A COUNTING PROBLEM IN ERGODIC THEORY AND EXTRAPOLATION FOR ONE-SIDED WEIGHTS

By

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Abstract. The purpose of this paper is to prove that, given a dynamical system $(X, \mathcal{M}, \mu, \tau)$ and $0 < q < 1$, the Lorentz spaces $L^{1,q}(\mu)$ satisfy the so-called Return Times Property for the Tail, contrary to what happens in the case $q = 1$. In fact, we consider a more general case than in previous papers since we work with a σ-finite measure μ and a transformation τ which is only Cesàro bounded. The proof uses the extrapolation theory of Rubio de Francia for one-sided weights. These results are of independent interest and can be applied to many other situations.

1 Introduction

Initially, let us consider a finite dynamical system $(X, \mathcal{M}, \mu, \tau)$; that is, a finite measure space with an invertible measure-preserving transformation τ on X. The following result is usually referred to as Bourgain's Return Times Theorem ([7], [8]).

Theorem 1.1. *Let $(X, \mathcal{M}, \mu, \tau)$ be a finite dynamical system and $f \in L^\infty(\mu)$. Then there exists $X_0 \subset X$ of full measure such that for all $x_0 \in X_0$, all finite dynamical systems $(Y, \mathcal{C}, \nu, S)$ and all $g \in L^1(\nu)$, the sequence of averages*

$$B_n g(y) = \frac{1}{n} \sum_{i=0}^{n-1} f(\tau^i x_0) g(S^i y)$$

converges for almost every $y \in Y$ (ν).

We say that $L^\infty(\mu)$ satisfies RTT or simply write $L^\infty(\mu) \in RTT$.

Theorem 1.1 gives no information in the case $f \in L^1(\mu)$; however, it is known that if the Return Times Theorem holds for f, then the so-called Return Times

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Property for the Tail holds for f; that is, for all $x_0 \in X_0$, all dynamical systems $(Y, \mathcal{C}, \nu, S)$ and all $g \in L^1(\nu)$, the sequence

$$R_n g(y) = \frac{1}{n} f(\tau^n x_0) g(S^n y)$$

converges to 0 for almost every $y \in Y$ (ν). In this case, we say that $f \in RTP$; and, if X is a space such that $f \in RTP$ for every $f \in X$, we say that X satisfies the RTP or simply write $X \in RTP$. In particular,

$$X \in RTT \quad \text{implies} \quad X \in RTP.$$

Using this necessary condition and the following result, it was proved in [5] (see also [4]) that the Return Times Theorem does not hold, in general, for $L^1(\mu)$.

Theorem 1.2 ([3] Theorem 8)**.** *Let* $\{c_n\}$ *be a sequence of nonnegative numbers such that* $\lim_{n\to\infty} c_n = 0$. *Then, the following two statements are equivalent:*
(a) $\sup_n \frac{1}{n}\#\{k : c_k > \frac{1}{n}\} < +\infty$;
(b) *for all finite dynamical systems* $(Y, \mathcal{C}, \nu, S)$ *and all* $g \in L^1(\nu)$*, the sequence* $c_n g(S^n y)$ *converges to* 0 *for almost every* $y \in Y$ (ν).

Now, it is known that given $f \in L^1(\mu)$, the sequence $c_n = f(\tau^n x)/n$ converges to 0 a.e. x, and hence

$$f \in RTP \quad \text{if and only if} \quad Nf(x) := \sup_{n\in\mathbb{N}} \frac{1}{n} N_{\frac{1}{n}} f(x) < +\infty \quad \text{a.e. } x,$$

where, for $\alpha > 0$,

$$N_\alpha f(x) = \#\left\{k \geq 1 : \frac{|f(\tau^k x)|}{k} > \alpha\right\}.$$

It was proved in [5] that if the measure space is nonatomic and the transformation is ergodic, there exists $f \in L^1(\mu)$ such that $Nf(x) = +\infty$ a.e.; consequently, under the mentioned hypotheses, the Return Times Property for the Tail and the Return Times Theorem do not hold for $L^1(\mu)$ functions; that is,

$$L^1(\mu) \notin RTP \quad \text{and} \quad L^1(\mu) \notin RTT.$$

The conclusion of our discussion is that the study of the finiteness of Nf is a key point in the Return Times theorems.

The example in [5] shows that, in general, N does not map $L^1(\mu)$ into $L^{1,\infty}(\mu)$. However, Assani [2] proved that $N : L\log L(\mu) \longrightarrow L^1(\mu)$, where

$$L\log L(\mu) = \left\{f : \|f\|_{L\log L(\mu)} = \int_0^1 f_\mu^*(t)\Big(1 + \log^+ \frac{1}{t}\Big)dt < \infty\right\}.$$

Recall that the decreasing rearrangement of f with respect to μ is

$$f_\mu^*(t) = \inf\{y > 0 : \lambda_f^\mu(y) \le t\},$$

with $\lambda_f^\mu(y) = \mu(\{x : |f(x)| > y\})$ the distribution function of f with respect to μ. Hence $Nf(x) < +\infty$ a.e. x, for every $f \in L\log L(\mu)$; and therefore, $L\log L(\mu) \in RTP$. Some years later, Demeter and Quas [13] proved that $N : L\log\log L(\mu) \longrightarrow L^{1,\infty}(\mu)$, and hence

$$L\log\log L(\mu) \in RTP, \tag{1.1}$$

where

$$L\log\log L(\mu) = \left\{ f : \|f\|_{L\log\log L(\mu)} = \int_0^1 f_\mu^*(t)\Big(1 + \log^+\log^+\frac{1}{t}\Big)dt < \infty \right\}.$$

Finally, it was observed in [13] that if X is an Orlicz (or Lorentz) space strictly larger than $L\log\log\log L(\mu)$, then $X \notin RTP$, and hence $X \notin RTT$.

