STABILITY OF THE UNIQUE CONTINUATION FOR THE WAVE OPERATOR VIA TATARU INEQUALITY: THE LOCAL CASE

By

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Abstract. In 1995, Tataru proved a Carleman-type estimate for linear operators with partially analytic coefficients that is generally used to prove the unique continuation of those operators. In this paper, we use this inequality to study the stability of the unique continuation in the case of the wave equation with coefficients independent of time. We prove a logarithmic estimate in a ball whose radius has an explicit dependence on the C^1-norm of the coefficients and on the other geometric properties of the operator.

1 Introduction

We consider the wave operator in $\mathbb{R}^{n+1}$,

$$P(y, D) = -D_0^2 + \sum_{j,k=1}^{n} g^{jk}(x) D_j D_k + \sum_{j=1}^{n} h^j(x) D_j + q(x), \tag{1.1}$$

where $y = (t, x) \in \mathbb{R} \times \mathbb{R}^n$ are the time-space variables, $D_0 = -i\partial_t$, $D_j = -i\partial_{x_j}$. The coefficients $g^{jk} \in C^1(\mathbb{R}^n)$ are real and independent of time, and $[g^{jk}]$ is a symmetric positive-definite matrix. The coefficients $h^j, q \in L^\infty(\mathbb{R}^n)$ are complex valued and independent of time.

An operator $P(y, D)$ is said to have the **unique continuation property** if every solution u of $Pu = 0$ in a connected open set $\Omega \subset \mathbb{R}^{n+1}$, and vanishing on an open subset $B \subset \Omega$, vanishes in Ω. In [20], Tataru proved for the first

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time the unique continuation property for (1.1) across every non-characteristic C^2-hypersurface with no limitation to the normal direction. The result is valid for a larger class of linear operators, where the pseudo-convexity condition across a surface is fulfilled for the cotangent vectors with $\xi_0 = 0$, and it has been extended to the case of coefficients analytic in time [7, 18, 21]. The key point of these results is a Carleman-type estimate involving an exponential pseudo-differential operator.

Much is known about the consequences of this property on the uniqueness of a corresponding Cauchy problem. Actually, the unique continuation property has proved to be instructive in many areas of mathematics, e.g., in studying the uniqueness for linear and nonlinear PDEs, together with their blow up or traveling wave solutions [6], in studying the Anderson localization [4], in control theory to get controllability results [22], in inverse problems, to obtain uniqueness and stability estimates [12]. Concerning the continuous dependence of the unique continuation property, that is, its stability, fewer results are known. The elliptic and parabolic cases have been studied in several settings, where use was made either of Carleman estimates or some versions of the Three Ball Theorem; see [1] for a review of the results.

To our knowledge, the hyperbolic case like (1.1) is still open for arbitrary domains and arbitrary matrix-valued coefficients $g^{jk}(x)$, while there exist results for particular coefficients or domains; see [19]. This is maybe related to the difficulty of using standard Carleman estimates for hyperbolic operators to prove unique continuation close to the characteristic directions; that is the reason Tataru's work was so important in this field. The aim of the present work is then to prove a stability estimate for the unique continuation of the operator $P(y, D)$. We focus on the local case and formulate an explicit stability estimate for the inhomogeneous operator $Pu = f$ that can be alternatively reformulated in terms of a boundary value problem.

Let $\Omega \subset \mathbb{R}^{n+1}$ be a connected open set, and consider a non-characteristic oriented hypersurface S written as the level set of the function $\psi : \Omega \to \mathbb{R}$, $S = \{y \in \Omega; \psi(y) = 0\}$. Assume that $Pu = f$ in a ball $\Omega_1 := B(y_0, 2R)$. Moreover, let $\mathrm{supp}(u) \subset \Omega_2 := \{y \in \Omega; \psi(y) \leq 0\}$ with $\|u\|_{H^1(\Omega_1)} \leq C_1$, and let $\|f\|_{L^2(\Omega_1)} \leq \epsilon_1$ for some small $\epsilon_1 > 0$. The stable unique continuation is based upon an estimate like

$$\|u\|_{L^2(\Omega_3)} \leq \Upsilon(C_1, \epsilon_1) \tag{1.2}$$

for some ball $\Omega_3 := B(y_0, r)$ contained in Ω_1, where the right hand side tends to 0 as $\epsilon_1 \to 0$. Our aim is to prove (1.2) with a function Υ that has an explicit form that depends on constants related to geometrical properties of Ω_3, Ω_1, and

Ω and the norm of the coefficients g^{jk} in $C^1(\overline{\Omega})$. We consider the case where the domains Ω_1 and Ω_3 are balls centred in $y_0 \in \Omega$, and we find a logarithmic function Υ dependent on the radii R and r of the balls and on the norms of g^{jk}, h^j, q and ψ. In the paper [3], we use the local stability estimate to prove (1.2) for quite general domains. As in the elliptic case, many possible applications can be derived out of it. In particular, we plan to use inequality (1.2) to obtain an explicit modulus of continuity for the inverse problem for the wave operator on manifolds. This would improve the existing inverse stability results for Riemannian manifolds, which are currently based either on compactness-type arguments (see [2, 15]) or on very strong geometrical conditions for the coefficients (see, e.g., [5, 13, 14]).

In the unpublished manuscript [23], Tataru suggested the possibility of obtaining a stability estimate by using Gevrey-class localizers to improve the estimates of u for low temporal frequencies. Here, we develop that idea by employing properties of subharmonic functions (see Lemma 2.7) and performing the explicit estimate of the radii r and R and the constants. Of fundamental importance is the possibility of linking the positive lower bound for r to the geometric parameters of the domain to assure that the estimate are valid close to the characteristic surfaces of the operator. We begin by introducing some assumptions.

Assumption A1. Let Ω be a connected open subset of $\mathbb{R} \times \mathbb{R}^n$. Let $P(y, D)$ be the wave operator (1.1), with $g^{jk}(x) \in C^1(\Omega)$, $h^j, q \in L^\infty(\Omega)$. Let

$$S = \{y \in \Omega; \psi(y) = 0\}$$

be a $C^{2,\rho}$-smooth oriented hypersurface, which is non-characteristic in Ω, for some fixed $\rho \in (0, 1)$. We assume that $u \in H^1(\Omega)$ is supported in $\{y; \psi(y) \leq 0\} \cap \Omega$, and $P(y, D)u \in L^2(\Omega)$.

Assumption A2. Let $A(D_0)$ be a pseudo-differential operator with symbol $a(\xi_0)$, $0 \leq a \leq 1$, where $a \in C_0^\infty(\mathbb{R})$ is a smooth localizer supported in $|\xi_0| \leq 2$ and equal to 1 in $|\xi_0| \leq 1$. Furthermore, let $a \in G_0^{1/\alpha}(\mathbb{R})$ for a fixed $\alpha \in (0, 1)$, here $G_0^{1/\alpha}$ is the set of Gevrey functions of class $1/\alpha$ with compact support defined in Definition‘4.1. We also define the smooth localizer $b(y)$, supported in $|y| \leq 2$, $0 \leq b \leq 1$ and equal to 1 in $|y| \leq 1$.

The main results of the paper are the following two theorems. The first one is a stability estimate of exponential type for the low temporal frequencies.

Theorem 1.1. *Under Assumptions A1 and A2, let $y_0 \in S$ with $\psi'(y_0) \neq 0$, and let $b \in G_0^{1/\alpha_1}(\mathbb{R}^{n+1})$ be a Gevrey functions of class $1/\alpha_1$ with compact support such that $0 < \alpha \leq \alpha_1 < 1$. Then there exist constants $0 < r < 2r \leq R$ and balls*

$B(y_0, r) \subset B(y_0, 2R) \subset \Omega$ centered at y_0 of radius r and $2R$, respectively, such that for $\mu \geq 1$ there exist constants $c_{129}, c_{131}, c_{132}$ such that if

$$\|u\|_{H^1(B_{2R})} = 1, \quad \|Pu\|_{L^2(B_{2R})} < 1, \quad \left\|A\Big(\frac{D_0}{\mu}\Big)b\Big(\frac{y-y_0}{R}\Big)Pu\right\|_0 \leq e^{-\mu^\alpha},$$

then, for all $\omega \leq \mu^\alpha/(3c_{131})$,

$$\left\|A\Big(\frac{D_0}{\omega}\Big)b\Big(\frac{y-y_0}{r}\Big)u\right\|_{H^1} \leq c_{129}e^{-c_{132}\mu^{\alpha\cdot\alpha_1}}.$$

The radii r and R are defined in Table 1 below, while the coefficients c_k are computed in the proof of the theorem.

The second result is a log-stability estimate in a ball, valid for all the temporal frequencies; see Figure 1 below for the construction.

Theorem 1.2. *Under Assumption A1, for each $y_0 \in S$ with $\psi'(y_0) \neq 0$, there exist constants r and R, with $0 < 2r \leq R$, and balls $B(y_0, r) \subset B(y_0, 2R) \subset \Omega$ centered in y_0 of radius r and $2R$, respectively, such that*

$$\|u\|_{L^2(B(y_0,r))} \leq c_{111}\frac{\|u\|_{H^1(B(y_0,2R))}}{\ln\Big(1+\frac{\|u\|_{H^1(B(y_0,2R))}}{\|Pu\|_{L^2(B(y_0,2R))}}\Big)}.$$

The radii r and R and coefficient c_{111} are defined in Table 1 below. Moreover, for each $m \in (0, 1]$,

$$\|u\|_{H^{1-m}(B(y_0,r))} \leq c_{111}^m\frac{\|u\|_{H^1(B(y_0,2R))}}{\Big(\ln\Big(1+\frac{\|u\|_{H^1(B(y_0,2R))}}{\|Pu\|_{L^2(B(y_0,2R))}}\Big)\Big)^m}.$$

As a consequence, one can find in a domain $\Omega_0 \subseteq \Omega$ a uniform radius

$$r_0 = r_0\big(|\psi'|_{C^{1,\rho}(\Omega_0)}, |g^{jk}|_{C^1(\Omega_0)}, \min_{y\in\Omega_0}|p(y,\psi')|, \min_{y\in\partial\Omega_0}|y_0-y|\big) > 0$$

such that $r \geq r_0$.

Theorems 1.1 and 1.2 are proved in Section 2. In Section 3, we compute the related parameters R, r, and c_k, which depend on the constants of the Carleman estimate of Theorem 2.1 and a particular geometric construction. The Appendix is devoted to the main definitions used in the article. Finally, we observe that even if we study the wave equation, the same method can be generalized to ultrahyperbolic operators of the type $-|D_a|^2 + g^{jk}(x_b)D_kD_j$, where the variable $y = (x_a, x_b)$ has a different splitting and where x_a corresponds to the conormal direction for the pseudo-convexity condition.

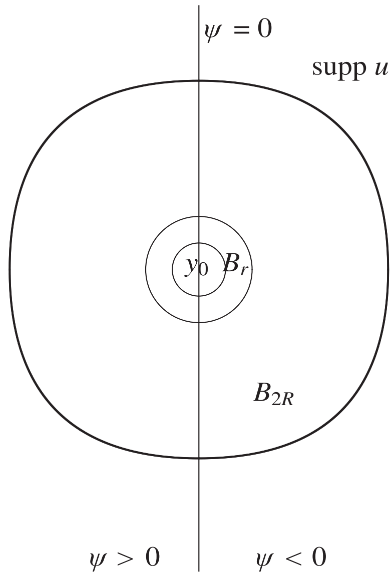

Figure 1. The domains of the stability estimate

2 Proof of the stability estimate

Notation. We start by introducing some notation and definitions used in the rest of the article. We consider $y = (t, x) \in \mathbb{R} \times \mathbb{R}^n$ a time-space variable and call $\xi = (\xi_0, \widetilde{\xi})$ its Fourier dual variable. Recall that the exponential pseudo-differential operator in Theorem 2.1 is defined as

$$e^{-\epsilon|D_0|^2/2\tau}v = \mathcal{F}^{-1}_{\xi_0\to t}e^{-\epsilon\xi_0^2/2\tau}\mathcal{F}_{t'\to\xi_0}v,$$

with $\mathcal{F}$ and $\mathcal{F}^{-1}$ representing, respectively, the Fourier transform and its inverse. Then $e^{-\epsilon|D_0|^2/2\tau}$ is an integral operator with kernel

$$\left(\frac{\tau}{2\pi\epsilon}\right)^{1/2}e^{-\tau|t'-t|^2/2\epsilon}.$$

We also define $A(D_0)$ to be a pseudo-differential operator with symbol $a(\xi_0)$, $0 \le a \le 1$, where $a \in C_0^\infty(\mathbb{R})$ is a smooth localizer supported in $|\xi_0| \le 2$ and equal to 1 in $|\xi_0| \le 1$. Hence we can write

$$A\left(\frac{\beta|D_0|}{\omega}\right)v = \mathcal{F}^{-1}_{\xi_0\to t}a\left(\frac{\beta|\xi_0|}{\omega}\right)\mathcal{F}_{t'\to\xi_0}v,$$

and the integral kernel is

$$\left(\frac{\omega}{2\pi\beta}\right)^{1/2}\widehat{a}\left(\frac{\omega|t'-t|}{\beta}\right).$$

We often work under Assumption A2, where the symbol a is of Gevrey class. The smooth localizer $b(y)$ is supported in $|y| \leq 2$ and equal to 1 in $|y| \leq 1$.

The norm of the Sobolev space H^s_τ is defined as

$$\|u\|_{s,\tau} = \|(|\xi|^2+\tau^2)^{s/2}\mathcal{F}_{y\to\xi}u\|_{L^2},$$

and the space H^s corresponds to the case $\tau = 1$.

According to our notation, the positive coefficients, denoted by c_x with $x \geq 100$, are defined just once, independently of the variables μ, τ and are computed explicitly in terms of the coefficients of the operator (1.1) and the geometric parameters. This is essential to recover finally the value of c_{111} and the radii R, r in Theorem 1.2.

We next introduce the Tataru inequality proved in [20] in the version presented by Hörmander [7] and adapted to the wave operator. In the Appendix, one can find the definition of conormally strongly pseudo-convex function or surface, and Gevrey function. According to Definition 4.6 and the splitting $y = (t, x)$, the conormal bundle in $\mathbb{R}^{n+1}$ with respect to the foliation x =const is defined as

$$N^*F := \{(y,\xi) \in T^*\mathbb{R}^{n+1} : \xi = (\xi_0, \widetilde{\xi}) \text{ and } \xi_0 = 0\}$$

and its fibre in y_0 is Γ_{y_0}.

Theorem 2.1. *Let Ω be an open subset of $\mathbb{R} \times \mathbb{R}^n$. Let $P(y, D)$ be the wave operator* (1.1)*, with $g^{jk}(x) \in C^1(\Omega)$, $h^j, q \in L^\infty(\Omega)$. For $\rho \in (0,1)$, let $y_0 \in \Omega$ and $\psi \in C^{2,\rho}(\Omega)$ be real-valued and such that $\psi'(y_0) \neq 0$. Let $S = \{y; \psi(y) = 0\}$ be an oriented hypersurface non-characteristic in y_0 for which there always exists $\lambda > 1$ such that $\phi(y) = \exp(\lambda\psi)$ is a conormally strongly pseudo-convex function with respect to P at y_0. Then there exists a real-valued quadratic polynomial f defined in* (3.3) *with proper $\sigma > 0$ and a ball $B_{R_2}(y_0)$ such that $f(y) < \phi(y)$ whenever $y \in B_{R_2} - \{y_0\}$ and $f(y_0) = \phi(y_0)$. Moreover, f is a conormally strongly pseudo-convex function with respect to P in B_{R_2}. Thus there exist ϵ_0, τ_0, $c_{1,T}$, $c_{2,T}$, and R, such that*

$$\|e^{-\epsilon|D_0|^2/2\tau}e^{\tau f}u\|_{1,\tau} \leq c_{1,T}\,\tau^{-1/2}\|e^{-\epsilon|D_0|^2/2\tau}e^{\tau f}P(y,D)u\|_0 + c_{2,T}e^{-\tau R_2^2/4\epsilon}\|e^{\tau f}u\|_{1,\tau}$$

for small enough $\epsilon < \epsilon_0$ and large enough $\tau > \tau_0$. Here, $u \in H^1_{loc}(\Omega)$, with $P(y,D)u \in L^2(\Omega)$ and $\mathrm{supp}(u) \subset B_R(y_0)$.

Remarks 2.2. (1) The explicit estimate for the coefficients ϵ_0, τ_0, $c_{1,T}$,$c_{2,T}$, σ, R_2, and R and their dependence upon the parameters of the problem have never been found. In this paper, we provide proper estimates, which are summarized in Table 1 of Section 3.1. Notice that this is possible under the condition that $\psi \in C^{2,\rho}(\Omega)$ instead of the usual condition $\psi \in C^2(\Omega)$. Furthermore we assume that S is not characteristic in y_0 and consequently not characterictic in a domain $\Omega_0 \subseteq \Omega$. This assumption is not required in [20, 7], where only the strongly pseudo-convexity of S in Γ_{y_0} is assumed. In Remark 3.1 we underscore this difference with an alternative condition on ψ. In any case, for the practical computations of the values in Table 1, we prefer to work in the stronger setting of Theorem 2.1.

(2) Our wave operator can be seen in two ways:

(H) an hyperbolic operator with constant-in-time and real-valued coefficients for the principal part, or

(E) an operator whose principal symbol is elliptic in the set $\Gamma_\Omega \subset N^*F$.

In the latter case, Tataru inequality (see [20]) is sharper. Here we prefer to consider just the case (H).

(3) Finally, some improvements to the assumption on the coefficients of (1.1) may be made. For example, taking $\Omega_x \subset \mathbb{R}^n$ to be the smooth domain of definition of $q(x)$, we can assume $q \in L^n(\Omega_x)$ for $n \geq 3$, $q \in L^{2+\epsilon}(\Omega_x)$ for $n = 2$, $q \in L^\infty(\Omega_x)$ for $n = 1$. Of course, by changing the localizers, we can reformulate the result in another kind of domain, e.g., a cylinder.

(4) After the submission of this article, a different kind of result by Vessella appeared in the framework of the strong unique continuation; see [24].

We now proceed with the detailed proof of Theorems 1.1 and 1.2. A first step is the following lemma, introducing a property used often in this section.

