Γ-CONVERGENCE OF THE ENERGY FUNCTIONALS FOR THE VARIABLE EXPONENT $p(\cdot)$-LAPLACIAN AND STABILITY OF THE MINIMIZERS WITH RESPECT TO INTEGRABILITY

By

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Abstract. We investigate the non-homogeneous modular Dirichlet problem $\Delta_{p(\cdot)}u(x) = f(x)$ (where $\Delta_{p(\cdot)}u(x) = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u(x))$ from the functional analytic point of view and we prove the stability of the solutions $(u_{p_i})_i$ of the equation $\Delta_{p_i(\cdot)}u_{p_i(\cdot)} = f$ as $p_i(\cdot) \to q(\cdot)$ via Gamma-convergence of sequence of appropriate functionals.

1 Introduction

The study of variational problems on Banach spaces leading to non-linear differential equations demands the developments of new techniques. In this paper, we focus on generalizations of the classical Dirichlet problem

$$\Delta u(x) = f(x), \quad x \in \Omega,$$
$$u = 0, \quad x \in \partial\Omega,$$

in which the classical Laplacian is replaced with the $p(\cdot)$—Laplacian with variable p. The main interest of our paper is then to study the stability with respect to $p(\cdot)$ of the problem

$$\Delta_{p(\cdot)}u_{p(\cdot)} = f(x), \quad x \in \Omega, \qquad u = 0, \quad x \in \partial\Omega, \tag{1.1}$$

where $\Delta_{p(\cdot)}u_{p(\cdot)} = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u(x))$. The $p(\cdot)$-Laplacian and the variable-exponent spaces that underlie its study are not only interesting from the point of view of their incipient applications, which include fluid dynamics and image processing (see [3],[5] and [6]), but also because the theory constitutes a living example of a non-linear situation involving non-rearrangement invariant spaces that can be used as a testing ground for new techniques in non-linear analysis.

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The solutions of a differential equation involving an operator T on a given domain Ω depend mainly on Ω, the particular structure of T, and the underlying function space. An enormous amount of work has been devoted to the analysis of stability of a solution with respect to changes in the domain and in the inner structure of the differential operator; see [2], [17] and the references therein. In this work, we present the appropriate definition of the problem (1.1) and then show that the solutions are stable under perturbations of the integrability of the underlying space.

The paper is organized as follows. In Section 2, we review the general properties of generalized Lebesgue spaces. Section 3 is devoted to a brief presentation of the $p(\cdot)-$Laplacian, and Sections 4 and 5 contain results about the stability of the solutions for the non-homogeneous modular Dirichlet problem (1.1).

2 Modulars and generalized Lebesgue spaces

Throughout this paper, $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded, Lipschitz domain. For a measurable function $p : \Omega \to \mathbb{R}^n$, we set

$$p_- = \inf_{\Omega} p(x), \quad p_+ = \sup_{\Omega} p(x),$$

and consider the family of admissible exponent functions

$$\mathcal{P} = \{p : \Omega \to \mathbb{R},\ p \text{ Borel-measurable},\ 1 < p_- \leq p_+ < \infty\}. \tag{2.1}$$

We denote by $L^{p(\cdot)}(\Omega)$ the set of all real-valued, Borel measurable functions on Ω for which

$$\rho_{p(\cdot)}(f) := \int_{\Omega} |f(x)|^{p(x)}\, dx < \infty.$$

The function ρ_p is a convex monotone modular on $L^{p(\cdot)}(\Omega)$, and

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

defines a norm under which $L^{p(\cdot)}(\Omega)$ is a uniform convex reflexive Banach space; see [22]. It is apparent that the latter coincides with the usual Lebesgue $L^p(\Omega)$ norm when p is constant; accordingly, we refer to the family of $L^{p(\cdot)}(\Omega)$ for p as in (2.1) as the generalized Lebesgue class in Ω. The generalized Sobolev class in Ω can be defined analogously, namely,

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}$$

endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

We denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ by $W_0^{1,p(\cdot)}(\Omega)$ and furnish it with the norm

$$\|v\|_{W_0^{1,p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|\nabla v(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

The reader is referred to [9] and [18] for an exhaustive treatment of variable-exponent Lebesgue-Sobolev spaces; throughout this work, we use the results therein extensively, frequently without explicit notice. We highlight the inequalities

$$\min\left\{\rho_{p(\cdot)}^{\frac{1}{p+}}(w), \rho_{p(\cdot)}^{\frac{1}{p-}}(w)\right\} \le \|w\|_{p(\cdot)} \le \max\left\{\rho_{p(\cdot)}^{\frac{1}{p+}}(w), \rho_{p(\cdot)}^{\frac{1}{p-}}(w)\right\}, \tag{2.2}$$

which are valid for any $w \in L^{p(\cdot)}(\Omega)$; see [19]. The estimate of the norm of the embedding $L^{q(\cdot)} \hookrightarrow L^{p(\cdot)}$ resulting from the application of Hölder's inequality turns out to be too coarse for the present analysis; accordingly, we include the following version of a more refined estimate first observed in [13, Lemma 4.1] (see also [19]).

Lemma 2.1. *For p, q in $\mathcal{P}$ with $p < q < p + \epsilon$ a.e. in Ω and a Borel-measurable function $f : \Omega \to \mathbb{R}$,*

$$\int_\Omega |f(x)|^{p(x)}\, dx \le \epsilon|\Omega| + \epsilon^{-\epsilon} \int_\Omega |f(x)|^{q(x)}\, dx.$$

Corollary 2.2. *Let $p(\cdot)$ and $q(\cdot)$ be as above. Then, for the norm $\|E_{p,q}\|$ of the embedding $E_{p,q} : L^{q(\cdot)}(\Omega) \to L^{p(\cdot)}(\Omega)$,*

$$\|E_{p,q}\| \le \epsilon^{-\epsilon} + \epsilon|\Omega|.$$

To ensure the validity of the Sobolev embedding theorems needed in the sequel, we assume from now on that the exponent p is continuous in the closure of Ω, $p \in C(\overline{\Omega})$. In particular, this condition on the variable exponent is sufficient for the (bounded) embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ to be compact; see [18]. For the benefit of the reader, we conclude the section with a definition and results to which we refer to in the body of the paper. In the sequel, we use the standard notation X^* to denote the dual of a Banach space X. We write the action of $\Lambda \in X^*$ on $x \in X$ as $\langle x, \Lambda\rangle$.

Definition 2.3. Let X be a Banach space. An operator $T : X \to X^*$ is said to be of **type** M if for any weakly-convergent sequence $x_n \rightharpoonup x$ such that $T(x_n) \rightharpoonup f$ and

$$\limsup \langle x_n, T(x_n)\rangle \le \langle x, f\rangle, \tag{2.3}$$

one has $T(x) = f$. T is said to be **hemicontinuous** if for any fixed $x, y \in X$, the real-valued function $s \to \langle y, T(x + sy)\rangle$ is continuous.

Theorem 2.4. *Let X be a reflexive Banach space and $T : X \to X^*$ be hemicontinuous and monotone. Then T is of type M.*

Proof. For fixed $y \in X$, (x_n), x and f as in Definition 2.3, the assumed monotonicity of T yields $0 \le \langle x_n - y, T(x_n) - T(y)\rangle$ for all n; hence, from (2.3), we have $\langle x - y, T(y)\rangle \le \langle x - y, f\rangle$. In particular, for any $z \in X$ and $n \in \mathbb{N}$,

$$\left\langle z, T\left(x - (z/n)\right)\right\rangle \le \langle z, f\rangle ,$$

which, in conjunction with hemicontinuity, immediately yields $\langle z, T(x)\rangle \le \langle z, f\rangle$ for all $z \in X$. This implies $T(x) = f$, as claimed. □

We also need the following result.

