ENTIRE FUNCTIONS OF EXPONENTIAL TYPE REPRESENTED BY PSEUDO-RANDOM AND RANDOM TAYLOR SERIES

By

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To Alex Eremenko on the occasion of his birthday

Abstract. We study the influence of the multipliers $\xi(n)$ on the angular distribution of zeroes of the Taylor series

$$F_\xi(z) = \sum_{n\ge 0} \xi(n)\frac{z^n}{n!}.$$

We show that the distribution of zeroes of F_ξ is governed by certain autocorrelations of the sequence ξ. Using this guiding principle, we consider several examples of random and pseudo-random sequences ξ and, in particular, answer some questions posed by Chen and Littlewood in 1967.

As a by-product, we show that if ξ is a stationary random integer-valued sequence, then either it is periodic, or its spectral measure has no gaps in its support. The same conclusion is true if ξ is a complex-valued stationary ergodic sequence that takes values in a uniformly discrete set.

1 Introduction

In this work, we consider entire functions of exponential type represented by the Taylor series

$$F_\xi(z) = \sum_{n\ge 0} \xi(n)\frac{z^n}{n!}, \quad \xi\colon \mathbb{Z}_+ \to \mathbb{C}.$$

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We are interested in the influence of the multipliers $\xi(n)$ on the angular distribution of zeroes of the function F_ξ. This question belongs to a "terra incognita" in the theory of entire functions that contains no general results going in this direction but several interesting examples. These examples include

(a) random independent identically distributed $\xi(n)$ (Littlewood–Offord [16], Kabluchko–Zaporozhets [11]),
(b) $\xi(n) = e(qn^2)$ with quadratic irrationality q (Nassif [18], Littlewood [15]) and, more generally, arbitrary irrational q (Eremenko–Ostrovskii [7]),
(c) $\xi(n) = e(n(\log n)^\beta)$ with $\beta > 1$, and $e(n^\beta)$ with $1 < \beta < 3/2$ (Chen–Littlewood [5]),
(d) uniformly almost periodic $\xi(n)$ (Levin [14, Chapter VI, §7]).

Here and elsewhere, $e(t) = e^{2\pi t \mathrm{i}}$.

In this work, we consider the following four sequences ξ:

(i) $\xi(n) = e(Q(n))$, where $Q(x) = \sum_{k\ge 2} q_k x^k$ is a polynomial with real coefficients q_k, at least one of which is irrational;
(ii) $\xi(n) = e(n^\beta)$, where $\beta \ge \frac{3}{2}$ is non-integer;
(iii) $\xi(n)$ is a stationary sequence with a mild decay of the maximal correlation coefficient;
(iv) $\xi(n)$ is a stationary Gaussian sequence.

In cases (i), (ii), and (iii), using some potential theory, we reduce the question on the asymptotic distribution of zeroes of F_ξ to certain lower bounds for the exponential sums

$$W_R(\theta) = \sum_{|n|\le N} \xi(n+R) e(n\theta) e^{-\frac{n^2}{2R}}$$

when $R \gg 1$ and N has the size $R^{\frac{1}{2}+\varepsilon}$; see Lemmas 4.2.1 and 4.3.1. These lower bounds, in turn, depend on the behaviour of the autocorrelations

$$m \mapsto \frac{1}{N}\sum_{n=1}^{N} \xi(n+R)\,\overline{\xi(n+m+R)}e(m\theta).$$

In case (iv) (similarly to the almost-periodic case (d)), the zero set of F_ξ has an angular density that, generally speaking, is not constant, as in the cases (i), (ii) and (iii). This density is determined by the spectrum of the sequence ξ, that is, after all, also by the autocorrelations between the elements of ξ.