Lemma 2.3. *Let $A(D_0)$ be a pseudo-differential operator with symbol $a(\xi_0)$, where $a \in C_0^\infty(\mathbb{R})$ is a smooth localizer supported in $|\xi_0| \leq 2$ and equal to 1 in $|\xi_0| \leq 1$. Assume that $f(y) \in C_0^\infty(\mathbb{R}^{n+1}) \cap G_0^{1/\alpha}(\mathbb{R}_t^1)$, where $0 < \alpha < 1$.*

a) *For every $\mu > 0$, $\beta_1 > 2$, and $v \in L^2(\mathbb{R}^{n+1})$ there exist constants c_{106} and c_{107}, independent of μ, such that*

$$\left\| A\left(\frac{\beta_1 D_0}{\mu}\right) f(y) \left(1 - A\left(\frac{D_0}{\mu}\right)\right) v \right\|_0 \leq c_{107} e^{-c_{106}\mu^\alpha} \|v\|_0. \tag{2.1}$$

b) *For every $h \in C_0^\infty(\mathbb{R}^{n+1})$ satisfying $h \equiv 1$ on* supp(f)*,*

$$\left\| A\left(\frac{\beta_1 D_0}{\mu}\right) fhv \right\|_0 \leq \|f\|_\infty \left\| A\left(\frac{D_0}{\mu}\right) h(y) v \right\|_0 + c_{107} e^{-c_{106}\mu^\alpha} \|hv\|_0. \tag{2.2}$$

c) *For* $v \in H^m(\mathbb{R}^{n+1})$,, $m \geq 1$, (2.1) *holds in* $H^m(\mathbb{R}^{n+1})$ *under the additional condition* $D_x^m f(y) \in G_0^{1/\alpha}(\mathbb{R}_t^1)$,

$$\left\| A\Big(\frac{\beta_1 D_0}{\mu}\Big) f \left(1 - A\Big(\frac{D_0}{\mu}\Big)\right) v \right\|_m \leq c_{108} e^{-c_{106}\mu^\alpha} \|v\|_m. \tag{2.3}$$

Proof. a) On the set $\operatorname{supp}[(1 - a(\xi_0/\mu))a(\beta_1 \xi_0^1/\mu)]$, one obtains $|\xi_0^1 - \xi_0|^\alpha \geq (\mu - 2\mu/\beta_1)^\alpha$; and the assumption $f(t, .) \in G_0^{1/\alpha}(\mathbb{R}_t)$ implies

$$|\mathcal{F}_{t' \to (\xi_0^1 - \xi_0)}[f(t', x)]| \leq c_3 e^{-c_{117}|\xi_0^1 - \xi_0|^\alpha} \leq c_3 e^{-2c_{106}\mu^\alpha} e^{-c_{117}|\xi_0^1 - \xi_0|^\alpha/2},$$

uniformly in x on every compact set $K \subset \mathbb{R}^n$ for some $c_3 = c_3(\alpha, K)$, $c_{117} = c_{117}(\alpha, K)$ and $c_{106} = c_{117}(1 - 2/\beta_1)^\alpha/4$, We then estimate the operator

$$A\Big(\beta_1 \frac{D_0}{\mu}\Big) f(\cdot) \left(1 - A\Big(\frac{D_0}{\mu}\Big)\right)$$

on the Fourier space, obtaining

$$\begin{aligned}
&\left\| a\left(\frac{\beta_1 \xi_0^1}{\mu}\right) \mathcal{F}_{t' \to \xi_0^1} \left(f(t', x) \left(\mathcal{F}^{-1}_{\xi_0 \to t'} \left(1 - a\frac{\xi_0}{\mu}\right) \mathcal{F}_{t \to \xi_0}[v] \right)\right) \right\|_0^2 \\
&\qquad = \left\| a\left(\frac{\beta_1 \xi_0^1}{\mu}\right) \left(\int_{\mathbb{R}} \left(1 - a\Big(\frac{\xi_0}{\mu}\Big)\right) \mathcal{F}_{t' \to (\xi_0^1 - \xi_0)}[f(t', x)] \mathcal{F}_{t \to \xi_0}[v]\, d\xi_0 \right) \right\|_0^2 \\
&\qquad \leq c_3 \int_{\mathbb{R}^{n+1}} dx d\xi_0^1 \left(\int_{\mathbb{R}} \left(1 - a\Big(\frac{\xi_0}{\mu}\Big)\right) a\left(\frac{\beta_1 \xi_0^1}{\mu}\right) e^{-2c_{106}\mu^\alpha} \right. \\
&\qquad\qquad \left. \cdot\, e^{-c_{117}|\xi_0^1 - \xi_0|^\alpha/2} |\mathcal{F}_{t \to \xi_0}[v](\xi_0, x)|\, d\xi_0 \right)^2 \\
&\qquad \leq c_3 \int_{\mathbb{R}^{n+1}} dx d\xi_0^1 \left\| \left(1 - a\Big(\frac{\xi_0}{\mu}\Big)\right) a\Big(\frac{\beta_1 \xi_0^1}{\mu}\Big) \right. \\
&\qquad\qquad \cdot\, e^{-2c_{106}\mu^\alpha} e^{-c_{117}|\xi_0^1 - \xi_0|^\alpha/2} \|^2_{L^2(d\xi_0)} \left\| \mathcal{F}_{t \to \xi_0}[v](\xi_0, x) \right\|^2_{L^2(d\xi_0)} \\
&\qquad \leq c_3 e^{-4c_{106}\mu^\alpha} \left\| \left(1 - a\Big(\frac{\xi_0}{\mu}\Big)\right) a\Big(\frac{\beta_1 \xi_0^1}{\mu}\Big) \right. \\
&\qquad\qquad \cdot\, e^{-c_{117}|\xi_0^1 - \xi_0|^\alpha/2} \|^2_{L^2(d\xi_0 d\xi_0^1)} \left\| \mathcal{F}_{t \to \xi_0}[v](\xi_0, x) \right\|^2_{L^2(d\xi_0 dx)} \\
&\qquad \leq c_{107}^2 e^{-2c_{106}\mu^\alpha} \|v\|_0^2,
\end{aligned}$$

with

$$c_{107} = c_3 \left(\frac{8}{\beta_1} \Gamma\Big(\frac{1}{\alpha}\Big) \frac{1}{\alpha(c_{117})^{1/\alpha}} \frac{1}{(\alpha c_{106})^{1/\alpha}} \right)^{1/2}$$

and where we have applied in the last step the inequalities

$$\left\|\left(1-a\frac{\xi_0}{\mu}\right)a\left(\beta_1\frac{\xi_0^1}{\mu}\right)e^{-c_{117}|\xi_0^1-\xi_0|^\alpha/2}\right\|^2_{L^2(d\xi_0 d\xi_0^1)} \le \frac{8}{\beta_1}\Gamma\left(\frac{1}{\alpha}\right)\frac{1}{\alpha(c_{117})^{1/\alpha}}\mu,$$

$$\mu e^{-c_{106}\mu^\alpha} \le \frac{1}{(\alpha c_{106})^{1/\alpha}}.$$

b) To prove the inequality, we observe that

$$\begin{aligned}\left\|A\left(\frac{\beta_1 D_0}{\mu}\right)fhv\right\|_0 &\le \left\|A\left(\frac{\beta_1 D_0}{\mu}\right)fA\left(\frac{D_0}{\mu}\right)hv\right\|_0\\ &\quad+\left\|A\left(\frac{\beta_1 D_0}{\mu}\right)f\left(1-A\left(\frac{D_0}{\mu}\right)\right)hv\right\|_0,\end{aligned}$$

where the first term is bounded by $\|f\|_\infty\|A(D_0/\mu)h(y)v\|_0$. We can apply the estimate a) to the second term on the right hand side.

c) The extension to H^m of a) follows from

$$D^\zeta f\left(1-A\left(\frac{D_0}{\mu}\right)\right)v = \sum_{v:v\le\zeta}\binom{\zeta}{v}\left(D^{\zeta-v}f\right)\left(\left(1-A\left(\frac{D_0}{\mu}\right)\right)(D^v v)\right).$$

By hypothesis, every derivative $D^{\zeta-v}f$ belongs to $G_0^{1/\alpha}(\mathbb{R}_t)$; hence we consider each derivative as a new function g having the same Gevrey-parameters c_3, c_{117} as f. We then apply $A(\beta_1 D_0/\mu)$ and repeat the computations of step a), replacing v with $D^v v$. The coefficient $c_{108} = c_m c_{107}$ is a proper multiple in m of c_{107}. $\square$

Another technical lemma is the following result.

Lemma 2.4. *Let $\varphi(y)$ be a second order polynomial in $y = (t, x) \in \mathbb{R}\times\mathbb{R}^n$. If $\chi(s) \in G_0^{1/\alpha_1}(\mathbb{R})$ for $\alpha_1 \in (0, 1)$, then $e^{\tau\varphi(y)}\chi(\varphi(y)) \in G^{1/\alpha_1}(\mathbb{R}^{n+1})$. If* supp$(\chi)$ = $[-8\delta, \delta]$*, then for a cut-off function $b((y-y_0)/(2R)) \in G_0^{1/\alpha_1}(\mathbb{R}^{n+1})$, there exist constants c_{122} and c_{123} such that*

$$\left|\mathcal{F}_{t\to\xi_0}\left[e^{\tau\varphi(y)}\chi(\varphi(y))b\left(\frac{y-y_0}{2R}\right)\right]\right| \le c_{122}e^{\tau\delta-c_{123}|\xi_0|^{\alpha_1}}.$$

Proof. By assumption, both $\varphi(y)$ and $e^{\tau s}$ are analytic functions (i.e., in G^1), while $\chi \in G_0^{1/\alpha_1}(\mathbb{R})$. Since $G^1 \subset G^{1/\alpha_1}$ and both G^1 and G^{1/α_1} are rings by [8, Proposition 8.4.1], we deduce that $e^{\tau s}\chi(s) \in G^{1/\alpha_1}(\mathbb{R})$. Moreover, $e^{\tau s}\chi(s)$ has compact support, since χ does. Let us write χ as $\chi(s) = \chi_1(s/\delta)$, where χ_1 has the properties of Definition 4.4 with associated coefficient c_{1X}. By assumption, for $z \in \mathbb{C}$, E = supp$(\chi) = [-8\delta, \delta]$, $c_{119} = \delta c_{1X}(\alpha_1)$, $B = \delta^{\alpha_1}c_{2X}(\alpha_1)$, and H_E as in Definition 4.1,

$$|(\mathcal{F}_{s\to z}\chi(s))| \le c_{119}\exp(H_E(\operatorname{Im} z) - B|\operatorname{Re} z|^{\alpha_1}).$$

Consequently, for $\xi \in \mathbb{R}$,

$$\mathcal{F}_{s\to\xi}\,(e^{\tau s}\chi(s)) = \mathcal{F}_{s\to\xi+i\tau}(\chi(s));$$

and for $\tau > 0$,

$$|\mathcal{F}_{s\to\xi}\,(e^{\tau s}\chi(s))| \le c_{119}\exp(H_E(\tau) - B|\xi|^{\alpha_1}) = c_{119}\exp(\delta\tau - B|\xi|^{\alpha_1}).$$

Hence we can estimate the derivatives:

$$\begin{aligned}
|\partial_s^k e^{\tau s}\chi(s)| &= \left|\int_{\mathbb{R}} e^{i\xi s}(i\xi)^k(\mathcal{F}_{s'\to\xi}\, e^{\tau s'}\chi(s'))(\xi)d\xi\right| \le \int_{\mathbb{R}} c_{119}|\xi|^k\exp(\delta\tau - B|\xi|^{\alpha_1})d\xi\\
&= 2\pi c_{119}e^{\tau\delta}B^{-\frac{(k+1)}{\alpha_1}}\frac{1}{\alpha_1}\Gamma\left(\frac{k+1}{\alpha_1}\right)\\
&\le \frac{2\pi c_{119}}{\alpha_1}e^{\tau\delta}\Gamma(2)B^{-\frac{(k+1)}{\alpha_1}}\left(\frac{k+1}{\alpha_1}\right)^{-1}\left(\frac{k+1}{\alpha_1}\right)^{\frac{k+1}{\alpha_1}} \le c_{121}^{k+1}e^{\tau\delta}k^{k/\alpha_1},
\end{aligned}$$

where, by Stirling formula,

$$\begin{aligned}
(k+1)^{(k+1)} &\le e^{k+1}\Gamma(k+2) \le ke^{k+1}k^k e^{-1}(2\pi k)^{1/2},\\
c_{121} &= (2\pi)^{1+1/(2\alpha_1)}c_{119}\Gamma(2)\left(e^3\alpha_1^{-3}\right)^{1/\alpha_1}B^{-1/\alpha_1}, \quad \text{and}\\
\Gamma\left(\frac{k+1}{\alpha_1}\right) &= \left(\frac{k+1}{\alpha_1}-1\right)\cdots\left(\frac{k+1}{\alpha_1}-p\right)\Gamma\left(\frac{k+1}{\alpha_1}-p\right) \le \Gamma(2)\left(\frac{k+1}{\alpha_1}-1\right)^{\frac{k+1}{\alpha_1}-1}
\end{aligned}$$

with $p = [(k+1)/\alpha_1] - 1$. We now recall that the composition of a Gevrey function with an analytic map is still a Gevrey function, therefore obtaining that $e^{\tau\varphi}\chi(\varphi(y)) \in G^{1/\alpha_1}$. Since $\varphi(y)$ is a second order polynomial, $|\partial_t^k\varphi(y)|_{C^0(B_R)} \le c_{118}(R)$ for $k = 0, 1, 2$; so, without loss of generality, we take $c_{118} \ge 1$. Considering the composition with φ we obtain by induction and setting $m(s) = e^{\tau s}\chi(s)$, for $k \ge 0$, we obtain

$$\partial_t^k m(\varphi(y)) = \sum_{r\in J}\frac{k!}{(2r-k)!(k-r)!}(\partial_s^r m)_{s=\varphi(y)}(\partial_t\varphi)^{2r-k}\left(\frac{\partial_t^2\varphi}{2}\right)^{k-r},$$

where $J = \{r \ge 0 :\ 2r \ge k \ge r\}$ and

$$\begin{aligned}
|\partial_t^k e^{\tau\varphi(y)}\chi(\varphi(y))| &\le c_{121}^{k+1}e^{\tau\delta}\sum_{r\in J}\frac{k!}{2^{k-r}(2r-k)!(k-r)!}c_{118}^{r+1}r^{r/\alpha_1}\\
&\le c_{121}^{k+1}e^{\tau\delta}c_{118}^{k+1}\sum_{r\in J}\binom{k}{r}\frac{r!}{2^{k-r}(2r-k)!}r^{r/\alpha_1} \le (e^{\tau\delta}c_{121}^{k+1}c_{118}^{k+1})2^k k^{k/\alpha_1},
\end{aligned}$$

with $\frac{r!}{(2r-k)!}r^{r/\alpha_1} \le k^{k/\alpha_1}$, for r admissible. For the product, applying (4.1) and setting $c_{122} = \max\{4c_{118}c_{121}, c_{1X}/R\}$, we obtain

$$\partial_t^k b((y-y_0)/(2R)) \le R^{-k}c_{1X}^{1+k}k^{k/\alpha_1},\ \partial_t^k[m(\varphi(y))b((y-y_0)/(2R))] \le e^{\tau\delta}c_{122}^{2+k}k^{k/\alpha_1}.$$

Consider the partial Fourier transform $\mathcal{F}_{t\to\xi_0}$ in time of $\partial_t^k(e^{\tau\varphi}\chi(\varphi)b((y-y_0)/(2R))$; from the estimate above, it follows that

$$|\xi_0|^k|\mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)b((y-y_0)/(2R)))| \le e^{\tau\delta}c_{122}^{k+1}k^{k/\alpha_1}.$$

This implies that

$$|\mathcal{F}_{t\to\xi_0}[e^{\tau\varphi}\chi(\varphi)]| \le e^{\tau\delta}c_{122}^{k+1}\frac{k^{k/\alpha_1}}{|\xi_0|^k} \le c_{122}e^{\tau\delta-k} \le c_{122}e^{\tau\delta-c_{123}|\xi_0|^{\alpha_1}+1},$$

where for each ξ_0 we have chosen k to be the largest integer such that $c_{122}|\xi_0|^{-1}k^{1/\alpha_1} < e^{-1}$. Since $k > [e^{-1}c_{122}^{-1}|\xi_0|]^{\alpha_1} - 1$, we get the result once we choose $c_{123} = (ec_{122})^{-\alpha_1}$. □

In Theorem 2.1, we referred to the radius R that is defined in Table 1 as $R := qR_2$ with $q = \frac{1}{4}\left(16+\frac{1}{16}\right)^{-1/2}$ and where R_2 in the same table is computed in terms of the pseudo-convexity constants introduced in Subsection 3.1. Using those quantities, one can introduce the geometric construction of Figure 2. Let $f(y)$ be the second order polynomial defined in Theorem 2.1, with $\phi = e^{\lambda\psi}$ and $y_0 \in \Omega$. Recall that $f(y_0) = \phi(y_0)$.

Proposition 2.5. *Let δ be a positive constant such that*

$$0 < \delta \le n|\phi''|_{C^{0,\rho}}q^2R_2^{2+\rho}/8, \tag{2.4}$$

and

$$\varphi(y) := f(y) - f(y_0) = \sum_{0<|\upsilon|\le 2}\frac{(\partial^\upsilon\phi)(y_0)\,(y-y_0)^\upsilon}{\upsilon!} - \sigma|y-y_0|^2. \tag{2.5}$$

Then

$$\{y \in B(y_0,R_2) : \phi \le \phi(y_0)\} \cap \{y \in B(y_0,R_2) : \varphi(y) \ge -8\delta\} \subset B(y_0,R).$$

In addition, let

$$0 < r \le \frac{n|\phi''|_{C^{0,\rho}(B_{R_2})}q^2R_2^{2+\rho}}{2|\phi'|_{C^0(B_{R_2})} + 10n|\phi''|_{C^{0,\rho}(B_{R_2})}R_2^{1+\rho}}. \tag{2.6}$$

Then $B(y_0,2r) \subset \{y : |\varphi(y)| \le \delta\}$.

We postpone the proof of Proposition 2.5 to the end of Section 3.

In the following lemma, we show how an exponential decay for the L^2-norm of a proper localization of Pu is transmitted to the right hand side of Tataru inequality.