Theorem 2.5 ([23, Theorem 2.1])**.** *, Let X be a separable and reflexive Banach space, and let $T : X \to X^*$ be of type M and bounded. If for some $f \in X^*$ there exists $\varepsilon > 0$ for which $\langle x, T(x)\rangle > \langle x, f\rangle$ for every $x \in X$ with $\|x\| > \varepsilon$, then f belongs to the range of T.*

Lemma 2.6. *For $x, y \in \mathbb{R}^n$ and constant p,*

$$\frac{1}{2}\left[\left(|x|^{p-2} - |y|^{p-2}\right)(|x|^2 - |y|^2) + (|x|^{p-2} + |y|^{p-2})|x - y|^2\right] = (|x|^{p-2}x - |y|^{p-2}y)\cdot(x - y) \tag{2.4}$$

Proof. This follows by straightforward calculation:

$$\begin{aligned}(|x|^{p-2} - |y|^{p-2})(|x|^2 - |y|^2) + |x - y|^2(|x|^{p-2} + |y|^{p-2})\\ = (x - y)\left((x + y)(|x|^{p-2} - |y|^{p-2}) + (x - y)(|x|^{p-2} + |y|^{p-2})\right)\\ = 2(x - y)(|x|^{p-2}x - |y|^{p-2}y).\end{aligned}$$

□

3 The $p(\cdot)$-Laplacian

In analogy with the constant exponent case, for $u \in W_0^{1,p(\cdot)}(\Omega)$, we define the Dirichlet $p(\cdot)$-Laplacian $\Delta_{p(\cdot)}(u)$ corresponding to a Borel-measurable function $p : \Omega \to [1, \infty)$ by

$$\Delta_{p(\cdot)}(u) = \operatorname{div}\left(|\nabla u(x)|^{p(x)-2}\nabla u(x)\right).$$

More specifically, $\Delta_{p(\cdot)}$ is the (non-linear) operator

$$\Delta_{p(\cdot)} : W_0^{1,p(\cdot)}(\Omega) \to \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$$

such that

$$\langle h, \Delta_{p(\cdot)}u\rangle = -\int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla h(x)\,dx$$

for all $u, h \in W_0^{1,p(\cdot)}(\Omega)$.

A few remarks are in order. The functional $F : W_0^{1,p(\cdot)}(\Omega) \to [0,\infty)$ defined by

$$F(w) = \int_\Omega \frac{|\nabla w(x)|^{p(x)}}{p(x)}\,dx \tag{3.1}$$

is convex, weakly-lower semicontinuous and Frechét differentiable with (see [7]), Frechét derivative given by $F' = \Delta_{p(\cdot)}$. It is obvious that when p is constant, the functional (3.1) is a multiple of the p^{th} power of the Sobolev norm, namely, $F(w) = \frac{1}{p}\|\nabla w\|^p$. For variable $p(\cdot)$, the consideration of the derivative of the norm, as opposed to the derivative of the modular (which is essentially the definition we have adopted for the $p(\cdot)$-Laplacian), leads to a different differential operator; see [10] and [11].

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $p : \Omega \to \mathbb{R}$ a Borel-measurable function satisfying $1 < p_- \le p(x) \le p_+ < \infty$ a.e. in Ω. Then the operator $\Delta_p(\cdot) : W_0^{1,p(\cdot)}(\Omega) \to \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$ is bounded, hemicontinuous and monotone. Also, $\Delta_{p(\cdot)}$ is of type M.*

Proof. Let $S \subset W_0^{1,p(\cdot)}$ be bounded, say $\sup\{\|\nabla u\|_p, u \in S\} \le C$. For $u \in S$ and w in the unit ball of $W_0^{1,p(\cdot)}(\Omega)$,

$$\langle w, \Delta_{p(\cdot)}u\rangle = \int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla w(x)\,dx. \tag{3.2}$$

Taking absolute values in (3.2) and invoking the variable exponent form of Hölder's inequality [18] makes it clear that

$$\sup\{\|\Delta_{p(\cdot)}u\|_{\left(W_0^{1,p(\cdot)}(\Omega)\right)^*}, u \in S\} \le C\left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right),$$

which shows that $\Delta_{p(\cdot)}$ is bounded.

For the proof of hemicontinuity, fix $t \in \mathbb{R}$. For $|s| < |t| + \frac{1}{2}$ and $1 < p(x) \le 2$,

$$|\nabla(u+sv)(x)|^{p(x)-1} \le |\nabla u(x)|^{p(x)-1} + |s|^{p(x)-1}|\nabla v(x)|^{p(x)-1}, \tag{3.3}$$

whereas for $p(x) > 2$,

$$|\nabla(u+sv)(x)|^{p(x)-1} \leq 2^{p(x)-2}(|\nabla u(x)|^{p(x)-1} + |s|^{p(x)-1}|\nabla v(x)|^{p(x)-1}). \tag{3.4}$$

On the other hand, it follows by definition that

$$\langle v, \Delta_{p(\cdot)}(u+sv)\rangle = \int_\Omega |\nabla(u+sv)(x)|^{p(x)-2}\nabla(u+sv)(x)\cdot\nabla v(x)\,dx. \tag{3.5}$$

In view of (3.3) and (3.4), the integrand in (3.5) is bounded above by

$$\begin{aligned}(|\nabla u|^{p(x)-1}|\nabla v| + |s|^{p(x)-1}|\nabla v|^{p(x)})\chi_{\{x:1<p(x)\leq 2\}}\\ + 2^{p(x)-2}\left(|\nabla u|^{p(x)-1}|\nabla v| + |s|^{p(x)-1}|\nabla v|^{p(x)}\right)\chi_{\{x:p(x)\geq 2\}},\end{aligned}$$

which is integrable by virtue of Hölder's inequality. A straightforward application of Lebesgue's Dominated Convergence Theorem yields the hemicontinuity of $\Delta_{p(\cdot)}$.

The proof of monotonicity relies ultimately on the identity (2.4). Observe that for constant $p \geq 2$ and $x, y \in \mathbb{R}^n$,

$$|x-y|^p = |x-y|^{p-2}|x-y|^2 \leq 2^{p-3}|x-y|^2(|x|^{p-2} + |y|^{p-2})$$

which, when combined with the identity (2.4), yields the estimate

$$|x-y|^p \leq 2^{p-2}(|x|^{p-2}x - |y|^{p-2}y)\cdot(x-y). \tag{3.6}$$

On the other hand, for $1 < p \leq 2$ (with the obvious provision that $x \neq 0$ and $y \neq 0$),

$$(p-1)|x-y|^2(1+|x|+|y|)^{p-2} \leq (|x|^{p-2}x - |y|^{p-2}y)(x-y). \tag{3.7}$$

Inequality (3.7) follows from

$$\begin{aligned}(|x|^{p-2}x - |y|^{p-2}y)(x-y) &= (x-y)\int_0^1 \frac{d}{dt}\left(|y+t(x-y)|^{p-2}(y+t(x-y))\right)dt\\ &= |x-y|^2\int_0^1 |y+t(x-y)|^{p-2}dt\\ &\quad + (p-2)\int_0^1 |y+t(x-y)|^{p-4}\left((y+t(x-y))(x-y)\right)^2 dt\\ &\geq (p-1)|x-y|^2\int_0^1 |y+t(x-y)|^{p-2}dt\\ &\geq (p-1)|x-y|^2(1+|x|+|y|)^{p-2}.\end{aligned}$$

For fixed u and v in $W_0^{1,p(\cdot)}(\Omega)$, the definition of $\Delta_{p(\cdot)}$ yields

$$(3.8)\quad \langle u-v, \Delta_{p(\cdot)}(u)-\Delta_{p(\cdot)}(v)\rangle = \int_\Omega (|\nabla u(x)|^{p(x)-2}\nabla u(x) - |\nabla v(x)|^{p(x)-2}\nabla v(x))\cdot\nabla(u-v)(x)\,dx.$$

The desired conclusion is obtained by splitting the integral in the right-hand side of (3.8) into one over $\{x : 1 < p(x) \le 2\}$ and one over $\{x : p(x) \ge 2\}$ and applying the inequalities (3.7) and (3.6). Since $W_0^{1,p(\cdot)}(\Omega)$ is reflexive and separable, $\Delta_{p(\cdot)}$ is of type M by virtue of Theorem 2.4 and the results just proved. □

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, Lipschitz domain and $p : \Omega \to \mathbb{R}$ a Borel-measurable function satisfying $1 < p_- \le p(x) \le p_+ < \infty$ a.e. in Ω. Then the $p(\cdot)$-Laplacian is a homeomorphism of $W_0^{1,p(\cdot)}(\Omega)$ onto its dual $\left(W_0^{1,p(\cdot)}(\Omega)\right)^*$.*

Proof. The surjectivity of $\Delta_{p(\cdot)}$ is derived from Theorem 2.5, injectivity follows from Poincare's inequality and inequalities (3.6) and (3.7), and the continuity of $\Delta_{p(\cdot)}^{-1}$ ensues from a functional-analytic argument coupled with inequalities (3.6) and (3.7).

