Qr} & \text{if } r \ge 1. \end{cases} \tag{2.11}$$

Proof. Since $|x|_\varrho = |x^{-1}|_\varrho$ by left-invariance of ϱ (cf. [55, Section III.4, p. 40]),

$$\int_G f(|x|_\varrho)\,\mathrm{d}\mu(x) = \int_G f(|x^{-1}|_\varrho)\,\mathrm{d}\mu(x) = \int_G f(|x|_\varrho) m(x)\,\mathrm{d}\mu(x).$$

Moreover, by formulas (2.8) and (2.9),

$$\begin{aligned} \int_G f(|x|_\varrho)\,\mathrm{d}\mu(x) &= V_N Q \int_{-\infty}^\infty \int_0^\infty f(\operatorname{arccosh}(\cosh u + e^{-u}s^2/2)) s^{Q-1}\,\mathrm{d}s\,\mathrm{d}u \\ &= V_N Q 2^{Q/2-1} \int_{-\infty}^\infty \int_0^\infty f(\operatorname{arccosh}(\cosh u + s)) e^{Qu/2} s^{Q/2-1}\,\mathrm{d}s\,\mathrm{d}u \\ &= V_N Q 2^{Q/2-1} \int_0^\infty f(r)\sinh r \int_{-r}^r e^{Qu/2}(\cosh r - \cosh u)^{Q/2-1}\,\mathrm{d}u\,\mathrm{d}r \end{aligned}$$

(in the last step, we have used the change of variable $s = \cosh r - \cosh u$). One can evaluate the inner integral in the last formula explicitly to obtain

$$\int_{-r}^r e^{Qu/2}(\cosh r - \cosh u)^{Q/2-1}\,\mathrm{d}u = 2^{Q/2}\frac{\Gamma(Q/2)^2}{\Gamma(Q)}\sinh^{Q-1} r.$$

This gives (2.10), and (2.11) follows once we take $f = \chi_{[0,r)}$. □

Similar computations give expressions for weighted integrals of radial functions that are useful in the sequel. We define the weight w on G by $w(z,u) = |z|_N^Q$.

Proposition 2.9. *There exists a constant* $C_Q > 0$ *such that*

$$\int_G m(x) f(|x|_\varrho) w(x)\,\mathrm{d}\mu(x) \le C_Q \int_G f(|x|_\varrho)|x|_\varrho\,\mathrm{d}\mu(x) \tag{2.12}$$

for all Borel functions $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$. *Moreover,*

$$\int_{B_\varrho(0,r)} (1+w)^{-1}\,\mathrm{d}\mu \lesssim \begin{cases} r^{Q+1} & \text{if } 0 < r \le 1, \\ r^2 & \text{if } r \ge 1. \end{cases} \tag{2.13}$$

Proof. A simple modification of the proof of Proposition 2.8 gives the integral formula

$$(2.14)\quad \int_G m(x)f(|x|_\varrho)w(x)\,\mathrm{d}\mu(x) = 2^{Q-1}V_N Q\int_0^\infty f(r)\sinh r\int_{-r}^r (\cosh r - \cosh u)^{Q-1}\,\mathrm{d}u\,\mathrm{d}r.$$

Since $\int_{-r}^r (\cosh r - \cosh u)^{Q-1}\,\mathrm{d}u \lesssim r\sinh^{Q-1} r$, the estimate (2.12) follows by comparison of (2.10) and (2.14).

As for (2.13), this is clear by (2.11) in the case $r \leq 1$. For $r \geq 1$,

$$\begin{aligned}\int_{B_\varrho(0,r)} (1+w)^{-1}\,\mathrm{d}\mu &= V_N Q\int_{-\infty}^\infty\int_0^\infty \chi_{[0,r)}\left(\operatorname{arccosh}\left(\cosh u + \frac{e^{-u}s^2}{2}\right)\right)\frac{s^{Q-1}}{1+s^Q}\,\mathrm{d}s\,\mathrm{d}u\\ &\sim \int_{-r}^r\int_0^{2e^u(\cosh r - \cosh u)} \frac{s^{Q/2-1}}{1+s^{Q/2}}\,\mathrm{d}s\,\mathrm{d}u\\ &\lesssim \int_{-r}^r\int_0^{2e^{2r}} \frac{1}{1+s}\,\mathrm{d}s\,\mathrm{d}u \sim r^2,\end{aligned}$$

and we are done. □

3 Calderón–Zygmund theory

3.1 Abstract Calderón–Zygmund theory. It is well known [9, 10] that in spaces of homogeneous type, integrable functions admit a Calderón–Zygmund decomposition and that in this context, the classical Calderón–Zygmund theory for singular integrals and the theory of Hardy and *BMO* spaces [48] can be generalized. However, because of exponential volume growth, the group G under consideration is not a space of homogeneous type, and a further generalization of the Calderón–Zygmund theory is necessary. This generalization was introduced by Hebisch and Steger [29] and further developed by Vallarino [51]. Here we summarize some of the results of this theory that are used in the sequel.

Definition 3.1. A **CZ-space** is a metric measure space (X, d, μ) such that there exist a positive constant κ_0 and a family $\mathcal{R}$ of measurable subsets of X with the following properties: for all $R \in \mathcal{R}$, there exist $x \in X$ and $r > 0$ such that

(i) $R \subset B(x, \kappa_0 r)$;

(ii) $\mu(R^*) \leq \kappa_0 \mu(R)$, where $R^* = \{x \in X : d(x, R) < r\}$.

Moreover, for all $f \in L^1(X)$ and for all $\alpha > \kappa_0\|f\|_1/\mu(X)$ ($\alpha > 0$ if $\mu(X) = \infty$), there exist a decomposition $f = g + \sum_{i\in\mathbb{N}} b_i$ and sets $R_i \in \mathcal{R}$ such that

(iii) $\|g\|_\infty \leq \kappa_0 \alpha$;
(iv) $\operatorname{supp} b_i \subset R_i$ and $\int b_i \, \mathrm{d}\mu = 0$ for all $i \in \mathbb{N}$;
(v) $\sum_i \mu(R_i) \leq \kappa_0 \|f\|_1/\alpha$;
(vi) $\sum_i \|b_i\|_1 \leq \kappa_0 \|f\|_1$.

The constant κ_0 is called the **CZ-constant** of (X, d, μ). A decomposition $f = g + \sum_{i\in\mathbb{N}} b_i$ which has properties (iii)-(vi) of Definition 3.1 is said to be a **Calderón–Zygmund decomposition** of f at height α. The elements of the family $\mathcal{R}$ are called **admissible sets** and, for each $R \in \mathcal{R}$, the point $x \in X$ and the number $r > 0$ satisfying properties (i)-(ii) of Definition 3.1 are called, respectively, the **center** and the **radius** of R.

Note that the above definition of CZ-space is more restrictive than the definition of "Calderón–Zygmund space" given by Hebisch and Steger in [29]. Hence the following boundedness theorem for a class of linear operators on CZ-spaces is a consequence of [29, Theorem 2.1].

Theorem 3.2. *Let (X, d, μ) be a CZ-space. Let T be a linear operator bounded on $L^2(X)$ such that $T = \sum_{j\in\mathbb{Z}} T_j$, where*
(i) *the series converges in the strong topology of operators on $L^2(X)$;*
(ii) *every T_j is an integral operator with kernel K_j;*
(iii) *there exist positive constants b, B, ε and $c > 1$ such that*

$$\int_X |K_j(x,y)|\big(1 + c^j d(x,y)\big)^{\varepsilon} \mathrm{d}\mu(x) \leq B \quad \text{for all } y \in X,$$
$$\int_X |K_j(x,y) - K_j(x,z)| \mathrm{d}\mu(x) \leq B\big(c^j d(y,z)\big)^{b} \quad \text{for all } y, z \in X.$$

Then T extends from $L^1(X) \cap L^2(X)$ to an operator of weak type $(1,1)$ and bounded on $L^p(X)$, for $1 < p \leq 2$.

In [51] it was noticed that if a CZ-space satisfies a certain additional condition, then one can develop an H^1-*BMO* theory on it.

Definition 3.3. We say that the CZ-space (X, d, μ) with family of admissible sets $\mathcal{R}$ satisfies condition (C) if there exists a subfamily $\mathcal{R}'$ of $\mathcal{R}$ such that
(i) if $R_1, R_2 \in \mathcal{R}'$ are such that $R_2 \cap R_1 \neq \varnothing$, then either $R_1 \subset R_2$ or $R_2 \subset R_1$;
(ii) for each $R \in \mathcal{R}$, there exists $R' \in \mathcal{R}'$ such that $R \subset R'$.

Suppose now that (X, d, μ) is a CZ-space with family of admissible sets $\mathcal{R}$ that satisfies condition (C). We introduce an atomic Hardy space H^1 and a space of bounded mean oscillation functions on X as follows.

Definition 3.4. An **atom** is a function $a \in L^1(X)$ such that

(i) a is supported in an admissible set $R \in \mathcal{R}$,

(ii) $\|a\|_2 \le \mu(R)^{-1/2}$,

(iii) $\int_S a \, \mathrm{d}\mu = 0$.

Definition 3.5. The Hardy space $H^1(X)$ is the Banach space

$$H^1(X) = \Big\{ f \in L^1(X) : f = \sum_j \lambda_j a_j, \ a_j \text{ atoms}, \ \lambda_j \in \mathbb{C}, \ \sum_j |\lambda_j| < \infty \Big\}$$

endowed with the norm

$$\|f\|_{H^1} = \inf \Big\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j, \ a_j \text{ atoms}, \ \lambda_j \in \mathbb{C} \Big\}.$$

We denote by $H^1_{\mathrm{fin}}(X)$ the subspace of $H^1(X)$ of finite linear combinations of atoms.

Definition 3.6. The space $\mathcal{BMO}(X)$ is the space

$$\mathcal{BMO}(X) = \Big\{ f \in L^2_{\mathrm{loc}}(X) : \sup_{R \in \mathcal{R}} \Big(\frac{1}{\mu(R)} \int_R |f - f_R|^2 \, \mathrm{d}\mu \Big)^{1/2} < \infty \Big\},$$

where $f_R = \frac{1}{\mu(R)} \int_R f \, \mathrm{d}\mu$. The space $BMO(X)$ is the quotient space of $\mathcal{BMO}(X)$ modulo constant functions, and, endowed with the norm

$$\|f\|_{BMO} = \sup_{R \in \mathcal{R}} \Big(\frac{1}{\mu(R)} \int_R |f - f_R|^2 \, \mathrm{d}\mu \Big)^{1/2},$$

is a Banach space.