2 Main results

2.1. We start with the cases when the zeroes of F_ξ have the uniform angular distribution.

Definition 1. We say that the sequence $\xi\colon \mathbb{Z}_+ \to \mathbb{C}$ is an ***L*-sequence** if

$$\frac{\log|F_\xi(tz)|}{t} \underset{t\to\infty}{\longrightarrow} |z|\,, \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{C})\,. \tag{2.1.1}$$

Since $L^1_{\mathrm{loc}}(\mathbb{C})$ convergence implies convergence in the sense of distributions, and the Laplacian is continuous in the distributional topology, (2.1.1) yields

$$\frac{1}{t}\,\Delta\log|F_\xi(tz)| \underset{t\to\infty}{\longrightarrow} \Delta|z| = \mathrm{d}r\otimes\mathrm{d}\theta, \quad z = re^{\mathrm{i}\theta}, \tag{2.1.2}$$

in the sense of distributions, with $rdr\otimes\mathrm{d}\theta$ being planar Lebesgue measure. Denoting by $n_F(r;\theta_1,\theta_2)$ the number of zeroes (counted with multiplicities) of the entire function F in the sector $\{z\colon 0\le|z|\le r, \theta_1\le\arg(z)<\theta_2\}$ and recalling that $\frac{1}{2\pi}\,\Delta\log|F|$ is the sum of point masses at zeroes of F, we can rewrite (2.1.2) in a more traditional form: for every $\theta_1<\theta_2$,

$$n_{F_\xi}(r;\theta_1,\theta_2) = \frac{(\theta_2-\theta_1+o(1))\,r}{2\pi} \quad \text{as } r\to\infty\,. \tag{2.1.3}$$

Theorem 1. *Suppose that $Q(x)=\sum_{k=2}^d q_kx^k$ is a polynomial with real coefficients q_k and that at least one of the coefficients is irrational. Then $\xi(n)=e(Q(n))$ is an L-sequence.*

For $Q(x)=qx^2$, q being a quadratic irrationality, this is a result of Nassif [18] and Littlewood [15]. For arbitrary irrational q's, this was proven by Eremenko and Ostrovskii [7]. It seems that the methods used in these works cannot be extended to polynomials Q of degree higher than 2. According to Chen and Littlewood [5], "many lines of experience converge to show that there can be nothing doing if $\Lambda(n)\succ n^2$" (in their notation, $\xi(n)=e(\Lambda(n))$, and $\Lambda(n)\succ n^2$ means that $\Lambda(n)/n^2\to\infty$).

Theorem 2. *For any non-integer $\beta>1$, the sequence $\xi(n)=e(n^\beta)$ is an L-sequence.*

As we have already mentioned, the result in case $1<\beta<3/2$ is due to Chen and Littlewood [5]. They used the Poisson summation combined with the saddle point approximation and obtained much more accurate information about the asymptotic location of zeroes of the function F_ξ. They write, "The gap $3/2\le\beta<2$ presents a most interesting unsolved problem."

2.2. Now, we turn to the case when $\xi\colon \mathbb{Z} \to \mathbb{C}$ is a stationary sequence of random variables (formally, we need only the restriction of ξ on $\mathbb{Z}_+$, but due to stationarity, this restriction determines a unique extension of ξ onto $\mathbb{Z}$). As usual, stationarity means that, for every positive integer k, every choice of integers $n_1, \dots, n_k$, and every integer m, the k-tuples of random variables

$$\langle \xi(n_1), \dots, \xi(n_k) \rangle, \quad \langle \xi(n_1+m), \dots, \xi(n_k+m) \rangle$$

are equidistributed. In what follows, we deal only with stationary sequences having a finite second moment. Then the sequence $m \mapsto \mathbb{E}\{\xi(0)\overline{\xi(m)}\}$ is positive-definite and therefore is the Fourier transform of a non-negative measure $\rho \in M_+(\mathbb{T})$. Here and elsewhere, $\mathbb{T} = \{e^{\mathrm{i}\theta}\colon |\theta| \le \pi\}$ is the unit circle. We call ρ the **spectral measure** of ξ. Then the **spectrum** $\sigma(\xi)$ of ξ is the support of the measure ρ. Note that we do not require that $\mathbb{E}\xi(0) = 0$. The definition of the spectral measure we use here differs from the one which is more customary in the theory of stationary processes [10] by the atom at $\theta = 0$ with the mass $|\mathbb{E}\xi(0)|^2$.