We proceed to the details of the proof. Fix $f \in \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$. For $u \in W_0^{1,p(\cdot)}(\Omega)$ with

$$\|\nabla u\|_{p(\cdot)} > \max\left\{1, \|f\|^{1/(p_--1)}_{(W_0^{1,p(\cdot)}(\Omega))^*}\right\},$$

we have

$$1 = \int \frac{|\nabla u(x)|^{p(x)}}{\|\nabla u\|^{p(x)}_{p(\cdot)}}\,dx \le \frac{1}{\|\nabla u\|^{p_-}_{p(\cdot)}}\int |\nabla u(x)|^{p(x)}dx;$$

thus for such u,

$$\langle u, \Delta_{p(\cdot)}u\rangle = \int |\nabla u(x)|^{p(x)}dx \ge \|\nabla u\|^{p_-}_{p(\cdot)} = \|\nabla u\|^{p_--1}_{p(\cdot)}\|\nabla u\| > \|f\|_{(W_0^{1,p(\cdot)}(\Omega))^*}\|\nabla u\|_{p(\cdot)},$$

which, by virtue of Theorem 2.5, implies that f is in the range of $\Delta_{p(\cdot)}$, i.e., $\Delta_{p(\cdot)}$ is surjective.

To prove the injectivity of $\Delta_{p(\cdot)}$, consider $u, v \in W_0^{1,p(\cdot)}(\Omega)$ with $\Delta_{p(\cdot)}(u) = \Delta_{p(\cdot)}(v)$. We estimate the modular of $\nabla u - \nabla v$ as

$$(3.9)\quad \int_\Omega |\nabla u(x)-\nabla v(x)|^{p(x)}dx = \int_{\{x:p(x)>2\}} |(\nabla u-\nabla v)(x)|^{p(x)}dx + \int_{\{x:1<p(x)\le 2\}} |(\nabla u-\nabla v)(x)|^{p(x)}dx.$$

According to (3.6), the first integral in (3.9) is bounded by

$$2^{p_+-2}\int_\Omega(|\nabla u(x)|^{p(x)-2}\nabla u(x)-|\nabla v(x)|^{p(x)-2}\nabla v(x))\cdot(\nabla u-\nabla v)(x)\,dx = 2^{p_+-2}\left\langle\Delta_{p(\cdot)}(u)-\Delta_{p(\cdot)}(v),u-v\right\rangle=0. \tag{3.10}$$

Denote the second integral in (3.9) by I, i.e., set

$$I=\int_{\{x:1<p(x)<2\}}\frac{|(\nabla u-\nabla v)(x)|^{p(x)}(1+|\nabla u(x)|+|\nabla v(x)|)^{p(x)(2-p(x))/2}}{(1+|\nabla u(x)|+|\nabla v(x)|)^{p(x)(2-p(x))/2}}\,dx. \tag{3.11}$$

Utilizing the modular Hölder inequality, the right-hand side of (3.11) can be seen to be bounded above by

$$2\left\|\frac{|\nabla u-\nabla v|^p}{(1+|\nabla u|+|\nabla v|)^{p(2-p)/2}}\chi_{\{x:1<p(x)<2\}}\right\|_{2/p(\cdot)} \times\left\|(1+|\nabla u|+|\nabla v|)^{p(2-p)/2}\right\|_{2/(2-p(\cdot))}.$$

Hence, from the norm-modular inequalities (2.2), setting

$$C(u,v)=\left\|(1+|\nabla u|+|\nabla v|)^{p(2-p)/2}\right\|_{2/(2-p(\cdot))}$$

and

$$J=\int_\Omega\frac{|(\nabla u-\nabla v)(x)|^2}{(1+|\nabla u(x)|+|\nabla v(x)|)^{(2-p(x))}}\,dx$$

one concludes that

$$I\le C(u,v)\max\left\{J^{\frac{p_+}{2}},J^{\frac{p_-}{2}}\right\}. \tag{3.12}$$

Estimates (3.7) and (3.12) imply that

$$I\le C(u,v)\left\langle u-v,\Delta_{p(\cdot)}(u)-\Delta_{p(\cdot)}(v)\right\rangle^\alpha=0 \tag{3.13}$$

for some positive constant α. From (3.9), (3.10), and (3.13), it follows that $u-v$ is constant. Poincare's inequality now implies that $u=v$.

It remains to verify the continuity of the inverse operator $\Delta_{p(\cdot)}^{-1}$. To that end, set $\Delta_{p(\cdot)}=T$ and suppose that $\Delta_{p(\cdot)}(v_n)\to\Delta_{p(\cdot)}(u)$ for $(v_n)_n\subset W_0^{1,p(\cdot)}(\Omega)$. Were the sequence $\{v_n\}_n$ unbounded, one could extract a subsequence $\{u_n\}_n$ with $\|u_n\|_{p(\cdot)}>n$. Set $w_n=1/\|\nabla u_n\|_{p(\cdot)}u_n$ and observe that for arbitrary $\phi\in W_0^{1,p(\cdot)}(\Omega)$ with $\|\nabla\phi\|_{p(\cdot)}\le 1$, the equality

$$|\langle\phi,T(w_n)\rangle|=\left|\int_\Omega\frac{1}{\|\nabla u_n\|_{p(\cdot)}^{p(x)-1}}|\nabla u_n(x)|^{p(x)-2}\nabla u_n(x)\cdot\nabla\phi(x)\,dx\right|,$$

yields

$$
\begin{aligned}
| \langle \phi, T(w_n) \rangle | &\le \frac{1}{\|\nabla u_n\|_{p(\cdot)}^{p_- -1}} \left| \int_\Omega |\nabla u_n(x)|^{p(x)-2} \nabla u_n(x) \cdot \nabla \phi(x)\, dx \right| \\
&\le \frac{1}{\|\nabla u_n\|_{p(\cdot)}^{p_- -1}} \|T(u_n)\|_{(W^1_{p(\cdot)}(\Omega))^*}.
\end{aligned}
$$

This last estimate implies that

$$
\|T(w_n)\|_{(W^{1,p(\cdot)}(\Omega))^*} \to 0 \quad \text{as } n \to \infty. \tag{3.14}
$$

On the other hand, it follows automatically by definition that

$$
\|T(w_n)\| \ge \langle w_n, T(w_n) \rangle = \int_\Omega |\nabla w_n(x)|^{p(x)}\, dx = 1,
$$

which contradicts (3.14). Consequently, the sequence $\{v_n\}_n$ must be bounded in $W_0^{1,p(\cdot)}(\Omega)$.