For more details on the spaces $H^1(X)$ and $BMO(X)$ we refer the reader to [51]. In particular, the space $BMO(X)$ can be identified with the dual of $H^1(X)$ [51, Theorem 3.9].

Proposition 3.7. (i) *For each g in $BMO(X)$, the functional Λ defined on $H^1_{\mathrm{fin}}(X)$ by*

$$\Lambda(f) = \int fg \, \mathrm{d}\mu \quad \textit{for all } f \in H^1_{\mathrm{fin}}(X)$$

extends to a bounded functional on $H^1(X)$. Furthermore, there exists a constant C such that

$$\|\Lambda\|_{(H^1(X))^*} \le C \|f\|_{BMO}.$$

(ii) *For each bounded linear functional Λ on $H^1(X)$, there exists a function g in $BMO(X)$ such that*

$$\Lambda(f) = \int gf \, \mathrm{d}\mu \quad \textit{for all } f \in H^1_{\mathrm{fin}}(X), \quad \textit{and} \quad \|g\|_{BMO} \le C \|\Lambda\|_{(H^1(X))^*},$$

with C independent of g and Λ.

Moreover, the following H^1-L^1 boundedness result holds for singular integral operators on CZ-spaces.

Theorem 3.8 ([51, Theorem 3.10].). *Let (X, d, μ) be a CZ-space which satisfies condition* (C). *If T is a linear operator which satisfies the hypotheses of Theorem 3.2, then T is bounded from $H^1(X)$ to $L^1(X)$.*

3.2 Calderón–Zygmund theory on (G, ϱ, μ).

We prove that the space (G, ϱ, μ) is a CZ-space as defined in the previous subsection. This fact was already announced and proved by Hebisch in [26] for a more general class of amenable Lie groups, including the groups we consider here. However, for our groups, the construction of the Calderón–Zygmund decomposition becomes more transparent than the one given in [26], and it is worthwhile to see the explicit construction in our setting. Moreover, our construction allows us to show that the CZ-space (G, ϱ, μ) satisfies condition (C). Consequently, a theory of Hardy spaces can be developed on G.

The difficulty in the construction is in finding a suitable family $\mathcal{R}$ of admissible sets on G. We cannot use balls as in the classical case, because their measure increases exponentially, and condition (ii) of Definition 3.1 is not satisfied. To define admissible sets, we adapt to the sub-Riemannian distance the ideas of [29] and [52].

Christ [6, Theorem 11] proved the existence of a family of dyadic sets in a space of homogeneous type, which can be formulated for the stratified group N as follows.

Theorem 3.9. *There exist constants $\eta, C_N > 1$, an integer $J \geq 2$, a collection of Borel subsets $Q_\alpha^k \subset N$, and points $n_\alpha^k \in N$, where $k \in \mathbb{Z}$, $\alpha \in I_k$ and I_k is a countable index set, such that, for all $k \in \mathbb{Z}$,*

(i) $|N - \bigcup_{\alpha \in I_k} Q_\alpha^k| = 0$;
(ii) $B_N(n_\alpha^k, C_N^{-1}\eta^k) \subset Q_\alpha^k \subset B_N(n_\alpha^k, C_N\eta^k)$ *for all* $\alpha \in I_k$;
(iii) $Q_\alpha^k \cap Q_\beta^k = \varnothing$ *for all* $\alpha, \beta \in I_k$ *with* $\alpha \neq \beta$;
(iv) *for all* $\alpha \in I_k$, Q_α^k *has at most* J *subsets of the form* Q_β^{k-1} *for* $\beta \in I_{k-1}$;
(v) *for all* $\ell \leq k$ *and* $\beta \in I_\ell$, *there is a unique* $\alpha \in I_k$ *such that* $Q_\beta^\ell \subset Q_\alpha^k$;
(vi) *for all* $\ell \leq k$, $\alpha \in I_k$, *and* $\beta \in I_\ell$, *either* $Q_\alpha^k \cap Q_\beta^\ell = \varnothing$ *or* $Q_\beta^\ell \subset Q_\alpha^k$.

Let us fix a system of dyadic sets Q_α^k, points n_α^k, index sets I_k, and constants η, C_N, J in accordance with Theorem 3.9. Further, let us fix positive constants M

and r_0 such that

$$1 < r_0 < 2\log 2, \tag{3.1}$$

$$M > 1, \tag{3.2}$$

$$e^{r_0} e^{2M} r_0 \le e^{2Mr_0}, \tag{3.3}$$

$$6M > \log\eta - \log 2 + \frac{r_0}{2}, \tag{3.4}$$

$$\eta e^{4Mr_0} < 2e^{8M} \inf\{re^{-r/2} : r_0 < r \le 2r_0\}, \tag{3.5}$$

$$\eta < 4e^{(4M-1)r_0}. \tag{3.6}$$

We define admissible sets as the product of dyadic sets in N and intervals in A as follows.

Definition 3.10. An **admissible set** in G is a set of the form

$$Q^k_\alpha \times (u_0 - r, u_0 + r),$$

where $k \in \mathbb{Z}$, $\alpha \in I_k$, $u_0 \in \mathbb{R}$, $r > 0$ are such that

$$\begin{aligned} re^{2M} e^{u_0} &\le \eta^k < 4re^{8M} e^{u_0} \text{ if } 0 < r \le r_0, \\ e^{2Mr} e^{u_0} &\le \eta^k < 4e^{8Mr} e^{u_0} \text{ if } r > r_0. \end{aligned} \tag{3.7}$$

A **small admissible set** is an admissible set corresponding to a parameter $r \in (0, r_0]$, and a **big admissible set** is an admissible set corresponding to a parameter $r \in (r_0, \infty)$. We denote by $\mathcal{R}$ the family of all admissible sets in G.

Proposition 2.7 allows us to obtain precise relations between balls and "rectangles" on G, which are important in the following.

Proposition 3.11. *There exists a positive constant C_1 such that*

(i) $B_N\big(0_N, 4C_N e^{8M} r\big) \times (-r, r) \subset B_\varrho(0_G, C_1 r)$ *for every* $r \in (0, \infty)$,

(ii) $B_N\big(0_N, 4C_N e^{8Mr}\big) \times (-r, r) \subset B_\varrho(0_G, C_1 r)$ *for every* $r \in (r_0, \infty)$,

(iii) $B_\varrho(0_G, r) \subset B_N\big(0_N, e^r\big) \times (-r, r)$ *for every* $r \in (0, \infty)$,

(iv) $B_\varrho(0_G, r) \subset B_N\big(0_N, C_1 r\big) \times (-r, r)$ *for every* $r \in (0, r_0]$.

Proof. We first prove (i). If $(z, u) \in B_N\big(0_N, 4C_N e^{8M} r\big) \times (-r, r)$, then, by formula (2.8),

$$\begin{aligned} \varrho\big((z,u), 0_G\big) &< \operatorname{arccosh}\left(\cosh r + \frac{16 e^r C_N^2 e^{16M} r^2}{2}\right) \\ &\le \operatorname{arccosh}\cosh(C_1 r), \end{aligned}$$

for a sufficiently large C_1 and for every $r \in (0, \infty)$.

We now prove (ii). If $(z,u) \in B_N\big(0_N, 4C_N e^{8Mr}\big) \times (-r,r)$, then, by formula (2.8),

$$\varrho\big((z,u),0_G\big) < \operatorname{arccosh}\left(\cosh r + \frac{16 e^r C_N^2 e^{16Mr}}{2}\right) \le \operatorname{arccosh}\cosh(C_1 r),$$

for a sufficiently large C_1 and for every $r \in (r_0, \infty)$.

We now consider any point $(z,u) \in B_\varrho(0_G, r)$. By formula (2.8), it is obvious that $\cosh u < \cosh r$, and then $u \in (-r,r)$. Suppose now that $|z| \ge e^r$. Then

$$\varrho\big((z,u),0_G\big) \ge \operatorname{arccosh}\left(1 + \frac{e^{-r}e^{2r}}{2}\right) \ge \operatorname{arccosh}\cosh r = r.$$

Then $|z| < e^r$, and (iii) is proved. Take now any point $(z,u) \in B_\varrho(0_G, r)$, and suppose that $|z| \ge C_1 r$. Then

$$\varrho\big((z,u),0_G\big) \ge \operatorname{arccosh}\left(1 + \frac{e^{-r}C_1^2 r^2}{2}\right) \ge \operatorname{arccosh}\cosh r = r$$

for every $r \in (0, r_0]$, if C_1 is chosen sufficiently large. Then $|z| < C_1 r$, and (iv) is proved. □

We now investigate some properties of admissible sets.

Proposition 3.12. *There exists a positive constant C^* such that, for every admissible set $R = Q_\alpha^k \times (u_0 - r, u_0 + r)$,*

(i) *$R \subset B_\varrho\big((n_\alpha^k, u_0), C_1 r\big)$, where C_1 is the constant which appears in Proposition 3.11;*

(ii) *$\mu\big(R^*\big) \le C^* \mu\big(R\big)$, where $R^* = \{(z,u) \in G : \varrho\big((z,u),R\big) < r\}$.*

Proof. *Case* $0 < r \le r_0$. By Theorem 3.9 and Definition 3.10,

$$\begin{aligned} R \subset B_N\big(n_\alpha^k, 4C_N e^{8M} e^{u_0} r\big) \times (u_0 - r, u_0 + r) \\ = (n_\alpha^k, u_0) \cdot B_N\big(0_N, 4C_N e^{8M} r\big) \times (-r, r). \end{aligned}$$

By Proposition 3.11, $B_N\big(0_N, 4C_N e^{8M} r\big) \times (-r,r) \subset B_\varrho(0_G, C_1 r)$, which implies (i).