In this paper (see, e.g., Theorem 2.9), we weaken the previous assumptions since we work with σ-finite measures and τ is an invertible measurable transformation Cesàro bounded in $L^1(\mu)$; that is, there exists $C > 0$ such that

$$\sup_{n\ge1}\left\|\frac{1}{n}\sum_{i=0}^{n-1} f\circ\tau^i\right\|_{L^1(\mu)} \le C\|f\|_{L^1(\mu)}.$$

Let now $0 < p, q \le \infty$ and $L^{p,q}(\mu)$ be the Lorentz space defined as the space of measurable functions such that

$$\|f\|_{L^{p,q}(\mu)} = \left(q\int_0^\infty y^{q-1}\lambda_f^\mu(y)^{\frac{q}{p}}dy\right)^{1/q} = \left(\frac{q}{p}\int_0^\infty f_\mu^*(t)^q t^{\frac{q}{p}-1}dt\right)^{1/q} < \infty$$

if $q < \infty$; and, if $q = \infty$,

$$\|f\|_{L^{p,\infty}(\mu)} = \sup_{y>0} y\lambda_f^\mu(y)^{1/p} = \sup_{t>0} t^{1/p} f_\mu^*(t) < \infty.$$

Recall that, if $q < r < 1$, then $L^{1,q}(\mu) \subset L^{1,r}(\mu) \subset L^1(\mu)$.

Our goal is to study the Return Times Property for the Tail. Our main results, Theorem 2.9 and Corollary 2.10, show that for every $0 < q < 1$,

$$L^{1,q}(\mu) \in RTP. \tag{1.2}$$

In fact, we prove that if $N^*f(x) = \sup_{\alpha>0}\alpha N_\alpha f(x)$ (an operator bigger than N), then $N^* : L^{1,q}(\mu) \longrightarrow L^{1,\infty}(\mu)$ is a bounded operator. The interesting part of the proof of this result is that it uses a new technique, developed in [10], and based on the Rubio de Francia extrapolation theory. We need to extend this theory to the case of one-sided weights and do so in Section 3.

Remark 1.3. Let μ be a non-atomic probability measure. If f is a measurable function such that

$$f^*_\mu(t) = \frac{\chi_{(0,1)}(t)}{t(1+\log^+\frac{1}{t})(1+\log^+\log^+\frac{1}{t})^3},$$

then $f \in L\log\log L(\mu) \setminus \cup_{0<q<1} L^{1,q}(\mu)$. On the other hand, $L^{1,q}(\mu)$ is not embedded in $L\log\log L(\mu)$ since

$$A := \sup_{f\downarrow} \frac{\int_0^\infty f(t)(1+\log^+\log^+\frac{1}{t})dt}{\left(\int_0^1 f(t)^q t^{q-1} dt\right)^{1/q}} < \infty.$$

But it is known (see, e.g., [11]) that

$$A = \sup_{0<r<1} \frac{\int_0^r (1+\log^+\log^+\frac{1}{t})dt}{\left(\int_0^r t^{q-1} dt\right)^{1/q}} = \infty.$$

Therefore, (1.1) and (1.2) are independent results, since they provide nonrelated metric spaces B such that $B \in RTP$.

Finally, as is usual, we let $|E|$ stand for the Lebesgue measure of the set E, and if μ is the measure $d\mu = u(x)\,dx$, then λ^μ_f and f^*_μ are written λ^u_f and f^*_u. Moreover, if the measure is clearly understood, we simply write λ_f and f^*. If the set X is the set of integers $\mathbb{Z}$ and the measure μ is the counting measure, the Lorentz spaces are denoted by $\ell^{p,q}$; and if the measure on the integers is given by a density $u = \{u_n\}_{n\in\mathbb{Z}}$ then, we write $\ell^{p,q}(u)$. In this case, for a sequence $a = \{a_n\}_{n\in\mathbb{Z}}$,

$$\|a\|_{\ell^{p,q}(u)} = \left(q\int_0^\infty \left(\sum_{\{n\in\mathbb{Z}:a_n>y\}} u_n\right)^{\frac{q}{p}-1} y^{q-1}dy\right)^{1/q}$$

if $q < \infty$; and if $q = \infty$,

$$\|a\|_{\ell^{p,\infty}(u)} = \sup_{y>0} y\left(\sum_{\{n\in\mathbb{Z}:a_n>y\}} u_n\right)^{1/p}.$$

By a positive constant C, we mean a constant independent of all important parameters. The expression $A \lesssim B$ indicates that there exists a constant C such that $A \le CB$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 A counting problem for Cesàro bounded transformations

From now on, $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $\tau : X \to X$ an invertible measurable transformation whose inverse is measurable and (two-sided)-nonsingular, i.e., $\mu(E) = 0$ if and only if $\mu(\tau^{-1}E) = 0$. We emphasize that it is easy to adapt the proof in [3] of Theorem 1.2 to the case of σ-finite measures, while Theorem 1.1 may fail in this case; see [14].

Let us now consider the ergodic maximal operator

$$M_\tau f(x) = \sup_{n\geq 1} A_n|f|(x), \qquad A_n f = \frac{1}{n}\sum_{i=0}^{n-1} f\circ\tau^i.$$

We need the following result, which can be found in [23] and in [27].

Theorem 2.1. *Let* $1 \leq p < +\infty$ *and assume that* τ *is Cesàro bounded in* $L^p(\mu)$*; that is,* $\sup_{n\geq 1} \|A_n f\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}$ *for all* $f \in L^p(\mu)$*. Then*

(a) *if* $p = 1$,

$$\|M_\tau f\|_{L^{1,\infty}(\mu)} \lesssim \|f\|_{L^1(\mu)}, \quad \textit{for all } f \in L^1(\mu);$$

(b) *if* $1 < p < +\infty$,

$$\|M_\tau f\|_{L^p(\mu)} \leq C_p\|f\|_{L^p(\mu)}, \quad \textit{for all } f \in L^p(\mu).$$

Definition 2.2. Let $1 \leq p < \infty$. An operator T is said to be of **restricted weak type** (p, p) if there exists a constant $C > 0$ such that for all measurable sets E,

$$\|T\chi_E\|_{L^{p,\infty}(\mu)} \leq C\mu(E)^{1/p}.$$

The least of all possible constants C is denoted by $\|T\|_{p,rest}$.

The operator N^* is closely related to the ergodic maximal operator since, if A is a measurable set, then $N^*(\chi_A) \leq M_\tau(\chi_A)$. Hence, using Theorem 2.1, we immediately have the following result.

Corollary 2.3. *Under the hypotheses of Theorem 2.1,* N^* *is of restricted weak type* (p, p).