Next we write

$$
\begin{aligned}
&\int_\Omega |(\nabla v_n - \nabla v)(x)|^{p(x)} dx \\
&\qquad = \int_{p \ge 2} |(\nabla v_n - \nabla v)(x)|^{p(x)} dx + \int_{1<p<2} |(\nabla v_n - \nabla v)(x)|^{p(x)} dx.
\end{aligned} \tag{3.15}
$$

By inequality (3.6), the first term in the right-hand-side of equality (3.15) is bounded by

$$
\begin{aligned}
&\left| 2^{p_+ -2} \langle v_n - v, T(v_n) - T(v) \rangle \right| \\
&\qquad \le 2^{p_+ -2} \|T(v_n) - T(v)\|_{(W_0^{1,p(\cdot)}(\Omega))^*} \|v_n - v\|_{W_0^{1,p(\cdot)}(\Omega)},
\end{aligned} \tag{3.16}
$$

and the right-hand side above tends to 0 as $n \to \infty$ since $T(v_n) - T(v) \to 0$ and $\{v_n\}_n$ is bounded. The second term in (3.15) can be expressed as

$$
\int_{1<p<2} \frac{|(\nabla v_n - \nabla v)(x)|^{p(x)}}{(1 + |\nabla v(x)| + |\nabla v_n(x)|)^{p(x)(2-p(x))/2}} (1 + |\nabla v(x)| + |\nabla v_n(x)|)^{p(x)(2-p(x))/2}\, dx,
$$

which a straightforward application of the generalized Hölder inequality shows to be dominated by

$$
\left\| (1 + |\nabla v| + |\nabla v_n|)^{p(.)(2-p(.))/2} \right\|_{2/(2-p(.))} \left\| \frac{|\nabla v_n - \nabla v|^{p(.)}}{(1 + |\nabla v| + |\nabla v_n|)^{p(.)(2-p(.))/2}} \right\|_{2/p(\cdot)}.
$$

Invoking again the boundedness of the sequence $\{v_n\}_n$, we see immediately that the left factor above is bounded by a constant independent of n. Set

$$
I_n = \int_\Omega |\nabla(v_n - v)(x)|^2 (1 + |\nabla v(x)| + |\nabla v_n(x)|)^{p(x)-2} dx.
$$

With n chosen so large that

$$\|T(v_n) - T(v)\|_{\left(W_0^{1,p(\cdot)}(\Omega)\right)^*} < (p_- - 1)/(\sup_n \|v_n\| + \|v\|),$$

inequality (3.7) yield $I_n < 1$. Consequently, using inequalities (2.2), one concludes that

$$\left\| \frac{|\nabla v_n - \nabla v|^{p(\cdot)}}{(1 + |\nabla v| + |\nabla v_n|)^{p(\cdot)(2-p(\cdot))/2}} \right\|_{2/p(\cdot)} \leq \max\left\{ I_n^{p_+/2}, I_n^{p_-/2} \right\} = I_n^{p_-/2}.$$

A further application of inequality (3.7) for such n implies

$$\begin{aligned} I_n^{\frac{p_-}{2}} &\leq \frac{1}{p_- - 1} \left| \langle v_n - v, T(v_n) - T(v) \rangle \right| \\ &\leq \frac{1}{p_- - 1} \|T(v_n) - T(v)\|_{(W_0^{1,p(\cdot)}(\Omega))^*} \|\nabla(v_n - v)\|_{p(.)} \\ &\leq \frac{1}{p_- - 1} (\sup_n \|\nabla v_n\|_{p(.)} + \|\nabla v\|_{p(.)}) \|T(v_n) - T(v)\|_{(W_0^{1,p(\cdot)}(\Omega))^*}. \end{aligned} \tag{3.17}$$

According to (3.15), the bounds (3.16) and (3.17) show that

$$\int_\Omega |(\nabla v - \nabla v_n)(x)|^{p(\cdot)}\, dx \to 0 \quad \text{as } n \to \infty,$$

and hence $\|v_n - v\|_{1,p(\cdot)} \to 0$ as $n \to \infty$.

Thus $\Delta_{p(\cdot)}^{-1} : \left(W_0^{1,p(\cdot)}(\Omega)\right)^* \to W_0^{1,p(\cdot)}(\Omega)$ is continuous. □

Theorem 3.2 can be rephrased as the following existence and uniqueness theorem of Fang-Zhang.

Theorem 3.3 ([15, Theorem 4.2]). *For each $f \in \left(W_0^{1,p(\cdot)}(\Omega)\right)^*$, there exists a unique solution $u \in W_0^{1,p(\cdot)}(\Omega)$ of the equation $\Delta_{p(\cdot)} u = f$.*

4 Stability with respect to integrability

In this section, the behaviour of the solutions to the Dirichlet problem for the $p(\cdot)$-Laplacian with respect to perturbations of the integrability is discussed. In the sequel, $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, and all integrability indexes are tacitly assumed to belong to $C(\overline{\Omega})$. We fix a non-decreasing sequence $\{p_k\}_{k\in\mathbb{N}} \subset C(\overline{\Omega})$ with $p \leq p_k \to q$ uniformly for each $k \in \mathbb{N}$. In the interest of facilitating the notation, in the sequel we will not explicitly indicate the dependence of variable exponents on x, i.e., the variable exponent $r(x)$ will be denoted as r whenever no confusion arises from doing so. As in the previous section, for any integrability index $m \in C(\overline{\Omega})$ we fix the notation $m_- = \inf_{\overline{\Omega}} m$ and $m_+ = \sup_{\overline{\Omega}} m$. We furthermore assume $1 < p_- = \inf_{\overline{\Omega}} p$ and $\sup_{\overline{\Omega}} q(x) = q_+ < \infty$.

Consider an arbitrary $f \in \big(W_0^{1,p(\cdot)}(\Omega)\big)^*$. For each k, let $u_k \in W_0^{1,p_k(\cdot)}(\Omega)$, $u_p \in W_0^{1,p(\cdot)}(\Omega)$, and $u_q \in W_0^{1,q(\cdot)}(\Omega)$ be, respectively, the unique solutions of

$$\Delta_{p_k}(u_{p_k}) = f, \quad \Delta_p(u_p) = f, \quad \text{and} \quad \Delta_q(u_q) = f \tag{4.1}$$

given by Theorem 3.3. The following is a variant of [13, Lemma 4.1].

Lemma 4.1. *Let $0 < \varepsilon < 1/e$ be arbitrary and N be large enough to guarantee that $\|p_i - p_j\|_\infty < \varepsilon$ if $i, j > N$. Fix k, j such that $N \leq k \leq j$. Then, for each u_{p_j} defined by* (4.1)*,*

$$\int_\Omega \frac{|\nabla u_{p_j}|^{p_k}}{p_k}\,dx \leq \varepsilon \int_\Omega \frac{1}{p_k}\,dx + \varepsilon^{-\varepsilon} \int_\Omega \frac{|\nabla u_{p_j}|^{p_j}}{p_k}\,dx.$$

Proof.

$$\int_\Omega \frac{|\nabla u_{p_j}|^{p_k}}{p_k}\,dx \leq \int_{|\nabla u_{p_j}|<\varepsilon} \frac{|\nabla u_{p_j}|^{p_k}}{p_k}\,dx + \int_{\varepsilon \leq |\nabla u_{p_j}|<1} \frac{|\nabla u_{p_j}|^{p_k}}{p_k}\,dx + \int_{|\nabla u_{p_j}|\geq 1} \frac{|\nabla u_{p_j}|^{p_k}}{p_k}\,dx. \tag{4.2}$$

A brief computation shows that the first integral on the left-hand side above is less than or equal to $\varepsilon \int_\Omega 1/p_k\,dx$. Using the conditions on ε, we can deal with the second integral in (4.2) as follows:

$$\int_{\varepsilon \leq |\nabla u_{p_j}|<1} \frac{|\nabla u_{p_j}|^{p_k - p_j} |\nabla u_{p_j}|^{p_j}}{p_k}\,dx \leq \int_{|\nabla u_{p_j}|<1} \frac{\varepsilon^{-\varepsilon} |\nabla u_{p_j}|^{p_j}}{p_k}\,dx;$$

On the other hand, it is clear that the last integral in (4.2) is less than or equal to $\int_{|\nabla u_{p_j}|\geq 1} |\nabla u_{p_j}|^{p_j}/p_k\,dx$. Substituting the last three inequalities in the expression (4.2) yields the lemma. □

We are now ready to inspect the effect of the variation of the integrability exponent on the solution of the Dirichlet problem.