We also need **the maximal correlation coefficient** of the sequence ξ

$$r(m) = r_\xi(m) \stackrel{\mathrm{def}}{=} \sup\left\{ \frac{\left|\mathbb{E}\{(x-\mathbb{E}x)\overline{(y-\mathbb{E}y)}\}\right|}{\sqrt{\mathbb{E}|x-\mathbb{E}x|^2\cdot\mathbb{E}|y-\mathbb{E}y|^2}} : x \in L^2_{(-\infty,0]},\, y \in L^2_{[m,+\infty)} \right\},$$

where $L^2_{(-\infty,0]}$ is the space of random variables measurable with respect to the σ-algebra generated by the set $\{\xi(n)\colon -\infty < n \le 0\}$ with finite second moment, and $L^2_{[m,+\infty)}$ is the space of random variables measurable with respect to the σ-algebra generated by the set $\{\xi(n)\colon m \le n < +\infty\}$ with finite second moment.

Theorem 3. *Let ξ be a bounded stationary sequence of random variables, and let the maximal correlation coefficient of ξ satisfy*

$$r(m) = O\big((\log m)^{-\kappa}\big), \quad m \to \infty, \tag{2.2.1}$$

with some $\kappa > 1$. Then, almost surely, ξ is an L-sequence.

2.3. Now we turn to the Gaussian stationary sequences ξ. In this case, the leading term of the asymptotics of $\log|F_\xi|$ is determined by the support of the spectral measure ρ of the sequence ξ; see Section 10.

We start with some preliminaries. For any set $\sigma \subset \mathbb{T}$, we denote by $\mathrm{ch}(\sigma)$ the closed convex hull of σ, and by

$$H_\sigma(z) \stackrel{\mathrm{def}}{=} \max_{\lambda\in\mathrm{ch}(\sigma)} \mathrm{Re}\left(z\overline{\lambda}\right) = \sup_{\lambda\in\sigma} \mathrm{Re}\left(z\overline{\lambda}\right)$$

the "Minkowski functional" of $\mathrm{ch}(\sigma)$. This function is subharmonic in $\mathbb{C}$ and homogeneous, that is, $H_\sigma(re^{\mathrm{i}\theta}) = h_\sigma(\theta)r$, where h_σ is the so-called **supporting function** of $\mathrm{ch}(\sigma)$. The distributional Laplacian of the function H_σ is $\Delta H_\sigma = \mathrm{d}r \otimes \mathrm{d}s_\sigma(\theta)$, where $\mathrm{d}s_\sigma(\theta) = (h_\sigma'' + h_\sigma)\,\mathrm{d}\theta$, the second derivative h_σ'' also being understood in the sense of distributions.

Definition 2. Let $\sigma \subset \mathbb{T}$. We say that the sequence ξ is an **$L(\sigma)$-sequence** if

$$\frac{\log|F_\xi(tz)|}{t} \underset{t\to\infty}{\longrightarrow} H_\sigma(z)\,, \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{C})\,. \tag{2.3.1}$$

Obviously, L-sequences are a special case of $L(\sigma)$-sequences that correspond to the case when the set σ is dense in $\mathbb{T}$.

In the language of entire function theory [14, Chapters II and III] (see also [1] for a modern treatment), this definition says that F_ξ is an entire function of completely regular growth in the Levin–Pfluger sense with the Phragmén–Lindelöf indicator h_σ. Condition (2.3.1) yields the angular asymptotics of zeroes of F_ξ:

$$n_{F_\xi}(r;\theta_1,\theta_2) = \frac{\big(s_\sigma(\theta_2) - s_\sigma(\theta_1) + o(1)\big)r}{2\pi}, \quad r \to \infty, \tag{2.3.2}$$

where $-\pi \le \theta_1 < \theta_2 \le \pi$ with an at most countable set of exceptional values of θ_1 and θ_2 that correspond to possible atoms of the measure s_σ; cf. (2.1.3). It also yields the Lindelöf-type symmetry condition, namely, the existence of the limit

$$\lim_{r\to\infty} \sum_{|z_n|\le r} \frac{1}{z_n}, \tag{2.3.3}$$

where the sum is taken over zeroes of F_ξ. In the reverse direction, for functions of exponential type, conditions (2.3.2) and (2.3.3) together yield (2.3.1).