Lemma 2.6. *Under Assumption A1, let $y_0 \in \Omega$ and φ be the quadratic polynomial of* (2.5)*. Let $0 < \alpha, \alpha_1 < 1$ and $\chi(s) \in G_0^{1/\alpha_1}(\mathbb{R})$ be a localizer supported in $[-8\delta, \delta]$ and equal to 1 in $[-7\delta, \delta/2]$. Let μ, $\delta > 0$, $b \in C_0^\infty(\mathbb{R}^{n+1})$, and $a \in C_0^\infty(\mathbb{R})$. Let $A(D_0)$ be a pseudo-differential operator with symbol a, and let $\mu_* = \min\{\mu^\alpha, \mu^{\alpha_1}\}$. If*

$$\|u\|_{H^1(B_{2R})} = 1, \quad \|Pu\|_{L^2(B_{2R})} < 1, \quad \textit{and} \quad \left\| A\Big(\frac{D_0}{\mu}\Big) b\Big(\frac{y-y_0}{R}\Big) Pu \right\|_0 \le e^{-\mu^\alpha},$$

then, for each $\tau \ge 0$, there exist constants c_{110}, c_{109} such that

$$\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi} \chi(\varphi) P(y,D)u\|_0 \le c_{110} e^{2\tau\delta - c_{109}\mu_*}.$$

Proof. Define $a_{\mu/3}(s) := a(3s/\mu)$, so $\operatorname{supp}(1 - a_{\mu/3}(\xi_0)) \subset \{|\xi_0| \ge \mu/3\}$. Then

$$\begin{aligned}
\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi} \chi(\varphi) P(y,D)u\|_0 &= \|e^{-\epsilon\xi_0^2/(2\tau)} \mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)P(y,D)u)\|_0 \\
&\le \|(1 - a_{\mu/3}(\xi_0)) e^{-\epsilon\xi_0^2/(2\tau)} \mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)P(y,D)u)\|_0 \\
&\quad + \|a_{\mu/3}(\xi_0) e^{-\epsilon\xi_0^2/(2\tau)} \mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)P(y,D)u)\|_0 \\
&=: I_1 + I_2.
\end{aligned}$$

By our construction, $b((y-y_0)/R) = 1$ on $\operatorname{supp}(\chi(\varphi)Pu)$; hence we can write $\chi(\varphi)P(y,D)u = \chi(\varphi)b((y-y_0)/R)P(y,D)u$. The first integral can be estimated for $\tau < c_{127}\mu$ as follows, where $c_{127} = \sqrt{\epsilon/(36\delta)}$:

$$\begin{aligned}
I_1 &\le e^{-\epsilon\mu^2/(18\tau)} \|\mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)P(y,D)u)\|_0 \le e^{-\epsilon\mu^2/(18\tau)} \|e^{\tau\varphi}\chi(\varphi)P(y,D)u\|_0 \\
&\le e^{-\epsilon\mu^2/(18\tau) + \tau\delta} \|\chi(\varphi)P(y,D)u\|_0 \le e^{-c_{127}\delta\mu} \left\| b\Big(\frac{y-y_0}{R}\Big) P(y,D)u \right\|_0,
\end{aligned}$$

where we have used the fact that $-\epsilon\mu^2/(18\tau) + \tau\delta \le -c_{127}\delta\mu$. Notice that the estimate for I_1 holds only for $\tau < c_{127}\mu$. For $\tau \ge c_{127}\mu$, we have

$$\begin{aligned}
\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi} \chi(\varphi) P(y,D)u\|_0 &\le e^{\delta\tau} \|\chi(\varphi)P(y,D)u\|_0 \\
&\le e^{2\delta\tau - c_{127}\delta\mu} \left\| b\Big(\frac{y-y_0}{R}\Big) P(y,D)u \right\|_0,
\end{aligned}$$

since $e^{\delta\tau} = e^{2\delta\tau-\delta\tau} \le e^{2\delta\tau - c_{127}\delta\mu}$. For the second integral, we get

$$\begin{aligned}
I_2 &= \left\| e^{-\epsilon\xi_0^2/(2\tau)} a_{\mu/3}(\xi_0)\mathcal{F}_{t\to\xi_0}\Big(e^{\tau\varphi}\chi(\varphi)b\Big(\frac{y-y_0}{R}\Big)P(y,D)u\Big)\right\|_0 \\
&\le \left\| A_{\mu/3}(D_0)e^{\tau\varphi}\chi(\varphi)b\Big(\frac{y-y_0}{R}\Big)P(y,D)u\right\|_0 \\
&\le \left\| A_{\mu/3}(D_0)e^{\tau\varphi}\chi(\varphi)b\Big(\frac{y-y_0}{2R}\Big)A_\mu(D_0)b\Big(\frac{y-y_0}{R}\Big)P(y,D)u\right\|_0 \\
&\quad + \left\| A_{\mu/3}(D_0)e^{\tau\varphi}\chi(\varphi)b\Big(\frac{y-y_0}{2R}\Big)(1-A_\mu(D_0))b\Big(\frac{y-y_0}{R}\Big)P(y,D)u\right\|_0 \\
&=: I_3 + I_4.
\end{aligned}$$

To estimate I_3, we apply the assumption and obtain

$$\begin{aligned}
\left\| A\Big(\frac{3D_0}{\mu}\Big)e^{\tau\varphi}\chi(\varphi)b\Big(\frac{y-y_0}{2R}\Big)A\Big(\frac{D_0}{\mu}\Big)b\Big(\frac{y-y_0}{R}\Big)Pu\right\|_0 &\le e^{\tau\delta}\left\| A\Big(\frac{D_0}{\mu}\Big)b\Big(\frac{y-y_0}{R}\Big)Pu\right\|_0 \\
&\le e^{\tau\delta-\mu^\alpha}.
\end{aligned}$$

To estimate I_4, we apply Lemmas 2.3 and 2.4. By the estimates for $f(y)$ at its derivatives in Step 3 of Section 3.1, we deduce

$$|\partial_t^k\varphi(y)| \le c_{118}(\phi) := 1 + |\phi'|_0(1+R_2) + 5n|\phi''|_{0,\rho}R_2^{\rho+1} + |\phi''|_0(1+R_2^2) + \sigma(2+R_2^2)$$

for $k = 0, 1, 2$. Lemma 2.4 and the properties of $e^{\tau\varphi}\chi(\varphi)$ imply that

$$|\mathcal{F}_{t'\to(\xi_0^1-\xi_0)}[e^{\tau\varphi}\chi(\varphi)(t',x)b((y'-y_0)/(2R))]| \le c_{122}e^{\tau\delta}e^{-c_{123}\mu^{\alpha_1}/(2\,3^{\alpha_1})}e^{-c_{123}|\xi_0^1-\xi_0|^{\alpha_1}/2},$$

since $|\xi_0^1 - \xi_0| \ge \mu - 2\mu/3 = \mu/3$ on $\operatorname{supp}[(1-a(\xi_0/\mu))a(3\xi_0^1/\mu)]$.

To estimate I_4, we apply Lemma 2.3.a); and, using the fact that

$$f = e^{\tau\varphi}\chi(\varphi)b\Big(\frac{y-y_0}{2R}\Big)$$

and recomputing the constants, we get

$$\begin{aligned}
&\left\| A\Big(\frac{3D_0}{\mu}\Big)e^{\tau\varphi}\chi(\varphi)b\Big(\frac{y-y_0}{2R}\Big)\Big(1-A\Big(\frac{D_0}{\mu}\Big)\Big)b\Big(\frac{y-y_0}{R}\Big)Pu\right\|_0^2 \\
&\qquad\qquad \le c_{110}^2 e^{2\tau\delta - c_{128}\mu^{\alpha_1}}\left\| b\Big(\frac{y-y_0}{R}\Big)Pu\right\|_0^2,
\end{aligned}$$

with $c_{128} = \frac{1}{3^{\alpha_1}2}c_{123}$ and

$$c_{110} = c_{122}\left(\frac{(8/3)\Gamma(1/\alpha_1)}{\alpha_1 c_{123}^{1/\alpha_1}(\alpha_1 c_{128})^{1/\alpha_1}}\right)^{1/2}.$$

Setting $c_{109} = \min(\sqrt{\epsilon\delta/36}, c_{128}/2, 1)$ we finally get the result. □

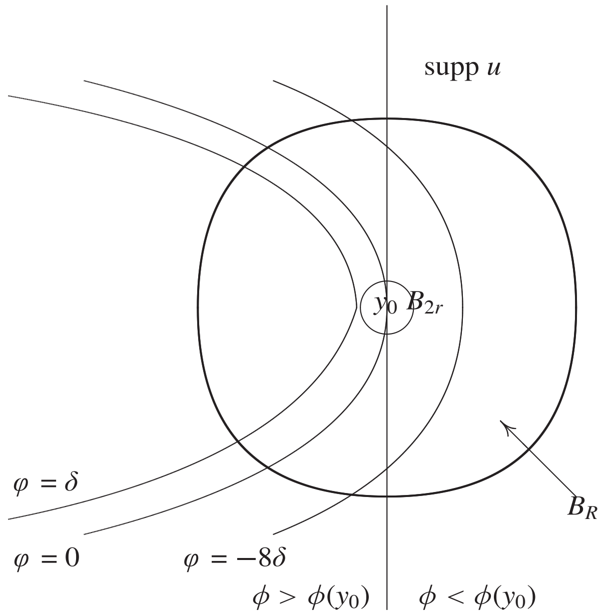

Figure 2. Geometric construction around y_0

We now prove Theorem 1.1, giving an estimate of inverse exponential type for the temporal frequencies $|\xi_0| \leq 2\omega$.

Proof of Theorem 1.1. Let $y_0 \in S$ be as in the Assumption A1. Then, by Theorem 2.1, there exists $\lambda > 1$ such that $\phi(y) = \exp(\lambda\psi)$ is a conormally strongly pseudo-convex function with respect to P in $\boldsymbol{\Omega}$. We introduce the function φ defined in (2.5) as the second order polynomial approximation of the conormally pseudo-convex function $\phi - \phi(y_0)$ around y_0, translated by $-\sigma|y-y_0|^2$. In Table 1, we found σ independent of y_0 such that φ also satisfies the conormally pseudo-convexity condition with respect to P in the ball $B(y_0, R_2)$. In Proposition 2.5 we also computed a δ independent of y_0 such that

$$\{y : \phi \leq \phi(y_0)\} \cap \{y : \varphi > -8\delta\} \subset B(y_0, R).$$

Given δ, we found $r > 0$ such that $B(y_0, 2r) \subset \{y : |\varphi(y)| \leq \delta\}$. Let $\chi_1 \in G_0^{1/\alpha_1}(\mathbb{R})$ be a smooth cut-off function which vanishes on $(-\infty, -8] \cup [1, \infty)$, equals 1 in $[-7, 1/2]$, and satisfies $0 \leq \chi_1 \leq 1$. Then, the scaled version $\chi(s) := \chi_1(s/\delta)$ of χ satisfies $P\chi(\varphi)u = \chi(\varphi)Pu + [P, \chi(\varphi)]u$; and, since u is supported in $\{y : \phi \leq \phi(y_0)\}$, it follows from Proposition 2.5 that

$$\operatorname{supp}(\chi(\varphi)u) \subset \{y : \phi(y) \leq \phi(y_0)\} \cap \{y : -8\delta < \varphi(y) < \delta\} \cap \{y : |y - y_0| \leq R\}.$$

Moreover, $[P, \chi(\varphi)]$ is a partial differential operator of order 1 and satisfies

$$\operatorname{supp}([P, \chi(\varphi)]u(y)) \subset \{y : -8\delta < \varphi(y) < -7\delta\}.$$

We now apply the estimate of Theorem 2.1 to χu to obtain

$$\begin{aligned}\tau \|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi} \chi(\varphi)u\|_{1,\tau}^2 \leq c_{1,T}^2 \|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi} \chi(\varphi)Pu\|_0^2 \\ + c_{1,T}^2 \|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi} [P, \chi(\varphi)]u\|_0^2 + c_{2,T}^2 \tau \|e^{\tau(\varphi-d)} \chi(\varphi)u\|_{1,\tau}^2\end{aligned}$$

for all $\tau > \tau_0$, where $d = R_2^2/(4\epsilon)$. We refer to Table 1 for all the involved parameters. According to our construction, δ is chosen such that $d > 8\delta$.

To estimate the first term at the right hand side, we apply Lemma 2.6. The second term can be bounded by

$$c_{1,T}^2 \|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi} [P, \chi(\varphi)]u\|_0^2 \leq c_{114} e^{-14\tau\delta} \|u\|_{H^1(B_{2R})}^2,$$

with

$$c_{114} = c_{1,T}^2 |g|_{C^1}^2 |\chi_1|_{C^2}^2 \left(1 + \frac{|\varphi'|_{C^0}^4}{\delta^4} + \frac{|\varphi''|_{C^0}^2}{\delta^2}\right),$$

since

$$\chi'(\varphi) = \chi_1'\left(\frac{\varphi}{\delta}\right)\frac{\varphi'}{\delta} \quad \text{and} \quad \chi''(\varphi) = \chi_1''\left(\frac{\varphi}{\delta}\right)\frac{\varphi' \cdot \varphi'}{\delta^2} + \chi_1'\left(\frac{\varphi}{\delta}\right)\frac{\varphi''}{\delta}.$$

Applying $\|\chi(\varphi)u\|_1^2 \leq (1 + |\chi_1'|_{C^0}^2/\delta^2)\|u\|_{H^1(B_{2R})}^2$, we see that the third term is such that

$$c_{2,T}^2 \tau \|e^{\tau(\varphi-d)} \chi(\varphi)u\|_{1,\tau}^2 \leq c_{2,T}^2 (|\varphi'|_{C^0}^2 + 1)\tau^3 e^{-14\tau\delta} \|\chi(\varphi)u\|_1^2 \leq c_{115}\, e^{-13\tau\delta},$$

with $c_{115} = c_{2,T}^2(|\varphi'|_{C^0}^2 + 1)(3^3 e^{-3}/\delta^3)(1 + |\chi_1'|_{C^0}^2/\delta^2)$. Since $\tau_0 \geq 1$ is such that $(1 + \tau_0)/2 \leq \tau$, we get

$$\frac{(1 + \tau_0)}{2} \|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi} \chi(\varphi)u\|_{1,\tau}^2 \leq c_{116} e^{4\delta\tau}(e^{-2c_{109}\mu^\alpha} + e^{-16\delta\tau}) \tag{2.7}$$

for all $\tau > \tau_0$, where $c_{116} := 3\max(c_{1,T}^2 c_{110}^2, c_{114}, c_{115})$. We want to extend the previous estimate to the complex upper half-plane. For $\tau \geq 0$, defining

$$N(\tau) := \frac{1}{2}(1 + \tau_0)\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi} \chi(\varphi)u\|_{1,\tau}^2,$$

we get

$$\begin{aligned}
N(\tau) &= \frac{1}{2}(1+\tau_0)\|\sqrt{|\xi|^2+\tau^2}\mathcal{F}_{t\to\xi_0}\mathcal{F}_{x\to\widetilde{\xi}}[\mathcal{F}^{-1}_{\xi_0\to t}e^{-\epsilon\xi_0^2/(2\tau)}\mathcal{F}_{t\to\xi_0}(e^{\tau\varphi}\chi(\varphi)u)]\|_0^2\\
&= \frac{1}{2}(1+\tau_0)\|\sqrt{|\xi|^2+\tau^2}e^{-\epsilon\xi_0^2/(2\tau)}\mathcal{F}_{t\to\xi_0}\mathcal{F}_{x\to\widetilde{\xi}}(e^{\tau\varphi}\chi(\varphi)u)\|_0^2\\
&= \frac{(1+\tau_0)}{2}\int_{\mathbb{R}^{n+1}} d\widetilde{\xi}d\xi_0(|\xi|^2+\tau^2)e^{-\epsilon\xi_0^2/(2\tau)}\\
&\quad \times \mathcal{F}_{y\to\xi}(e^{\tau\varphi}\chi(\varphi)u)\overline{e^{-\epsilon\xi_0^2/(2\tau)}\mathcal{F}_{y\to\xi}(e^{\tau\varphi}\chi(\varphi)u)}.
\end{aligned}$$

We first extend the estimate (2.7) to the interval $0 \le \tau \le \tau_0$. Set $c_{112} = (1+|\varphi'|^2_{C^0})$ and

$$c_{113} = \max\left\{c_{116}, c_{112}(1+\tau_0^3)\left(1+\frac{|\chi_1'|^2_{C^0}}{\delta^2}\right)e^{12\delta\tau_0}\right\}. \tag{2.8}$$

Since $\varphi \le 0$ on $\mathrm{supp}(\chi u)$, we have

$$\begin{aligned}
N(\tau) &\le \frac{1}{2}(1+\tau_0)\int_{\mathbb{R}^{n+1}} d\widetilde{\xi}d\xi_0\Big(|\xi\mathcal{F}_{y\to\xi}(e^{\tau\varphi}\chi(\varphi)u)|^2+\tau^2|\mathcal{F}_{y\to\xi}(e^{\tau\varphi}\chi(\varphi)u)|^2\Big)\\
&\le \frac{1}{2}(1+\tau_0)\int_{\mathbb{R}^{n+1}} dxdt\Big(|\nabla_y(e^{\tau\varphi(y)}\chi(\varphi)u)|^2+\tau_0^2|e^{\tau\varphi}\chi(\varphi)u|^2\Big)\\
&\le c_{112}(1+\tau_0^3)\|\chi(\varphi)u\|_1^2 \le c_{113}e^{-12\delta\tau_0} \le c_{113}e^{4\delta\tau}(e^{-2c_{109}\mu^\alpha}+e^{-16\delta\tau}).
\end{aligned} \tag{2.9}$$

We now consider $z\in\mathbb{C}$ with $\mathrm{Im}\,(z) > 0$ and rewrite the previous expression in the complex half-plane, replacing τ with $-iz$ and obtain

$$\begin{aligned}
N(-iz) :&= \frac{1+\tau_0}{2}\int_{\mathbb{R}^{n+1}} d\xi(|\xi|^2+|z|^2)e^{\frac{\epsilon\xi_0^2}{i2z}}\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)\overline{e^{\frac{\epsilon\xi_0^2}{i2z}}\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)}\\
&= \frac{1}{2}(1+\tau_0)\int_{\mathbb{R}^{n+1}} d\xi e^{-\frac{\epsilon\xi_0^2 Im\, z}{|z|^2}}(|\xi\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)|^2+|z\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)|^2)\\
&\le \frac{1}{2}(1+\tau_0)\int_{\mathbb{R}^{n+1}} d\xi(|\xi\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)|^2+|z\mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)|^2)\\
&= \frac{1}{2}(1+\tau_0)\int_{\mathbb{R}^{n+1}} dy\Big(|\nabla_y(e^{-iz\varphi}\chi(\varphi)u)|^2+|z(e^{-iz\varphi}\chi(\varphi)u)|^2\Big)\\
&= \frac{1}{2}(1+\tau_0)\int_{\mathbb{R}^{n+1}} dy\Big(|e^{-iz\varphi}[\nabla_y(-iz\varphi)\,(\chi(\varphi)u)+\nabla_y(\chi(\varphi)u)]|^2\\
&\qquad\qquad +|z|^2|e^{-iz\varphi}\chi(\varphi)u|^2\Big)\\
&\le \frac{1}{2}(1+\tau_0)c_{112}\Big((1+|z|^2)\|\chi(\varphi)u\|_0^2+\|\nabla_y\chi(\varphi)u\|_0^2\Big)\\
&\le \frac{1}{2}(1+\tau_0)c_{112}\,(1+|z|^2)\|\chi(\varphi)u\|_1^2 \le c_{113}(1+|z|^2).
\end{aligned} \tag{2.10}$$

In the following, we want to apply properties of subharmonic functions. We notice that the function $U(y,z) := e^{-\epsilon|D_0|^2/(-2iz)}e^{-iz\varphi}\chi(\varphi)u(y)$ is analytic in z and satisfies $\operatorname{Im} z > 0$, and that $N(-iz)$ is subharmonic in z, as it integral in one parameter of the sum of two squares of the absolute values of analytic functions.

Our aim is now to estimate the H^1 norm of $A(D_0/\omega)b((y-y_0)/R)u)(y)$, where $\omega = \mu^\alpha/\beta$, for some $\beta > 0$ to be determined. For η_1 of Gevrey class $1/\alpha_1$ with support in $[-4, 1]$ and equal to 1 in $[-3, 1/2]$, let $\eta(s) := \eta_1(s/\delta)$. To shorten the notation, let $\widetilde{\mu} = \mu^\alpha$ and $\widehat{\eta} = \mathcal{F}_{s\to z}\eta$. First define

$$F(y) := A\Big(\frac{\beta D_0}{\widetilde{\mu}}\Big)(\eta(\varphi)u)(y).$$

Because of the regularity of η, we can write the following foliation with respect to the level sets of φ:

$$\eta(\varphi)(y') = \int_{\mathbb{R}} \eta(s)\delta(s-\varphi(y'))\,ds = \int_{\mathbb{R}} \overline{\widehat{\eta}}(z)e^{-iz\varphi(y')}\,dz.$$

Recall that according to our construction, $\chi(\varphi) = 1$ on $\operatorname{supp}(\eta(\varphi)u)$, and $\eta(\varphi)u = \eta(\varphi)\chi(\varphi)u$. Consequently we rewrite F as

$$F(y) = A\Big(\beta\frac{D_0}{\widetilde{\mu}}\Big)(\eta(\varphi)\chi(\varphi)u)(y) = \int_{\mathbb{R}} \overline{\widehat{\eta}}(\overline{z})\Big(A\Big(\frac{\beta D_0}{\widetilde{\mu}}\Big)e^{-iz\varphi}\chi(\varphi)u\Big)(y)\,dz.$$

Recall that $A(\beta D_0/\widetilde{\mu})$ is an integral operator with kernel

$$k(t,t') = \frac{\widetilde{\mu}}{\beta}\widehat{a}\Big(\frac{\widetilde{\mu}}{\beta}(t'-t)\Big).$$

Hence the previous equality is justified by Fubini's theorem, because for $y' = (t', x)$ the integrand $|\overline{\widehat{\eta}}(\overline{z})k(t,t')e^{-iz\varphi(y')}\chi(\varphi(y'))u(y')|$ is bounded by the function $ce^{-|z|^{\alpha_1}}e^{-|t-t'|^\alpha}u(t',x) \in L^1(\mathbb{R}_z \times \mathbb{R}_{t'})$.