Theorem 4.2. *For u_{p_k}, and u_q as in the previous paragraph, there exists a subsequence (still denoted by $\{u_{p_k}\}$) such that $u_{p_k} \rightharpoonup u_q$, $u_{p_k} \to u_q$ strongly in $L^p(\Omega)$ and for which*

$$\lim_{k\to\infty} \int_\Omega |\nabla u_{p_k}|^{p_k}\,dx = -\int_\Omega f u_q\,dx = \int_\Omega |\nabla u_q|^q\,dx.$$

Proof. By definition of $\Delta_{p_k(\cdot)}$ and from inequalities (2.2), it immediately follows that either $\||\nabla u_{p_k}|\|_{p_k} \leq 1$ or

$$\||\nabla u_{p_k}|\|_{p_k}^{p_{k-}} \leq \int_\Omega |\nabla u_{p_k}|^{p_k}\,dx = -\langle f, u_{p_k}\rangle \leq \|f\|_{\left(W_0^{1,p(\cdot)}(\Omega)\right)^*} \cdot \||\nabla u_{p_k}|\|_{p(\cdot)}.$$

Young's inequality, in concert with [13, Lemma 4.1] implies that for every $\delta > 0$,

$$\begin{aligned} \||\nabla u_{p_k}|\|_{p_k}^{p_{k-}} &\leq \left(\left(\frac{\|f\|}{\delta}\right)^{p_{k-}/(p_{k-}-1)} \frac{p_{k-}-1}{p_{k-}} + \frac{(\||\nabla u_{p_k}|\|_{p(\cdot)}\delta)^{p_{k-}}}{p_{k-}} \right) \\ &\leq \left(\left(\frac{\|f\|}{\delta}\right)^{p_{k-}/(p_{k-}-1)} \frac{p_{k-}-1}{p_{k-}} \right. \\ &\qquad \left. + \frac{\left(\left(\|p-q\|_\infty^{-\|p-q\|_\infty} + \|p-q\|_\infty|\Omega|\right) \||\nabla u_{p_k}|\|_{p_k}\delta\right)^{p_{k-}}}{p_{k-}} \right). \end{aligned} \tag{4.3}$$

Since δ is arbitrary, inequality (4.3) shows that $(\||\nabla u_{p_k}|\|_{p_k(\cdot)})_{k\in\mathbb{N}}$ is bounded. As a consequence of the inequality

$$\||\nabla u_{p_k}|\|_{p(\cdot)} \leq \left(\|p-q\|_\infty^{-\|p-q\|_\infty} + \|p-q\|_\infty|\Omega|\right) \||\nabla u_{p_k}|\|_{p_k}$$

[13, Theorem 4.1], it is immediate that the sequence $\{u_{p_k}\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$. Thus, it has a weakly convergent subsequence, which we continue to denote by $\{u_{p_k}\}$. Let u be the weak limit of $\{u_{p_k}\}$ in $W_0^{1,p(\cdot)}(\Omega)$. We now show that $u = u_q$. To that goal, we fix an exponent $r \in C(\overline{\Omega})$ such that $r > p$, and choose k large enough so that $p_k > r$ and $\|q - p_k\|_\infty \leq \|q - r\|_\infty/2$ on Ω. The sequence $\{u_{p_{k+j}}\}_j$ is bounded in $W_0^{1,r(\cdot)}(\Omega)$, since according to [13, Lemma 4.1],

$$\||\nabla u_{p_{k+j}}|\|_{r(\cdot)} \leq \left(\|p-q\|_\infty^{-\|p-q\|_\infty} + \|p-q\|_\infty|\Omega|\right) \||\nabla u_{p_{k+j}}|\|_{p_{k+j}(\cdot)};$$

it is therefore clear that $\{u_{p_{k+j}}\}_j$ is weakly convergent in $W_0^{1,r(\cdot)}(\Omega)$. Denote the weak limit by $v \in W_0^{1,r(\cdot)}(\Omega)$. The uniqueness of the limit implies $u = v$, so $u \in \bigcap_{p\leq r<q} W_0^{1,r(\cdot)}(\Omega)$. Sobolev's embedding theorem allows us to assume without loss of generality that for $u_{p_k} \to u$ strongly in $L_{p_j}(\Omega)$ for $k \geq j$.

By virtue of the weak lower semicontinuity of the functional

$$I : W_0^{1,p_k(\cdot)}(\Omega) \to [0,\infty) \quad I(w) = \int_\Omega |\nabla w|^{p_k}\,dx,$$

for any fixed j, u is subject to the condition

$$\int_\Omega |\nabla u|^{p_k}\,dx \leq \liminf_{j\geq k} \int_\Omega |\nabla u_{p_j}|^{p_k}\,dx. \tag{4.4}$$

On the other hand, for $j \geq k$,

$$\int_\Omega |\nabla u_{p_j}|^{p_k}\,dx \leq \|p_k - p_j\|_\infty |\Omega| + \|p_k - p_j\|_\infty^{-\|p_j - p_k\|_\infty} \int_\Omega |\nabla u_{p_j}|^{p_j}\,dx \tag{4.5}$$

which is bounded independently of j by virtue of (4.3) and by the assumption on the sequence $\{p_k\}$. Fatou's Lemma yields

$$\int_\Omega \liminf_k |\nabla u|^{p_k}\,dx = \int_\Omega |\nabla u|^q\,dx \leq \liminf_k \int_\Omega |\nabla u|^{p_k}\,dx. \tag{4.6}$$

In view of (4.4) and (4.5), the right-hand-side of (4.6) is finite, so $u \in W_0^{1,q(\cdot)}(\Omega)$. For a given $\delta > 0$, fix positive numbers θ and γ such that $(1+\theta)(1+\gamma) < 1 + \delta/2$. Since for $\varepsilon < e^{-1}$ the function $\omega(\varepsilon) = \varepsilon^{-\varepsilon}$ decreases to 1 as $\varepsilon \to 0^+$, given the assumptions on the sequence $\{p_k\}$, there exists $M > 0$ such that for $M \leq k \leq j$,

$$\|p_j - p_k\|_\infty < \min\left\{\gamma, \frac{\delta}{2|\Omega|}p_-\right\}, \quad \|p_j - p_k\|_\infty^{-\|p_k - p_j\|_\infty} < 1 + \theta. \tag{4.7}$$

Notice that for such k and j,

$$\left\|\frac{p_j}{p_k} - 1\right\|_\infty < \gamma. \tag{4.8}$$

Next, Lemma 4.1 and (4.7), (4.8) guarantee that for $M \leq k \leq j$,

$$\begin{aligned}
\int_\Omega \frac{|\nabla u_{p_j}|^{p_k}}{p_k}\,dx &\leq \|p_k - p_j\|_\infty \int_\Omega \frac{1}{p_k}\,dx \\
&\quad + \|p_k - p_j\|_\infty^{-\|p_k - p_j\|_\infty} \int_\Omega \frac{|\nabla u_{p_j}|^{p_j}}{p_j}(1+\gamma)\,dx \\
&\leq \frac{\delta}{2} + (1+\theta)(1+\gamma) \int_\Omega \frac{|\nabla u_{p_j}|^{p_j}}{p_j}\,dx \\
&\leq \frac{\delta}{2} + (1 + \frac{\delta}{2}) \int_\Omega \frac{|\nabla u_{p_j}|^{p_j}}{p_j}\,dx.
\end{aligned} \tag{4.9}$$

For fixed $k \in \mathbb{N}$ and each $M \leq k$, the weak lower semicontinuity of the functional

$$I_k : W_0^{1,p(\cdot)}(\Omega) \to [0, \infty), \quad I_k(w) = \int_\Omega \frac{|\nabla w|^{p_k}}{p_k}\,dx,$$

yields

$$\int_\Omega \frac{|\nabla u|^{p_k}}{p_k}\,dx \leq \liminf_{j \geq k} \int_\Omega \frac{|\nabla u_{p_j}|^{p_k}}{p_k}\,dx.$$