Theorem 2.4. *Let* $1 < p < +\infty$ *and assume that* τ *is Cesàro bounded in* $L^p(\mu)$*. Then* $N^* : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\mu)$ *is bounded with*

$$\|N^*\|_{L^{p,1}(\mu)\to L^{p,\infty}(\mu)} \leq \frac{4p\|N^*\|_{p,rest}}{p-1}.$$

Proof. Let $f \in L^{p,1}(\mu)$. For each integer number i, define

$$E_i = \{x : 2^{i-1} < |f(x)| \leq 2^i\},$$

and $f_i = f\chi_{E_i}$. Since N^* is sublinear on disjointly supported functions and monotone,

$$N^* f \leq \sum_{i=-\infty}^{\infty} N^* f_i \leq \sum_{i=-\infty}^{\infty} 2^i N^*\chi_{E_i}.$$

Since $p > 1$ and N^* is of restricted weak type (p, p),

$$\begin{aligned} \|N^* f\|_{L^{p,\infty}(\mu)} &\le \frac{p}{p-1} \sum_{i=-\infty}^{\infty} 2^i \|N^* \chi_{E_i}\|_{L^{p,\infty}(\mu)} \le \frac{p\|N^*\|_{p,rest}}{p-1} \sum_{i=-\infty}^{\infty} 2^i \mu(E_i)^{1/p} \\ &\le \frac{4p}{p-1} \|N^*\|_{p,rest} \sum_{i=-\infty}^{\infty} \int_{2^{i-2}}^{2^{i-1}} \mu(\{x : |f(x)| > t\})^{1/p}\, dt \\ &= \frac{4p\|N^*\|_{p,rest}}{p-1} \|f\|_{L^{p,1}(\mu)}, \end{aligned}$$

and the result follows. □

Remark 2.5. If $1 < p < \infty$ and τ is an invertible measurable transformation which is Cesàro bounded in $L^p(\mu)$, then it is Cesàro bounded in $L^{p-\varepsilon}(\mu)$ for some ε, $0 < \varepsilon < p - 1$; see [23] and [27].

Using Remark 2.5, Theorem 2.4 and Marcinkiewicz's interpolation theorem (adapted to sublinear operators on disjointly supported functions), one can easily prove the boundedness on $L^p(\mu)$.

Theorem 2.6. *Let $1 < p < +\infty$ and assume that τ is Cesàro bounded in $L^p(\mu)$. Then $N^* : L^p(\mu) \longrightarrow L^p(\mu)$ is bounded.*

As mentioned in the introduction, we already know that N^* does not map $L^1(\mu)$ into $L^{1,\infty}(\mu)$. However, we can prove some boundedness result on $L^{1,q}(\mu)$ for every $0 < q < 1$. To this end, we first need to assume a claim and then prove a proposition.

Consider the continuous version of the operator N^* defined by

$$N^{*,c} f(x) = \sup_{\alpha>0} \alpha \left| \left\{ y > 0 : \frac{|f(x+y)|}{y} > \alpha \right\} \right|,$$

and recall the definition of the one-sided weights $u \in A_1^+$: a locally integrable positive function u belongs to A_1^+ if there exists $C > 0$ such that $M^- u(x) \le Cu(x)$, a.e. x, where

$$M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)| dy;$$

we denote by $\|u\|_{A_1^+}$ the infimum of such constants.

Claim. *Let $0 < q < 1$ and $u \in A_1^+$. Then*

$$N^{*,c} : L^{1,q}(u) \longrightarrow L^{1,\infty}(u) \tag{2.1}$$

is bounded with a constant less than or equal to $C_q \|u\|_{A_1^+}$, for some constant $C_q > 0$ depending only on q.

Section 3 is devoted to the proof of this claim.

Let us now take the dynamical system $(X, \mathcal{M}, \mu, \tau)$, where $X = \mathbb{Z}$ is the set of the integers, $\mathcal{M} = \mathcal{P}(X)$, μ is the counting measure ($\mu(E) = \#(E)$) and $\tau(i) = i + 1$. The **counting function** N^* associated to this dynamical system is denoted by $N^{*,d}$; that is, if f is a function defined on the integers,

$$N^{*,d} f(i) = \sup_{\alpha>0} \alpha \, \#\left\{k \geq 1 : \frac{|f(i+k)|}{k} > \alpha\right\}.$$

A **weight** $w \in A_1^+(\mathbb{Z})$ is a nonnegative function on the integers such that

$$m^- w(j) = \sup_{n\geq 1} \frac{1}{n} \sum_{i=0}^{n-1} w(j-i) \leq C w(j) \tag{2.2}$$

for all $j \in \mathbb{Z}$, where C is a positive constant independent of j and

$$\|w\|_{A_1^+(\mathbb{Z})} = \inf\{C > 0;\ (2.2) \text{ holds}\}.$$

Remark 2.7. Let w be a nonnegative function on $\mathbb{Z}$. Define $W : \mathbb{R} \to \mathbb{R}$ by $W(x) = w([x])$, where $[x]$ is the integer part of x. It is easy to see that $w \in A_1^+(\mathbb{Z})$ if and only if $W \in A_1^+$ and $\|w\|_{A_1^+(\mathbb{Z})} \approx \|W\|_{A_1^+}$.

Now, assuming the claim and using a discretization argument, we can prove the following result.

Proposition 2.8. *Let* $0 < q < 1$ *and let* $w \in A_1^+(\mathbb{Z})$*. Then* $N^{*,d} : \ell^{1,q}(w) \longrightarrow \ell^{1,\infty}(w)$ *is bounded with constant* $C_q \|w\|_{A_1^+(\mathbb{Z})}$*, for some constant* $C_q > 0$ *depending only on* q*.*

Proof. Let $f : \mathbb{Z} \to \mathbb{R}^+$ and define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = f([x])$. Observe that if $\alpha < f(i+k)/k$, $x \in (i, i+1)$ and $z \in (i+k, i+k+1)$, then

$$\begin{aligned} \alpha < \frac{f(i+k)}{k} &= \frac{F(z)}{k} = \frac{F(x+z-x)}{z-x} \frac{z-x}{k} \\ &\leq \frac{F(x+z-x)}{z-x} \frac{k+1}{k} \leq 2 \frac{F(x+z-x)}{z-x}. \end{aligned}$$