We say that the stationary sequence ξ is **Gaussian** if $\big(\mathrm{Re}\,\xi(n), \mathrm{Im}\,\xi(n)\big)$ are random normal vectors in $\mathbb{R}^2$ with non-zero covariance matrix (so that this definition includes also real-valued Gaussian stationary sequences).

Theorem 4. *Suppose ξ is a Gaussian stationary sequence with the spectrum $\sigma = \sigma(\xi)$. Then, almost surely, ξ is an $L(\sigma^*)$-sequence, where σ^* is the reflection of σ in the real axis.*

Comparing this result with Theorem 3, we note that if ξ satisfies condition (2.2.1), then the spectral measure ρ has a density $|f|^2$, where f belongs to the Hardy space $H^2(\mathbb{T})$. This follows from a classical result that goes back to Kolmogorov; see [10, Chapter XVII, § 1]. Since no function in $H^2(\mathbb{T}) \setminus \{0\}$ vanishes on an arc, it follows that for every Gaussian stationary sequence ξ satisfying (2.2.1), we have $\sigma(\xi) = \mathbb{T}$; and, almost surely, ξ is an L-sequence.

2.4. Theorems 3 and 4 have a counterpart for uniformly almost-periodic sequences found by Levin [14, Chapter VI, § 7], which we recall here.

Let $\xi\colon \mathbb{Z} \to \mathbb{C}$ be a uniformly almost-periodic sequence, that is, a uniform limit of trigonometric polynomials on $\mathbb{Z}$. Then the limit

$$\widehat{\xi}(e^{\mathrm{i}\lambda}) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\le N} \xi(n) e^{-\mathrm{i}\lambda n}$$

exists for every $e^{\mathrm{i}\lambda} \in \mathbb{T}$, and does not vanish for a non-empty at most countable set of $e^{\mathrm{i}\lambda}$. This set is called the **spectrum** of ξ, and the values $\widehat{\xi}(e^{\mathrm{i}\lambda})$ are called the **Fourier coefficients** of ξ; cf. [13, Section VI.5].

Theorem 5 (B. Ya. Levin)**.** *Suppose ξ is a uniformly almost-periodic sequence with the spectrum σ. Then ξ is an $L(\sigma^*)$-sequence, with σ^* being the reflection of σ in the real axis.*

The proof of this theorem, given in [14], is based upon deep results on the zero distribution of entire functions approximated by finite linear combinations of exponentials. For the reader's convenience, we include a proof of this theorem which is based on the same ideas as Levin's original proof but can be read independently of the theory developed in [14, Chapter VI].

2.5. Here, we briefly explain how Theorems 1–5 are related to a wealth of results which deal with the analytic continuation of the Taylor series

$$f_\xi(s) \stackrel{\text{def}}{=} \sum_{n\ge 0} \xi(n) s^n$$

through the boundary of the disk of convergence. A survey of these results obtained prior to 1955 can be found in [2]. First, observe that the function

$$w^{-1} f_\xi(w^{-1}) = \sum_{n\ge 0} \frac{\xi(n)}{w^{n+1}}$$

is the Laplace transform of F_ξ. Then, by Pólya's theorem (see [14, Theorem 33, Chapter I] or [2, Theorem 1.1.5]), the upper limit

$$H^{F_\xi}(z) = \limsup_{t\to\infty} \frac{\log |F_\xi(tz)|}{t} \tag{2.5.1}$$

is the Minkowski functional of the closed convex hull of the set of singularities of the function $w^{-1} f_\xi(w^{-1})$, reflected in the real axis. Hence, the results about analytic continuation of f_ξ provide information about *the upper limit* in (2.5.1) but not about *the existence of the limit* in (2.3.1).

For instance, the property that the unit circle is a natural boundary for the Taylor series f_ξ is equivalent to the property that the upper limit $H^F(z) \equiv 1$, but it cannot guarantee that ξ is an L-sequence.