Hence, as one sees from (4.9),

$$\int_\Omega \frac{|\nabla u|^{p_k}}{p_k}\,dx \leq \liminf_{j \geq k} \int_\Omega \frac{|\nabla u_{p_j}|^{p_k}}{p_k}\,dx \leq \liminf_{j \geq k}\left(\frac{\delta}{2} + (1 + \frac{\delta}{2}) \int_\Omega \frac{|\nabla u_{p_j}|^{p_j}}{p_j}\,dx\right),$$

i.e.,

$$\int_\Omega \frac{|\nabla u|^{p_k}}{p_k}\,dx \le \liminf_{j\ge k} \int_\Omega \frac{|\nabla u_{p_j}|^{p_j}}{p_j}\,dx. \tag{4.10}$$

Since $u \in W_0^{1,q(\cdot)}(\Omega)$, Lebesgue's Dominated Convergence Theorem yields

$$\lim_{k\to\infty} \int_\Omega \frac{|\nabla u|^{p_k}}{p_k}\,dx = \int_\Omega \frac{|\nabla u|^q}{q}\,dx.$$

By virtue of the minimizing property of u_{p_k}, it is clear that

$$\int_\Omega \frac{|\nabla u_{p_k}|^{p_k}}{p_k}\,dx + \int_\Omega f u_{p_k}\,dx \le \int_\Omega \frac{|\nabla u|^{p_k}}{p_k}\,dx + \int_\Omega f u\,dx,$$

which automatically leads to

$$\limsup \int_\Omega \frac{|\nabla u_{p_k}|^{p_k}}{p_k}\,dx \le \limsup \int_\Omega \frac{|\nabla u|^{p_k}}{p_k}\,dx = \int_\Omega \frac{|\nabla u|^q}{q}\,dx$$

which, in conjunction with (4.10), implies

$$\int_\Omega \frac{|\nabla u|^q}{q}\,dx = \lim \int_\Omega \frac{|\nabla u_{p_k}|^{p_k}}{p_k}\,dx. \tag{4.11}$$

Again the minimal character of u_{p_k} yields the inequality

$$\int_\Omega \frac{|\nabla u_{p_k}|^{p_k}}{p_k}\,dx + \int_\Omega f u_{p_k}\,dx \le \int_\Omega \frac{|\nabla u_q|^{p_k}}{p_k}\,dx + \int_\Omega f u_q\,dx,$$

which is valid for each k. Passing to the limits as $k \to \infty$ and taking into account (4.11) and the fact that $u_q \in W_0^{1,q(\cdot)}(\Omega)$, one obtains

$$\int_\Omega \frac{|\nabla u|^q}{q}\,dx + \int_\Omega f u\,dx \le \int_\Omega \frac{|\nabla u_q|^q}{q}\,dx + \int_\Omega f u_q\,dx,$$

i.e., u minimizes the functional

$$G : W_0^{1,q(\cdot)}(\Omega) \to [0,\infty)\ ,\ G(v) = \int_\Omega \frac{|\nabla v|^q}{q}\,dx + \int_\Omega f v\,dx. \tag{4.12}$$

In turn, (4.12) implies that $\Delta_{q(\cdot)}(u) = f$; and, since $u \in W_0^{1,q(\cdot)}(\Omega)$, Theorem 3.3 yields $u = u_q$. Hence, letting $k \to \infty$ in

$$\int_\Omega |\nabla u_{p_k}|^{p_k}\,dx = -\int_\Omega f u_{p_k}\,dx,$$

which holds by definition, we at once obtain

$$\lim_{k\to\infty} \int_\Omega |\nabla u_{p_k}|^{p_k}\,dx = -\int_\Omega f u\,dx = \int_\Omega |\nabla u_q|^q\,dx,$$

as claimed. □

Corollary 4.3. *In the terminology of Theorem 3.2, if* $\inf_\Omega q \geq 2$*, then for a subsequence of* $\{u_{p_j}\}$ *(still denoted by* $\{u_{p_j}\}$*),*

$$\int_\Omega |\nabla u_{p_j} - \nabla u_q|^{p_j}\, dx \to 0 \quad \text{as } j \to \infty.$$

If $n = 1$*, then the restriction* $\inf_\Omega q \geq 2$ *is not needed.*

Proof. Due to weak-lower semicontinuity and convexity, for fixed j,

$$\begin{aligned}\int_\Omega |\nabla u_q|^{p_j}\, dx &\leq \liminf_{j \leq k \to \infty} \int_\Omega \left| \frac{\nabla u_q}{2} + \frac{\nabla u_{p_k}}{2} \right|^{p_j} dx \\ &\leq \frac{1}{2} \int_\Omega |\nabla u_q|^{p_j}\, dx + \frac{1}{2} \liminf_{j \leq k \to \infty} \int_\Omega |\nabla u_{p_k}|^{p_j}\, dx.\end{aligned} \tag{4.13}$$

From [13, Theorem 4.1] and due to the convergence given in Theorem 4.2, we have

$$\begin{aligned}\liminf_{j \leq k \to \infty} \int_\Omega |\nabla u_{p_k}|^{p_j}\, dx &\leq \liminf_{k \to \infty} \Big(\|p_j - p_k\|_\infty^{-\|p_j - p_k\|_\infty} \\ &\qquad \cdot \int_\Omega |\nabla u_{p_k}|^{p_k}\, dx + \|p_j - p_k\|_\infty |\Omega| \Big) \\ &= \left(\|p_j - q\|_\infty^{-\|p_j - q\|_\infty} \int_\Omega |\nabla u_q|^q\, dx + \|p_j - q\|_\infty |\Omega| \right).\end{aligned}$$

Thus, letting $j \to \infty$ in (4.13) makes it clear that

$$\liminf_{j \leq k \to \infty} \int_\Omega \left| \frac{\nabla u_q}{2} + \frac{\nabla u_{p_k}}{2} \right|^{p_j} dx = \int_\Omega |\nabla u_q|^q\, dx.$$

The inequality (valid for $\inf_\Omega p_j \geq 2$)

$$\int_\Omega \left| \frac{\nabla u_q}{2} + \frac{\nabla u_{p_k}}{2} \right|^{p_j} dx + \int_\Omega \left| \frac{\nabla u_q}{2} - \frac{\nabla u_{p_k}}{2} \right|^{p_j} dx \leq \frac{1}{2} \int_\Omega |\nabla u_q|^{p_j}\, dx + \frac{1}{2} \int_\Omega |\nabla u_{p_k}|^{p_j}\, dx$$

(see [1],[12]) yields the first part of the corollary.

As to the 1-dimensional case, observe that there exists a positive constant c such that as long as $1 < s < S < \infty$ and $(\xi, r) \in (s, S) \times [-M, M]$,

$$\frac{1}{2} \left(|r + 1|^\xi + |r - 1|^\xi \right) - |r|^\xi \geq c. \tag{4.14}$$

Set $A = \{x \in (a, b) : |(u_{p_j} - u_q)'| > \delta |(u_{p_j} + u_q)'|\}$ and fix $\delta > 0$. Then if $x \in A$, the substitution

$$r = \frac{u_q'(x) + u_{p_j}'(x)}{u_q'(x) - u_{p_j}'(x)},$$

$\xi = p_j(x)$ in (4.14) yields, pointwise in Ω,

$$\frac{1}{2}\left(|u_q'|^{p_j} + |u_{p_j}'|^{p_j}\right) - \left|\frac{u_q' + u_{p_j}'}{2}\right|^{p_j} \geq c\left|\frac{u_q' - u_{p_j}'}{2}\right|^{p_j},$$

whence it follows by integrating on A that

$$\frac{1}{2}\left(\int_\Omega |u_q'|^{p_j}dx + \int_\Omega |u_{p_j}'|^{p_j}dx\right) - \int_\Omega \left|\frac{u_q' + u_{p_j}'}{2}\right|^{p_j} dx \geq c\int_A \left|\frac{u_q' - u_{p_j}'}{2}\right|^{p_j} dx. \tag{4.15}$$

From Lebesgue's Dominated Convergence Theorem and the proof of Theorem 4.2, it follows that

$$\lim_{j\to\infty}\int_\Omega |u_q'(x)|^{p_j}dx = \lim_{j\to\infty}\int_\Omega \left|\frac{u_q' + u_{p_j}'}{2}\right|^{p_j} dx = \int_\Omega |u_q'|^q\,dx.$$