Consequently,

$$\bigcup_{\left\{k\geq 1 : \frac{f(i+k)}{k} > \alpha\right\}} (i+k, i+k+1) \subset \left\{z > x : \frac{F(x+z-x)}{z-x} > \frac{\alpha}{2}\right\},$$

which implies

$$\#\left\{k \geq 1 : \frac{f(i+k)}{k} > \alpha\right\} \leq \left|\left\{y > 0 : \frac{F(x+y)}{y} > \frac{\alpha}{2}\right\}\right|;$$

and thus, $N^{*,d}f(i) \le 2N^{*,c}F(x)$ for all $x \in (i, i+1)$. It follows that

$$\|N^{*,d}f\|_{\ell^{1,\infty}(w)} \le 2\|N^{*,c}F\|_{L^{1,\infty}(W)}.$$

By Remark 2.7 and the claim, we obtain

$$\|N^{*,d}f\|_{\ell^{1,\infty}(w)} \lesssim C_q\|W\|_{A_1^+}\|F\|_{L^{1,q}(W)} \approx C_q\|w\|_{A_1^+(\mathbb{Z})}\|f\|_{\ell^{1,q}(w)},$$

as we wanted to prove. □

We are now ready to formulate and prove our first main result via a classical transference argument.

Theorem 2.9. *Let $0 < q < 1$ and assume that τ is Cesàro bounded in $L^1(\mu)$. Then $N^* : L^{1,q}(\mu) \longrightarrow L^{1,\infty}(\mu)$ is bounded.*

Proof. It suffices to prove the theorem for nonnegative measurable functions. Since τ is two-sided nonsingular, by the Radon-Nikodym theorem (see [27]), there exists a family of nonnegative measurable functions $\{J_i(x)\}_{i\in\mathbb{Z}}$ such that $J_{i+j}(x) = J_i(\tau^j x)J_j(x)$; and for all nonnegative measurable functions f,

$$\int_X f\,d\mu = \int_X f(\tau^i(x))J_i(x)\,d\mu(x).$$

Furthermore, it is known [27] that

$$h_x(i) = J_i(x) \in A_1^+(\mathbb{Z}), \quad \text{a.e. } x, \text{ and } \|h_x\|_{A_1^+(\mathbb{Z})} \le \sup_n \|A_n\|_{L^1(\mu)}. \tag{2.3}$$

Fix a natural number L and consider

$$(N_\alpha)_L f(x) = \#\left\{1 \le k \le L : \frac{|f(\tau^k x)|}{k} > \alpha\right\}$$

and the truncated operator $N_L^* f(x) = \sup_{\alpha>0}\alpha(N_\alpha)_L f(x)$. Then fix a nonnegative measurable function f. For any $x \in X$, let f^x be the function defined on $\mathbb{Z}$ by $f^x(i) = f(\tau^i x)$. Let $n \in \mathbb{N}$ and $\lambda > 0$. If $O_{\lambda,L} = \{x : N_L^* f(x) > \lambda\}$, then

$$\lambda\mu(O_{\lambda,L}) = \frac{1}{n+1}\int_X \lambda\sum_{i=0}^n \chi_{O_{\lambda,L}}(\tau^i x)h_x(i)\,d\mu(x).$$

It is easy to see that if $\chi_{O_{\lambda,L}}(\tau^i x) = 1$ $(0 \le i \le n)$, then $N^{*,d}(f^x\chi_{[0,n+L]})(i) > \lambda$. Then for all x,

$$\lambda\sum_{i=0}^n \chi_{O_{\lambda,L}}(\tau^i x)h_x(i) \le \lambda \sum_{\{i:N^{*,d}(f^x\chi_{[0,n+L]})(i)>\lambda\}} h_x(i) \le \|N^{*,d}(f^x\chi_{[0,n+L]})\|_{\ell^{1,\infty}(h_x)}.$$

Since, by (2.3), for a.e. $x \in X$, the functions $h_x \in A_1^+(\mathbb{Z})$ with a uniform constant, Proposition 2.8 gives $C_q > 0$ such that

$$\|N^{*,d}(f^x \chi_{[0,n+L]})\|_{\ell^{1,\infty}(h_x)} \leq C_q \|f^x \chi_{[0,n+L]}\|_{\ell^{1,q}(h_x)} \quad \text{a.e. } x.$$

Therefore,

$$\begin{aligned}
\lambda\mu(O_{\lambda,L}) &\leq \frac{C_q}{n+1} \int_X \|f^x \chi_{[0,n+L]}\|_{\ell^{1,q}(h_x)} d\mu(x) \\
&= \frac{C_q}{n+1} \int_X \left(q \int_0^\infty \left(t \sum_{\{i \in [0,n+L]: f^x(i)\} > t} h_x(i) \right)^q \frac{dt}{t} \right)^{1/q} d\mu(x).
\end{aligned}$$

Applying Minkowski's integral inequality, we obtain

$$\begin{aligned}
\lambda\mu(O_{\lambda,L}) &\leq \frac{C_q}{n+1} q^{\frac{1}{q}} \left(\int_0^\infty t^q \left(\int_X \sum_{\{i \in [0,n+L]: f^x(i) > t\}} h_x(i)\, d\mu(x) \right)^q \frac{dt}{t} \right)^{1/q} \\
&= \frac{C_q}{n+1} q^{\frac{1}{q}} \left(\int_0^\infty t^q \left(\sum_{i=0}^{n+L} \int_{\{x: f(\tau^i x) > t\}} h_x(i)\, d\mu(x) \right)^q \frac{dt}{t} \right)^{1/q} \\
&= C_q q^{\frac{1}{q}} \frac{n+L+1}{n+1} \left(\int_0^\infty (t\mu\{x : f(x) > t\})^q \frac{dt}{t} \right)^{1/q} \\
&= C_q \frac{n+L+1}{n+1} \|f\|_{L^{1,q}(\mu)}.
\end{aligned}$$

Letting $n \to +\infty$ and then $L \to +\infty$, we obtain the desired result. □

The final result of this section is our second main result.

Corollary 2.10. *Under the hypotheses of Theorem 2.9, $L^{1,q}(\mu) \in RTP$.*

Proof. The proof follows easily since for every $f \in L^{1,q}(\mu)$, $0 < q < 1$,

$$\lim_{n \to \infty} \frac{f(\tau^n x)}{n} = 0, \quad \text{a.e. } x,$$

because the averages converge a.e.; see [23] and [27]. □

With a similar argument we also obtain the following result.