Since $\eta \in C_0^\infty$, the Fourier-Laplace transform $\widehat{\eta}(z)$ is holomorphic for $z \in \mathbb{C}$, and hence $\overline{\widehat{\eta}}(\overline{z})$ is also holomorphic. We then need a good estimate for both $\overline{\widehat{\eta}}(\overline{z})$ and $A(\beta D_0/\widetilde{\mu})(e^{-iz\varphi}\chi(\varphi)u(y))$ in the upper half plane.

From the Gevrey class condition, we compute

$$|\widehat{\eta}(z)| = |\delta\widehat{\eta}_1(\delta z)| \le \delta c_{101}\exp(\delta \sup_{w\in\operatorname{supp}(\eta_1)}\langle w, \operatorname{Im} z\rangle - c_{102}\delta^{\alpha_1}|\operatorname{Re} z|^{\alpha_1}).$$

By considering the domain $\operatorname{Im}\overline{z} = -\operatorname{Im} z < 0$, we have

$$\begin{aligned}|\overline{\widehat{\eta}}(\overline{z})| &\le \delta c_{101}\exp(\delta \sup_{w\in[-4,1]}\langle w, \operatorname{Im}\overline{z}\rangle - c_{102}\delta^{\alpha_1}|\operatorname{Re}\overline{z}|^{\alpha_1})\\ &\le \delta c_{101}\exp(-4\delta\operatorname{Im}\overline{z} - c_{102}\delta^{\alpha_1}|\operatorname{Re} z|^{\alpha_1}),\end{aligned}$$

where $c_{101} = c_{101}(\alpha_1)$ is a given constant and $c_{102} = c_{102}(\alpha_1, c_{101})$. We now change path of integration in the upper half plane $\operatorname{Im} z > 0$:

$$F(y) = \int_{\Gamma_1 \cup \Gamma_2} \overline{\widehat{\eta}}(\overline{z}) A\Big(\beta \frac{D_0}{\widetilde{\mu}}\Big) \Big(e^{-iz\varphi} \chi(\varphi) u(y)\Big)\, dz,$$

where $\Gamma_1 = \{z \in \mathbb{R} : |z| \geq \frac{1}{\sqrt{2}} c_{130} \widetilde{\mu}\}$ and Γ_2 is the open rectangle inside the ball $|z| \leq c_{130}\widetilde{\mu}$ defined by

$$\begin{aligned}\Gamma_2 = &\Big\{z \in \mathbb{C} : \operatorname{Re} z = -\frac{1}{\sqrt{2}} c_{130}\widetilde{\mu},\ 0 \leq \operatorname{Im} z \leq \frac{1}{\sqrt{2}} c_{130}\widetilde{\mu}\Big\} \\ &\cup \Big\{z \in \mathbb{C} : |\operatorname{Re} z| \leq \frac{1}{\sqrt{2}} c_{130}\widetilde{\mu},\ \operatorname{Im} z = \frac{1}{\sqrt{2}} c_{130}\widetilde{\mu}\Big\} \\ &\cup \Big\{z \in \mathbb{C} : \operatorname{Re} z = \frac{1}{\sqrt{2}} c_{130}\widetilde{\mu},\ 0 \leq \operatorname{Im} z \leq \frac{1}{\sqrt{2}} c_{130}\widetilde{\mu}\Big\}.\end{aligned}$$

Hence

$$\begin{aligned}\|F\|_{H^1} \leq &\int_{\Gamma_1} |\overline{\widehat{\eta}}(\overline{z})| \Big\| A\Big(\frac{\beta D_0}{\widetilde{\mu}}\Big)(e^{-iz\varphi} \chi(\varphi) u(y)) \Big\|_{H^1} |dz| \\ &+ \int_{\Gamma_2} |\overline{\widehat{\eta}}(\overline{z})| \Big\| A\Big(\frac{\beta D_0}{\widetilde{\mu}}\Big)(e^{-iz\varphi} \chi(\varphi) u(y)) \Big\|_{H^1} |dz| := I_{\Gamma_1} + I_{\Gamma_2}.\end{aligned}$$

Along Γ_1, with $z = \operatorname{Re} z$, we have $|\widehat{\eta}(\overline{z})| \leq \delta c_{101} \exp(-c_{102}\delta^{\alpha_1}|z|^{\alpha_1})$ and

$$\begin{aligned}\Big\| A\Big(\beta \frac{D_0}{\widetilde{\mu}}\Big) e^{-iz\varphi} \chi(\varphi) u(y) \Big\|_{H^1}^2 &\leq \|e^{-iz\varphi}\chi(\varphi)u(y)\|_{H^1}^2 \\ &\leq 2(|z|^2 |\varphi'|_{C^0}^2 + 1) \|\chi(\varphi) u\|_{H^1}^2 \\ &\leq (|z|^2 + 1) c_{113}.\end{aligned}$$

The final estimate for I_{Γ_1} is

$$\begin{aligned}I_{\Gamma_1} &\leq 2\delta c_{101} \sqrt{c_{113}} \int_{\frac{1}{\sqrt{2}} c_{130}\widetilde{\mu}}^{+\infty} \sqrt{s^2+1}\, e^{-c_{102}\delta^{\alpha_1} s^{\alpha_1}} ds \\ &\leq 2\delta c_{101}\sqrt{c_{113}} e^{-c_{102}\delta^{\alpha_1}(\frac{1}{2\sqrt{2}} c_{130})^{\alpha_1} \widetilde{\mu}^{\alpha_1}} \int_{\mathbb{R}} \sqrt{s^2+1}\, e^{-c_{102}\delta^{\alpha_1} s^{\alpha_1}/2} ds \\ &\leq 2c_{101}\sqrt{c_{113}} e^{-c_{102}\delta^{\alpha_1}(\frac{1}{2\sqrt{2}} c_{130})^{\alpha_1} \widetilde{\mu}^{\alpha_1}} \int_{\mathbb{R}} \sqrt{(s/\delta)^2+1}\, e^{-c_{102} s^{\alpha_1}/2} ds.\end{aligned}$$

For I_{Γ_2}, we multiply and divide by the invertible operator $e^{\epsilon |D_0|^2/(2iz)}$, obtaining

$$\begin{aligned}I_{\Gamma_2} &= \int_{\Gamma_2} |\overline{\widehat{\eta}}(\overline{z})| \Big\| A\Big(\beta\frac{D_0}{\widetilde{\mu}}\Big) e^{-\epsilon|D_0|^2/(2iz)} e^{\epsilon|D_0|^2/(2iz)} e^{-iz\varphi} u(y)\|_{H^1} |dz| \\ &\leq \delta c_{101} \int_{\Gamma_2} e^{4\delta \operatorname{Im} z - c_{102}\delta^{\alpha_1}|\operatorname{Re} z|^{\alpha_1}} \Big\| A\Big(\beta\frac{D_0}{\widetilde{\mu}}\Big) e^{-\epsilon|D_0|^2/(2iz)} \Big\|_{\mathcal{B}(H^1)} \cdot \\ &\quad \cdot \|e^{-\epsilon|D_0|^2/(-2iz)} e^{-iz\varphi} \chi(\varphi) u(y)\|_{H^1} |dz|.\end{aligned}$$

In the region $\Gamma_2 \subset \{z : c_{130}\widetilde{\mu}/\sqrt{2} \le |z| \le c_{130}\widetilde{\mu}\}$, the norm in $\mathcal{B}(H^1)$ can be estimated independently of $\widetilde{\mu}$ via the Fourier symbol of the product:

$$\left|a\Big(\beta\frac{\xi_0}{\widetilde{\mu}}\Big)e^{-\epsilon\xi_0^2/(2iz)}\right| = \left|a\Big(\beta\frac{\xi_0}{\widetilde{\mu}}\Big)e^{\frac{\epsilon\xi_0^2\operatorname{Im}z}{2|z|^2}}\right| \le \exp\Big(\frac{\epsilon(2\widetilde{\mu})^2\operatorname{Im}z}{2\beta^2|c_{130}\widetilde{\mu}/\sqrt{2}|^2}\Big) = \exp\Big(\frac{4\epsilon Im\, z}{\beta^2c_{130}^2}\Big),$$

while the latter H^1-norm is related to the estimate (2.14) for $N(-iz)$:

$$\|e^{-\epsilon|D_0|^2/(-2iz)}e^{-iz\varphi}\chi(\varphi)u(y)\|_{H^1}^2 \le \frac{N(-iz)}{\min\{1,\widetilde{\mu}^2c_{130}^2/2\}} \le \frac{2c_{113}(1+|z|^2)}{\min\{1,\ c_{130}^2/2\}}e^{-10\delta Im\, z}$$

where we have used the estimate

$$(|\xi|^2+1) \le \frac{1}{\min\{1,\widetilde{\mu}^2c_{130}^2/2\}}(|\xi|^2+|z|^2)$$

in the first inequality and and the estimate $\widetilde{\mu} \ge 1$ in the second. Hence

$$\begin{aligned} I_{\Gamma_2} &\le \delta c_{101}\Big(\frac{2c_{113}(1+\widetilde{\mu}^2c_{130}^2)}{\min\{1,c_{130}^2/2\}}\Big)^{1/2}\int_{\Gamma_2} e^{4\delta\operatorname{Im}z-c_{102}\delta^{\alpha_1}|\operatorname{Re}z|^{\alpha_1}}e^{\frac{4\epsilon\operatorname{Im}z}{\beta^2c_{130}^2}}e^{-5\delta\operatorname{Im}z}|dz| \\ &\le \delta c_{101}\Big(\frac{2c_{113}(1+\widetilde{\mu}^2c_{130}^2)}{\min\{1,c_{130}^2/2\}}\Big)^{1/2}\int_{\Gamma_2} e^{-c_{102}\delta^{\alpha_1}|\operatorname{Re}z|^{\alpha_1}}e^{-\delta\operatorname{Im}z/2}|dz| \end{aligned}$$

where we choose ϵ and β such that $\epsilon \le \delta\beta^2c_{130}^2/8$. Actually, by our choice of c_{130}, this inequality can be written as

$$\epsilon \le \frac{9\beta^2}{2^{47}\delta}\min\Big(\frac{\epsilon\delta}{36},\frac{c_{123}^2}{4(3)^{2\alpha^1}},1\Big).$$

The latter relation is satisfied for any $\epsilon \le \epsilon_0$ and $\beta \ge c_{131}$, where

$$c_{131} = \max\Big\{\sqrt{2}(16)^6,\frac{\sqrt{2}(16)^6 3^{(\alpha_1-1)}\sqrt{\epsilon_0\delta}}{c_{123}},\frac{(16)^6\sqrt{\epsilon_0\delta}}{3\sqrt{2}}\Big\},$$

with ϵ_0 computed in Table 1.

Writing $z = x' + iy'$ we conclude the estimate

$$\begin{aligned} &\delta\int_{\Gamma_2} e^{-c_{102}\delta^{\alpha_1}|\operatorname{Re}z|^{\alpha_1}-\delta\operatorname{Im}z/2}|dz| \\ &\quad\le 2\delta\int_0^{\frac{c_{130}\widetilde{\mu}}{\sqrt{2}}} e^{-c_{102}\delta^{\alpha_1}\frac{(c_{130}\widetilde{\mu})^{\alpha_1}}{\sqrt{2}^{\alpha_1}}}e^{-\delta y'/2}dy' + \delta\int_{-\frac{c_{130}\widetilde{\mu}}{\sqrt{2}}}^{\frac{c_{130}\widetilde{\mu}}{\sqrt{2}}} e^{-c_{102}\delta^{\alpha_1}|x'|^{\alpha_1}}e^{-\delta\frac{c_{130}\widetilde{\mu}}{2\sqrt{2}}}dx' \\ &\quad\le 2\delta e^{-c_{102}\delta^{\alpha_1}\frac{(c_{130})^{\alpha_1}}{(\sqrt{2})^{\alpha_1}}\widetilde{\mu}^{\alpha_1}}\int_0^{+\infty} e^{-\delta y'/2}dy' + \delta e^{-\delta\frac{c_{130}}{2\sqrt{2}}\widetilde{\mu}}\int_{\mathbb{R}} e^{-c_{102}\delta^{\alpha_1}|x'|^{\alpha_1}}dx' \\ &\quad\le 2e^{-c_{102}\delta^{\alpha_1}\frac{(c_{130})^{\alpha_1}}{(\sqrt{2})^{\alpha_1}}\widetilde{\mu}^{\alpha_1}}\int_0^{+\infty} e^{-y'/2}dy'. + e^{-\delta\frac{c_{130}}{2\sqrt{2}}\widetilde{\mu}}\int_{\mathbb{R}} e^{-c_{102}|x'|^{\alpha_1}}dx'\,. \end{aligned}$$

Comparing the estimates for I_{Γ_1} and I_{Γ_2}, recalling that $e^{-c\widetilde{\mu}} \leq e^{-c\widetilde{\mu}^{\alpha_1}}$, and choosing the largest constants, we obtain the final estimate for $F(y)$

$$\left\| A\Big(\frac{\beta D_0}{\widetilde{\mu}}\Big)(\eta(\varphi)\chi(\varphi)u)(y) \right\|_{H^1} \leq c_{136} e^{-c_{137}\widetilde{\mu}^{\alpha_1}}, \tag{2.11}$$

with

$$c_{137} = \frac{1}{2} \min\left(c_{102}\delta^{\alpha_1} \frac{(c_{130})^{\alpha_1}}{(\sqrt{2})^{\alpha_1}},\ \delta \frac{c_{130}}{2\sqrt{2}},\ c_{102}\delta^{\alpha_1}\Big(\frac{1}{2\sqrt{2}} c_{130}\Big)^{\alpha_1} \right)$$

and

$$\begin{aligned} c_{136} &= 2c_{101}\sqrt{c_{113}} \int_{\mathbb{R}} \sqrt{(s/\delta)^2 + 1}\, e^{-c_{102}s^{\alpha_1}/2} ds \\ &\quad + c_{101}\left(\frac{2c_{113}(1 + c_{130}^2)}{\min\{1,\ c_{130}^2/2\}}\right)^{\frac{1}{2}} \left(2\int_0^{+\infty} e^{-y'/2} dy' + \int_{\mathbb{R}} e^{-c_{102}|x'|^{\alpha_1}} dx' \right). \end{aligned}$$

One can prove a similar estimate with $\eta(\varphi)$ replaced by $b((y - y_0)/r)$. We have chosen r such that

$$\operatorname{supp}\left(b\Big(\frac{y - y_0}{r}\Big) \right) \cap \operatorname{supp} u \subset \{y : \eta(\varphi(y)) = 1\} \cap \operatorname{supp} u,$$

and we write

$$\begin{aligned} A\Big(\frac{3\beta D_0}{\widetilde{\mu}}\Big) b\Big(\frac{y - y_0}{r}\Big) u(y) &= A\Big(\frac{3\beta D_0}{\widetilde{\mu}}\Big) b\Big(\frac{y - y_0}{r}\Big) A\Big(\frac{\beta D_0}{\widetilde{\mu}}\Big) \eta(\varphi) u(y) \\ &\quad + A\Big(\frac{3\beta D_0}{\widetilde{\mu}}\Big) b\Big(\frac{y - y_0}{r}\Big) \left[1 - A\Big(\frac{\beta D_0}{\widetilde{\mu}}\Big) \right] \eta(\varphi) u(y) := J_1 + J_2. \end{aligned}$$

From (2.11), J_1 has the desired estimate

$$\|J_1\|_1 \leq c_{136}\Big(1 + \frac{|b'|_{C^0}}{r}\Big) e^{-c_{137}\widetilde{\mu}^{\alpha_1}},$$

since $A(3\beta D_0/\widetilde{\mu}) b((y - y_0)r)$ is a bounded operator.

To estimate J_2, we apply Lemma 2.3.c), using the fact that $b \in G^{1/\alpha_1}(\mathbb{R}^{n+1})$, obtaining $\|J_2\|_1 \leq c_{134} e^{-c_{135}\widetilde{\mu}^{\alpha_1}}$, where

$$\begin{aligned} c_{134} &= r c_{1X} \left(\frac{8}{3} \Gamma\Big(\frac{1}{\alpha_1}\Big) \frac{1}{\alpha_1 (r^{\alpha_1} c_{2X})^{1/\alpha_1} (\alpha_1 c_{135})^{1/\alpha_1}} \right)^{1/2}, \\ c_{135} &= r^{\alpha_1} c_{2X} \frac{1}{23^{\alpha^1}}. \end{aligned}$$

We conclude the proof of Theorem 1.1 by choosing $c_{129} = \max\{c_{134}, c_{136}\}$, $c_{132} = \min\{c_{135}, c_{137}\}$. □

Next, we show in detail the estimate for the function N applied in the proof of Theorem 1.1.

Lemma 2.7. *Let $N(-iz)$ be defined as in* (2.10). *Then, for $z \in \mathbb{C} \cap \{\operatorname{Im} z \geq 0\}$, the function $N_1(-iz) := N(-iz)/|1-iz|^2$ satisfies*

(2.12)
$$N_1(-iz) \leq c_{113} e^{4\delta \operatorname{Im} z}(e^{-2c_{109}\widetilde{\mu}} + e^{-16\delta \operatorname{Im} z}), \quad z \in \mathbb{R} \cup \{\operatorname{Re} z = 0, \operatorname{Im} z \geq 0\},$$
$$N_1(-iz) \leq c_{113}, \quad \operatorname{Im} z > 0,$$

where c_{113} is given in (2.8) *and c_{109} is defined in Lemma 2.6. Therefore, there exists some constant c_{130}, independent of $\widetilde{\mu}$, such that*

$$N_1(-iz) \leq 2c_{113} e^{-10\delta Im\, z}, \quad |z| \leq c_{130}\widetilde{\mu},\ \operatorname{Im} z \geq 0, \tag{2.13}$$

with

$$c_{130} = \frac{3c_{109}}{4\delta}\left(\frac{1}{16}\right)^5.$$

Consequently, in the region $|z| \leq c_{130}\widetilde{\mu}$, $\operatorname{Im} z \geq 0$,

$$N(-iz) \leq 2c_{113}(1+|z|^2)e^{-10\delta Im\, z}. \tag{2.14}$$

Proof. Since $N_1(-iz) \leq N(-iz)$, the estimates (2.12) for N_1 follow from the equivalent estimates for N proved in (2.7), (2.9), and (2.10). To show (2.13) we first consider $z = x' + iy'$ in the region $x' > 0$, $y' > 0$. Here we define the analytic function

$$h(z) = e^{2i\delta z} e^{-8\delta i(z - C_1 \widetilde{\mu}^{(1-\kappa)} z^\kappa)},$$

where $z = |z|e^{i\theta}$, $z^\kappa = \exp(\kappa \ln z)$, with $\ln z = \ln|z| + i\theta$, $\theta \in [0, \pi/2]$, and C_1 is a constant to be determined. Taking $\kappa = 6/5$, so that $1 < \kappa < 2$ and κ is close to 1, we write $h(z)$ as

$$h(z) = \exp(2\delta(ix' - y')) \exp(-8\delta[-y' + C_1 \widetilde{\mu}^{1-\kappa}|z|^\kappa \sin(\kappa\theta)])$$
$$\cdot \exp(-8\delta i[x' - C_1\widetilde{\mu}^{1-\kappa}|z|^\kappa \cos(\kappa\theta)]),$$

and use h and its inverse to estimate N_1. Consider $N_1(-iz) = N_2(-iz)|h^{-1}(z)|^2$, where $N_2(-iz)$ is the subharmonic function in the first quadrant given by

$$N_2(-iz) := N_1(-iz)|h(z)|^2 = \int_{\mathbb{R}^{n+1}} (|\xi|^2 + |z|^2)\frac{|h(z)|^2}{|1-iz|^2} |e^{\frac{-\epsilon\xi_0^2}{-i2z}} \mathcal{F}_{y\to\xi}(e^{-iz\varphi}\chi(\varphi)u)|^2 d\widetilde{\xi} d\xi_0.$$

Observe the following.

a. On the real axis $y' = 0$, we have $|h(x')| = 1$; therefore, $N_2(-iz) \le N_1(-iz) \le 2c_{113}$.

b. On the positive imaginary axis $y' > 0$, $x' = 0$,

$$|h(iy')| = \exp(-2\delta y')\exp(a(y')),$$

with $a(y') := 8\delta(y' - C_1\widetilde{\mu}^{1-\kappa}(y')^\kappa s_\kappa)$, where $s_\kappa = \sin(\kappa\pi/2) > 1/2$.