Letting $j \to \infty$ in (4.15) makes it clear that

$$\int_A \left|\frac{u_q' - u_{p_j}'}{2}\right|^{p_j} dx \to 0 \quad \text{as}\,,\ j \to \infty$$

which through an easy calculation yields

$$\int_A |u' - u_{p_j}'|^p dx \to 0 \text{ as } \ j \to \infty.$$

On the other hand,

$$\int_{\Omega\setminus A} \left|\frac{u_q' - u_{p_j}'}{2}\right|^{p_j} dx \leq \delta^{p-}\int_\Omega \left|\frac{u_q' + u_{p_j}'}{2}\right|^{p_j} dx \leq \delta^{p-}.$$

Putting this all together, we have

$$\int_\Omega |u_q' - u_{p_j}'|^p dx \to 0 \quad \text{as } j \to \infty.$$

□

Consider next a non-increasing sequence $\{p_j\}_j$. We retain the terminology from the beginning of Section 4.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $\{p_k\}_k \subset C(\overline{\Omega})$ a non-increasing sequence converging to p uniformly on Ω. Let u_{p_k} for $k \in \mathbb{N}$ and u_p be as in* (4.1). *Assume that for some $r > p$, the higher integrability condition $\int_\Omega |\nabla u_p|^r\,dx < \infty$ holds* [1]*. Then there exists a subsequence of $\{u_{p_k}\}$ (still denoted by $\{u_{p_k}\}$) for which $u_{p_k} \rightharpoonup u_p$ and $u_{p_k} \to u_p$ strongly in $L^p(\Omega)$ and*

$$\int_\Omega |\nabla u_{p_k}|^{p_k}\,dx \to \int_\Omega |\nabla u_p|^p\,dx.$$

[1] This condition is satisfied, for example, if $\mathbb{R}^n \setminus \Omega$ is $p(\cdot)$-fat (see [16, Theorem 5.1])

Proof. Owing to inequality (4.3) and the argument thereafter, the sequence $\{u_{p_j}\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, and the numerical sequence $\{\int_\Omega |\nabla u_{p_j}|^{p_j}\,dx\}$ is bounded. Therefore, no generality is lost in assuming that $\{\int_\Omega |\nabla u_{p_j}|^{p_j}\,dx\}$ (and therefore also $\{\int_\Omega |\nabla u_{p_j}|^{p_j}/p_j\,dx\}$) is convergent unless it is finite, in which case the claim follows trivially since $\langle f, u_{p_j}\rangle \to \langle f, u\rangle$.

We point out that for fixed j, $\int_\Omega |\nabla u_{p_j}|^{p_k}\,dx \to \int_\Omega |\nabla u_{p_j}|^{p}\,dx$ as $j \le k \to \infty$. Taking $k \ge j$ and invoking the estimate

$$\int_\Omega |\nabla u_{p_j}|^{p_k}\,dx \le \|p_j - p_k\|_\infty^{-\|p_k - p_j\|_\infty} \int_\Omega |\nabla u_{p_j}|^{p_j}\,dx + \|p_j - p_k\|_\infty |\Omega|,$$

we see easily that

$$\int_\Omega |\nabla u_{p_j}|^{p}\,dx \le \|p_j - p\|_\infty^{-\|p_j - p\|_\infty} \int_\Omega |\nabla u_{p_j}|^{p_j}\,dx + \|p_j - p\|_\infty |\Omega|.$$

Weak-lower semicontinuity yields

$$\int_\Omega |\nabla u|^p\,dx \le \liminf_j \int_\Omega |\nabla u_{p_j}|^p\,dx \le \liminf_j \int_\Omega |\nabla u_{p_j}|^{p_j}\,dx = \lim_j \int_\Omega |\nabla u_{p_j}|^{p_j}\,dx.$$

In a similar fashion, via Lemma 4.1 one obtains

$$\int_\Omega \frac{|\nabla u|^p}{p}\,dx \le \lim_j \int_\Omega \frac{|\nabla u_{p_j}|^{p_j}}{p_j}\,dx.$$

In view of the above considerations, the minimality condition

$$\int_\Omega \frac{|\nabla u_{p_j}|^{p_j}}{p_j}\,dx + \langle f, u_{p_j}\rangle \le \int_\Omega \frac{|\nabla u_p|^{p_j}}{p_j}\,dx + \langle f, u_p\rangle, \tag{4.16}$$

implies

$$\int_\Omega \frac{|\nabla u|^p}{p}\,dx + \langle f, u\rangle \le \int_\Omega |\nabla u_p|^p\,dx + \langle f, u_p\rangle,$$

and the minimal character of u_p implies automatically that $u = u_p$. A further application of (4.16) yields

$$\lim_{j\to\infty} \int_\Omega \frac{|\nabla u_{p_j}|^{p_j}}{p_j}\,dx \le \int_\Omega \frac{|\nabla u|^p}{p}\,dx.$$

In conclusion,

$$\lim_{j\to\infty} \int_\Omega |\nabla u_{p_j}|^{p_j}\,dx \le \int_\Omega |\nabla u|^p\,dx. \qquad \square$$

Corollary 4.5. *In the terminology of Theorem 4.4, if* $\inf_\Omega p \ge 2$, *then for a subsequence of* $\{u_{p_j}\}$ *(still denoted by* $\{u_{p_j}\}$*),*

$$\int_\Omega |\nabla u_{p_j} - \nabla u_p|^p\,dx \to 0 \quad \textit{as } j \to \infty.$$

Corollary 4.5 can be obtained via arguments similar to those used in the proof of Corollary 4.3; its proof is omitted.

5 Γ-convergence

The concept of Γ-convergence was introduced by E. De Giorgi in the 1970's. Its flexibility and ubiquity in the direct methods of variational calculus have brought this idea to the main stage of mathematical research; see [8], [4] and the references therein. In this section, we explore the connection between the results in Section 4 and Γ-convergence.

Definition 5.1. Let X be a metric space. A sequence $\{F_k\}$ of functionals $F_k : X \to [0,\infty]$ is said to **Γ-converge** to $F : X \to [0,\infty]$ at $x \in X$ if and only if

(i) for any sequence $\{x_k\} \subseteq X$ with $x_k \to x$ as $k \to \infty$,

$$F(x) \le \liminf_{k\to\infty} F_k(x_k);$$

(ii) there exists a sequence $\{y_k\}_k \subseteq X$, converging to x for which

$$F(x) = \lim_{k\to\infty} F_k(y_k).$$

A sequence $\{F_k\}$ that Γ-converges to F at every $x \in X$ is said to **Γ-converge to** F **on** X. A sequence $\{F_k\}$ is said to be **equicoercive** if there exists a compact set $K \subset X$ such that $\inf_X F_k = \inf_K F_k$ for each $k \in \mathbb{N}$.