Corollary 2.11. *Under the hypotheses of Theorem 2.6, $L^p(\mu) \in RTP$.*

3 Extrapolation for one-sided weights

The purpose of this section is to prove the claim stated in the previous section. However, the results obtained here are of independent interest and can be applied to many other situations. The main idea is to obtain (2.1) for every $u \in A_1^+$, using the Rubio de Francia extrapolation argument recently developed in [10] adapted to the case of one-sided weights. We emphasize that we give only the proofs of those results which do not easily follow in the same way as in the two-sided case.

3.1 One-sided weights. As mentioned in the previous section, by a weight, we mean a locally integrable function $w \geq 0$. The good weights for the classical Hardy-Littlewood maximal operator M are the weights in the classes A_p of Muckenhoupt [25]. One-sided weights are the good weights for one-sided operators like the one-sided Hardy-Littlewood maximal functions, defined in $\mathbb{R}$ for locally integrable functions f by

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f| \quad \text{and} \quad M^-f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f|.$$

In general, a one-sided operator T^+ (respectively, T^-) in $\mathbb{R}$ is an operator such that the value of $T^+f(x)$ (respectively, $T^-f(x)$) depends only on the values of f in $[x, \infty)$ (respectively, $(-\infty, x]$); for examples of one-sided operators, see [1], [6], [16], [18], [19], [20], [29]. There are many classical operators in Real Analysis that are one-sided operators for which the class of weights is wider than the one of Muckenhoupt.

The one-sided A_p weights, $1 \leq p < \infty$, were introduced by E. Sawyer [28] and are defined as follows. We say that $w \in A_p^+$, $p > 1$, if

$$\|w\|_{A_p^+} = \sup_{h>0} \left(\frac{1}{h} \int_{x-h}^{x} w \right) \left(\frac{1}{h} \int_x^{x+h} w^{-\frac{1}{p-1}} \right)^{p-1} < \infty;$$

A_1^+ is defined as in Section 2. The following results about these weights can be found in [28] and [22].

1. The operator M^+ is of weak type $(1, 1)$ with respect to w if and only if $w \in A_1^+$
2. The operator M^+ is a bounded operator on $L^p(w)$, $p > 1$, if and only if $w \in A_p^+$.
3. If $M^-f < \infty$ a.e., then $(M^-f)^\delta \in A_1^+$ for every $0 < \delta < 1$ and $\|(M^-f)^\delta\|_{A_1^+} \leq C\frac{1}{1-\delta}$.
4. Factorization: $w \in A_p^+$ if an only if $w = u_0 u_1^{1-p}$ for some $u_0 \in A_1^+$ and $u_1 \in A_1^-$.

Of course, there are similar definitions and theorems for A_p^-, obtained by reversing the orientation of $\mathbb{R}$. All the results that we state in this section have a corresponding result for a reversed orientation of the real line.

Remarks 3.1. i) Unlike the two-sided case, f not identically 0 does not imply $M^+f > 0$ a.e.; therefore, one-sided weights can be 0 in a set of positive measure. Here we are adopting the usual conventions $\infty \cdot t = t \cdot \infty = \infty$, for $0 < t \leq \infty$, $0 \cdot \infty = \infty \cdot 0 = 0$, $\infty^{-1} = 0$ and $0^{-1} = \infty$.

ii) It follows from the definitions that if $w \in A_p^+$, there exist $-\infty \leq a \leq b \leq \infty$ such that $w = 0$ in $(-\infty, a)$, $w = \infty$ in (b, ∞), $0 < w < \infty$ in (a, b), $w \in L^1_{loc}(a, b)$ and, if $1 < p < \infty$, $w^{1-p'} \in L^1_{loc}(a, b)$, where $p' = \frac{p}{p-1}$ is the dual exponent of p; see [15]. Thus, when working with one-sided weights, we can assume without loss of generality that $(a, b) = \mathbb{R}$.

There is a vast amount of work dealing with one-sided weights and one-sided operators, extending the results for the standard cases to the one-sided case. Most of the time, this involves significant technical difficulties.

In this section, we extend the results in [10] to the setting of one-sided weights. We use them to prove the claim in Section 2. Other extrapolation results for one-sided weights can be found in [12], [21], [17], [9].

3.2 Restricted weak-type extrapolation.

Definition 3.2. Let $1 \leq p < \infty$. We say that a weight $w \in A_p^{\mathcal{R},+}$ if there exists a constant $C > 0$ such that for any three numbers $a < b < c$ and any measurable set $E \subset (b, c)$,

$$w((a, b)) \left(\frac{|E|}{c - a} \right)^p \leq C w(E);$$

we denote by $\|w\|_{A_p^{\mathcal{R},+}}$ the infimum of these constants.

It is easy to see that to check this condition, it suffices to check the condition only for $a < b < c$ such that $b - a = c - b$ (the constant appearing is $2^p C$). The following result gives a characterization of the class $A_p^{\mathcal{R},+}$.

Lemma 3.3 ([26, Lemma 3])**.** *Let $1 < p < +\infty$. Then $w \in A_p^{\mathcal{R},+}$ if and only if*

$$\sup_{a<b<c} \frac{\|\chi_{(a,b)}\|_{L^{p,1}(w)} \|w^{-1}\chi_{(b,c)}\|_{L^{p',\infty}(w)}}{c - a} < +\infty.$$

Theorem 3.4 ([24])**.** *Let $1 \leq p < \infty$ and let u be a weight. Then*

$$M^+ : L^{p,1}(u) \to L^{p,\infty}(u)$$

is bounded if and only if $u \in A_p^{\mathcal{R},+}$. Furthermore, if $1 < p < \infty$,

$$(p - 1)\|M^+\|_{L^{p,1}(u)\to L^{p,\infty}(u)} \lesssim \|u\|_{A_p^{\mathcal{R},+}} \leq \|M^+\|_{L^{p,1}(u)\to L^{p,\infty}(u)};$$

and, if $p = 1$,

$$\|M^+\|_{L^1(u)\to L^{1,\infty}(u)} \approx \|u\|_{A_1^+}.$$

Following the same steps as in the two-sided case, it is easy to see that the following result also holds.

Proposition 3.5. *For every* $\varepsilon > 0$,

$$A_p^+ \subset A_p^{\mathcal{R},+} \subset A_{p+\varepsilon}^+ \quad \text{and} \quad \|u\|_{A_p^{\mathcal{R},+}} \le \|u\|_{A_p^+}^{1/p}.$$

3.3 Construction of $A_p^{\mathcal{R},+}$ weights.

Lemma 3.6. *Suppose* $0 < M^+f(x) < \infty$ *a.e. Let* $a < c < d$ *be such that* $d - c = c - a$. *If* $g = f\chi_{(d,\infty)}$, *there exists* K, $0 < K < +\infty$, *such that* $K/2 \le M^+g(x) \le K$ *for all* $x \in (a, c)$.