2.6. Here, we mention two curious results which follow from Lemma 7.2.1 and which might be of an independent interest.

2.6.1. The first result sheds some light on the nature of very strong cancellations in Taylor series.

Suppose $\xi : \mathbb{Z} \to \mathbb{C}$ is a stationary sequence with the spectral measure ρ. Then, almost surely,

$$\limsup_{r\to\infty} \frac{\log |F_\xi(re^{\mathrm{i}\theta})|}{r} \le \max_{t\in \mathrm{spt}(\rho)} \cos(\theta + t).$$

In particular, $F_\xi(re^{\mathrm{i}\theta})$, almost surely, decays exponentially on some angle $A = \{z : \theta_1 < \arg(z) < \theta_2\}$, provided that the origin does not belong to the convex hull $\mathrm{ch}(\sigma)$ of the support of ρ. Just choose $A \subset \mathrm{ch}(\sigma)^O$. Here, C^O is the polar cone of a plane set C, $C^O = \{z \in \mathbb{C} : \mathrm{Re}\, z\bar{w} \le 0,\ w \in C\}$.

Note that this result is helpful when there are no special restrictions on the support of the spectral measure.

2.6.2. The second result says that in some situations such restrictions do exist. We say that a set $A \subset \mathbb{C}$ is **uniformly discrete** if

$$\inf\big\{|z - w| : z, w \in A, z \ne w\big\} > 0.$$

Theorem 6. *Suppose that $\xi : \mathbb{Z} \to \mathbb{Z}$ is a stationary integer-valued sequence. Let ρ be the spectral measure of ξ. Then either* $\mathrm{spt}(\rho) = \mathbb{T}$, *or the sequence ξ is periodic and* $\mathrm{spt}(\rho) \subset \{w : w^N = 1\}$ *for some $N \ge 1$.*

The same conclusion holds if $\xi : \mathbb{Z} \to A$ is an ergodic stationary sequence and the set A is uniformly discrete.

3 Subharmonic preliminaries

3.1. In this section, we systematically use the following facts on the local convergence of subharmonic functions; see, e.g., [8, Theorem 4.1.9] and [9, Theorem 3.2.12].

Proposition A. *Let $(v_j)_j$ be a sequence of subharmonic functions on the plane having a uniform upper bound on any compact set. Then*

(a) *if* (v_j) *does not converge to* $-\infty$ *uniformly on every compact set, then there is a subsequence* (v_{j_k}) *converging in* $L^1_{\rm loc}(\mathbb{C})$*;*
(b) *if* $v_j \to U$ *in* $L^1_{\rm loc}(\mathbb{C})$*, then* U *is equal almost everywhere to a subharmonic function;*
(c) *if* v *is a subharmonic function and* $v_j \to v$ *in* $L^1_{\rm loc}(\mathbb{C})$*, then*
  (i) $\limsup_{j\to\infty} v_j(z) \le v(z)$, $\quad z \in \mathbb{C}$, *with the two sides equal and finite almost everywhere, and*
  (ii) $\limsup_{j\to\infty} \sup_K v_j \le \sup_K v$ *for every compact set* K *in the plane.*

Now we recall several basic facts from Azarin's theory of limit sets of subharmonic functions [1]. In what follows, we deal only with entire functions F of exponential type, i.e., $|F(z)| \le Ae^{\tau|z|}$, $z \in \mathbb{C}$. Consider the family of subharmonic functions

$$u_t(z) = \frac{1}{t}\log|F(tz)|, \quad t \ge 1.$$

By Proposition A, this family is pre-compact in $L^1_{\rm loc}(\mathbb{C})$. For every $L^1_{\rm loc}(\mathbb{C})$-limit U of subharmonic functions u_{t_k} there is a unique subharmonic function u such that $U = u$ almost everywhere. We remark that u might not be the pointwise limit of the u_{t_k} (for instance, this limit might fail to be upper semi-continuous). Now each sequence $t_j \to \infty$ has a subsequence t_{j_k} such that $u_{t_{j_k}}$ converges in $L^1_{\rm loc}(\mathbb{C})$ to a subharmonic function v. By $\mathcal{L}(F)$ we denote the set of all limiting subharmonic functions v. The set $\mathcal{L}(F)$ is called **the limit set** of $\log|F|$. This set is invariant with respect to the multiplicative action of $\mathbb{R}_+$, that is, if $v \in \mathcal{L}(F)$, then for each $t > 0$,