Then $a(y')$ achieves its maximum at $y'_M = \widetilde{\mu}(C_1\kappa s_\kappa)^{1/(1-\kappa)}$ with the value

$$a(y'_M) = \frac{8\delta\widetilde{\mu}(\kappa-1)}{\kappa(C_1\kappa s_\kappa)^{1/(\kappa-1)}}.$$

We choose

$$C_1 \ge \frac{1}{\kappa s_\kappa}\Big(\frac{8(\kappa-1)}{\kappa c_{109}}\Big)^{\kappa-1}\delta^{\kappa-1},$$

so that $-c_{109}\widetilde{\mu} + a(y'_M) \le 0$; and consequently, by the estimates of N_1 and $|h|^2$,

$$\begin{aligned} N_2(-iz) &\le \Big(c_{113}e^{4\delta y'}(e^{-2c_{109}\widetilde{\mu}} + e^{-16\delta y'})\Big)e^{-4\delta y'}e^{2a(y')} \\ &= c_{113}(e^{2(-c_{109}\widetilde{\mu}+a(y))} + e^{-16\delta y'+2a(y')}) \\ &\le c_{113}(e^{2(-c_{109}\widetilde{\mu}+a(y'_M))} + e^{-16\delta C_1\widetilde{\mu}^{1-\kappa}|z|^\kappa s_\kappa}) \le 2c_{113}. \end{aligned}$$

c. In the region $y' > 0$, $x' > 0$,

$$|h(z)| = \exp(-2\delta y')\,\exp[8\delta|z|(\sin\theta - C_1\widetilde{\mu}^{1-\kappa}|z|^{\kappa-1}\sin(\kappa\theta))] \le e^{-2\delta y'}e^{c(\widetilde{\mu})}.$$

Indeed, for each $\theta \neq 0$, we can compute the maximum in $|z|$ of that expression and apply the inequality $1/2 < \sin(\kappa\theta)/\sin\theta \le \kappa$ to obtain

$$\max_{r\ge 0}[8\delta r(\sin\theta - C_1\widetilde{\mu}^{1-\kappa}r^{\kappa-1}\sin(\kappa\theta))] \le c(\widetilde{\mu}) := \frac{8\delta(\kappa-1)}{\kappa(\kappa C_1/2)^{1/(\kappa-1)}}\widetilde{\mu},$$

which implies $N_2(-iz) \le c_{113}e^{-4\delta y'}e^{2c(\widetilde{\mu})}$. To get rid of the $\widetilde{\mu}$ dependency in this estimate, we apply the Phragmen-Lindelöf Theorem 4.5 for subharmonic functions in the sector $x' \ge 0, y' \ge 0$ to obtain $N_2(-iz) \le 2c_{113}$ and note that c_{113} is independent of $\widetilde{\mu}$.

To prove (2.13), observe that for $\kappa = 6/5$, we have $\sin(\kappa\pi/2) > 1/2$; and the inequality

$$|h^{-1}(z)| = \exp(2\delta Im\, z)\,\exp(-8\delta Im\, z + 8\delta C_1\widetilde{\mu}^{1-\kappa}|z|^\kappa\sin(\kappa\theta)) \le \exp(-5\delta\,\mathrm{Im}\, z),$$

where

$$c_{130} := \frac{3c_{109}}{4\delta}\Big(\frac{1}{16}\Big)^5 \le \min_{\theta\in[0,\pi/2]}\Big(\frac{\sin\theta}{8C_1\sin(\kappa\theta)}\Big)^{\frac{1}{\kappa-1}}$$

is valid in the region $|z| = c_{130}\widetilde{\mu}$ with $\operatorname{Re} z \geq 0$, $\operatorname{Im} z \geq 0$, Indeed,

$$8C_1\widetilde{\mu}^{1-\kappa}(c_{130}\widetilde{\mu})^{\kappa-1}\sin(\kappa\theta) \leq \sin\theta,$$

which proves the derived estimate; consequently, (2.14) follows.

Next, we observe that the same estimate (2.13) can be obtained in the sector $\operatorname{Re} z \leq 0$, $\operatorname{Im} z \geq 0$ by the following arguments. In the region $x' < 0$, $y' > 0$, we set $z = x' + iy'$ and $w = -\overline{z} = -x' + iy'$ (belonging to the first quadrant), and define $N_3(-iw) := N_1(-i(-\overline{z}))$ and $N_4(-iw) := N_1(-i(-\overline{z}))|h(-\overline{z})|^2$. Notice that $h(w)$ is an antiholomorphic function in w, and therefore $|h(w)|$ is subharmonic. Also, $N_3(-iw)$ and $N_4(-iw)$ are subharmonic and satisfy the same estimates as $N_1(-iz)$, $N_2(-iz)$, respectively. We then apply the same procedure as in the first step with N_1, N_2 replaced by N_3, N_4, respectively. □

We now can complete the proof of the logarithmic stability estimate in Theorem 1.2.

Proof of Theorem 1.2. We consider the following two cases.

Case A: $\|Pu\|_{L^2(B_{2R})} \geq \|u\|_{H^1(B_{2R})}/e$. In this case, the estimate is trivial, as

$$\|u\|_{L^2(B_r)} \leq \|u\|_{H^1(B_{2R})} \leq \ln(1+e)\frac{\|u\|_{H^1(B_{2R})}}{\ln\left(1+\frac{\|u\|_{H^1(B_{2R})}}{\|Pu\|_{L^2(B_{2R})}}\right)}.$$

Case B: $\|Pu\|_{L^2(B_{2R})} < \|u\|_{H^1(B_{2R})}/e$. Without loss of generality, we take $\|u\|_{H^1(B_{2R})} = 1$. Our aim is to consider separetely estimates for low and high temporal frequencies.

Let $A(D_0)$ be a pseudo-differential operator with symbol $a(\xi_0)$, where $a \in G_0^{1/\alpha}(\mathbb{R})$ with $\alpha \in (0,1)$ is a smooth Gevrey class localizer that is supported in $|\xi_0| \leq 2$, equals 1 in $|\xi_0| \leq 1$ and satisfies $0 \leq a \leq 1$. The function $a(\beta\xi_0/\widetilde{\mu})$ is a scaled version of a, where $\widetilde{\mu} > 1$ is the parameter to be optimized, and $\beta > 0$ an adjusting constant. Let $b \in G_0^{1/\alpha_1}(\mathbb{R}^{n+1})$ with $0 < \alpha \leq \alpha_1$ be another localizer supported in B_2 equal to 1 in B_1 and satisfying $0 \leq b \leq 1$. Observe that according to our geometric construction,

$$B_r \subset \operatorname{supp} b\Big(\frac{y-y_0}{r}\Big) \subseteq B_{2r} \subset B_R \subset \operatorname{supp} b\Big(\frac{y-y_0}{R}\Big) \subseteq B_{2R}$$

(see Proposition 2.5), and hence $\|u\|_{L^2(B_r)} \leq \|b((y-y_0)/r)u\|_{L^2}$. Then we perform the splitting

$$b\Big(\frac{y-y_0}{r}\Big)u = A\Big(\frac{\beta D_0}{\widetilde{\mu}}\Big)b\Big(\frac{y-y_0}{r}\Big)u + \bigg(1 - A\Big(\frac{\beta D_0}{\widetilde{\mu}}\Big)\bigg)b\Big(\frac{y-y_0}{r}\Big)u.$$

For high temporal frequencies $|\xi_0| \geq \widetilde{\mu}/\beta$ we estimate as follows:

$$\begin{aligned}
\Big\|\Big(1-A\Big(\frac{\beta D_0}{\widetilde{\mu}}\Big)\Big)b\Big(\frac{y-y_0}{r}\Big)u(y)\Big\|_{L^2}^2 &= \Big\|\Big(1-a\Big(\frac{\beta\xi_0}{\widetilde{\mu}}\Big)\Big)\mathcal{F}_{t\to\xi_0}\Big(b\Big(\frac{y-y_0}{r}\Big)u(y)\Big)\Big\|_{L^2}^2 \\
&\leq \frac{\beta^2}{\widetilde{\mu}^2}\int_{|\xi_0|>\widetilde{\mu}/\beta}\int_{\mathbb{R}^n}\Big|\xi_0\mathcal{F}_{t\to\xi_0}\Big(b\Big(\frac{y-y_0}{r}\Big)u(t,x)\Big)\Big|^2 dx d\xi_0 \\
&\leq \frac{\beta^2}{\widetilde{\mu}^2}\Big\|b\Big(\frac{y-y_0}{r}\Big)u(y)\Big\|_{H^1}^2 \leq \frac{\beta^2}{\widetilde{\mu}^2}\Big(1+\frac{|b'|_{C^0}^2}{r^2}\Big)\|u(y)\|_{H^1(B_R)}^2.
\end{aligned}$$

For low temporal frequencies, we first choose $\widetilde{\mu}$ such that $\|Pu\|_{L^2(B_{2R})} = e^{-\widetilde{\mu}} < e^{-1}$. Then we take $\mu > 1$ such that $\widetilde{\mu} = \mu^\alpha$. Hence for A and b as above, we get

$$\Big\|A\Big(\frac{\zeta D_0}{\mu}\Big)b\Big(\frac{y-y_0}{R}\Big)Pu\Big\|_0 \leq \|Pu\|_{L^2(B_{2R})} = e^{-\widetilde{\mu}} \tag{2.15}$$

for all $\zeta > 0$.

For $\zeta = 1$ in (2.15), we can apply Theorem 1.1 to obtain

$$\Big\|A\Big(\frac{\beta D_0}{\widetilde{\mu}}\Big)b\Big(\frac{y-y_0}{r}\Big)u\Big\|_{L^2} \leq c_{129}e^{-c_{132}\widetilde{\mu}^{\alpha_1}}, \quad \beta \geq 3c_{131}. \tag{2.16}$$

By collecting the previous estimates for low and high temporal frequencies we conclude that, as $\widetilde{\mu} \geq 1$,

$$\begin{aligned}
\|u\|_{L^2(B_r)} &\leq \frac{\beta}{\widetilde{\mu}}\Big(1+\frac{|b'|_{C^0}^2}{r^2}\Big)^{1/2} + c_{129}e^{-c_{132}\widetilde{\mu}^{\alpha_1}} \leq \frac{c_{105}}{\widetilde{\mu}} = \frac{c_{105}}{-\ln(\|Pu\|_0)} \\
&\leq 2c_{105}\frac{\|u\|_{H^1(B_{2R})}}{\ln\Big(1+\frac{\|u\|_{H^1(B_{2R})}}{\|Pu\|_{L^2(B_{2R})}}\Big)},
\end{aligned}$$

where

$$c_{105} = \beta\Big(1+\frac{|b'|_{C^0}^2}{r^2}\Big)^{1/2} + c_{129}$$

and in the last step we have applied the inequality $\ln(y) \geq \ln(1+y)/2$ for $y = \|u\|_{H^1(B_{2R})}/\|Pu\|_{L^2(B_{2R})} > e$, and then returned to the original notation. Choosing $c_{111} = \ln(1+e) + 2c_{105}$, we obtain the result. $\square$

3 Geometric constants

3.1 Pseudoconvexity constants. In the following, we work under the following assumptions, derived from those in Theorem 1.2.

Assumption A3. Consider the case of the wave operator (1.1) with principal symbol $p(y,\xi) = -\xi_0^2 + \sum_{jk=1}^n g^{jk}(x)\xi_j\xi_k$, with $0 < a_1\delta^{jk} \le g^{jk}(x) \le b_1\delta^{jk}$, $a_1,\ b_1 > 0$. Let $\xi = (\xi_0, \widetilde{\xi}) \in \mathbb{R}\times\mathbb{R}^n$, where $|\widetilde{\xi}|^2 = \sum_{j=1}^n \xi_j^2$.

Assumption A4. We fix a function $\psi \in C^{2,\rho}(\mathbb{R}^{n+1})$, for some $\rho \in (0,1)$, such that $p(y,\psi'(y)) \neq 0$ and $\psi'(y) \neq 0$ in a domain $\Omega_0 \subseteq \Omega$, containing the point y_0 lying on the level set $S = \{y;\ \psi(y) = 0\}$. In particular, we assume that $|\psi'(y)| \ge C_l$ in Ω_0 for $C_l > 0$.

We use Einstein's convention for repeated indexes.

To get Tataru inequality, we proceed in three steps. The computed constants are listed in Table 1.

Step 1. Given a function $\psi \in C^{2,\rho}(\mathbb{R}^{n+1})$ satisfying Assumptions A3 and A4 in a domain Ω_0, we find positive constants M_2, M_1, M_P such that

$$
\begin{aligned}
(3.1)\quad M_2\xi_0^2 + M_1\Big(&\frac{|p(y,\xi+i\tau\psi'(y))|^2}{\tau^2+|\xi|^2} + |\langle p'_\xi(y,\xi+i\tau\psi'(y)),\psi'(y)\rangle|^2\Big)\\
&+\frac{\{\overline{p(y,\xi+i\tau\psi'(y))},p(y,\xi+i\tau\psi'(y))\}}{2i\tau} \ge M_P(\tau^2+|\xi|^2)
\end{aligned}
$$

for every $\xi \in \mathbb{R}\times\mathbb{R}^n$, $\xi \neq 0$, $\tau \in \mathbb{R}$. The previous inequality proves that the hypersurface $S = \{y;\ \psi(y) = 0\}$ is conormally strongly pseudo-convex with respect to P in Ω_0.

Step 2. For $\phi = e^{\lambda\psi}$, with y_0 on the level set $\phi(y) = 1$, we find $\lambda > 0$ such that the following inequality holds true

$$
\begin{aligned}
(3.2)\quad M_2\xi_0^2 + &\frac{M_1}{\min\{1,\lambda^2\phi^2(y)\}}\frac{|p(y,\xi+i\tau\phi'(y))|^2}{\tau^2+|\xi|^2}\\
&+\frac{1}{\lambda\phi(y)}\frac{\{\overline{p(y,\xi+i\tau\phi'(y))},p(y,\xi+i\tau\phi'(y))\}}{2i\tau}\\
&\ge M_P\min\{1,\lambda^2\phi^2(y)\}(\tau^2+|\xi|^2)
\end{aligned}
$$

for every $\xi \in \mathbb{R}\times\mathbb{R}^n$, $\xi \neq 0$, $\tau \in \mathbb{R}$. The previous inequality proves that the function ϕ is conormally strongly pseudo-convex with respect to P in Ω_0.

Step 3. We consider a perturbation of ϕ by the shifted second order polynomial centered at the point y_0,

$$
(3.3)\qquad f(y) = \sum_{|\upsilon|\le 2}\frac{(\partial^\upsilon\phi)(y_0)\,(y-y_0)^\upsilon}{\upsilon!} - \sigma|y-y_0|^2.
$$

In a ball $B(y_0,R_1) \subset \Omega_0$ where $f' \neq 0$, we define

$$
\phi_0 = \min_{y\in B(y_0,R_1)}\phi(y), \qquad \phi_M = \max_{y\in B(y_0,R_1)}\phi(y).
$$

We find σ and $R_2 > 0$ small enough so that in the ball $B(y_0, R_2)$,

$$f(y) < \phi(y) \quad \text{in } B(y_0,\ R_2)\backslash\{y_0\}$$

and

$$M_2\xi_0^2 + 2M_1\frac{|p(y,\xi+i\tau f'(y))|^2}{\tau^2+|\xi|^2} + \frac{\{\overline{p(y,\xi+i\tau f'(y))}, p(y,\xi+i\tau f'(y))\}}{(\lambda\phi_0)2i\tau} \geq \frac{1}{2}(\tau^2+|\xi|^2). \tag{3.4}$$

The previous inequality proves that f is conormally strongly pseudo-convex with respect to P in $B(y_0, R_2)$.