Proof. Let $f \ge 0$. Let $x, y \in (a, c)$, $x \le y$ and $h > 0$. Then

$$\frac{1}{h+d-x}\int_d^{d+h} f \le \frac{1}{h+d-y}\int_d^{d+h} f \le M^+g(y).$$

It follows that $M^+g(x) \le M^+g(y)$. Analogously,

$$\begin{aligned}\frac{1}{h+d-y}\int_d^{d+h} f &\le \frac{h+d-x}{h+d-y}\cdot\frac{1}{h+d-x}\int_d^{d+h} f \le \frac{h+d-x}{h+d-y}M^+g(x)\\ &= \Big(1+\frac{y-x}{h+d-y}\Big)M^+g(x) \le \Big(1+\frac{c-a}{d-c}\Big)M^+g(x)\\ &= 2M^+g(x).\end{aligned}$$

Therefore, $M^+g(y) \le 2M^+g(x)$. Taking $K = \sup_{x\in(a,c)} M^+g(x)$, we have the result. □

The following results (Theorem 3.7 and Corollary 3.8) are proved in a completely different way from the two-sided case.

Theorem 3.7. *Let* $1 < p < +\infty$. *Let* f *be a measurable function. Then* $w = (M^+f)^{1-p} \in A_p^{\mathcal{R},+}$, *with constant independent of* f.

Proof. We may assume that $0 < M^+f(x) < \infty$ a.e.; otherwise, we have to work only in the interval where $0 < M^+f(x) < \infty$. By Lemma 3.3, it suffices to prove that there exists $C > 0$ such that for all $a < b < c$ and for all $t > 0$,

$$\Big(\int_a^b w\Big)^{1/p} t \Big(\int_{\{x\in(b,c):w^{-1}(x)>t\}} w\Big)^{1/p'} \le C(c-a).$$

Let $a < b < c$ and d be such that $d-c = c-a$. Consider K and g as in Lemma 3.6. Then $w \le (M^+g)^{1-p}$ since $g \le f$.

If $0 < t \le (2K)^{p-1}$, then

$$\begin{aligned}
&\left(\int_a^b w\right)^{1/p} t\left(\int_{\{x\in(b,c):w^{-1}(x)>t\}} w\right)^{1/p'} \\
&\quad\le t\left(\int_a^b (M^+g)^{1-p}\right)^{1/p}\left(\int_{x\in(b,c)} (M^+g)^{1-p}\right)^{1/p'} \\
&\quad\le t\left(\int_a^b (K/2)^{1-p}\right)^{1/p}\left(\int_{x\in(b,c)} (K/2)^{1-p}\right)^{1/p'} \\
&\quad\le t(K/2)^{1-p}(b-a)^{1/p}(c-b)^{1/p'} \\
&\quad\le (2K)^{p-1}(K/2)^{(1-p)}(c-a) = 4^{p-1}(c-a).
\end{aligned}$$

If $t > (2K)^{p-1}$, then for all $x \in (b,c)$,

$$\begin{aligned}
M^+f(x) &\le M^+(f\chi_{(b,d)})(x) + M^+g(x) \le M^+(f\chi_{(b,d)})(x) + K \\
&< M^+(f\chi_{(b,d)})(x) + \frac{t^{1/(p-1)}}{2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\{x\in(b,c) : w^{-1}(x) > t\} &= \left\{x\in(b,c) : M^+f(x) > t^{1/(p-1)}\right\} \\
&\subset \left\{x\in(b,c) : (M^+(f\chi_{(b,d)})(x))^{p-1} > \frac{t}{2^{p-1}}\right\}.
\end{aligned}$$

Since M^+ is of weak type (1, 1) with respect to Lebesgue measure,

$$\begin{aligned}
t\left(\int_{\{x\in(b,c):w^{-1}(x)>t\}} w\right)^{1/p'} &\le t\left(\int_{\left\{x\in(b,c):(M^+(f\chi_{(b,d)})(x))^{p-1}>\frac{t}{2^{p-1}}\right\}} w\right)^{1/p'} \\
&\le t^{1/p}\left(t\int_{\left\{x\in(b,c):(M^+(f\chi_{(b,d)})(x))^{p-1}>\frac{t}{2^{p-1}}\right\}} (M^+(f\chi_{(b,d)}))^{1-p}\right)^{1/p'} \\
&\le t^{1/p}\left(\left|\left\{x\in(b,c) : M^+(f\chi_{(b,d)})(x) > \frac{t^{1/(p-1)}}{2}\right\}\right| 2^{p-1}\right)^{1/p'} \\
&= 2^{(p-1)/p'}t^{1/p}\left(\frac{2}{t^{1/(p-1)}}\int_b^d f\right)^{1/p'} = 2^{p-1}\left(\int_b^d f\right)^{1/p'}.
\end{aligned}$$

Consequently,

$$\left(\int_a^b w\right)^{1/p} t\left(\int_{\{x\in(b,c):w^{-1}(x)>t\}} w\right)^{1/p'} \leq \left(\int_a^b w\right)^{1/p} 2^{p-1}\left(\int_b^d f\right)^{1/p'}$$
$$\leq 2^{p-1}\left(\int_a^b w(M^+f)^{p-1}\right)^{1/p}(d-a)^{1/p'}$$
$$= 2^{p-1}(b-a)^{1/p}(2(c-a))^{1/p'} \leq 2^{p-1/p}(c-a). \qquad \square$$

Corollary 3.8. *Let f be a measurable function. Let $1 \leq p < \infty$. If $u \in A_1^+$, then $w = (M^+f)^{1-p}u \in A_p^{\mathcal{R},+}$ and*

$$\|w\|_{A_p^{\mathcal{R},+}} \lesssim \|u\|_{A_1^+}^{1/p}.$$

Proof. As before, we may assume that $0 < M^+f(x) < \infty$ a.e. Let $a < b < c$, and let $E \subset (b,c)$ be a measurable set. For $x \in (a,b)$, consider the decreasing sequence $\{z_n\}$ defined by

$$z_0 = c,$$
$$z_{n+1} = \frac{x+z_n}{2}, \quad n \in \mathbb{N}.$$

For $\alpha \in (0,1)$, $(M^+f)^\alpha \in A_1^-$, with constant depending only on α. Then, for every $n \in \mathbb{N}$,

$$\frac{1}{z_{n+1}-x}\int_x^{z_{n+1}} (M^+f)^\alpha \leq M^+(M^+f)^\alpha(x) \leq C(M^+f)^\alpha(x). \tag{3.1}$$