To prove (ii), we remark that $R^* = \bigcup_{(z,u)\in R} B_\varrho\big((z,u), r\big)$. By the left-invariance of the metric and Proposition 3.11, for every $(z,u) \in R$,

$$\begin{aligned} B_\varrho\big((z,u),r\big) &= (z,u) \cdot B_\varrho(0_G, r) \subset (z,u) \cdot B_N\big(0_N, C_1 r\big) \times (-r, r) \\ &= B_N\big(z, C_1 e^u r\big) \times (u - r, u + r) \\ &\subset B_N\big(n_\alpha^k, C_1 e^u r + C_N \eta^k\big) \times (u_0 - 2r, u_0 + 2r) \\ &\subset B_N\big(n_\alpha^k, C e^{u_0} r\big) \times (u_0 - 2r, u_0 + 2r), \end{aligned}$$

where $C = C_1 e^{r_0} + 4C_N e^{8M}$; we have applied the triangle inequality in N and the admissibility condition. This implies that $\mu(R^*) \lesssim e^{Qu_0} r^Q r \sim \eta^{Qk} r \sim \mu(R)$, which gives (ii).

Case $r > r_0$. To prove (i), note that by Theorem 3.9

$$R \subset B_N\left(n_\alpha^k, C_N \eta^k\right) \times (u_0 - r, u_0 + r),$$

which is contained in $B_N\left(n_\alpha^k, 4C_N e^{8Mr} e^{u_0}\right) \times (u_0 - r, u_0 + r)$, by the admissibility condition (3.7). By the left-invariance of the metric and Proposition 3.11

$$R \subset (n_\alpha^k, u_0) \cdot B_N\left(0_N, 4C_N e^{8Mr}\right) \times (-r, r) \subset (n_\alpha^k, u_0) \cdot B_\varrho(0_G, C_1 r) = B_\varrho\left((n_\alpha^k, u_0), C_1 r\right).$$

To prove (ii), we remark that $R^* = \bigcup_{(z,u)\in R} B_\varrho\left((z, u), r\right)$. By the left-invariance of the metric and Proposition 3.11, for every $(z, u) \in R$,

$$\begin{aligned} B_\varrho\left((z, u), r\right) &= (z, u) \cdot B_\varrho(0_G, r) \subset (z, u) \cdot B_\varrho\left(0_N, e^r\right) \times (-r, r) \\ &= B_N\left(z, e^u e^r\right) \times (u - r, u + r). \end{aligned}$$

Using the fact that $(z, u) \in R$ and the admissibility condition on R, we see that

$$(u - r, u + r) \subset (u_0 - 2r, u_0 + 2r)$$

and

$$B_N\left(z, e^u e^r\right) \subset B_N\left(z, e^{u_0+r} e^r\right) \subset B_N\left(n_\alpha^k, e^{u_0+2r} + C_N \eta^k\right) \subset B_N\left(n_\alpha^k, (1 + C_N)\eta^k\right).$$

Thus

$$R^* \subset B_N\left(n_\alpha^k, (1 + C_N)\eta^k\right) \times (u_0 - 2r, u_0 + 2r),$$

and so

$$\mu\left(R^*\right) \lesssim |B_N\left(n_\alpha^k, \eta^k\right)| r \sim \mu\left(R\right),$$

as required. □

We now define a way of splitting an admissible set into at most J disjoint admissible subsets, where J is the constant which appears in Theorem 3.9.

Definition 3.13. An admissible set $R = Q_\alpha^k \times (u_0 - r, u_0 + r)$ is called **strongly admissible** if (3.7) also holds with $k - 1$ in place of k, i.e., if

$$\begin{aligned} re^{2M} e^{u_0} &\le \eta^{k-1} < 4re^{8M} e^{u_0} \quad \text{when } 0 < r \le r_0, \\ e^{2Mr} e^{u_0} &\le \eta^{k-1} < 4e^{8Mr} e^{u_0} \quad \text{when } r > r_0. \end{aligned}$$

Note that the upper bound for η^{k-1} in the above inequalities is automatically satisfied, because R is admissible and $\eta^{k-1} < \eta^k$; the additional requirement for R to be strongly admissible is the lower bound for η^{k-1}.

Definition 3.14. Let $R = Q_\alpha^k \times (u_0 - r, u_0 + r)$ be admissible. If R is strongly admissible, we define the **children** of R to be the sets of the form $Q_\beta^{k-1} \times (u_0 - r, u_0 + r)$, where $\beta \in I_{k-1}$ and $Q_\beta^{k-1} \subset Q_\alpha^k$. If R is not strongly admissible, we define the **children** of R to be the sets $Q_\alpha^k \times (u_0 - r, u_0)$ and $Q_\alpha^k \times (u_0, u_0 + r)$. We denote by $\mathfrak{C}(R)$ the set of the children of R.

Definition 3.15. Let E be a measurable subset of a measure space. A **quasi-partition** of E is an at most countable family of non-negligible, pairwise disjoint measurable subsets of E whose union has full measure in E.

Lemma 3.16. *Let $C_2 = \max\{2, (C_N^2 \eta)^Q\}$. Then, for all admissible sets R,*

(i) *R has at most J children,*
(ii) *$\mathfrak{C}(R)$ is a quasi-partition of R,*
(iii) *$C_2^{-1}\mu(R) \le \mu(R') \le \mu(R)$ for all $R' \in \mathfrak{C}(R)$,*
(iv) *all the children of R are admissible.*

Proof. Let $R = Q_\alpha^k \times (u_0 - r, u_0 + r)$. Since R is admissible, (3.7) holds. Suppose that R is strongly admissible. Then the children of R are admissible too. Moreover, from the properties of dyadic sets given by Theorem 3.9, it is clear that properties (i), (ii), and (iii) hold.

Suppose now that R is not strongly admissible. Then, for $r \le r_0$,

$$re^{2M} e^{u_0} \le \eta^k < \eta r e^{2M} e^{u_0}, \tag{3.8}$$

while, for $r > r_0$,

$$e^{2Mr} e^{u_0} \le \eta^k < \eta e^{2Mr} e^{u_0}. \tag{3.9}$$

Moreover the children of R, i.e., $R_1 = Q_\alpha^k \times (u_0 - r, u_0)$ and $R_2 = Q_\alpha^k \times (u_0, u_0 + r)$, are "centered" at $(n_\alpha^k, u_0 - r/2)$ and $(n_\alpha^k, u_0 + r/2)$, respectively. It is clear that properties (i), (ii), and (iii) hold.

We prove that R_1 and R_2 are admissible: to do so, we distinguish three cases.

Case $r \le r_0$. In this case, R is a small admissible set, and we must prove that R_1, R_2 are both small admissible sets, because $r/2 \le r_0$. Notice that

$$e^{u_0 - r/2} e^{2M} \frac{r}{2} \le e^{u_0 + r/2} e^{2M} \frac{r}{2} \le \frac{1}{2} \eta^k e^{r/2} \le \frac{1}{2} \eta^k e^{\frac{r_0}{2}} \le \eta^k,$$

since $r_0 \le 2 \log 2$ by (3.1). Moreover,

$$\eta^k < \eta e^{u_0} e^{2M} r < 4 e^{8M} e^{u_0 - r/2} \frac{r}{2} < 4 e^{8M} e^{u_0 + r/2} \frac{r}{2},$$

since $\eta < 2e^{6M} e^{-\frac{r_0}{2}}$ by condition (3.4). This proves that R_1 and R_2 are admissible.

Case $r_0 < r \le 2r_0$. In this case, R is a big admissible set, and we must prove that R_1, R_2 are both small admissible sets, because $r/2 \le r_0$. Notice that

$$e^{u_0 - r/2} e^{2M} \frac{r}{2} \le e^{u_0 + r/2} e^{2M} \frac{r}{2} \le e^{u_0} e^{2Mr} \le \eta^k,$$

since $e^{r_0} e^{2M} r_0 \le e^{2Mr_0}$ by condition (3.3). Moreover,

$$\eta^k < \eta e^{u_0} e^{2Mr} < 4e^{8M} e^{u_0 - r/2} r/2 < 4e^{8M} e^{u_0 + r/2} r/2,$$

since $\eta e^{4Mr_0} < 2e^{8M} \inf_{r_0 < r \le 2r_0} r e^{-r/2}$ by condition (3.5). This proves that R_1 and R_2 are admissible.

Case $r > 2r_0$. In this case, R is a big admissible set, and we must prove that R_1, R_2 are both big admissible sets, because $r/2 > r_0$. Notice that

$$e^{u_0 - r/2} e^{2Mr/2} \le e^{u_0 + r/2} e^{2Mr/2} \le e^{u_0} e^{2Mr} \le \eta^k,$$

since $M > 1/2$ by (3.2). Moreover,

$$\eta^k < \eta e^{u_0} e^{2Mr} < 4e^{8Mr/2} e^{u_0 - r/2} < 4e^{8Mr/2} e^{u_0 + r/2},$$

since $\eta < 4e^{(4M-1)r_0}$ by condition (3.6). This proves that both R_1 and R_2 are admissible. □

By adapting the proof of [51, Lemma 3.16], we can construct a quasi-partition of G in big admissible sets whose measure is as large as we want.

Lemma 3.17. *For all $\sigma > 0$, there exists a quasi-partition $\mathcal{P}$ of G in big admissible sets whose measure is greater than σ.*

Proof. Choose $r_1 > r_0$ and $k_1 \in \mathbb{Z}$ such that $e^{2Mr_1} \le \eta^{k_1} < 4e^{8Mr_1}$. Then the sets $R^1_\alpha = Q^{k_1}_\alpha \times (-r_1, r_1)$, $\alpha \in I_{k_1}$, are a quasi-partition of $N \times (-r_1, r_1)$ made up of big admissible sets. It is possible to choose k_1 and r_1 in such a way that $|B_N(0_N, C_N^{-1}\eta^{k_1})| 2r_1 > \sigma$, so that $\mu(R^1_\alpha) > \sigma$ for all $\alpha \in I_{k_1}$.

Suppose that a quasi-partition of $N \times (r_1 + \cdots + 2r_{n-1}, r_1 + \cdots + 2r_{n-1} + 2r_n)$, made up of big admissible sets of measure greater than σ, has been constructed. Choose $r_{n+1} > r_0$ and $k_{n+1} \in \mathbb{Z}$ such that $e^{2Mr_{n+1}} e^{u_{n+1}} \le \eta^{k_{n+1}} < 4e^{8Mr_{n+1}} e^{u_{n+1}}$, where $u_{n+1} = r_1 + \cdots + 2r_n + r_{n+1}$. Then the sets $R^{n+1}_\alpha = Q^{k_{n+1}}_\alpha \times (u_{n+1} - r_{n+1}, u_{n+1} + r_{n+1})$, $\alpha \in I_{k_{n+1}}$, are a quasi-partition of $N \times (r_1 + \cdots + 2r_n, r_1 + \cdots + 2r_n + 2r_{n+1})$ made up of big admissible sets. It is possible to choose k_{n+1} and r_{n+1} in such a way that $|B_N(0_N, C_N^{-1}\eta^{k_{n+1}})| 2r_{n+1} > \sigma$, so that $\mu(R^{n+1}_\alpha) > \sigma$ for all $\alpha \in I_{k_{n+1}}$.