(3.1.1) $$\text{the function } v_t(z) = t^{-1}v(tz) \text{ also belongs to } \mathcal{L}(F).$$

Since F is an entire function of exponential type, every function $v \in \mathcal{L}(F)$ satisfies

$$v(z) \le \tau|z|, \quad z \in \mathbb{C}.$$

The **homogeneous indicator** H^F of F is the upper envelope of functions in $\mathcal{L}(F)$:

$$H^F(z) \stackrel{\rm def}{=} \sup_{v\in\mathcal{L}(F)} v(z), \quad z \in \mathbb{C}.$$

Then $H^F(re^{\mathrm{i}\theta}) = h^F(\theta)r$, where

$$h^F(\theta) = \sup_{v\in\mathcal{L}(F)} v(e^{\mathrm{i}\theta}), \quad -\pi \le \theta \le \pi$$

is the **Phragmén–Lindelöf indicator** of F. An equivalent (more traditional) definition of h^F is

$$h^F(\theta) = \limsup_{r\to\infty}\frac{\log|F(re^{\mathrm{i}\theta})|}{r}.$$

In particular, this definition gives that h^F is continuous; see [14, Chapter I, Sections 15, 16]. To verify the equivalence, we need to check that for every θ,

$$A \overset{\text{def}}{=} \sup_{v\in\mathcal{L}(F)} v(e^{\mathrm{i}\theta}) = \limsup_{r\to\infty} u_r(e^{\mathrm{i}\theta}) \overset{\text{def}}{=} B.$$

First, we can choose a subharmonic function v and a sequence $r_k \to \infty$ such that

$$\lim_{k\to\infty} u_{r_k}(e^{\mathrm{i}\theta}) = B, \quad u_{r_k} \to v \ \text{ in } \ L^1_{\mathrm{loc}}(\mathbb{C}).$$

By Proposition A, $\lim_{k\to\infty} u_{r_k}(e^{\mathrm{i}\theta}) \le v(e^{\mathrm{i}\theta}) \le A$, and we conclude that $A \ge B$. In the opposite direction, if $B < A$, then again by Proposition A, there exist a subharmonic function v, a sequence $r_k \to \infty$, and a neighbourhood U of $e^{\mathrm{i}\theta}$ such that $\sup_{z\in U} \limsup_{k\to\infty} u_{r_k}(z) < v(e^{\mathrm{i}\theta})$ and for almost all z in U, $\limsup_{k\to\infty} u_{r_k}(z) = v(z)$. This contradicts the subharmonicity of v.

The homogeneous indicator H^F is the Minkowski functional of a convex compact set called the **indicator diagram** I^F of F, see [14, Chapter I, Section 19].

The ray $\{\arg(z) = \theta\}$ is called a **ray of completely regular growth** of the function F if the set $\mathcal{L}(F)$ restricted to that ray is a singleton. Then

$$v(re^{\mathrm{i}\theta}) = H^F(re^{\mathrm{i}\theta}) = h^F(\theta)r, \quad v \in \mathcal{L}(F). \tag{3.1.2}$$

By continuity of the Phragmén–Lindelöf indicator, *the set of the rays of completely regular growth is closed.* Clearly, the function F has completely regular growth in $\mathbb{C}$ if it has a completely regular growth on every ray. Hence, it suffices to verify condition (3.1.2) on a dense set of rays.

3.2.

Definition 3. A sequence $R_j \uparrow \infty$ is **thick** if $\lim_{j\to\infty} R_{j+1}/R_j = 1$.