Now, from Hölder's inequality, we get that for every $q > 1$,

$$(z_{n+1}-x)^q \leq \left(\int_x^{z_{n+1}} (M^+f)^\alpha\right)\left(\int_x^{z_{n+1}} (M^+f)^{-\alpha/(q-1)}\right)^{q-1}.$$

Therefore,

$$\frac{z_{n+1}-x}{\int_x^{z_{n+1}} (M^+f)^\alpha} \leq \left(\frac{1}{z_{n+1}-x}\int_x^{z_{n+1}} (M^+f)^{-\alpha/(q-1)}\right)^{q-1}.$$

Take $q = 1 + \frac{\alpha}{p-1}$. Then the last inequality and (3.1) give

$$C(M^+f(x))^{-\alpha} \leq \left(\frac{1}{z_{n+1}-x}\int_x^{z_{n+1}} (M^+f)^{1-p}\right)^{\frac{\alpha}{p-1}}.$$

Raising to the power $\frac{p-1}{\alpha p} > 0$, using that $(M^+f)^{1-p} \in A_p^{\mathcal{R},+}$ with $x < z_{n+1} < z_n$ and $E_n = E \cap (z_{n+1}, z_n)$, and taking into account that $z_n - x = 2(z_n - z_{n+1})$ and $z_{n+1} - x = z_n - z_{n+1}$, we get

$$(M^+f(x))^{\frac{1-p}{p}} |E_n| \leq 2C(z_n - z_{n+1})^{(p-1)/p} \left(\int_{E_n} (M^+f)^{1-p} \right)^{1/p}.$$

Summing in $n \in \mathbb{N}$ and using Hölder's inequality with exponents (p', p), we obtain

$$\begin{aligned}(M^+f(x))^{\frac{1-p}{p}} |E| &= (M^+f(x))^{\frac{1-p}{p}} \sum_{n=0}^{\infty} |E_n| \\ &\leq C \left(\sum_{n=0}^{\infty} (z_n - z_{n+1}) \right)^{(p-1)/p} \left(\sum_{n=0}^{\infty} \int_{E_n} (M^+f)^{1-p} \right)^{1/p} \\ &= C(c-x)^{(p-1)/p} \left(\int_E (M^+f)^{1-p} \right)^{1/p}.\end{aligned}$$

Therefore,

$$(M^+f(x))^{1-p} |E|^p \leq C(c-a)^{p-1} \int_E (M^+f)^{1-p} \tag{3.2}$$

for almost every $x \in (a, b)$. Now, since $u \in A_1^+$,

$$\frac{1}{c-a} \int_a^b u(x)dx \leq \frac{1}{y-a} \int_a^y u(x)dx \leq M^- u(y) \leq \|u\|_{A_1^+} u(y)$$

for almost every $y \in E$. Then, multiplying in (3.2) by $u(x)$ and integrating in (a, b), we get

$$\begin{aligned}|E|^p \int_a^b (M^+f(x))^{1-p} u(x)dx &\leq C(c-a)^p \frac{1}{c-a} \int_a^b u(x)dx \int_E (M^+f)^{1-p}(y)dy \\ &\lesssim \|u\|_{A_1^+} (c-a)^p \int_E (M^+f)^{1-p}(y) u(y)dy,\end{aligned}$$

i.e., $w = (M^+f)^{1-p} u \in A_p^{\mathcal{R},+}$. □

We now define the class of weights that we use to extrapolate.

Definition 3.9. Let $1 \leq p < \infty$. We say that a weight w belongs to $\widehat{A}_p^+$ if there exist $f \in L^1_{loc}$ and $u \in A_1^+$ such that $w = (M^+f)^{1-p} u$, with $\|w\|_{\widehat{A}_p^+} = \inf \|u\|_{A_1^+}^{1/p}$.

By Corollary 3.8, $\widehat{A}_p^+ \subset A_p^{\mathcal{R},+}$ and $\|w\|_{A_p^{\mathcal{R},+}} \lesssim \|w\|_{\widehat{A}_p^+}$.

The following distribution inequality is used in the proof of our first extrapolation result. Its proof follows exactly the same pattern as that of [10, Proposition 2.10], and we omit it.

Proposition 3.10. *Let u be a weight, f and g nonnegative functions, $\gamma > 0$ and $1 \le p < p_0$. Then*

$$\lambda_g^u(y) \le \lambda_{M^+f}^u(\gamma y) + \gamma^{p_0-p}\frac{y^{p_0}}{y^p}\int_{\{x:g(x)>y\}}(M^+f)^{p-p_0}(x)u(x)dx.$$

Now we state the extrapolation result, which also follows the same pattern as the two-sided case.

Theorem 3.11. *Let T be an operator such that*

$$\|Tf\|_{L^{p_0,\infty}(v)} \le \varphi_{p_0}(\|v\|_{\widehat{A}^+_{p_0}})\|f\|_{L^{p_0,1}(v)}$$

for some $p_0 > 1$ and every $v \in \widehat{A}^+_{p_0}$, where φ_{p_0} is an increasing function on $(0,\infty)$. Then, for every $1 \le p < p_0$ and every $v \in \widehat{A}^+_p$,

$$\|Tf\|_{L^{p,\infty}(v)} \le C\|v\|_{\widehat{A}^+_p}^{1-p/p_0}\varphi_{p_0}(C\|v\|_{\widehat{A}^+_p}^{p/p_0})\|f\|_{L^{p,p/p_0}(v)}.$$

In particular, T is of restricted weak-type (p,p) with respect to v.

4 Proof of the Claim

Using Theorem 3.4 and the fact that $N^{*,c}\chi_E = M^+\chi_E$, and arguing as in the proof of Theorem 2.4, we easily have that for every $1 < p < +\infty$ and every $u \in A_p^{\mathcal{R},+}$, $N^{*,c} : L^{p,1}(u) \longrightarrow L^{p,\infty}(u)$ is bounded with

$$\|N^{*,c}\|_{L^{p,1}(u)\to L^{p,\infty}(u)} \lesssim \frac{\|u\|_{A_p^{\mathcal{R},+}}}{p-1};$$

hence, by Theorem 3.11, we finally obtain (2.1).