Proof of Step 1. We recall that

$$\begin{aligned}
p(y,\xi+i\tau\psi'(y)) &= p(y,\xi) - \tau^2 p(y,\psi') + i\tau\{p,\psi\}\\
|p(y,\xi+i\tau\psi'(y))|^2 &= |p(y,\xi) - \tau^2 p(y,\psi')|^2 + \tau^2|\{p,\psi\}|^2\\
&= |p(y,\xi)|^2 + \tau^4|p(y,\psi')|^2 - 2\tau^2 p(y,\xi)p(y,\psi') + \tau^2|\{p,\psi\}|^2\\
\langle p'_\xi(y,\xi+i\tau\psi'(y)),\psi'(y)\rangle &= \{p,\psi\}(y,\xi) + i2\tau p(y,\psi')\\
|\langle p'_\xi(y,\xi+i\tau\psi'(y)),\psi'(y)\rangle|^2 &= |\{p,\psi\}(y,\xi)|^2 + 4\tau^2|p(y,\psi')|^2.
\end{aligned}$$

We have to estimate the quantities

$$\begin{aligned}
I_{1,\psi} &:= \frac{|p(y,\xi+i\tau\psi'(y))|^2}{\tau^2+|\xi|^2} + |\langle p'_\xi(y,\xi+i\tau\psi'(y)),\psi'(y)\rangle|^2,\\
I_{2,\psi} &:= \frac{\{\overline{p(y,\xi+i\tau\psi'(y))}, p(y,\xi+i\tau\psi'(y))\}}{2i\tau}\\
&= \{p,\{p,\psi\}\}(y,\xi) + \tau^2\{p,\{p,\psi\}\}(y,\psi'(y)),
\end{aligned}$$

where the last equality holds for our second order wave operator. For the second term we get, by setting $a^{00} = -1$, $a^{j0} = 0$, $a^{jk} = g^{jk}$, $j,k = 1\ldots n$,

$$\begin{aligned}
I_{2,\psi} = {}& \sum_{l,m=0}^{n}\xi_l\xi_m\Big(4\sum_{j,k=0}^{n}a^{jl}\psi''_{jk}a^{km} + 4\sum_{j,k=0}^{n}a^{jl}\partial_{x_j}a^{km}\psi'_k - 2\sum_{j,k=0}^{n}\partial_{x_j}a^{lm}a^{kj}\psi'_k\Big)\\
&+ \tau^2\sum_{l,m=0}^{n}\psi'_l\psi'_m\Big(4\sum_{j,k=0}^{n}a^{jl}\psi''_{jk}a^{km} + 2\sum_{j,k=0}^{n}a^{jl}\partial_{x_j}a^{km}\psi'_k\Big) \geq -C_3(|\xi|^2+\tau^2),
\end{aligned}$$

where C_3 is defined by

$$\begin{aligned}
&\max_{y\in\Omega_0}\Big(4\sum_{j,k}a^{jl}\psi''_{jk}a^{km} + 4\sum_{j,k}a^{jl}\partial_{x_j}a^{km}\psi'_k - 2\sum_{j,k}\partial_{x_j}a^{lm}a^{kj}\psi'_k\Big)(2+\psi'_l\psi'_m)\\
&\qquad\leq 20(1+n^2|g^{jl}|^2_{C^1})|\psi'|_{C^1}(1+|\psi'|^2_{C^0}) := C_3.
\end{aligned}$$

For the first term, we get

$$\begin{aligned}
I_{1,\psi} &= \frac{|p(y,\xi)|^2}{\tau^2+|\xi|^2} + \tau^2|p(y,\psi')|^2\Big(4+\frac{\tau^2}{\tau^2+|\xi|^2}\Big) - 2\frac{\tau^2}{\tau^2+|\xi|^2}p(y,\xi)p(y,\psi') \\
&\quad + |\{p,\psi\}|^2\Big(1+\frac{\tau^2}{\tau^2+|\xi|^2}\Big) \\
&\geq \frac{|p(y,\xi)|^2}{\tau^2+|\xi|^2}(1-\omega) + \tau^2|p(y,\psi')|^2\Big(4|\xi|^2+\Big(5-\frac{1}{\omega}\Big)\tau^2\Big)\frac{1}{\tau^2+|\xi|^2} \\
&\quad + |\{p,\psi\}|^2\Big(1+\frac{\tau^2}{\tau^2+|\xi|^2}\Big) \\
&\geq \frac{1}{\tau^2+|\xi|^2}\Big(|p(y,\xi)|^2(1-\omega) + 4|p(y,\psi')|^2|\xi|^2\tau^2 + |p(y,\psi')|^2\Big(5-\frac{1}{\omega}\Big)\tau^4 \\
&\quad + |\{p,\psi\}|^2(2\tau^2+|\xi|^2)\Big);
\end{aligned}$$

Observe that by Young's inequality,

$$2p(y,\xi)\tau^2 p(y,\psi') \leq \omega|p(y,\xi)|^2 + \frac{1}{\omega}\tau^4|p(y,\psi')|^2,$$

where we choose $\omega \in (0,1)$, so that $(4 \geq)\ 5-\frac{1}{\omega} > 0$. We now split the estimate into two parts.

Case 1: $p(y,\xi) > 0$. In this case,

$$\begin{aligned}
|p(y,\xi)| &= p(y,\xi) = -\xi_0^2 + \sum_{kj} g^{kj}\xi_k\xi_j \geq a_1|\widetilde{\xi}|^2 - \xi_0^2, \\
|p(y,\xi)|^2 &= (-\xi_0^2 + \sum_{kj} g^{kj}\xi_k\xi_j)^2 \geq (a_1|\widetilde{\xi}|^2 - \xi_0^2)(-\xi_0^2 + \sum_{kj} g^{kj}\xi_k\xi_j) \\
&= \xi_0^4 + a_1|\widetilde{\xi}|^2(\sum_{kj} g^{kj}\xi_k\xi_j) - \xi_0^2[\sum_{kj} g^{kj}\xi_k\xi_j + a_1|\widetilde{\xi}|^2] \\
&\geq \xi_0^4 + a_1^2|\widetilde{\xi}|^4 - (b_1+a_1)|\widetilde{\xi}|^2\xi_0^2.
\end{aligned}$$

Our aim is to find M_2, M_1, M_P such that $M_2\xi_0^2 + M_1 I_{1,\psi} + I_{2,\psi} \geq M_P(\tau^2+|\xi|^2)$. Then

$$\begin{aligned}
M_2\xi_0^2 + M_1 I_{1,\psi} + I_{2,\psi} &\geq M_2\xi_0^2 - C_3(\tau^2+|\xi|^2) + M_1\Big[\frac{|p(y,\xi)|^2}{\tau^2+|\xi|^2}(1-\omega) \\
&\quad + \frac{\tau^2|p(y,\psi')|^2}{\tau^2+|\xi|^2}\Big(4|\xi|^2+\big(5-\frac{1}{\omega}\big)\tau^2\Big) + |\{p,\psi\}|^2\Big(1+\frac{\tau^2}{\tau^2+|\xi|^2}\Big)\Big] \\
&\geq \frac{1}{\tau^2+|\xi|^2}\Big(M_2(\tau^2\xi_0^2+|\xi|^2\xi_0^2) + M_1(1-\omega)[a_1^2|\widetilde{\xi}|^4+\xi_0^4-(a_1+b_1)|\widetilde{\xi}|^2\xi_0^2] \\
&\quad + M_1|p(y,\psi')|^2\big[4\tau^2|\xi|^2+\big(5-\frac{1}{\omega}\big)\tau^4\big] - C_3(\tau^4+|\xi|^4+2\tau^2|\xi|^2)\Big) \\
&\geq M_P(\tau^2+|\xi|^2).
\end{aligned}$$

To solve the last inequality we have to solve the following system of seven inequalities:

$$\begin{aligned}\Big(M_1(1-\omega)a_1^2 - C_3\Big)\,|\widetilde{\xi}|^4 &\geq M_P|\widetilde{\xi}|^4\\ \Big(M_1(1-\omega) + M_2 - C_3\Big)\,\xi_0^4 &\geq M_P\xi_0^4\\ \Big(M_2 - (b_1+a_1)M_1(1-\omega) - 2C_3\Big)\,|\widetilde{\xi}|^2\xi_0^2 &\geq 2M_P|\widetilde{\xi}|^2\xi_0^2\\ \Big(4M_1|p(y,\psi')|^2 - 2C_3\Big)\,\tau^2|\widetilde{\xi}|^2 &\geq 2M_P\tau^2|\widetilde{\xi}|^2\\ \Big(4M_1|p(y,\psi')|^2 - 2C_3 + M_2\Big)\,\tau^2\xi_0^2 &\geq 2M_P\tau^2\xi_0^2\\ \Big(M_1|p(y,\psi')|^2\Big(5-\frac{1}{\omega}\Big) - C_3\Big)\,\tau^4 &\geq M_P\tau^4.\end{aligned}$$

<u>Case 2</u>: $p(y,\xi) \leq 0$. In this case,

$$|p(y,\xi)| = -p(y,\xi) = \xi_0^2 - \sum_{kj} g^{kj}\xi_k\xi_j \geq 0 \;\Rightarrow\; \xi_0^2 \geq \sum_{kj} g^{kj}\xi_k\xi_j \geq a_1|\widetilde{\xi}|^2.$$

Once again we seek M_2, M_1, and M_P such that $M_2\xi_0^2 + M_1 I_{1,\psi} + I_{2,\psi} \geq M_P(\tau^2 + |\xi|^2)$. Then

$$\begin{aligned}M_2\xi_0^2 + M_1 I_{1,\psi} + I_{2,\psi} \geq{}& M_1\Big[\frac{\tau^2|p(y,\psi')|^2}{\tau^2+|\xi|^2}\Big(4|\xi|^2 + \Big(5-\frac{1}{\omega}\Big)\tau^2\Big)\\ &+ |\{p,\psi\}|^2\Big(1+\frac{\tau^2}{\tau^2+|\xi|^2}\Big)\Big] + M_2\Big(\frac{\xi_0^2}{2}+\frac{\xi_0^2}{2}\Big) - C_3(\tau^2+|\xi|^2)\\ \geq{}& \frac{1}{\tau^2+|\xi|^2}\Big(M_1|p(y,\psi')|^2\Big[4\tau^2|\xi|^2 + \Big(5-\frac{1}{\omega}\Big)\tau^4\Big]\\ &- C_3(\tau^4+|\xi|^4+2\tau^2|\xi|^2) + M_2\Big(\frac{\xi_0^2}{2}+\frac{a_1|\widetilde{\xi}|^2}{2}\Big)(\tau^2+|\xi|^2)\Big)\\ \geq{}& M_P(\tau^2+|\xi|^2).\end{aligned}$$

To solve the last inequality, we have to solve the following system of three inequalities:

$$\begin{aligned}\Big(4M_1|p(y,\psi')|^2 - 2C_3 + \frac{M_2}{2}\min\{a_1,1\}\Big)\,\tau^2|\xi|^2 &\geq 2M_P\tau^2|\xi|^2\\ \Big(M_1|p(y,\psi')|^2\Big(5-\frac{1}{\omega}\Big) - C_3\Big)\,\tau^4 &\geq M_P\tau^4\\ \Big(\frac{M_2}{2}\min\{a_1,1\} - C_3\Big)\,|\xi|^4 &\geq M_P|\xi|^4.\end{aligned}$$

From Cases 1 and 2, we obtain two systems of inequalities for the coefficients; by choosing $\omega = 1/2$ and solving them, we obtain the pseudo-convexity estimate

(3.1) with M_1 and M_2 shown in Table 1 and with M_P a free parameter to be set in the following. $\square$

Remarks 3.1. 1. Notice that the estimate is valid also in the limit $\tau \to 0$. Indeed, for $\xi \neq 0$

$$\begin{aligned} M_2\xi_0^2 + M_1 I_{1,\psi} + I_{2,\psi} &= M_2\xi_0^2 + M_1\Big(\frac{|p(y,\xi)|^2}{|\xi|^2} + |\{p,\psi\}|^2\Big) + \{p,\{p,\psi\}\} \\ &\geq M_2\xi_0^2 - C_3|\xi|^2 + M_1\frac{|p(y,\xi)|^2}{|\xi|^2} \geq M_P|\xi|^2. \end{aligned}$$

2. From the constraint on M_1, one can understand the reason for the assumption $p(y,\psi') \neq 0$. Actually, as observed in [7] and by other authors, in the case $p(y_0,\psi'(y_0)) = 0$, the estimate (3.1) is still possible if $\{p,\{p,\psi\}\}(y_0,\psi'(y_0)) > 0$. Indeed, in that case, there exist positive constants C_4 and C_5 such that $I_{2,\psi} \geq C_5\tau^2 - C_4|\xi|^2$; and one can proceed as above to get (3.1) with different coefficients.

Proof of Step 2. Let $\phi(y) = e^{\lambda\psi(y)}$, $\tau_1 = \tau\lambda\phi(y)$, and recall that

$$\tau\phi'(y) = \tau\lambda\phi(y)\psi'(y) = \tau_1\psi'(y), \quad \phi''(y) = \lambda\phi(y)(\psi''(y) + \lambda\psi'(y)\otimes\psi'(y)),$$

where $\phi'(y) \neq 0$ in Ω_0. Then, for $\tau \neq 0$ (see [7, Lemma 4.2]),

$$\begin{aligned} \frac{\{\overline{p(y,\xi+i\tau\phi'(y))}, p(y,\xi+i\tau\phi'(y))\}}{\lambda\phi(y)(2i\tau)} &= \frac{1}{2i\tau_1}\{\overline{p(y,\xi+i\tau_1\psi'(y))}, p(y,\xi+i\tau_1\psi'(y))\} \\ &\quad + \lambda|\langle p'_\xi(y,\xi+i\tau_1\psi'(y)), \psi'(y)\rangle|^2, \end{aligned}$$

where on the right hand side one has first to perform the derivatives and then substitute τ_1 (which, consequently, must not be seen as a function of y and τ in the bracket). In the case $\tau = 0$,

$$\frac{\{p,\{p,\phi\}\}(y,\xi)}{\lambda\phi(y)} = \{\{p,\{p,\psi\}\}(y,\xi) + \lambda|\langle p'_\xi(y,\xi), \psi'(y)\rangle|^2.$$

Hence, for $\lambda \geq M_1$, substituting in (3.1) the variables τ_1, ξ, we obtain

$$\tau_1^2 + |\xi|^2 \geq \min\left(1, \lambda^2\phi^2(y)\right)(\tau^2 + |\xi|^2),$$

and finally (3.2). $\square$

Proof of Step 3. For simplicity, we now consider λ and a domain $B(y_0,R_1)$ on which $\phi_0 = e^{-1} \leq \phi(y) \leq e = \phi_M$ and $\min\left(1, \lambda^2\phi^2(y)\right) = 1$. Since $|\psi(y) - \psi(y_0)| \leq |\psi'|_{C^0(\Omega_0)}R_1$, we choose

$$R_1 \leq \min\left\{1, \min_{\Omega_0}|y_0 - y|, \frac{1}{\lambda|\psi'|_{C^0(\Omega_0)}}\right\}, \quad \lambda \geq e. \tag{3.5}$$

We then rewrite f as

$$\begin{aligned} f(y) &= \phi(y_0) + \sum_{j=1}^{n} \partial_j \phi(y_0)(x_j - x_{0,j}) + \partial_t \phi(y_0)(t - t_0) \\ &\quad + \frac{1}{2} \sum_{j,k=1}^{n} \partial^2_{j,k} \phi(y_0)(x_j - x_{0,j})(x_k - x_{0,k}) \\ &\quad + \sum_{j=1}^{n} \partial^2_{j,t} \phi(y_0)(x_j - x_{0,j})(t - t_0) \\ &\quad + \frac{1}{2} \partial^2_t \phi(y_0)(t - t_0)^2 - \sigma|x - x_0|^2 - \sigma|t - t_0|^2 \end{aligned}$$

and its derivatives, identifying ∂_t with ∂_0 and denoting the Kroenecker symbol by δ_{ab}, as

$$\begin{aligned} f'_j(y) &= \phi'_j(y_0) + \sum_{h=1}^{n} \phi''_{jh}(y_0)(x_h - x_{0h}) \\ &\qquad + \phi''_{tj}(y_0)(t - t_0) - 2\sigma\big((x_j - x_{0j})(1 - \delta_{0j}) + (t - t_0)\delta_{0j}\big) \\ f''_{jm}(y) &= \phi''_{jm}(y_0) - 2\sigma\delta_{jm}, \quad j, m \in \{0, 1, \dots, n\}. \end{aligned}$$

First we require $f' \neq 0$ in the ball $|y - y_0| \leq R_2$ to satisfy

$$\begin{aligned} |f'(y)| &\geq |\phi'(y_0)| - |\phi''(y_0)||y - y_0| - 2\sigma|y - y_0| \\ &\geq |\phi'(y_0)| - |\phi''(y_0)|R_2 - 2\sigma R_2 \geq \frac{|\phi'(y_0)|}{2}, \end{aligned}$$

which implies the constraint on R_2

$$|\phi''(y_0)|R_2 + 2\sigma R_2 \leq \frac{|\phi'(y_0)|}{2}. \tag{3.6}$$

In order to pass from (3.2) to (3.4), we compute

$$\begin{aligned} |p(y, \xi + i\tau\phi'(y))|^2 &= |p(y,\xi) - \tau^2 p(y, f') + \tau^2(p(y, f') - p(y, \phi'))|^2 \\ &\quad + \tau^2|\{p, f\} + (\{p, \phi - f\})|^2 \\ &\leq 2|p(y,\xi) - \tau^2 p(y, f')|^2 + 2\tau^4|p(y, \phi') - p(y, f')|^2 \\ &\quad + 2\tau^2|\{p, f\}|^2 + 2\tau^2|(\{p, \phi - f\})|^2 \\ &\leq 2|p(y, \xi + i\tau f'(y))|^2 + 2\tau^4|p(y, \phi') - p(y, f')|^2 + 2\tau^2|(\{p, \phi - f\})|^2 \\ &\leq 2|p(y, \xi + i\tau f'(y))|^2 + 2\tau^4\eta_1 + 2\tau^2|\xi|^2\eta_2 \\ &\leq 2|p(y, \xi + i\tau f'(y))|^2 + \eta_2(1 + |\phi'|^2_{C^0} + |f'|^2_{C^0})\,(\tau^2 + |\xi|^2)^2, \end{aligned}$$

where η_1 and η_2 are defined by

$$
\begin{aligned}
|p(y,\phi') - p(y,f')|^2 &= \left| -(\phi_t')^2 + \sum_{jk=1}^{n} g^{jk}\phi_j'\phi_k' + (f_t')^2 - \sum_{jk=1}^{n} g^{jk} f_j' f_k' \right|^2 \\
&\le 2|f_t' - \phi_t'|^2(|\phi_t'| + |f_t'|)^2 + 2|\sum_{jk=1}^{n} g^{jk}((\phi_j' - f_j')\phi_k' + f_j'(-f_k' + \phi_k'))|^2 \\
&\le 4(1 + n^4|g^{jk}|^2_{C^0})(|\phi'|^2_{C^0} + |f'|^2_{C^0})|f' - \phi'|^2_{C^0} := \eta_1
\end{aligned}
$$

and

$$
\begin{aligned}
|\{p,\phi - f\}|^2 &= |2\xi_0(f_t' - \phi_t') + 2\sum g^{jk}\xi_j(\phi_k' - f_k')|^2 \\
&\le 8(1 + n^4|g^{jk}|^2_{C^0})|f' - \phi'|^2_{C^0}|\xi|^2 \ := \ \eta_2|\xi|^2.
\end{aligned}
$$

Next we have

$$
\begin{aligned}
&\frac{\{\overline{p(y,\xi + i\tau\phi'(y))}, p(y,\xi + i\tau\phi'(y))\}}{2i\tau} = \{p,\{p,\phi\}\}(y,\xi) + \tau^2\{p,\{p,\phi\}\}(y,\phi'(y)) \\
&\quad \le \{p,\{p,f\}\}(y,\xi) + \tau^2\{p,\{p,f\}\}(y,f'(y)) + |\{p,\{p,\phi - f\}\}(y,\xi)| \\
&\quad + \tau^2|\{p,\{p,\phi - f\}\}(y,\phi'(y))| + \tau^2|\{p,\{p,f\}\}(y,\phi'(y)) - \{p,\{p,f\}\}(y,f'(y))| \\
&\quad \le \{p,\{p,f\}\}(y,\xi) + \tau^2\{p,\{p,f\}\}(y,f'(y)) + \eta_3|\xi|^2 + \eta_4\tau^2 + \eta_5\tau^2,
\end{aligned}
$$

where $\eta_3,\ \eta_4,\ \eta_5$, are defined by

$$
\begin{aligned}
\{p,\{p,\phi - f\}\}(y,\xi) &= 4(\phi_{tt}'' - f_{tt}'')\xi_0^2 + \sum_{l,m=1}^{n} \xi_l\xi_m\Big(4\sum_{j,k=1}^{n} g^{jl}(\phi_{jk}'' - f_{jk}'')g^{km} \\
&\qquad + 4\sum_{j,k=1}^{n} g^{jl}\partial_{x_j}g^{km}(\phi_k' - f_k') - 2\sum_{j,k=1}^{n}\partial_{x_j}g^{lm}g^{kj}(\phi_k' - f_k')\Big) \\
&\le 4|\phi'' - f''|_{C^0}\xi_0^2 \\
&\quad + \Big(4|\phi'' - f''|_{C^0}|g^{jl}|^2_{C^0}n^4 + 6|g^{jl}\partial_{x_j}g^{km}|_{C^0}n^4|\phi' - f'|_{C^0}\Big)|\widetilde{\xi}|^2 \\
&\le 10(1 + n^4|g^{jl}|^2_{C^1})\big(|\phi'' - f''|_{C^0} + |\phi' - f'|_{C^0}\big)|\xi|^2 := \eta_3|\xi|^2.
\end{aligned}
$$