Iterating this process, we get a quasi-partition of $N \times (-r_1, \infty)$ made up of big admissible sets with measure greater than σ. By a similar procedure, we get a quasi-partition of $N \times (-\infty, -r_1)$ made up of big admissible sets with measure greater than σ, as required. □

Lemma 3.16 shows that we can iteratively consider children, grandchildren, great grandchildren, etc., i.e., **descendants** of an admissible set, and all descendants are admissible. In this way, we can also define subsequent refinements of a quasi-partition of G in admissible sets. Namely, let $\mathcal{P}$ be a quasi-partition of G in admissible sets, and define $\mathfrak{D}^n(\mathcal{P})$ iteratively for all $n \in \mathbb{N}$ by

$$\mathfrak{D}^0(\mathcal{P}) = \mathcal{P}, \quad \mathfrak{D}^{n+1}(\mathcal{P}) = \bigcup_{R \in \mathfrak{D}^n(\mathcal{P})} \mathfrak{C}(R).$$

Finally, define $G_{\mathcal{P}} = \bigcap_{n\in\mathbb{N}} \bigcup \mathfrak{D}^n(\mathcal{P})$ and $\mathfrak{D}(\mathcal{P}) = \bigcup_{n\in\mathbb{N}} \mathfrak{D}^n(\mathcal{P})$. The set $\mathfrak{D}(\mathcal{P})$ is the set of descendants of elements of $\mathcal{P}$.

Lemma 3.18. *Let $\mathcal{P}$ be a quasi-partition of G in admissible sets. Then*

(i) *for all $n \in \mathbb{N}$, $\mathfrak{D}^n(\mathcal{P})$ is a quasi-partition of G in admissible sets;*
(ii) *for all $R, R' \in \mathfrak{D}(\mathcal{P})$, either $R \cap R' = \varnothing$ or $R \subset R'$ or $R' \subset R$;*
(iii) *$G_{\mathcal{P}}$ has full measure in G;*
(iv) *for all $x \in G_{\mathcal{P}}$ and $n \in \mathbb{N}$, there is a unique $R_x^n \in \mathfrak{D}^n(\mathcal{P})$ such that $x \in R_x^n$;*
(v) *for all $x \in G_{\mathcal{P}}$ and all neighborhoods U of x, there exists $n \in \mathbb{N}$ such that $R_x^n \subset U$.*

Proof. Item (i) is an immediate consequence of Lemma 3.16, and (iii) follows because $G_{\mathcal{P}}$ is a countable intersection of sets of full measure in G.

As for (ii), take $R \in \mathfrak{D}^n(\mathcal{P})$ and $R' \in \mathfrak{D}^{n'}(\mathcal{P})$ for some $n, n' \in \mathbb{N}$. If $R \cap R' \neq \varnothing$, then $n \neq n'$. Suppose that $n < n'$. Then, by construction, R' is descendant of exactly one element $R'' \in \mathfrak{D}^n(\mathcal{P})$. Consequently, either $R'' = R$ and therefore $R' \subset R$, or $R'' \cap R = \varnothing$, in which case also $R' \cap R = \varnothing$.

As for (iv), clearly, since x belongs to the union of $\mathfrak{D}^n(\mathcal{P})$ and $\mathfrak{D}^n(\mathcal{P})$ is a quasi-partition of G, there exists a unique set $R_x^n \in \mathfrak{D}^n(\mathcal{P})$ such that $x \in R_x^n$. In fact, from the construction, it is clear that R_x^{n+1} is a child of R_x^n for all $n \in \mathbb{N}$. In particular, the sets R_x^n for fixed x form a decreasing sequence as n grows in $\mathbb{N}$; and, at each step, in the passage from $R_x^n = Q_\alpha^k \times (u_0 - r, u_0 + r)$ to its child R_x^{n+1}, either the first factor Q_α^k is replaced by one of its children Q_β^{k-1}, or the second factor $(u_0 - r, u_0 + r)$ is halved.

To prove (v), it is thus sufficient to show that each of these two alternatives occurs infinitely many times, i.e., that for infinitely many n, R_x^n is strongly admissible and for infinitely many n, R_x^n is not strongly admissible. In fact, in this case, the diameter of both projections of R_x^n onto the two factors N and A of G tends to 0 as $n \to \infty$.

In search of a contradiction, suppose that, for all n greater than some n_0, R_x^n is strongly admissible. This means that, if $R_x^{n_0} = Q_\alpha^k \times (u_0 - r, u_0 + r)$, then

R_x^n has the form $Q_{\alpha_n}^{k+n_0-n} \times (u_0 - r, u_0 + r)$ for all $n \geq n_0$, where $\alpha_n \in I_{k+n_0-n}$. Since the R_x^n are all admissible, (3.7) must hold when k is replaced by ℓ for all integers $\ell \leq k$, while u_0 and r remain the same, and, allowing ℓ to tends to $-\infty$, one obtains a contradiction. Similarly one obtains a contradiction by assuming that, for all $n \geq n_0$, R_x^n is not strongly admissible: in this case, one would have $\eta^k < 4(2^{-\ell} r) e^{8M} e^{u_0+r}$ for fixed k,u_0,r and for all sufficiently large ℓ, which is clearly impossible. □

For all quasi-partitions $\mathcal{P}$ of G in admissible sets, we define the maximal operator $M^{\mathcal{P}}$ as follows: for all functions f in $L^1_{\mathrm{loc}}(G)$ and $x \in G$,

$$M^{\mathcal{P}} f(x) = \begin{cases} \sup\limits_{\substack{R \in \mathfrak{D}(\mathcal{P}) \\ R \ni x}} \dfrac{1}{\mu(R)} \displaystyle\int_R |f| \, \mathrm{d}\mu & \text{if } x \in \bigcup \mathfrak{D}(\mathcal{P}), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.19. *Let $\mathcal{P}$ be a quasi-partition of G in admissible sets. Then*

(i) *$M^{\mathcal{P}} f$ is measurable for all $f \in L^1_{\mathrm{loc}}(G)$, and*

$$M^{\mathcal{P}}(\lambda f + \lambda' f') \leq |\lambda| M^{\mathcal{P}} f + |\lambda'| M^{\mathcal{P}} f' \tag{3.10}$$

for all $\lambda, \lambda' \in \mathbb{C}$ and $f, f' \in L^1_{\mathrm{loc}}(G)$;

(ii) *the maximal operator $M^{\mathcal{P}}$ is of weak type* (1, 1)*;*

(iii) *for all $f \in L^1_{\mathrm{loc}}(G)$, $|f| \leq M^{\mathcal{P}} f$ almost everywhere.*

Proof. (i). $M^{\mathcal{P}} f = \sup_{n \in \mathbb{N}} M_n^{\mathcal{P}} f$, where

$$M_n^{\mathcal{P}} f(x) = \begin{cases} \dfrac{1}{\mu(R_x^n)} \displaystyle\int_{R_x^n} |f| \, \mathrm{d}\mu & \text{if } x \in \bigcup \mathfrak{D}^n(\mathcal{P}), \\ 0 & \text{otherwise,} \end{cases}$$

and the sets R_x^n are defined as in Lemma 3.18. Clearly, the $M_n^{\mathcal{P}} f$ are measurable, and consequently, $M^{\mathcal{P}} f$ is measurable too. The inequality (3.10) is clear by the definition.

(ii). Let f be in $L^1(G)$, and let $\alpha > 0$. Consider the set $\Omega_\alpha = \{M^{\mathcal{P}} f > \alpha\}$. For each point $x \in \Omega_\alpha$ let R_x be the largest set (in the sense of inclusion) in $\mathfrak{D}(\mathcal{P})$ that contains x such that the average of $|f|$ on R_x is greater than α. If $\mathcal{S} = \{R_x : x \in \Omega_\alpha\}$, then $\mathcal{S}$ is a partition of Ω_α made up of elements of $\mathfrak{D}(\mathcal{P})$. Thus,

$$\mu\left(\Omega_\alpha\right) = \sum_{R \in \mathcal{S}} \mu(R) \leq \frac{1}{\alpha} \sum_{R \in \mathcal{S}} \int_R |f| \, \mathrm{d}\mu \leq \frac{\|f\|_1}{\alpha}.$$

(iii). By (ii) and standard arguments (cf. [49, Theorem II.3.12] or [16, Theorems 2.2 and 2.10]), it is sufficient to consider the case where f is continuous. In this case,

$$M^{\mathcal{P}} f(x) \geq \lim_{n\to\infty} \frac{1}{\mu(R_x^n)} \int_{R_x^n} |f| \, \mathrm{d}\mu = |f(x)|$$

for all $x \in G_{\mathcal{P}}$, by Lemma 3.18(v), and $G_{\mathcal{P}}$ has full measure by Lemma 3.18(iii). □

Now we are able to construct the Calderón–Zygmund decomposition of an integrable function on G.

Theorem 3.20. *The space* (G, ϱ, μ) *with the family* $\mathcal{R}$ *of admissible sets is a CZ-space which satisfies condition* (C).

Proof. By Proposition 3.12, the family $\mathcal{R}$ of admissible sets in G satisfies conditions (i)-(ii) of Definition 3.1.

Let now f be in $L^1(G)$, and let $\alpha > 0$. Our purpose is to construct a Calderón–Zygmund decomposition of f at height α. Let $\mathcal{P}$ be a quasi-partition of G in big admissible sets whose measure is greater than $\|f\|_1/\alpha$ (it exists, by Lemma 3.17). For each R in $\mathcal{P}$, we have $\frac{1}{\mu(R)} \int_R |f| \, \mathrm{d}\mu < \alpha$.