Remark 4.1. If T is a quasi-sub-linear operator which satisfies the same hypothesis of Theorem 3.11, we also get that it is possible to extrapolate up to a space quite near to $L^1(u)$, for $u \in A_1^+$. Namely, for every $\varepsilon > 0$,

$$\|Tf\|_{L(\log L)^\varepsilon(u)} \le C\|f\|_{L^{1,\infty}_{loc}(u)},$$

with constant $C \lesssim \frac{1}{\varepsilon}\|u\|_{A_1^+}^{1-1/p_0}\varphi_{p_0}(\|u\|_{A_1^+}^{1/p_0})$, where the spaces $L(\log L)^\varepsilon(u)$ and $L^{1,\infty}_{loc}(u)$ are defined by the conditions

$$\|f\|_{L(\log L)^\varepsilon(u)} = \int_0^\infty f_u^*(t)\left(1+\log^+\frac{1}{t}\right)^\varepsilon dt < \infty,$$

$$\|f\|_{L^{1,\infty}_{loc}(u)} = \sup_{0<t\le 1} tf_u^*(t) < \infty,$$

respectively. The proof of this fact follows the same pattern as in [10].

REFERENCES

[1] H. Aimar, L. Forzani, and F. J. Martín-Reyes, *On weighted inequalities for singular integrals*, Proc. Amer. Math. Soc. **125** (1997), 2057–2064.

[2] I. Assani, *Convergence of the p-series for stationary sequences*, New York J. Math. **3A** (1997/98), 15–30.

[3] I. Assani, *Strong laws for weighted sums of independent identically distributed random variables*, Duke Math J. **88** (1997), 217–246.

[4] I. Assani, Z. Buczolich, and D. Mauldin, *Counting and convergence in ergodic theory*, Acta Univ. Carolin. Math. Phys. **45** (2004), 5–21.

[5] I. Assani, Z. Buczolich, and D. Mauldin, *An L^1 counting problem in ergodic theory*, J. Anal. Math. **95** (2005), 221–241.

[6] A. L. Bernardis, M. Lorente, F. J. Martín-Reyes, M. T. Martínez, A. de la Torre, and J. L. de la Torre, *Differential transforms in weighted spaces*, J. Fourier Anal. Appl. **12** (2006), 83–103.

[7] J. Bourgain, *Return time sequences of dynamical systems*, unpublished preprint, 1988.

[8] J. Bourgain, H. Furstenberg, Y. Katznelson, and D. Ornstein, *Appendix to J. Bourgain: Pointwise ergodic theorems for arithmetic sets*, Inst. Hautes Études Sci. Publ. Math. **69** (1989), 5–45.

[9] M. Carro, J. Duoandikoetxea, and M. Lorente, *Weighted estimates in a limited range with applications to the Bochner-Riesz operators*, Indiana Univ. Math. J. **61** (2012), 1485–1511.

[10] M. Carro, L. Grafakos, and J. Soria, *Weighted weak-type (1,1) estimates via Rubio de Francia extrapolation*, J. Funct. Anal. **269** (2015), 1203–1233.

[11] M. Carro and J. Soria, *Weighted Lorentz spaces and the Hardy operator*, J. Funct. Anal. **112** (1993), 480–494.

[12] D. Cruz-Uribe, J. Martell, and C. Pérez, *Weights, Extrapolation and the Theory of Rubio de Francia*, Birkhäuser/Springer Basel AG, Basel, 2011.

[13] C. Demeter and A. Quas, *Weak-L^1 estimates and ergodic theorems*, New York J. Math. **10** (2004), 169–174.

[14] M. Lacey, *The return time theorem fails on infinite measure-preserving systems*, Ann. Inst. H. Poincaré Probab. Statist. **33** (1997), 491–495.

[15] Q. Lai, *A note on the weighted norm inequality for the one-sided maximal operator*, Proc. Amer. Math. Soc. **124** (1996), 527–537.

[16] M. Lorente, J. Martell, M. Riveros and A. de la Torre, *Generalized Hörmander's conditions, commutators and weights*, J. Math. Anal. Appl. **342** (2008), 1399–1425.

[17] M. Lorente and M. Riveros, *Two extrapolation theorems for related weights and applications*, Math. Inequal. Appl. **10** (2007), 643–660.

[18] M. Lorente, M. Riveros, and A. de la Torre, *Weighted estimates for singular integral operators satisfying Hörmander's conditions of Young type*, J. Fourier Anal. Appl. **11** (2005), 479–509.

[19] M. Lorente, M. Riveros, and A. de la Torre, *On the Coifman type inequality for the oscillation of one-sided averages*, J. Math. Anal. Appl. **336** (2007), 577–592.

[20] M. Lorente and A. de la Torre, *Weighted and ergodic theorems for series of differences of convolutions*, J. Fourier Anal. Appl. **14** (2008), 39–59.

[21] R. Macías and M. Riveros, *One-sided extrapolation at infinity and singular integrals*, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), 1081–1102.

[22] F. Martín-Reyes, S. Ortega, and A. de la Torre, *Weighted inequalities for one-sided maximal functions*, Trans. Amer. Math. Soc. **319** (1990), 517–534.

[23] F. Martín-Reyes and A. de la Torre, *The dominated ergodic estimate for mean bounded, invertible, positive operators*, Proc. Amer. Math. Soc. **104** (1988), 69–75.

[24] F. Martín-Reyes and A. de la Torre, *Some weighted inequalities for general one-sided maximal operators*, Studia Math. **122** (1997), 1–14.

[25] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[26] P. Ortega Salvador, *Weighted Lorentz norm inequalities for the one-sided Hardy-Littlewood maximal functions and for the maximal ergodic operator*, Canad. J. Math. **46** (1994), 1057–1072.

[27] R. Sato, *On the weighted ergodic properties of invertible Lamperti operators*, Math. J. Okayama Univ. **40** (1998), 147–176.

[28] E. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. **297** (1986), 53–61.

[29] A. de la Torre and J. Torrea, *One-sided discrete square function*, Studia Math. **156** (2003), 243–260.

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