Analogously, setting $\xi = \phi'(y)$, we have

$$
\{p,\{p,\phi - f\}\}(y,\phi') \le \eta_3|\phi'|^2_{C^0} := \eta_4.
$$

Then, replacing $\phi - f$ with f and ξ with ϕ' or f' in the computations for η_3, we obtain

$$
\begin{aligned}
&|\{p,\{p,f\}\}(y,\phi'(y)) - \{p,\{p,f\}\}(y,f'(y))| \le 4|f''|_{C^0}(|\phi'|_{C^0} + |f'|_{C^0})|\phi' - f'|_{C^0} \\
&\qquad + \Big(4|f''|_{C^0}|g^{jl}|^2_{C^0}n^4 + 6|g^{jl}\partial_{x_j}g^{km}|_{C^0}n^4|f'|_{C^0}\Big)(|\phi'|_{C^0} + |f'|_{C^0})|\phi' - f'|_{C^0} \\
&\quad \le 10|f'|_{C^1}(1 + n^4|g^{jl}|^2_{C^1})(|\phi'|_{C^0} + |f'|_{C^0})|\phi' - f'|_{C^0} := \eta_5.
\end{aligned}
$$

In summary,

$$\begin{aligned}
M_2\xi_0^2 + \frac{M_1}{\min\{1,\lambda^2\phi^2\}}&\left(\frac{2|p(y,\xi+i\tau f'(y))|^2 + 2\eta_1\tau^4 + 2\eta_2\tau^2|\xi|^2}{\tau^2+|\xi|^2}\right)\\
&+\frac{\{\overline{p(y,\xi+i\tau f'(y))}, p(y,\xi+i\tau f'(y))\}}{(\lambda\phi)2i\tau} + \frac{1}{\lambda\phi}(\eta_3|\xi|^2 + (\eta_4+\eta_5)\tau^2)\\
&\geq M_P\min\{1,\lambda^2\phi^2\}(\tau^2+|\xi|^2)\,.
\end{aligned}$$

Without loss of generality, we can take $M_P = 1$, while on the ball $B_{R_2}(y_0) \subset B(y_0, R_1)$ we also have $\min\{1,\lambda^2\phi_0^2\} = 1$. Then

$$\begin{aligned}
M_2\xi_0^2 + 2M_1\frac{|p(y,\xi+i\tau f'(y))|^2}{\tau^2+|\xi|^2} &+ \frac{\{\overline{p(y,\xi+i\tau f'(y))}, p(y,\xi+i\tau f'(y))\}}{(\lambda\phi)2i\tau} +\\
&\geq (\tau^2+|\xi|^2) - \eta_2 M_1(1+|\phi'|^2_{C^0}+|f'|^2_{C^0})(\tau^2+|\xi|^2)\\
&\quad - \frac{\eta_3}{\lambda\phi}|\xi|^2 - \frac{1}{\lambda\phi}(\eta_3|\phi'|^2_{C^0}+\eta_5)\tau^2\\
&\geq \Big(1-\eta_2 M_1(1+|\phi'|^2_{C^0}+|f'|^2_{C^0})\\
&\quad - \frac{\eta_3}{\lambda\phi}(1+|\phi'|^2_{C^0}+|f'|_{C^1}(|\phi'|_{C^0}+|f'|_{C^0}))\Big)(\tau^2+|\xi|^2)\\
&:= M_R\,(\tau^2+|\xi|^2),
\end{aligned}$$

where we have used the fact that $\eta_5 \leq \eta_3|f'|_{C^1}(|\phi'|_{C^0}+|f'|_{C^0})$. Furthermore, on σ, we must set the constraint $f < \phi$ for $y \neq y_0$. Define $v(s) = \phi(p(s))$ and $p(s) = y_0 + s(y-y_0)$. Then there exists $q \in (0,1)$ such that $v(1) = v(0) + v'(0) + \frac{1}{2}v''(q)$; hence

$$\begin{aligned}
\left|v(1)-v(0)-v'(0)-\frac{1}{2}v''(0)\right| &= \frac{1}{2}|v''(q)-v''(0)|\\
&= \left|\sum_{|\zeta|=2}\frac{1}{\zeta!}(\partial^\zeta\phi(p(q)) - \partial^\zeta\phi(y_0))\,(y-y_0)^\zeta\right|\\
&\leq c_T|y-y_0|^{\rho+2},
\end{aligned}$$

$$|\phi(y) - \sum_{|\zeta|\leq 2}\frac{1}{\zeta!}(\partial^\zeta\phi)(y_0)\,(y-y_0)^\zeta| \leq c_T|y-y_0|^{\rho+2} \quad \text{for } c_T = n\max_{|\zeta|=2}|\partial^\zeta\phi|_{C^{0,\rho}}.$$

On the set $|y-y_0| \leq R_2$, $y \neq y_0$, we now consider the inequality

$$f(y)-\phi(y) \leq -\sigma|y-y_0|^2 + c_T|y-y_0|^{\rho+2} \leq -(\sigma - c_T R_2^\rho)|y-y_0|^2 < 0.$$

This is satisfied by taking

$$\sigma := 2c_T R_2^\rho = 2n|\phi''|_{C^{0,\rho}(B_{R_2}(y_0))}R_2^\rho. \tag{3.7}$$

Since $R_2^{1+\rho} \le R_2$, with this choice and C_l as in Assumption A2, the constraint (3.6) becomes

$$(|\phi''|_0 + 4n|\phi''|_{0,\rho})R_2 \le \lambda C_l/2. \tag{3.8}$$

Hence, the main quantities can be estimated as follows[1]:

$$\begin{aligned}
|\phi'|_{C^0(B_{R_2})} &\le \lambda\phi_M|\psi'|_{C^0},\\
|\phi''|_{C^0(B_{R_2})} &\le \lambda\phi_M(|\psi''|_{C^0} + \lambda|\psi'|^2_{C^0}),\\
|\phi|_{C^{0,\rho}(B_{R_2})} &\le |\phi'|_{C^0}|\psi|_{C^{0,\rho}} \le \lambda\phi_M|\psi|_{C^{0,1}}R_2^{1-\rho},\\
|\phi''|_{C^{0,\rho}(B_{R_2})} &\le \lambda|\phi\psi''|_{0,\rho} + \lambda^2|\phi\psi'\circ\psi'|_{0,\rho}\\
&\le \lambda\phi_M|\psi''|_{0,\rho} + \lambda^2\phi_M|\psi|_{0,1}|\psi''|_0R_2^{1-\rho}\\
&\quad + 2\lambda^2\phi_M|\psi'|_0|\psi'|_{0,1}R_2^{1-\rho} + \lambda^3\phi_M|\psi|_{0,1}|\psi'|_0^2R_2^{1-\rho},\\
|\phi' - f'|_{C^0(B_{R_2})} &\le \sup_j |\sum_{k=0}^n(\partial^k\phi'_j(p(\widetilde{q})) - \partial^k\phi'_j(y_0))\,(y_k - y_{0,k})| + 2\sigma|y - y_0|\\
&\le n|\phi''|_{0,\rho}|y-y_0|^{1+\rho} + 2\sigma|y-y_0| \le 5n|\phi''|_{0,\rho}R_2^{1+\rho},\\
|\phi'' - f''|_{C^0(B_{R_2})} &\le |\phi''|_{C^{0,\rho}}|y-y_0|^\rho + 2\sigma \le (2n+1)|\phi''|_{C^{0,\rho}}R_2^\rho,\\
|f'|_{C^0(B_{R_2})} &\le |\phi'|_{C^0} + |\phi' - f'|_{C^0} \le |\phi'|_{C^0} + 5n|\phi''|_{0,\rho}R_2^{1+\rho}\\
|f''|_{C^0(B_{R_2})} &\le |\phi''|_{C^0} + 2\sigma = |\phi''|_{C^0} + 4n|\phi''|_{C^{0,\rho}}R_2^\rho.
\end{aligned}$$

We end up with the estimates above

$$\begin{aligned}
\eta_2 &\le c(|g^{jk}|_{C^0})\,|\phi' - f'|^2_{C^0},\\
\eta_3 &\le c(|g^{jk}|_{C^1})\,(|\phi' - f'|_{C^0} + |\phi'' - f''|_{C^0}),\\
\eta_5 &\le \eta_3|f'|_{C^1}(|\phi'|_{C^0} + |f'|_{C^0}).
\end{aligned}$$

Let $c_{100}(g) = 10(1 + n^4|g^{jk}|^2_{C^1(B(y_0,R_1))})$, which is the largest constant entering in the estimates for η_j. Then, for $R_2 < 1$,

$$\begin{aligned}
M_R :=\ & 1 - c_{100}(g)\Big[|\phi' - f'|^2_{C^0}M_1(1 + |\phi'|^2_{C^0} + |f'|^2_{C^0})\\
&+ (|\phi' - f'|_{C^0} + |\phi'' - f''|_{C^0})\frac{1}{\lambda\phi_0}\big(1 + |\phi'|^2_{C^0} + |f'|_{C^1}(|\phi'|_{C^0} + |f'|_{C^0})\big)\Big]\\
&\ge 1 - c_{100}(g)\Big[((5n)^2R_2^{2(1+\rho)}|\phi''|^2_{C^{0,\rho}})M_1(1 + 5|\phi'|^2_{C^0})\\
&+ (10nR_2^\rho|\phi''|_{C^{0,\rho}})\frac{1}{\lambda\phi_0}\big(1 + |\phi'|^2_{C^0} + (2|\phi'|_{C^0} + |\phi''|_{C^0} + 4n|\phi''|^2_{0,\rho}R_2^\rho)(3|\phi'|_{C^0})\big).\Big]
\end{aligned}$$

[1] Unless specified otherwise, the C^0, C^1, C^2, and $C^{0,\rho}$ norms of ψ and g^{jk} refer to operators on the given domain $B(y_0, R_1)$, while the norms for ϕ and f refer to operators on the smaller ball $|y - y_0| \le R_2$, with radius $R_2 \le R_1$ to be determined.

In the last step, we have used the constraint on B_{R_2} $|f'|_{C^0} \leq 2|\phi'|_{C^0}$, which is a consequence of (3.8).

Defining the term $|\lambda\psi|_{max}$ by

$$|\phi''|_{C^{0,\rho}} \leq \phi_M \max(\lambda|\psi''|_{0,\rho},\ \lambda^2|\psi|_{0,1}|\psi''|_0,\ \lambda^3|\psi|_{0,1}|\psi'|_0^2) := |\lambda\psi|_{max},$$

we can refine condition (3.8) and add an extra conditions on R_2^ρ (which is qualitatively equivalent to $|f''|_{C^0} \leq 2|\phi''|_{C^0}$):

$$\begin{aligned}(3.9)\quad & (\lambda\phi_M(|\psi''|_{C^0} + \lambda|\psi'|_{C^0}^2) + 4n|\lambda\psi|_{max})R_2 \leq \lambda C_l/2,\\ & 4n|\phi''|_{0,\rho}R_2^\rho \leq 4n|\lambda\psi|_{max}R_2^\rho \leq \lambda\phi_M(|\psi''|_{C^0} + \lambda|\psi'|_{C^0}^2),\end{aligned}$$

where we have applied the previous estimates to the norms of ϕ' and ϕ''. By including the numeric constants into c_{100}, we can then write

$$\begin{aligned} M_R \geq 1 - c_{100}(g)\Big[&|\lambda\psi|_{max}^2 R_2^{2(1+\rho)} M_1(1+\lambda^2\phi_M^2|\psi'|_0^2)\\ &+ |\lambda\psi|_{max}R_2^\rho \frac{1}{\lambda\phi_0}\left(1+\lambda^2\phi_M^2|\psi'|_0^2 + \lambda^2\phi_M^2(|\psi'|_0|\psi''|_0 + \lambda|\psi'|_0^3)\right)\Big].\end{aligned}$$

We first require R_2 to satisfy

$$\begin{aligned} c_{100}(g)|\lambda\psi|_{max}^2 R_2^{2(1+\rho)} M_1(1+\lambda^2\phi_M^2|\psi'|_0^2) &\leq \frac{1}{4},\\ c_{100}(g)|\lambda\psi|_{max}R_2^\rho \frac{1}{\lambda\phi_0}\left(1+\lambda^2\phi_M^2|\psi'|_0^2 + \lambda^2\phi_M^2(|\psi'|_0|\psi''|_0 + \lambda|\psi'|_0^3)\right) &\leq \frac{1}{4}.\end{aligned}$$

Then we add the previous two constraints (3.9). The resulting upper bound for R_2 is in Table 1. □

We collect in the following table all the constants computed in Step 1, 2, 3 and in the following sections [2]. In case of special geometries for which ψ is given explicitly, the constraints in the table can be improved.

[2]Unless specified otherwise, the C^0, C^1,C^2, $C^{0,\rho}$ norms of ψ and g^{jk} refer to operators on the domain $B(y_0, R_1)$, while the other norms for ϕ and f refer operators on the smaller ball $B_{R_2} : |y-y_0| \leq R_2$.

Name		Limit Value
C_3	$\geq$	$20(1+n^2\|g^{jk}\|^2_{C^1(\Omega_0)})\|\psi'\|_{C^1(\Omega_0)}(1+\|\psi'\|^2_{C^0(\Omega_0)})$
M_1	$\geq$	$(M_P+C_3)\max_{y\in\Omega_0}\left\{\frac{2}{a_1^2},\ \frac{1}{2\|p(y,\psi')\|^2}\right\}$
M_2	$\geq$	$\frac{2}{\min\{1,a_1\}}(M_P+C_3)+\frac{(b_1+a_1)}{2}M_1$
M_P	$\leq$	1
λ	$\geq$	$\max\{M_1, e, \frac{2\|\psi''\|_{C^0(\Omega_0)}}{C_l^2}\}$
ϕ_0	$\geq$	e^{-1}
ϕ_M	$\leq$	e
R_1	$\leq$	$\min\{1, \min_{y\in\partial\Omega_0}\|y_0-y\|, \frac{1}{\lambda\|\psi'\|_{C^0(\Omega_0)}}\}$
R_2	$\leq$	$\min\left\{R_1,\ \left(\frac{C_l}{2\phi_M(\|\psi''\|_{C^0(B(R_1))}+\lambda\|\psi'\|^2_{C^0(B(R_1))})}\right),\ \left(\frac{\lambda\phi_M(\|\psi''\|_{C^0(B(R_1))}+\lambda\|\psi'\|^2_{C^0(B(R_1))})}{4n\|\lambda\psi\|_{max}}\right)^{\frac{1}{\rho}},\right.$ $\left(\frac{1}{4c_{100}(g)\|\lambda\psi\|^2_{max}M_1(1+\lambda^2\phi_M^2\|\psi'\|^2_{C^0(B(R_1))}}\right)^{\frac{1}{2+2\rho}},$ $\left.\left(\frac{\lambda\phi_0}{4c_{100}(g)\|\lambda\psi\|_{max}\left(1+\lambda^2\phi_M^2\|\psi'\|^2_{C^0(B(R_1))}+\lambda^2\phi_M^2(\|\psi'\|_{C^0(B(R_1))}\|\psi''\|_{C^0(B(R_1))}+\lambda\|\psi'\|^3_{C^0(B(R_1))}\right)}\right)^{\frac{1}{\rho}}\right\}$
σ	$\geq$	$2n\|\phi''\|_{C^{0,\rho}(B_{R_2})}R_2^{\rho}$
ϵ_0	$\leq$	$\frac{1}{2n\|f''\|_{C^0(B_{R_2})}}$
τ_0	$\geq$	$\max\left\{1,\ 64\left(4M_1+\frac{1}{4\lambda\phi_0}\right)\right.$ $\left.\cdot\left(\|f''\|^2_{C^0}(1+n^2\|g^{jk}\|_{C^0})^2+n\|h\|^2_{L^\infty}(2+2\|f'\|^2_{C^0})+2\|q\|^2_{L^\infty}\right)\right\}$
R	$\leq$	$\frac{1}{4}\left(16+\frac{1}{16}\right)^{-1/2}R_2$
δ	$\leq$	$n\frac{1}{32}\left(16+\frac{1}{16}\right)^{-1}\|\phi''\|_{C^{0,\rho}(B_{R_2})}R_2^{2+\rho}$
r	$\leq$	$\frac{n\|\phi''\|_{C^{0,\rho}(B_{R_2})}\frac{1}{4}\left(16+\frac{1}{16}\right)^{-1}R_2^{2+\rho}}{\|\phi'\|_{C^0(B_{R_2})}+5n\|\phi''\|_{C^{0,\rho}(B_{R_2})}R_2^{1+\rho}}$
r_0	$\leq$	$\frac{n\lambda^2C_l^2\frac{1}{4}\left(16+\frac{1}{16}\right)^{-1}R_2^{2+\rho}}{2e\left(\|\phi'\|_{C^0(B_{R_2})}+5n\|\phi''\|_{C^{0,\rho}(B_{R_2})}R_2^{1+\rho}\right)}$
$c_{1,T}$	$\geq$	$\sqrt{4\left(\frac{4M_1}{\tau_0}+\frac{1}{4(\lambda\phi_0)}\right)}$
$c_{2,T}$	$\geq$	$(\frac{1}{2}+\sqrt{2M_2})(1+\frac{2\|\chi_1'\|_{C^0}}{\tau_0\kappa})+\frac{c_{1,T}}{\sqrt{\tau_0}}c_{133}$
c_{111}	$\geq$	$\ln(1+e)+6c_{131}\left(1+\frac{\|b'\|^2_{C^0}}{r^2}\right)^{1/2}+2c_{129}$

Table 1. Table for the constants computed under Assumptions A3 and A4 with the notation of Step 1, 2, 3 at the beginning of the section.

The coefficient c_{133} is defined and derived in Subsection 3.2; c_{129} and c_{131} are defined and derived in the proof of Theorem 1.1.