Let $\mathcal{B} = \{R \in \mathfrak{D}(\mathcal{P}) : \mu(R)^{-1} \int_R |f| \, \mathrm{d}\mu \geq \alpha\}$. We define the family $\mathcal{C}$ of the stopping sets as follows:

$$\mathcal{C} = \{R \in \mathcal{B} : R' \notin \mathcal{B} \text{ for all } R' \in \mathfrak{D}(\mathcal{P}) \text{ such that } R \subsetneq R'\}.$$

By Lemma 3.18(ii), it is clear that the elements of $\mathcal{C}$ are pairwise disjoint. On the other hand, $\bigcup \mathcal{C} = \bigcup \mathcal{B}$; therefore,

$$M^{\mathcal{P}} f(x) \leq \alpha \quad \text{for all } x \in \Omega, \tag{3.11}$$

where Ω is the complement of $\bigcup \mathcal{C}$ in G. Further, $R \in \mathcal{B}$ for all $R \in \mathcal{C}$. Hence $R \notin \mathcal{P}$; consequently, R is the child of some $R' \in \mathfrak{D}(\mathcal{P}) \setminus \mathcal{B}$. Therefore,

$$\alpha \leq \mu(R)^{-1} \int_R |f| \, \mathrm{d}\mu \leq C_2 \mu(R')^{-1} \int_{R'} |f| \, \mathrm{d}\mu < C_2 \alpha, \tag{3.12}$$

by Lemma 3.16(iii).

For $E \in \mathcal{C}$, define

$$g = \sum_{E \in \mathcal{C}} \Big(\frac{1}{\mu(E)} \int_E f \, \mathrm{d}\mu \Big) \chi_E + f \chi_\Omega \quad \text{and} \quad b_E = \Big(f - \frac{1}{\mu(E)} \int_E f \, \mathrm{d}\mu \Big) \chi_E.$$

By (3.12), it follows that $|g| \leq C_2 \alpha$ on each set $E \in \mathcal{C}$. Moreover, by (3.11) and Proposition 3.19(iii),

$$|g(x)| = |f(x)| \leq \alpha \quad \text{for a.a. } x \in \Omega.$$

Each function b_E is supported in E and has average zero. Moreover,

$$\sum_{E\in\mathcal{C}} \|b_E\|_1 \leq 2\sum_{E\in\mathcal{C}} \int_E |f|\,\mathrm{d}\mu \leq 2\|f\|_1.$$

Finally, again by (3.12) and the disjointness of $\mathcal{C}$,

$$\sum_{E\in\mathcal{C}} \mu(E) \leq \frac{1}{\alpha}\sum_{E\in\mathcal{C}} \int_E |f|\,\mathrm{d}\mu \leq \frac{1}{\alpha}\|f\|_1.$$

Thus $f = g + \sum_{E\in\mathcal{C}} b_E$ is a Calderón–Zygmund decomposition of the function f at height α. The CZ-constant of the space is $\kappa_0 = \max\{C_1, C_2, C^*\}$.

To conclude the proof, we construct a family of admissible sets $\mathcal{R}'$ which satisfies condition (C). For all $k \in \mathbb{Z}^+$, set $r_k = \frac{k}{2M}\log\eta$. Clearly, $e^{2Mr_k} \leq \eta^k < 4e^{8Mr_k}$, and $r_k \to \infty$ as $k \to \infty$, so $r_k \geq r_0$ if $k \geq k_0$, say. Consequently, for all $k \geq k_0$ and $\alpha \in I_k$, the sets

$$R^k_\alpha = Q^k_\alpha \times (-r_k, r_k) \tag{3.13}$$

are admissible. Set $\mathcal{R}' = \{R^k_\alpha : k \geq k_0, \alpha \in I_k\}$ and note that the following properties hold.

(i) If $R^k_\alpha \cap R^\ell_\beta \neq \varnothing$ and $k > \ell$, then $R^\ell_\beta \subset R^k_\alpha$.

(ii) If $R = Q^\ell_\beta \times (u_0 - r, u_0 + r)$ is an admissible set, then there exist $k \geq k_0$ and $\alpha \in I_k$ such that $R \subset R^k_\alpha$. Indeed, we may choose $k \geq \max\{\ell, k_0\}$ such that $(u_0 - r, u_0 + r) \subset (-r_k, r_k)$. In this case, there exists $\alpha \in I_k$ such that $Q^\ell_\beta \subset Q^k_\alpha$.

Thus condition (C) is satisfied. □

Since by Theorem 3.20 the space (G, ϱ, μ) with the family $\mathcal{R}$ of admissible sets satisfies condition (C), we can define a Hardy space $H^1(G)$ and a space $BMO(G)$ as in Definitions 3.5 and 3.6. By using the geometric properties of (G, ϱ, μ) and the properties of admissible sets, one can easily check that all the results obtained in [53] and [35] for Hardy and BMO spaces on $ax + b$-groups can be proved also in our setting, with only slight changes in their proofs; see, e.g., [5] for definition and discussion of the real and complex interpolation methods.

Proposition 3.21. (i) *(John–Nirenberg inequality) There exist positive constants γ and D such that*

$$\mu\big(\{x \in R : |g(x) - g_R| > s\|g\|_{BMO}\}\big) \leq De^{-\gamma s}\mu(R)$$

for all $s > 0$, $R \in \mathcal{R}$, and $g \in BMO(G)$.

(ii)
$$\big(H^1(G), L^2(G)\big)_{\theta,p} = L^p(G),$$
where $\theta \in (0,1)$*,* $\frac{1}{p} = 1 - \frac{\theta}{2}$ *and* $\big(\cdot,\cdot\big)_{\theta,p}$ *denotes the interpolation space obtained by the real method;*

(iii)
$$\big(H^1(G), L^2(G)\big)_{[\theta]} = L^p(G),$$
where $\theta \in (0,1)$*,* $\frac{1}{p} = 1 - \frac{\theta}{2}$*, and* $\big(\cdot,\cdot\big)_{[\theta]}$ *denotes the interpolation space obtained by the complex method;*

(iv)
$$\big(L^2(G), BMO(G)\big)_{\theta,p} = L^p(G),$$
where $\theta \in (0,1)$*,* $\frac{1}{p} = \frac{1-\theta}{2}$*;*

(v)
$$\big(L^2(G), BMO(G)\big)_{[\theta]} = L^p(G),$$
where $\theta \in (0,1)$ *and* $\frac{1}{p} = \frac{1-\theta}{2}$.

4 The sub-Laplacian Δ, its heat kernel, and its spectral multipliers

4.1 The sub-Laplacian. Let Δ be the sub-Laplacian defined in (1.1). We recall now some well-known properties of Δ, that are common to all left-invariant sub-Laplacians on Lie groups; see, e.g., [55], [36], and references therein for further details.

Since the horizontal distribution on G is bracket-generating, Δ is hypoelliptic [31]. Moreover, Δ is essentially self-adjoint and positive with respect to the right Haar measure; in fact, for all $f, g \in C_c^\infty(G)$,

$$\langle \Delta f, g\rangle = \sum_{j=0}^{q} \langle X_j f, X_j g\rangle, \tag{4.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of $L^2(G)$.

In particular, Δ extends uniquely to a positive self-adjoint operator on $L^2(G)$; and, for all bounded Borel functions $F : \mathbb{R}_0^+ \to \mathbb{C}$, the operator $F(\Delta)$ is a convolution operator with kernel $k_{F(\Delta)}$; see (1.2). By means of the convolution formula, when $k_{F(\Delta)} \in L^1_{\mathrm{loc}}(G)$, we can interpret $F(\Delta)$ as an integral operator with integral kernel $K_{F(\Delta)}$ given by

$$K_{F(\Delta)}(x,y) = k_{F(\Delta)}(y^{-1}x)m(y) \quad \text{for a.a. } x, y \in G. \tag{4.2}$$

In the sequel, we will often make use of some properties of differential equations associated with Δ. First of all, we have finite propagation speed [40, 13] for solutions of the wave equation:

$$\operatorname{supp}(\cos(t\sqrt{\Delta})f) \subset \{x \in G : \varrho(x, \operatorname{supp} f) \leq t\}$$

for all $f \in L^2(G)$ and all $t \geq 0$.

Moreover, since Δ is associated to the Dirichlet form (4.1) and annihilates constants, the heat kernel $t \mapsto h_t = k_{e^{-t\Delta}}$ is a semigroup of probability measures on G [32]. By hypoellipticity of $\partial_t + \Delta$, the distribution $(t, x) \mapsto h_t(x)$ is in fact a smooth function on $\mathbb{R}^+ \times G$; and, from the above discussion, it follows that

$$h_t * h_{t'} = h_{t+t'}, \quad h_t \geq 0, \quad \|h_t\|_1 = 1$$

(semigroup of probability measures) and

$$h_t^* = h_t, \quad h_t(x) \leq m(x)^{1/2} h_t(0)$$

(self-adjointness and positivity on L^2). It is also possible to obtain "Gaussian-type" estimates for h_t and its left-invariant derivatives: for all $p \in [1, \infty]$, $\alpha = (\alpha_0, \dots, \alpha_q) \in \mathbb{N}^{1+q}$, and $b \geq 0$, there exist $C, \omega \geq 0$ such that

$$\|e^{b|\cdot|_\varrho} X^\alpha h_t\|_p \leq C t^{-(Q+1)/(2p')-|\alpha|/2} e^{\omega t}, \tag{4.3}$$

where $p' = p/(p-1)$, $X^\alpha = X_0^{\alpha_0} \cdots X_q^{\alpha_q}$, and $|\alpha| = \alpha_0 + \dots + \alpha_q$; see, e.g., [55], [50], or [36, Theorem 2.3(f)]. However, these estimates are of little use for t large.

4.2 L^1 gradient heat kernel estimates. The heat kernel h_t associated to Δ can be expressed in terms of the heat kernel h_t^N associated to the sub-Laplacian $\Delta^N = -\sum_{j=1}^q \breve{X}_j^2$ on N (see [43, §3] or [19, §2]):

$$h_t(z, u) = \int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh u}{\xi}\right) h^N_{e^u \xi/2}(z) \,\mathrm{d}\xi, \tag{4.4}$$

where

$$\Psi_t(\xi) = \frac{\xi^{-2}}{\sqrt{4\pi^3 t}} \exp\left(\frac{\pi^2}{4t}\right) \int_0^\infty \sinh\theta \sin\frac{\pi\theta}{2t} \exp\left(-\frac{\theta^2}{4t} - \frac{\cosh\theta}{\xi}\right) \mathrm{d}\theta. \tag{4.5}$$

This formula was used in the aforementioned works to obtain L^1-estimates for the heat kernel h_t at complex times $t = 1 + i\tau$, $\tau \in \mathbb{R}$. Here we show that the same formula can be used to obtain L^1-estimates for the horizontal gradient of the heat kernel h_t at real times $t \in \mathbb{R}$.