Notice that the second condition in Definition 5.1 is satisfied automatically if $\lim_{k\to\infty} F_k(x) = F(x)$. For a bounded Lipschitz domain Ω and $s \in C(\overline{\Omega})$, fix $f \in \left(W_0^{1,s}(\Omega)\right)^*$; and for $s \le r \in C(\overline{\Omega})$, define $F_r : L^r(\Omega) \to [0,\infty]$ by

$$F_r(u) = \begin{cases} \int_\Omega \frac{|\nabla u|^r}{r}\,dx + \langle f, u\rangle, & \text{if } u \in W^{1,r}(\Omega) \\ \infty & \text{otherwise.} \end{cases} \tag{5.1}$$

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $p, q \in C(\overline{\Omega})$ and $f \in \left(W_0^{1,p}(\Omega)\right)^*$.*

(i) *If $\{p_j\}$ is a non-increasing sequence in $C(\overline{\Omega})$ converging uniformly in Ω to its infimum p, then the sequence $\{F_{p_j}\}$ is equicoercive in $L^p(\Omega)$ and $F_p(u) = \Gamma lim_{j\to\infty} F_{p_j}(u)$ at every $u \in L^p(\Omega)$ for which there exists $\epsilon > 0$ such that $\int_\Omega |\nabla u|^{p+\epsilon}\,dx < \infty$.*

(ii) *If $\{p_j\}$ is a non-decreasing sequence in $C(\overline{\Omega})$ converging uniformly in Ω to its supremum q with $1 < \inf_{\overline{\Omega}} p \le \sup_{\overline{\Omega}} q < \infty$, then the sequence $\{F_{p_j}\}$ of functionals defined on $L^p(\Omega)$ via (5.1) is equicoercive in $L^p(\Omega)$ and Γ-converges to F_q in $L^p(\Omega)$.*

Proof. For the proof of (i), we observe that it is clear that the infimum of each F_{p_j} is attained in $W_0^{1,p_j}(\Omega)$. It is well known that the Fréchet derivative of F_{p_j} is the functional on $W_0^{1,p_j}(\Omega)$ given by $h \to \int_\Omega |\nabla u|^{p_j-2}\nabla u \nabla h\, dx + \langle f, h\rangle$; by virtue of Theorem 3.3, the unique minimizer of F_{p_j} in $W_0^{1,p_j}(\Omega)$ is u_{p_j}. Owing to Sobolev's embedding theorem, the closure $\tilde{K}$ in $L^p(\Omega)$ of the set $K = \{u_{p_j}, j \in \mathbb{N}\}$ is compact. Also it follows from the preceding discussion that $\inf_{L^p} F_{p_j} = F(u_{p_j}) = \inf_{\tilde{K}} F_{p_j}$ for each $j \in \mathbb{N}$. The sequence $\{F_{p_j}\}$ is thus equicoercive; see [8].

Consider a sequence $\{v_j\}$ in $L^p(\Omega)$ that converges to $v \in L^p(\Omega)$. If

$$L = \liminf_{j\to\infty} \left(\int_\Omega \frac{|\nabla v_j|^{p_j}}{p_j}\, dx + \langle f, v_j\rangle \right) < \infty,$$

one can extract from $\{v_j\}$ a subsequence $\{w_j\}$ for which

$$\int_\Omega \frac{|\nabla w_j|^{p_j}}{p_j}\, dx + \langle f, w_j\rangle \to L.$$

Since $p \le p_j$, it is easy to see that

$$\int_\Omega \frac{|\nabla w_j|^{p_j}}{p_j}\, dx \le C \left(1 + \|f\|_{\left(W_0^{1,p_j}(\Omega)\right)^*} \|\nabla w_j\|_{W_0^{1,p_j}(\Omega)} \right)$$

for some positive constant C and each $j \in \mathbb{N}$. It follows from inequalities (2.2) and Young's inequality that if $\int_\Omega |\nabla w_j|^{p_j}\, dx > 1$, then

$$\||\nabla w_j|\|_{p_j}^{p_{j-}} \le \left(C + \left(\frac{\|f\|}{\delta}\right)^{p_{j-}/(p_{j-}-1)} \frac{p_{j-}-1}{p_{j-}} + \frac{(\||\nabla w_j|\|_{p_j}\delta)^{p_{j-}}}{p_{j-}} \right)$$

for every $\delta > 0$. Thus the sequence $\{w_j\}$ is bounded in $W_0^{1,p}(\Omega)$. Owing to the reflexivity of $W_0^{1,p}(\Omega)$ in conjunction with Sobolev's embedding theorem, one readily concludes that $v \in W_0^{1,p}(\Omega)$. Therefore, without loss of generality one can consider $w_j \rightharpoonup v$ in $W_0^{1,p}(\Omega)$. Moreover, $v \in W_0^{1,p_k}(\Omega)$ for sufficiently large $k \in \mathbb{N}$, which follows immediately from the higher integrability assumption on the gradient.

Weak-lower semicontinuity and the appropriate version of Lemma 4.1 imply

$$\begin{aligned}
\int_\Omega \frac{|\nabla v|^p}{p}\, dx &\le \liminf_{j\to\infty} \int_\Omega \frac{|\nabla w_j|^p}{p}\, dx \\
&\le \liminf_{j\to\infty} \left(\|p-p_j\|_\infty^{-\|p-p_j\|_\infty} \int_\Omega \frac{|\nabla w_j|^{p_j}}{p_j}\, dx + \|p-p_j\|_\infty |\Omega| \right) \\
&= \liminf_{j\to\infty} \left(\|p-p_j\|_\infty^{-\|p-p_j\|_\infty} \left(\int_\Omega \frac{|\nabla w_j|^{p_j}}{p_j}\, dx + \langle f, w_j\rangle \right) \right) \\
&\quad + \liminf_{j\to\infty} \left(\|p-p_j\|_\infty |\Omega| - \|p-p_j\|_\infty^{-\|p-p_j\|_\infty} \langle f, w_j\rangle \right) \\
&= L - \langle f, v\rangle. \qquad (5.2)
\end{aligned}$$

It is evident from the preceding argument that if $\int_\Omega |\nabla v|^p/p\,dx = \infty$, then

$$\liminf_{j\to\infty}\left(\int_\Omega \frac{|\nabla v_j|^{p_j}}{p_j}\,dx + \langle f, v_j\rangle\right) = \infty;$$

if this is the case, then obviously

$$F_p(v) = \infty = \liminf_{j\to\infty}\left(\int_\Omega \frac{|\nabla v_j|^{p_j}}{p_j}\,dx + \langle f, v_j\rangle\right).$$

It then follows from the latter and (5.2) that if $v_j \to v$ in $L^p(\Omega)$, then also $F_{p_j}(v_j) \to F_p(v)$ in $[0,\infty]$. The proof of the Γ-convergence is completed by observing that for each $u \in L^p(\Omega)$,

$$\lim_{j\to\infty} F_{p_j}(u) = \lim_{j\to\infty}\int_\Omega \frac{|\nabla u|^{p_j}}{p_j}\,dx + \langle f, u\rangle = \int_\Omega \frac{|\nabla u|^p}{p}\,dx + \langle f, u\rangle = F_p(u)$$

if $u \in W_0^{1,p}(\Omega)$, or else $\lim_{j\to\infty} F_{p_j}(u) = \infty = F_p(u)$.

The proof of (ii) follows along similar lines and is omitted. □

Theorems 4.2 and 4.4 follow from Theorem 5.2 via the general theory of Γ-convergence; see [8, Corollary 7.20] . Rather than deploying the full theoretical details of Γ-convegence, we have opted for the ad hoc argument presented in Section 4 in this particular case for the benefit of those readers not familiar with Γ-convergence. For constant exponent p, Theorem 4.2 and 4.4 were treated in [21], though not using any Γ-convergence. We hope this section will elicit further interest in the application of Γ-convergence in the framework of non-rearrangement invariant function spaces.

6 Final Comments

The stability question for constant exponent p was addressed by Lindqvist in [21]. For the variable exponent case, the stability of the homogeneous problem for a general operator was studied in [14], [16] for a certain class of domains. More specifically, it is shown that in the notation of Section 4, under the assumption of log-Hölder continuity of the sequence $\{p_i\}_i$ (see [14, Theorem 7.3]) and for $f \in W_0^{1,p(\cdot)(1+\delta)}(\Omega)$ for some $\delta > 0$, if for each $i \in \mathbb{N}$

$$\begin{cases} \Delta_{p_i(\cdot)} = 0, \\ u_i|_{\partial\Omega} = 0 \end{cases} \tag{6.1}$$

with $u_i - f \in W_0^{1,p_i(\cdot)}(\Omega)$, a subsequence of $\{u_i\}_i$ converges strongly to the solution u of

$$\begin{cases} \Delta_{p(\cdot)} = 0 \\ u|_{\partial\Omega} = 0 \end{cases} \tag{6.2}$$

with with $u - f \in W_0^{1,p(\cdot)}(\Omega)$. We remark that the stronger assumptions upon the exponents yield higher regularity and stronger convergence results of the sequence of solutions. Similar results are obtained in [16] for $\inf_\Omega p_i \to 1$.

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