Lemma 3.2.1. *Let F be an entire function of exponential type. Let $h^F(\theta) \le \kappa$ for some $\theta \in [-\pi, \pi]$. Suppose that there exist a thick sequence $R_j \uparrow \infty$ and a sequence $\theta_j \to \theta$ such that*

$$\liminf_{j\to\infty} \frac{1}{R_j} \log|F(R_j e^{\mathrm{i}\theta_j})| \ge \kappa. \tag{3.2.1}$$

Then $h^F(\theta) = \kappa$ and F has completely regular growth on the ray $\{\arg(z) = \theta\}$.

Proof. Suppose that there exists a function $v \in \mathcal{L}(F)$ such that $v(e^{\mathrm{i}\theta}) < \kappa$. Since v is subharmonic (in particular, upper semi-continuous), in a small compact neighbourhood U of $e^{\mathrm{i}\theta}$ we have $\sup_U v < \kappa$. Next,

$$\frac{1}{t_k} \log|F(t_k z)| \to v(z) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{C})$$

for some sequence $t_k \to \infty$. By Proposition A, we obtain

$$\limsup_{k\to\infty} \sup_{z\in U} \frac{1}{t_k} \log|F(t_k z)| < \kappa,$$

and hence

$$\limsup_{k\to\infty} \frac{1}{t_k} \log|F(t_k z_k)| < \kappa,$$

provided that $z_k \to e^{\mathrm{i}\theta}$.

Now, we choose j_k such that $R_{j_k} \le t_k < R_{j_k+1}$, and put $\tau_k = t_k^{-1} R_{j_k}$ and $z_k = \tau_k e^{\mathrm{i}\theta_k}$. Then $\tau_k \to 1$ (this is the place where we use thickness of the sequence R_j), and therefore, $z_k \to e^{\mathrm{i}\theta}$. Thus,

$$\limsup_{k\to\infty} \frac{1}{R_{j_k}} \log|F(R_{j_k} e^{\mathrm{i}\theta_k})| = \limsup_{k\to\infty} \frac{1}{\tau_k t_k} \log|F(t_k z_k)| < \kappa,$$

which is a contradiction. □

3.3. The following lemma is a variation on the theme of the maximum principle. It is needed for the proof of Theorem 5.

Lemma 3.3.1. *Let F be an entire function of exponential type, and let $\sigma \subset \mathbb{T}$. Suppose that*

(i) $h^F \le h_\sigma$ *everywhere on* $[-\pi, \pi]$*;*
(ii) $h^F = h_\sigma = 1$ *everywhere on* σ*;*
(iii) *F has completely regular growth on the set of rays* $\{z\colon \arg(z) \in \sigma\}$.

Then $h^F = h_\sigma$ everywhere, and F has completely regular growth in $\mathbb{C}$.

Proof. If σ is dense on $\mathbb{T}$, then the statement is obvious; so we concentrate on the case when σ is not dense in $\mathbb{T}$.

Let I^F be the indicator diagram of F. By condition (i), $I^F \subseteq \mathrm{ch}(\sigma)$. By the definition of the convex hull, $\mathrm{ch}(\sigma)$ is the smallest convex compact that contains the set σ. By condition (ii), $\sigma \subseteq I_F$. Hence, $I^F = \mathrm{ch}(\sigma)$, that is, $h^F = h_\sigma$ everywhere.

Let $S = \{\theta\colon h^F(\theta) < 1\}$. The set S is a union of disjoint open intervals. Let $J = (\alpha, \beta)$ be one of these intervals, i.e., $h_\sigma(\alpha) = h_\sigma(\beta) = 1$, while $h_\sigma < 1$ everywhere on (α, β). For $\theta \in \overline{J}$, we have

$$h^F(\theta) = \max\bigl(\cos(\theta-\alpha), \cos(\theta-\beta)\bigr) = \begin{cases} \cos(\theta-\alpha), & \alpha \le \theta \le \frac{1}{2}(\alpha+\beta), \\ \cos(\theta-\beta), & \frac{1}{2}(\alpha+\beta) \le \theta \le \beta. \end{cases}$$