For a (smooth) function f on G, define the horizontal gradient $\nabla_H f(x) \in H_x G$ at $x \in G$ by

$$g_x(\nabla_H f(x), v) = (\mathrm{d}f)_x(v) \quad \text{for all } v \in H_x G,$$

where $(\mathrm{d}f)_x$ is the differential of f at x. It is easily seen that

$$|\nabla_H f(x)|_g^2 = g_x(\nabla_H f(x), \nabla_H f(x)) = \sum_{j=0}^{q} |X_j f(x)|^2. \tag{4.6}$$

We use the following technical lemma repeatedly to estimate the L^1-norm of $|\nabla_H h_t|_g$.

Lemma 4.1. *For all $\alpha, \theta \geq 0$,*

$$\int_{\mathbb{R}} \int_0^\infty \frac{\cosh(\alpha u)}{\xi^{2+\alpha}} \exp\left(-\frac{\cosh\theta + \cosh u}{\xi}\right) \mathrm{d}\xi \, \mathrm{d}u \leq C_\alpha \begin{cases} e^{-\theta} & \text{if } \alpha > 0, \\ e^{-\theta}(1+\theta) & \text{if } \alpha = 0. \end{cases}$$

Proof. The inner integral in ξ is convergent and, after a rescaling, is equal to a constant times $(\cosh\theta + \cosh u)^{-1-\alpha} \cosh(\alpha u)$. Consequently, the integral in u is controlled by a constant times

$$\int_0^\infty (e^\theta + e^u)^{-1-\alpha} e^{\alpha u} \, \mathrm{d}u = e^{-\theta} \int_{e^{-\theta}}^\infty (1+v)^{-1-\alpha} v^{\alpha-1} \, \mathrm{d}v,$$

and the conclusion follows. □

Proposition 4.2. *There exists $C > 0$ such that*

$$\left\| |\nabla_H h_t|_g \right\|_1 \leq C t^{-1/2} \quad \textit{for all } t \in \mathbb{R}^+.$$

Proof. By (4.6), it suffices to show that $\|X_j h_t\|_1 \leq C t^{-1/2}$ for all $j \in \{0, 1, \ldots, q\}$ and $t \in \mathbb{R}^+$. These estimates are already known in the case $t \leq 1$; see (4.3). Therefore, throughout the rest of the proof, we assume that $t \geq 1$.

Note that, by homogeneity considerations, the corresponding estimates for the heat kernel on N are easily shown to hold for all times. In fact,

$$\|h_s^N\|_{L^1(N)} = 1, \quad \|\check{X}_j h_s^N\|_{L^1(N)} = C_j s^{-1/2} \tag{4.7}$$

for all $s \in \mathbb{R}^+$ and $j \in \{1, \ldots, q\}$; see, e.g., [17, Proposition (1.75)]. These equations, together with the formula (4.4), are the main ingredient of our proof.

We consider first the case $j > 0$. Recall that $X_j = e^u \check{X}_j$. Then, by (4.4) and differentiation under the integral sign,

$$X_j h_t(z, u) = \int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh u}{\xi}\right) e^u \check{X}_j h^N_{e^u \xi/2}(z) \, \mathrm{d}\xi.$$

Therefore, by (4.7),

$$\|X_j h_t\|_1 \lesssim \int_{\mathbb{R}} \int_0^\infty |\Psi_t(\xi)| \frac{e^{u/2}}{\xi^{1/2}} \exp\left(-\frac{\cosh u}{\xi}\right) \mathrm{d}\xi \, \mathrm{d}u.$$

Since $t \geq 1$, by (4.5), the above integral is controlled by a constant times

$$t^{-1/2} \int_0^\infty \sinh\theta \left|\sin \frac{\pi\theta}{2t}\right| \exp\left(-\frac{\theta^2}{4t}\right) \times \int_{\mathbb{R}} \int_0^\infty \frac{e^{u/2}}{\xi^{2+1/2}} \exp\left(-\frac{\cosh\theta + \cosh u}{\xi}\right) \mathrm{d}\xi \, \mathrm{d}u \, \mathrm{d}\theta.$$

By applying Lemma 4.1 (with $\alpha = 1/2$), we can control the integral in u by a constant times $e^{-\theta}$. Hence

$$\|X_j h_t\|_1 \lesssim t^{-1/2} \int_0^\infty \frac{\sinh\theta}{e^\theta} \frac{\theta}{t} \exp\left(-\frac{\theta^2}{4t}\right) \mathrm{d}\theta \lesssim t^{-1/2}.$$

For $j = 0$ we have instead, again by (4.4),

$$X_0 h_t(z,u) = -\int_0^\infty \Psi_t(\xi) \frac{\sinh u}{\xi} \exp\left(-\frac{\cosh u}{\xi}\right) h^N_{e^u\xi/2}(z) \, \mathrm{d}\xi + \int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh u}{\xi}\right) \frac{\partial}{\partial u}[h^N_{e^u\xi/2}(z)] \, \mathrm{d}\xi = I_1 + I_2.$$

The L^1-norm of the first summand I_1 can be controlled analogously as above (here the first identity in (4.7) is used and Lemma 4.1 is applied with $\alpha = 1$). For the second term I_2, we need some further manipulation.

Note that $\partial_u[h^N_{e^u\xi/2}(z)] = \xi\partial_\xi[h^N_{e^u\xi/2}(z)]$. Hence, by integration by parts,

$$I_2 = -\int_0^\infty \frac{\partial}{\partial\xi}[\xi\Psi_t(\xi)] \exp\left(-\frac{\cosh u}{\xi}\right) h^N_{e^u\xi/2}(z) \, \mathrm{d}\xi - \int_0^\infty \Psi_t(\xi) \frac{\cosh u}{\xi} \exp\left(-\frac{\cosh u}{\xi}\right) h^N_{e^u\xi/2}(z) \, \mathrm{d}\xi = I_3 + I_4.$$

The term I_4 can be controlled in the same way as I_1. As for I_3, we observe, by (4.5), that

$$
\begin{aligned}
&\frac{\partial}{\partial \xi}[\xi \Psi_t(\xi)] \\
&\quad = \frac{\exp\left(\frac{\pi^2}{4t}\right)}{\xi^2\sqrt{4\pi^3 t}} \int_0^\infty \sinh\theta \sin\frac{\pi\theta}{2t} \exp\left(-\frac{\theta^2}{4t} - \frac{\cosh\theta}{\xi}\right)\left(\frac{\cosh\theta}{\xi} - 1\right) \mathrm{d}\theta \\
&\quad = -\frac{\exp\left(\frac{\pi^2}{4t}\right)}{\xi^2\sqrt{4\pi^3 t}} \int_0^\infty \cosh\theta \sin\frac{\pi\theta}{2t} \exp\left(-\frac{\theta^2}{4t}\right) \frac{\partial}{\partial\theta}\left[\exp\left(-\frac{\cosh\theta}{\xi}\right)\right] \mathrm{d}\theta - \Psi_t(\xi) \\
&\quad = \frac{\exp\left(\frac{\pi^2}{4t}\right)}{\xi^2\sqrt{4\pi^3 t}} \int_0^\infty \frac{\partial}{\partial\theta}\left[\cosh\theta \sin\frac{\pi\theta}{2t} \exp\left(-\frac{\theta^2}{4t}\right)\right] \exp\left(-\frac{\cosh\theta}{\xi}\right) \mathrm{d}\theta - \Psi_t(\xi) \\
&\quad = \frac{\exp\left(\frac{\pi^2}{4t}\right)}{2t\xi^2\sqrt{4\pi^3 t}} \int_0^\infty \cosh\theta \left[\pi\cos\frac{\pi\theta}{2t} - \theta\sin\frac{\pi\theta}{2t}\right] \exp\left(-\frac{\theta^2}{4t}\right) \exp\left(-\frac{\cosh\theta}{\xi}\right) \mathrm{d}\theta.
\end{aligned}
$$

Consequently, since $t \geq 1$, by (4.7) the L^1-norm of I_3 is bounded by a constant times

$$
\begin{aligned}
&t^{-3/2} \int_0^\infty \cosh\theta \left|\pi\cos\frac{\pi\theta}{2t} - \theta\sin\frac{\pi\theta}{2t}\right| \exp\left(-\frac{\theta^2}{4t}\right) \\
&\qquad\qquad \times \int_{\mathbb{R}} \int_0^\infty \xi^{-2} \exp\left(-\frac{\cosh\theta + \cosh u}{\xi}\right) \mathrm{d}\xi\, \mathrm{d}u\, \mathrm{d}\theta.
\end{aligned}
$$

By applying Lemma 4.1 (with $\alpha = 0$), we can control the integral in u by a constant times $e^{-\theta}(1+\theta)$; hence

$$
\|I_3\|_1 \lesssim t^{-3/2} \int_0^\infty \frac{\cosh\theta}{e^\theta}(1+\theta^2/t)(1+\theta)\ \exp\left(-\frac{\theta^2}{4t}\right) \mathrm{d}\theta \lesssim t^{-1/2},
$$

and we are done. □

4.3 The Plancherel measure and weighted L^2-estimates. By abstract nonsense (see, e.g., [36, Theorem 3.10] for a quite general statement), one can show that there exists a Plancherel measure associated with Δ, i.e., a positive Borel measure σ_Δ on $\mathbb{R}_0^+$ whose support is the $L^2(G)$-spectrum of Δ, such that

$$
\|k_{F(\Delta)}\|_2^2 = \int_{\mathbb{R}_0^+} |F(\lambda)|^2 \,\mathrm{d}\sigma_\Delta(\lambda) \tag{4.8}
$$

for all bounded Borel functions $F : \mathbb{R}_0^+ \to \mathbb{C}$.