3.2 Tataru inequality for the wave operator. We now go quickly through [7] to compute the coefficients of the inequality in Theorem 2.1. We decompose the wave operator (1.1) into the sum of its principal part P_2 and the lower order part P_1:

$$\begin{aligned} P_2(y, D) &= -D_0^2 + g^{jk}(x) D_j D_k, \\ P_1(y, D) &= h^j(x) D_j + q(x), \end{aligned}$$

and then split the conjugate operator

$$P(y, D + i\tau f'(y)) = e^{\tau f(y)} P(y, D) e^{-\tau f(y)}$$

into the sum of its principal part P_3 and the lower order part P_4:

$$\begin{aligned} P_3(y, D, \tau) &= P_2(y, D) + \tau^2((f_0')^2 - g^{jk} f_j' f_k') + 2i\tau(-f_0' D_0 + g^{jk} f_j' D_k), \\ P_4(y, D, \tau) &= -\tau(f_0'' - g^{jk} f_{jk}'') + P_1(y, D + i\tau f'). \end{aligned}$$

The principal symbol of $P(y, D)$ and $P(y, D + i\tau f')$ are, respectively,

$$\begin{aligned} p(y, \xi) &= -\xi_0^2 + g^{jk}(x)\xi_j\xi_k, \\ p(y, \xi + i\tau f') &= p(y, \xi) - \tau^2 p(y, f') + i\tau\{p, f\}. \end{aligned}$$

Since f is a quadratic function and the coefficients g^{jk} are time independent, we can write

$$\begin{aligned} e^{-\epsilon|D_0|^2/(2\tau)} e^{\tau f} P(y, D) u &= e^{-\epsilon|D_0|^2/(2\tau)} P(y, D + i\tau f') e^{\tau f} u \\ &= (y, D - \epsilon f'' \cdot (D_0, 0) + i\tau f') e^{-\epsilon|D_0|^2/(2\tau)} e^{\tau f} u. \end{aligned}$$

Let $\vec{D} = D - \epsilon f'' \cdot (D_0, 0)$ and $\vec{\xi}_j = \xi_j - \epsilon f_{j0}'' \xi_0$, $j = 0, 1, \dots, n$. For ϵ such that $2n\epsilon|f''|_{C^0} \le 1$, we have $|\vec{\xi}_j|^2 \le 2|\xi_j|^2 + 2\epsilon^2|f''|_{C^0}^2 \xi_0^2$ and $|\xi|^2/2 \le |\vec{\xi}|^2 \le 2|\xi|^2$. Since $p(y, \xi + i\tau f')$ is the symbol of $P_3(y, D, \tau)$, $p(y, \widehat{\xi} + i\tau f')$ is the symbol of $P_3(y, \vec{D}, \tau)$. Now substitute $\vec{\xi}$ for ξ into the (3.4), which then becomes, for $V \in C_0^\infty(B(y_0, R_2))$,

$$2M_2 \| |D_0| V \|^2 + 4M_1 \| P_3(y, \vec{D}, \tau) V \|_{-1,\tau}^2 + \frac{\mathrm{Im}\langle \mathrm{Re}(P_3(y, \vec{D}, \tau))V, \mathrm{Im}(P_3(y, \vec{D}, \tau))V \rangle}{(\lambda\phi_0) 2\tau} \ge \frac{1}{4} \|V\|_{1,\tau}^2.$$

Since $\|W\|_0^2 \ge \tau^2 \|W\|_{-1,\tau}^2$, we see that $\|P_3 W\|_0^2 \ge 2\mathrm{Im}\langle (\mathrm{Re}P_3)W, (\mathrm{Im}P_3)W \rangle$; and for $\tau \ge 1$, we get

$$2M_2 \| |D_0| V \|^2 + \left(\frac{4M_1}{\tau} + \frac{1}{4(\lambda\phi_0)} \right) \frac{\|P_3(y, \vec{D}, \tau) V\|_0^2}{\tau} \ge \frac{1}{4} \|V\|_{1,\tau}^2. \tag{3.10}$$

We now estimate the error term E_1:

$$\begin{aligned}
E_1 :&= \|P(y,\vec{D}+i\tau f')V - P_3(y,\vec{D},\tau)V\|_0^2 \\
&= \|-\tau(f_0'' - g^{jk}f_{jk}'')V + P_1(y,\vec{D}+i\tau f')V\|_0^2 \\
&\le 2\tau^2\||f''|_{C^0}(1+n^2|g^{jk}|_{C^0})V\|_0^2 + 2\|h^jD_jV - \epsilon h^j f_{0j}''D_0V\|_0^2 \\
&\quad + 2\tau^2\|(n|h|_{L^\infty}|f'|_{C^0} + |q|_{L^\infty})V\|_0^2 \\
&\le 4\Big(|f''|_{C^0}^2(1+n^2|g^{jk}|_{C^0})^2 + n|h|_{L^\infty}^2(2+2|f'|_{C^0}^2) + 2|q|_{L^\infty}^2\Big)\|V\|_{1,\tau}^2.
\end{aligned}$$

Now choose $\tau_0 > 1$ such that

$$\frac{2}{\tau_0}\Big(4M_1 + \frac{1}{4\lambda\phi_0}\Big)E_1 \le \frac{1}{8}\|V\|_{1,\tau}^2$$

and let

$$c_{1,T} := \sqrt{4\Big(\frac{4M_1}{\tau_0} + \frac{1}{4(\lambda\phi_0)}\Big)}.$$

From (3.10) and $\|P_3(\vec{D})v\|^2 \le 2E_1 + 2\|P(\vec{D})v\|^2$, we have after multiplying by 2 and squaring,

$$\sqrt{2M_2}\||D_0|V\|_0 + c_{1,T}\frac{\|P(y,\vec{D}+i\tau f')V\|_0}{\sqrt{\tau}} \ge \frac{1}{2}\|V\|_{1,\tau} \tag{3.11}$$

for $\tau \ge \tau_0$, Now consider $u \in H^1(B_{\kappa/4})$ and define $v := e^{-\epsilon|D_0|^2/(2\tau)}e^{\tau f}u$, $V := \chi_1(t/(2\kappa))v$, with χ_1 as in (4.4) with $N = 1$, $B_1 = [-1,1]$ $B_2 = [-2,2]$. Then

$$\operatorname{supp}(V) \subset \{y : |t| \le 4\kappa, |x| \le \kappa/4\} \subset \{y : |y-y_0| \le R_2\},$$

with $\kappa = \big(16 + \frac{1}{16}\big)^{-1/2}R_2$. From [7, Lemma 3.4] (see also [11, Lemma 2.79]), we have

$$\begin{aligned}
&\Big\|P(y,\vec{D}+i\tau f')V - \chi_1\Big(\frac{t}{2\kappa}\Big)e^{-\epsilon|D_0|^2/(2\tau)}e^{\tau f}P(y,D)u\Big\|_0 \\
&\qquad = \Big\|\Big[P(y,\vec{D}+i\tau f'), \chi_1\Big(\frac{t}{2\kappa}\Big)\Big]v\Big\|_0 \\
&\qquad \le c_{133}\Big\|\Big(1-\chi_1\Big(\frac{t}{\kappa}\Big)\Big)(\nabla+\tau)v\Big\|_0 \le c_{133}e^{-\tau\kappa^2/(4\epsilon)}\|e^{\tau f}u\|_{1,\tau} \\
\||D_0|V\|_0 &\le \big\||D_0|v\big\|_0 + \frac{2|\chi_1'|_{C^0}}{\kappa}\Big\|\Big(1-\chi_1\Big(\frac{t}{\kappa}\Big)\Big)v\Big\| \\
&\le \frac{2\kappa\tau}{\epsilon}\|v\|_0 + \Big(1 + \frac{2|\chi_1'|_{C^0}}{\tau_0\kappa}\Big)e^{-\tau\kappa^2/(4\epsilon)}\|e^{\tau f}u\|_{1,\tau},
\end{aligned}$$

and

$$\|v\|_{1,\tau} \leq \|V\|_{1,\tau} + \left(1 + \frac{2|\chi_1'|_{C^0}}{\tau_0 \kappa}\right) e^{-\tau\kappa^2/(4\epsilon)} \|e^{\tau f} u\|_{1,\tau}$$

for $\tau \geq \tau_0$ and

$$c_{133} = 2(1 + n^2 |g^{jk}|_{C^0}) \Big(\frac{|\chi_1''|_{C^0}}{\tau_0 \kappa^2} + \frac{|\chi_1'|_{C^0}}{\kappa} (1 + |f'|_{C^0} + \frac{|h|_{L^\infty}}{\tau_0}) \Big).$$

As last step, we use the above relations to estimate the terms of (3.11) and notice that according to our choice of the parameters,

$$\sqrt{2M_2} \frac{2\kappa}{\epsilon_0} < \frac{1}{4}.$$

Therefore, we obtain, for $\tau > \tau_0$, the Tataru inequality of Theorem 2.1 with coefficients as in Table 1.

Remark 3.2. According to the computations above, ϵ cannot be smaller that ϵ_0, since this affects the size of R_2 and τ.

3.3 Proof of Proposition 2.5. In the previous subsection, we considered $u \in H^1(B_R)$, with the radius R is defined as $R := qR_2$ with

$$q = \frac{1}{4}\left(16 + \frac{1}{16}\right)^{-1/2},$$

and R_2 given in Table 1. Let us compute δ such that I_B is inside the ball, i.e.,

$$I_B := \{y \in B(y_0, R_2) : f(y) - \phi(y_0) \geq -8\delta\} \cap \{y \in B(y_0, R_2); \phi \leq \phi(y_0)\} \subset B(y_0, R).$$

By assumption, $f(y) - \phi(y) < -c_T R_2^\rho |y - y_0|^2$ in $B(y_0, R_2) - \{y_0\}$. Moreover, in I_B, $f(y) - \phi(y_0) \leq f(y) - \phi(y)$. Hence, the limit case is reached along the boundary $\{y : |y - y_0| = R\}$, where $f(y) - \phi(y_0) < -c_T q^2 R_2^{2+\rho}$. Define δ such that $-c_T q^2 R_2^{2+\rho} \leq -8\delta$, i.e.,

$$\delta = c_T q^2 R_2^{2+\rho}/8 = n|\phi''|_{C^{0,\rho}} q^2 R_2^{2+\rho}/8 \,.$$

Under this condition, $I_B \subset B(y_0, R)$.

In order to compute the smaller radius r, we apply a rougher estimate, using the definition of f. In the region $\{y : |f - \phi(y_0)| \leq \delta\} \cap \{y : |y - y_0| \leq 2r\}$,

$$|f(y) - \phi(y_0)| \leq |f'|_{C^0(B_{R_2})} |y - y_0| \leq |f'|_{C^0} 2r \leq \delta.$$

Hence the radius r must satisfy $r \leq \delta/2|f'|_{C^0(B_{R_2})}$, which is guaranteed by

$$r \leq \frac{n|\phi''|_{C^{0,\rho}(B_{R_2})} q^2 R_2^{2+\rho}}{2|\phi'|_{C^0(B_{R_2})} + 10n|\phi''|_{C^{0,\rho}(B_{R_2})} R_2^{1+\rho}} \quad (\leq R_2/10).$$

By hypothesis, $\phi'(y_0) \neq 0$, hence the denominator does not vanish.

Choosing $\lambda > 2|\psi''|_{C^0(\Omega_0)}/C_l^2$ and applying $\psi'(y) > C_l$, we obtain

$$\phi''(y) = \phi\lambda(\psi'' + \lambda\psi' \times \psi') \geq e^{-1}\lambda^2 C_l^2/2$$

in B_{R_1}. Consequently, $\phi''(y) \neq 0$ and $|\phi''|_{C^{0,\rho}(B_{R_2})} > C_\rho$, with $C_\rho := e^{-1}\lambda^2 C_l^2/2 > 0$. We get the uniform lower bound for r in B_{R_1}

$$r_0 \leq \frac{nC_\rho q^2 R_2^{2+\rho}}{2|\phi'|_{C^0(B_{R_2})} + 10n|\phi''|_{C^{0,\rho}(B_{R_2})} R_2^{1+\rho}}.$$

4 Appendix

We recall results on Gevrey class functions that are used in the article. The references are [8, 19].

Definition 4.1. Let L_s be an increasing sequence of positive numbers such that

$$L_0 = 1, \quad s \leq L_s, \quad L_{s+1} \leq CL_s,$$

for some constant $C > 1$. We denote by C^L the set of all $u \in C^\infty(X)$ (with $X \subset \mathbb{R}^N$ open subset) for which for every compact set $K \subset X$ there exists a constant C_K such that

$$|D^\zeta u(x)| \leq C_K(C_K L_{|\zeta|})^{|\zeta|}, \quad x \in K,$$

for all multi-indices ζ. By Stirling's formula, we can replace $|\zeta|^{|\zeta|}$ by $|\zeta|!$. $C^L(X)$ is a ring which is closed under differentiation. If $f : Y \to X$ is an analytic map from the open set $Y \subset \mathbb{R}^N$ to the open set $X \subset \mathbb{R}^N$, the composition with f defines the map $f^* : C^L(Y) \to C^L(X)$, $f^*u = u \circ f$. The class $C^L(X)$ with $L_s = (s+1)^m$ and $m > 1$ is called the **Gevrey class of order** m and is denoted by $G^m(X)$. If $m = 1$, $G^1(X)$ is the set of real analytic functions in X.

We denote the set $G^m(\mathbb{R}^N) \cap C_0^\infty(\mathbb{R}^N)$ by $G_0^m(\mathbb{R}^N)$. For $m > 1$, one has $\sum 1/k^m < \infty$, and it follows from [8, Theorem1.4.2] that G_0^m is large enough to contain cut-off functions; of course, it is an algebra. In particular, let $f, g \in G^m(\mathbb{R}^N)$, and let $K \subset R^N$ be a compact set. Then, by calling the constants

C_K for f and g $c_{1,f}$ and $c_{1,g}$, respectively, we get $fg \in G^m(\mathbb{R}^N)$ such that for $c_P = \max\{c_{1,f}, c_{1,g}\}$,

$$|D^\kappa(f(x)g(x))| \leq 2^{|\kappa|} c_P^{|\kappa|+2} |\kappa|^{m|\kappa|}, \quad x \in K. \tag{4.1}$$

Definition 4.2. For a compact subset E of $\mathbb{R}^N$, we define the **supporting function of** E H_E as ([8, (4.3.1), p. 105])

$$H_E(\xi) = \sup_{x \in E} \langle x, \xi \rangle, \quad \xi \in \mathbb{R}^N.$$

In the present paper, we use the Paley-Wiener-Schwartz Theorem for Gevrey class functions often. For reference, we give the following statement, proved in [9] for a proper subset $\gamma_0^m(\mathbb{R}^N)$ of $G_0^m(\mathbb{R}^N)$. The theorem can also be reformulated for $\phi \in G_0^m$ with substitution of the phrase "to every $B > 0$, there exists a constant C_B such that" with "there exist positive constants B and C such that". The proof is the same.

Theorem 4.3 ([9, Theorem 12.7.4, p. 137])**.** *An entire function $\Phi(\zeta)$, $\zeta \in \mathbb{C}^N$, is the Fourier-Laplace transform of a function $\phi \in \gamma_0^m(\mathbb{R}^N)$ with support in the compact convex set K with supporting function H_K if and only if for every $B > 0$ there exists a constant C_B such that*

$$|\Phi(\zeta)| \leq C_B \exp(H_K(Im\zeta) - B|Re\zeta|^{1/m}), \quad \zeta \in \mathbb{C}^N.$$

In particular, we can introduce the main properties of the Gevrey class localizers used in the paper.

Definition 4.4. Define $\chi_1 \in G_0^m(\mathbb{R}^N)$ and $\chi_\delta(v) := \chi_1(v/\delta)$ such that $\chi_1 = 1$ in a ball $B_1 \subset \mathbb{R}^N$, $\chi_1 = 0$ outside a larger ball B_2, and $0 \leq \chi_1 \leq 1$. Hence $\mathcal{F}_{v\to\zeta}\chi_\delta(v) = \delta^N \mathcal{F}_{v\to\delta\zeta}\chi_1(v)$ and

$$\begin{aligned} &|D^\kappa \chi_1(v)| \leq c_{1X}^{|\kappa|+1} |\kappa|^{m|\kappa|}, \quad v \in B_2, \\ &|\mathcal{F}_{v\to\zeta}\chi_1(v)| \leq c_{1X} \exp(H_{B_2}(\mathrm{Im}\,\zeta) - c_{2X}|\mathrm{Re}\,\zeta|^{1/m}), \quad \zeta \in \mathbb{C}, \\ &|\mathcal{F}_{v\to\zeta}\chi_\delta(v)| \leq \delta^N c_{1X} \exp(\delta H_{B_2}(\mathrm{Im}\,\zeta) - c_{2X}\delta^{1/m}|\mathrm{Re}\,\zeta|^{1/m}), \quad \zeta \in \mathbb{C}, \end{aligned} \tag{4.2}$$

with $c_{1X} = c_{1X}(m)$ a given number and $c_{2X} = 1/(eNc_{1X})^{1/m}$.

We now present the Phragmen-Lindelöf Theorem for subharmonic functions in a sector $D \subset \mathbb{C}$ used in Lemma 2.7.

Theorem 4.5 ([16, Chapter 7.3.])**.** *Let D be an angle of opening π/λ, and let $u(z)$ be a function subharmonic in this angle, satisfying an asymptotic estimate*

$$u(z) < |z|^\rho, \quad a.e., \quad \rho < \lambda,$$

and bounded by a constant M on the boundary of the angle. Then $u(z) \le M$ inside the full angle D.

We now recall the concept of conormal pseudo-convexity for operators as given in [20, 21]. We can represent a C^2-oriented hypersurface S as level set surface $S := \{y : \psi(y) = 0\}$ of a C^2-function ψ such that $\psi' \neq 0$ on S.

Definition 4.6. Decompose the coordinates of $\mathbb{R}^N$ into $y = (y', y'')$. The **conormal bundle** of the foliation F of $\mathbb{R}^N$ with the surfaces y'' =const is the set

$$N^*F := \{(y, \xi) \in T^*\mathbb{R}^N : \xi = (\xi', \xi'') \text{ and } \xi' = 0\}.$$

The **reduction** of N^*F **to a subset** $K \subset \mathbb{R}^N$ is defined by

$$\Gamma_K := \{(y, \xi) \in T^*K : \xi' = 0\};$$

the **fibre** of N^*F **in** y_0 is defined by

$$\Gamma_{y_0} := \{(y_0, \xi) \in N^*F\}.$$

Let $P(y, D)$ be a partial differential operator of order m with smooth coefficients. Denote its principal symbol by $p(y, \xi)$.

Definition 4.7. Let S be a smooth oriented hypersurface which is a level surface of a C^2-function ψ, and let $y_0 \in S$ be such that $\psi'(y_0) \neq 0$. We say that S is **conormally strongly pseudo-convex** with respect P at y_0 if

$$\operatorname{Re}\{\overline{p}, \{p, \psi\}\}(y_0, \xi) > 0 \tag{4.3}$$

on $p(y_0, \xi) = \{p, \psi\}(y_0, \xi) = 0$, $0 \neq \xi \in \Gamma_{y_0}$;

$$\frac{\{\overline{p(y, \xi + i\tau\psi'(y))}, p(y, \xi + i\tau\psi'(y))\}}{2i\tau} > 0 \tag{4.4}$$

on $y = y_0$ such that $0 \neq \xi \in \Gamma_{y_0}$, $\tau > 0$, and

$$p(y_0, \xi + i\tau\psi'(y_0)) = \{p(y, \xi + i\tau\psi'(y)), \psi(y)\}(y = y_0) = 0.$$

Definition 4.8. A C^2-real-valued function ψ is **conormally strongly pseudo-convex** with respect to P at y_0 if

$$\operatorname{Re}\{\overline{p}, \{p, \psi\}\}(y_0, \xi) > 0 \tag{4.5}$$

on $p(y_0, \xi) = 0$, $0 \neq \xi \in \Gamma_{y_0}$;

$$\frac{\{\overline{p(y, \xi + i\tau\psi'(y))}, p(y, \xi + i\tau\psi'(y))\}}{2i\tau} > 0 \tag{4.6}$$

on $y = y_0$ such that $p(y_0, \xi + i\tau\psi'(y_0)) = 0$, $0 \neq \xi \in \Gamma_{y_0}$, $\tau > 0$.

Hence, the term **conormally strongly pseudo-convex** means strongly pseudo-convex in N^*F or in a subset Γ_K. Definition 4.7 implies that for a sufficiently small neighborhood Ω_0 of y_0, there exist constants such that an inequality like (3.1) holds, while Definition 4.8 implies that the inequality (3.2) holds for the function $\phi = e^{\lambda\psi}$.

For second order differential operators, the definitions above are simpler. In particular, for the wave operator (1.1), the conditions are void for noncharacteristic surfaces ψ, as shown in Section 3.1; see also Remark 3.1.

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