In the case N is abelian, G is a real hyperbolic space, and the Plancherel measure σ_Δ can be explicitly computed via spherical analysis (cf. [11, 12]); namely, there exists $c_\Delta \in \mathbb{R}^+$ such that

$$
\int_{\mathbb{R}_0^+} F(\lambda)\,\mathrm{d}\sigma_\Delta(\lambda) = c_\Delta \int_{\mathbb{R}} F(s^2)|\mathbf{c}_Q(s)|^{-2}\,\mathrm{d}s, \tag{4.9}
$$

where $\mathbf{c}_Q$ is the Harish-Chandra function for the $(Q+1)$-dimensional real hyperbolic space (see, e.g., [22, Theorem IV.6.14]), so

$$|\mathbf{c}_Q(s)|^{-2} \sim \begin{cases} |s|^2 & \text{for } |s| \text{ small,} \\ |s|^Q & \text{for } |s| \text{ large.} \end{cases}$$

In the case N is nonabelian, spherical analysis can no longer be applied directly to the functional calculus of Δ. Nevertheless, as we show, the above formula for the Plancherel measure remains valid.

Let $\mathfrak{J}$ be the set of functions $\mathbb{R}_0^+ \to \mathbb{C}$ that are finite linear combinations of decaying exponentials $\lambda \mapsto e^{-t\lambda}$ ($t \in \mathbb{R}^+$). Note that $\mathfrak{J}$ is uniformly dense in $C_0(\mathbb{R}_0^+)$ by the Stone–Weierstrass Theorem. The following fundamental observation is in [24, Lemma (1.10)]; here we provide an alternative proof, using (4.4).

Proposition 4.3. *For all $F \in \mathfrak{J}$ and all $u \in \mathbb{R}$, there exists a bounded Borel function $M_{F,u} : \mathbb{R}_0^+ \to \mathbb{C}$ such that*

$$k_{F(\Delta)}(\cdot, u) = k_{M_{F,u}(\Delta^N)}, \tag{4.10}$$

and $M_{F,u}$ depends neither on the stratified group N nor on the sub-Laplacian Δ^N.

Proof. By linearity, it is sufficient to consider the case $F(\lambda) = e^{-t\lambda}$. However in this case, if we set

$$M_{F,u}(\lambda) = \int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh u}{\xi}\right) \exp(-e^u \xi\lambda/2)\, d\xi,$$

then (4.10) follows from the formula (4.4) for the heat kernel $h_t = k_{e^{-t\Delta}}$. Note that the above expression for $M_{F,u}$ depends only on t and u and does not depend on the particular choice of N or Δ^N. □

Let $\Delta^{\mathbb{R}^Q}$ be the Laplacian on $\mathbb{R}^Q$ and $\tilde{\Delta} = -\partial_u^2 + e^{2u}\Delta^{\mathbb{R}^Q}$ be the corresponding Laplacian on $\tilde{G} = \mathbb{R}^Q \rtimes \mathbb{R}$. Homogeneity and finite propagation speed properties of Δ^N and $\Delta^{\mathbb{R}^Q}$ yield the following result.

Proposition 4.4 (See [47, formula (3) and Lemme 2])**.** *For all $a \geq 0$, there is $C \in \mathbb{R}^+$ such that, for all bounded Borel functions $f : \mathbb{R} \to \mathbb{C}$,*

$$\int_N |z|_N^a |k_{f(\Delta^N)}(z)|^2\, dz \leq C \int_{\mathbb{R}^Q} |z|_{\mathbb{R}^Q}^a |k_{f(\Delta^{\mathbb{R}^Q})}(z)|^2\, dz,$$

with equality if $a = 0$.

Corollary 4.5. *For all $a \geq 0$, there is $C \in \mathbb{R}^+$ such that, for all bounded Borel functions $F : \mathbb{R} \to \mathbb{C}$,*

$$\int_G |z|_N^a |k_{F(\Delta)}(z,u)|^2 \, \mathrm{d}\mu(z,u) \leq C \int_{\tilde{G}} |z|_{\mathbb{R}^Q}^a |k_{F(\tilde{\Delta})}(z,u)|^2 \, \mathrm{d}\tilde{\mu}(z,u), \tag{4.11}$$

with equality if $a = 0$, where $\mathrm{d}\tilde{\mu}(z,u) = \mathrm{d}z\,\mathrm{d}u$ is the right Haar measure on $\tilde{G}$.

Proof. In the case $F \in \mathfrak{J}$, the above inequality (or equality if $a = 0$) follows immediately from a combination of Propositions 4.3 and 4.4. The general case is then given by density. □

By comparing the case $a = 0$ of Corollary 4.5 with the characterization (4.8) of the Plancherel measure, we obtain immediately the following result.

Corollary 4.6. *For an arbitrary stratified group N of homogeneous dimension Q, the Plancherel measure σ_Δ is given by* (4.9) *for some constant $c_\Delta \in \mathbb{R}^+$. In particular, the L^2-spectrum of Δ is $\mathbb{R}_0^+$ and, for all Borel functions $F : \mathbb{R} \to \mathbb{C}$,*

$$\|k_{F(\sqrt{\Delta})}\|_2 \sim \left(\int_0^\infty |F(\lambda)|^2 (\lambda^3 + \lambda^{Q+1}) \frac{\mathrm{d}\lambda}{\lambda} \right)^{1/2}.$$

5 The multiplier theorem

In this section, we prove Theorem 1.1. To do so, we need some preliminary estimates of the L^1-norm of the convolution kernels of spectral multipliers of Δ.

Proposition 5.1. *There exists a positive constant C such that, for all $r > 0$ and all even bounded Borel functions $F : \mathbb{R} \to \mathbb{C}$ whose Fourier transform $\hat{F}$ is supported in $[-r, r]$,*

$$\|k_{F(\sqrt{\Delta})}\|_1 \leq C \min\{r^{(Q+1)/2}, r^{3/2}\} \|k_{F(\sqrt{\Delta})}\|_2.$$

Proof. Note that, since Δ satisfies finite propagation speed, $\operatorname{supp} k_{F(\sqrt{\Delta})} \subset \overline{B_\varrho(0,r)}$; see, e.g., [14, Lemma 1.2]. Then, if $r \leq 1$, by Hölder's inequality and (2.11),

$$\|k_{F(\sqrt{\Delta})}\|_1 \lesssim r^{(Q+1)/2} \|k_{F(\sqrt{\Delta})}\|_2,$$

and we are done.

If $r \geq 1$, then, by Hölder's inequality and (2.13),

$$\|k_{F(\sqrt{\Delta})}\|_1 \lesssim r \left(\|k_{F(\sqrt{\Delta})}\|_2 + \|k_{F(\sqrt{\Delta})} w^{1/2}\|_2 \right), \tag{5.1}$$

where the weight w is given by $w(z,u) = |z|_N^Q$. Therefore, by applying Corollary 4.5 with $a = Q$, we obtain

$$\|k_{F(\sqrt{\Delta})} w^{1/2}\|_2 \lesssim \|k_{F(\sqrt{\tilde\Delta})} \tilde w^{1/2}\|_{L^2(\tilde G)}, \tag{5.2}$$

where $\tilde w$ is the analogous weight on $\tilde G = \mathbb{R}^Q \rtimes \mathbb{R}$. By spherical analysis on real hyperbolic spaces, if $\tilde m$ is the modular function on $\tilde G$, then $k_{F(\sqrt{\tilde\Delta})} = \tilde m^{1/2}\phi_F$ for some radial function ϕ_F on $\tilde G$; see, e.g., [11, Proposition 1.2] and [1, p. 148]. Moreover, if $\tilde\varrho$ denotes the left-invariant Riemannian distance on $\tilde G$, then $\operatorname{supp}\phi_F = \operatorname{supp} k_{F(\sqrt{\tilde\Delta})} \subset \overline{B_{\tilde\varrho}(0,r)}$, because $\tilde\Delta$ satisfies finite propagation speed too. We can then apply (2.12) and (2.10) to obtain

(5.3)

$$\begin{aligned}\|k_{F(\sqrt{\tilde\Delta})}\tilde w^{1/2}\|_{L^2(\tilde G)} &= \|\tilde w^{1/2}\, \tilde m^{1/2}\phi_F\|_{L^2(\tilde G)} \lesssim \|\phi_F\tilde\varrho(\cdot,0_{\tilde G})^{1/2}\|_{L^2(\tilde G)}\\ &\lesssim r^{1/2}\|\phi_F\|_{L^2(\tilde G)} = r^{1/2}\|\phi_F\tilde m^{1/2}\|_{L^2(\tilde G)} = r^{1/2}\|k_{F(\sqrt{\tilde\Delta})}\|_{L^2(\tilde G)}\\ &\sim r^{1/2}\|k_{F(\sqrt{\Delta})}\|_2,\end{aligned}$$

where the last step is given by Corollary 4.5 in the case $a = 0$. The conclusion follows once we combine (5.2) and (5.3) and plug the resulting inequality into (5.1). □

The next lemma shows that every function f supported in $[1/2, 2]$ may be written as sum of functions whose Fourier transforms have compact support.

Lemma 5.2 (See [24, Lemma (1.3)]). *Let $f \in L^2(\mathbb{R})$ be even and supported in $[-2,2]$. Then there exist even functions f_ℓ, $\ell \in \mathbb{N}$, such that*

(i) $f = \sum_{\ell=0}^\infty f_\ell$;
(ii) $\operatorname{supp}\hat f_\ell \subset [-2^\ell, 2^\ell]$;
(iii) *for all $\alpha,\beta,s \in \mathbb{R}_0^+$,*

$$\int_0^\infty |f_\ell(\lambda)|^2(\lambda^\alpha + \lambda^\beta)\,\mathrm{d}\lambda \le C_{\alpha,\beta,s}2^{-2s\ell}\|f\|^2_{H^s(\mathbb{R})}.$$

Let f_t denote the dilated of f defined by $f_t = f(t\cdot)$. Then

(i') $f_t = \sum_\ell f_{\ell,t}$, *where* $f_{\ell,t} = f_\ell(t\cdot)$;
(ii') $\operatorname{supp}\hat f_{\ell,t} \subset [-2^\ell t, 2^\ell t]$;
(iii') *for all $\alpha,\beta,s \in \mathbb{R}_0^+$,*

$$\int_0^\infty |f_{\ell,t}(\lambda)|^2(\lambda^\alpha + \lambda^\beta)\,\mathrm{d}\lambda \le C_{\alpha,\beta,s}\max\{t^{-(\alpha+1)}, t^{-(\beta+1)}\}2^{-2s\ell}\|f\|^2_{H^s(\mathbb{R})}.$$

Proposition 5.3. *Let $F \in L^2(\mathbb{R})$ be supported in $[-4,4]$. Then*

$$\sup_{y\in G}\int_G |K_{F(t\Delta)}(x,y)|\big(1+t^{-1/2}\varrho(x,y)\big)^\varepsilon\,\mathrm{d}\mu(x) \le C_{s,\varepsilon}\|F\|_{H^s} \tag{5.4}$$

for all $\varepsilon \in \mathbb{R}_0^+$ *and* $s, t \in \mathbb{R}^+$ *satisfying one of the following conditions:*

- $t \geq 1$ *and* $s > 3/2 + \varepsilon$;
- $t \leq 1$ *and* $s > (Q+1)/2 + \varepsilon$.

Proof. First we observe that, for all $y \in G$, by (4.2) and the left-invariance of the metric ϱ,

$$\int_G |K_{F(t\Delta)}(x,y)|\big(1 + t^{-1/2}\varrho(x,y)\big)^{\varepsilon} \,\mathrm{d}\mu(x) = \int_G |k_{F(t\Delta)}(y^{-1}x)| m(y)\big(1 + t^{-1/2}\varrho(y^{-1}x, 0_G)\big)^{\varepsilon} \,\mathrm{d}\mu(x) = \int_G |k_{F(t\Delta)}(x)|\big(1 + t^{-1/2}\varrho(x, 0_G)\big)^{\varepsilon} \,\mathrm{d}\mu(x).$$

Define $f(\lambda) = F(\lambda^2)$ for all $\lambda \in \mathbb{R}$. The function f is even and supported in $[-2, 2]$, and $F(t\Delta) = f(t^{1/2}\sqrt{\Delta})$ for all $t \in \mathbb{R}^+$. Moreover,

$$\|f\|_{H^s} \lesssim \|F\|_{H^s}. \tag{5.5}$$

Let $f = \sum_{\ell=0}^{\infty} f_\ell$ be the decomposition given by Lemma 5.2. Since $f(t^{1/2}\cdot) = \sum_\ell f_{\ell,t^{1/2}}$ and $\operatorname{supp} \hat{f}_{\ell,t^{1/2}} \subset [-2^\ell t^{1/2}, 2^\ell t^{1/2}]$, we can apply Proposition 5.1 to each function $f_{\ell,t^{1/2}}$ and sum these estimates. Namely, by finite propagation speed, Proposition 5.1, Corollary 4.6, Lemma 5.2(iii'), and (5.5),

$$\begin{aligned}
&\int_G |k_{f_{\ell,t^{1/2}}(\sqrt{\Delta})}(x)|\big(1 + t^{-1/2}\varrho(x, 0_G)\big)^{\varepsilon} \,\mathrm{d}\mu(x) \\
&\quad\lesssim (1 + t^{-1/2} 2^\ell t^{1/2})^{\varepsilon} \|k_{f_{\ell,t^{1/2}}(\sqrt{\Delta})}\|_1 \\
&\quad\lesssim 2^{\ell\varepsilon} \min\{(2^\ell t^{1/2})^{(Q+1)/2}, (2^\ell t^{1/2})^{3/2}\} \|k_{f_{\ell,t^{1/2}}(\sqrt{\Delta})}\|_2 \\
&\quad\lesssim 2^{\ell\varepsilon} \min\{(2^\ell t^{1/2})^{(Q+1)/2}, (2^\ell t^{1/2})^{3/2}\} \Big(\int_0^\infty |f_{\ell,t^{1/2}}(\lambda)|^2 (\lambda^2 + \lambda^Q) \,\mathrm{d}\lambda\Big)^{1/2} \\
&\quad\lesssim 2^{\ell\varepsilon} \min\{(2^\ell t^{1/2})^{(Q+1)/2}, (2^\ell t^{1/2})^{3/2}\} \max\{t^{-3/4}, t^{-(Q+1)/4}\} 2^{-\ell s} \|F\|_{H^s}.
\end{aligned}$$

In the case $t \geq 1$,

$$\int_G |k_{F(t\Delta)}(x)|\big(1 + t^{-1/2}\varrho(x, 0_G)\big)^{\varepsilon} \,\mathrm{d}\mu(x) \leq C_s \|F\|_{H^s} \sum_{\ell \geq 0} 2^{\ell(\varepsilon + 3/2 - s)},$$

and the series on the right-hand side converges. since $s > 3/2 + \varepsilon$.

In the case $t \leq 1$,

$$\begin{aligned}
&\int_G |k_{F(t\Delta)}(x)|\big(1 + t^{-1/2}\varrho(x, 0_G)\big)^{\varepsilon} \,\mathrm{d}\mu(x) \\
&\qquad\lesssim \|F\|_{H^s} \left(t^{3/4 - (Q+1)/4} \sum_{\ell : 2^\ell \geq t^{-1/2}} 2^{\ell(\varepsilon + 3/2 - s)} + \sum_{\ell : 2^\ell < t^{-1/2}} 2^{\ell\left(\varepsilon + (Q+1)/2 - s\right)} \right),
\end{aligned}$$

and the term in parentheses is finite and bounded above uniformly in t, since $s > (Q+1)/2 + \varepsilon$. ☐

We denote by R_y the right translation operator defined by

$$R_y f(x) = f(xy) \quad \text{for all } f : G \to \mathbb{C} \text{ and } x, y \in G.$$

Lemma 5.4. *For all $f \in L^1(G)$ and $y, z \in G$,*

$$\|R_y f - R_z f\|_1 \le \varrho(y,z) \left\| |\nabla_H f|_g \right\|_1 .$$

This follows, since the proof of [55, Lemma VIII.1.1] applies also to non-unimodular groups.

Proposition 5.5. *Let $F \in L^2(\mathbb{R})$ be supported in* $[-4, 4]$. *Then*

$$\int_G |K_{F(t\Delta)}(x,y) - K_{F(t\Delta)}(x,z)| \, \mathrm{d}\mu(x) \le C_s t^{-1/2} \varrho(y,z) \|F\|_{H^s} \tag{5.6}$$

for all $y, z \in G$ and $s, t \in \mathbb{R}^+$ satisfying one of the following conditions:

- *$t \ge 1$ and $s > 3/2$,*
- *$t \le 1$ and $s > (Q+1)/2$.*

Proof. Since F can be split into its real and imaginary parts, it is not restrictive to assume that F is real-valued. In particular, the operator $F(t\Delta)$ is self-adjoint, and

$$\begin{aligned}\int_G |K_{F(t\Delta)}(x,y) - K_{F(t\Delta)}(x,z)| \, \mathrm{d}\mu(x) &= \int_G |K_{F(t\Delta)}(y,x) - K_{F(t\Delta)}(z,x)| \, \mathrm{d}\mu(x) \\ &= \int_G |k_{F(t\Delta)}(x^{-1}y) - k_{F(t\Delta)}(x^{-1}z)| m(x) \, \mathrm{d}\mu(x) \\ &= \int_G |k_{F(t\Delta)}(xy) - k_{F(t\Delta)}(xz)| \, \mathrm{d}\mu(x) \\ &= \|R_y k_{F(t\Delta)} - R_z k_{F(t\Delta)}\|_1 .\end{aligned}$$

Define the function $\phi(\lambda) = F(\lambda) e^{-\lambda}$ for all $\lambda \in \mathbb{R}$. Then $k_{F(t\Delta)} = k_{\phi(t\Delta)} * h_t$ and, by Young's inequality,

$$\|R_y k_{F(t\Delta)} - R_z k_{F(t\Delta)}\|_1 \le \|k_{\phi(t\Delta)}\|_1 \|R_y h_t - R_z h_t\|_1 .$$

Note now that, under our assumptions on t and s, it follows from Proposition 5.3 that

$$\|k_{\phi(t\Delta)}\|_1 \lesssim \|\phi\|_{H^s} \lesssim \|F\|_{H^s} .$$

On the other hand, by Lemma 5.4 and Proposition 4.2,

$$\|R_y h_t - R_z h_t\|_1 \le \varrho(y,z) \left\| |\nabla_H h_t|_g \right\|_1 \lesssim t^{-1/2} \varrho(y,z),$$

and the conclusion follows. ☐

We can finally prove our main result.

Proof of Theorem 1.1. Choose $\varepsilon > 0$ such that $s_0 > \frac{3}{2}+\varepsilon$ and $s_\infty > \frac{Q+1}{2}+\varepsilon$. Let F be as in the statement of the theorem. It is not restrictive to assume that F is real-valued, so $F(\Delta)$ is self-adjoint. For each $j \in \mathbb{Z}$, define the function

$$F_j(\lambda) = F(2^j\lambda)\psi(\lambda) \quad \text{for all } \lambda \in \mathbb{R}^+,$$

where ψ is as in (1.3). Then $F(\Delta) = \sum_{j\in\mathbb{Z}} F_j(2^{-j}\Delta)$ in the sense of strong convergence of operators on $L^2(G)$, because the L^2-spectrum of Δ is $\mathbb{R}_0^+$ and $\{0\}$ has null spectral measure. Since each function F_j is supported in $[1/4, 4]$, we may apply estimates (5.4) and (5.6) to F_j and $t = 2^{-j}$ to obtain

$$\sup_{y\in G}\int_G |K_{F_j(2^{-j}\Delta)}(x,y)|\big(1+2^{j/2}\varrho(x,y)\big)^\varepsilon \,\mathrm{d}\mu(x) \lesssim \begin{cases} \|F\|_{0,s_0} & \text{for all } j \le 0, \\ \|F\|_{\infty,s_\infty} & \text{for all } j > 0; \end{cases} \tag{5.7}$$

and, for all $y, z \in G$,

$$\int_G |K_{F_j(2^{-j}\Delta)}(x,y) - K_{F_j(2^{-j}\Delta)}(x,z)| \,\mathrm{d}\mu(x) \lesssim \begin{cases} 2^{j/2}\varrho(y,z)\|F\|_{0,s_0} & \text{for all } j \le 0 \\ 2^{j/2}\varrho(y,z)\|F\|_{\infty,s_\infty} & \text{for all } j > 0. \end{cases} \tag{5.8}$$

Thus the operator $F(\Delta)$ satisfies the hypotheses of Theorem 3.2 and consequently is of weak type $(1,1)$, bounded on $L^p(G)$ for all $p \in (1,2]$ and, by duality, for all $p \in [2,\infty)$. By Theorem 3.8 it follows that $F(\Delta)$ is also bounded from $H^1(G)$ to $L^1(G)$, and a duality argument gives the boundedness from $L^\infty(G)$ to $BMO(G)$. □

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