Consider the angle $\alpha \le \arg(z) \le \frac{1}{2}(\alpha+\beta)$. In this angle the indicator h^F is trigonometric, and F has a completely regular growth on the boundary ray $\arg(z) = \alpha$. Moreover, $(h^F)'(\alpha+0) = 0$ and $(h^F)'(\alpha-0) = 0$. The first relation is obvious. To see that the second relation holds, we consider two cases: (i) α is not an isolated point of $[-\pi,\pi]\setminus S$, and (ii) α is an isolated point of $[-\pi,\pi]\setminus S$. In the first case, there is a sequence $\theta_\ell \uparrow \alpha$ such that $h^F(\theta_\ell) = h^F(\alpha) = 1$. On each interval $(\theta_\ell, \theta_{\ell+1})$, we have

$$0 \le 1 - h^F(\theta) \le O\big((\theta_{\ell+1}-\theta)^2\big) \le O\big((\alpha-\theta)^2\big).$$

Hence, $(h^F)'(\alpha-0) = 0$. In the second case, this relation is obvious, since α is a maximum point of a trigonometric function. Thus, the indicator h^F is C^1-smooth at $\theta = \alpha$, and we are in the assumptions of Levin's theorem on entire functions with trigonometric Phragmén–Lindelöf indicator [14, Theorem 7, Chapter III]. By this theorem, F has completely regular growth in the angle $\big\{\alpha \le \theta \le \frac{1}{2}(\alpha+\beta)\big\}$. Similarly, F has completely regular growth in the angle $\big\{\frac{1}{2}(\alpha+\beta) \le \theta \le \beta\big\}$. This proves Lemma 3.3.1. □

It is worth mentioning that Levin's theorem used in the proof of Lemma 3.3.1 can be deduced from Hopf's boundary maximum principle for non-positive subharmonic functions vanishing on a part of the boundary.

4 Exponential sums

4.1. For a bounded sequence $\xi\colon \mathbb{Z}_+ \to \mathbb{C}$, introduce the exponential sum

$$W_R(\theta) = \sum_{|n|\le N} \xi(n+R)e(n\theta)e^{-\frac{n^2}{2R}},$$

where R and N are large integer parameters such that $N = R^{1/2}\log R + O(1)$. In principle, any choice of N in the range $R^{\frac{1}{2}}\sqrt{\log R} \ll N \ll_\varepsilon R^{\frac{1}{2}+\varepsilon}$ would suffice for our purposes. Here, $x(R) \ll_\varepsilon y(R,\varepsilon)$ means that for every $\varepsilon > 0$, $x(R) = o(y(R,\varepsilon))$ as $R \to \infty$.

Lemma 4.1.1. *Let*

$$F_\xi(z) = \sum_{n\ge 0} \xi(n)\frac{z^n}{n!}$$

with a bounded sequence $\xi\colon \mathbb{Z}_+ \to \mathbb{C}$. Then, for each $\varepsilon > 0$,

$$|F_\xi(Re(\theta))| \ge \mu(R)\big[|W_R(\theta)| - C_\varepsilon R^\varepsilon\big],$$

where $\mu(R) = e^R/\sqrt{2\pi R}$.

Proof. First, we estimate the tails

$$\Big(\sum_{0\le n<R-N}+\sum_{n>R+N}\Big)|\xi(n)|\frac{R^n}{n!}.$$

Put $N_1=R-N$, $N_2=R+N$. These sums are bounded by

$$O(1)\sum_{0\le n\le N_1-1}\frac{R^n}{n!}\quad\text{and}\quad O(1)\sum_{n\ge N_2+1}\frac{R^n}{n!}$$

correspondingly. Note that the sequence $n\mapsto R^n/n!$ increases for $0\le n\le N_1-1$ and decreases for $n\ge N_2+1$. For $0\le n\le N_1-1$, we have

$$\frac{R^n}{n!}\;:\;\frac{R^{n+1}}{(n+1)!}=\frac{n+1}{R}\le 1-\frac{N}{R},$$

while, for $n\ge N_2+1$,

$$\frac{R^{n+1}}{(n+1)!}\;:\;\frac{R^n}{n!}=\frac{R}{n+1}<\frac{R}{N_2}=\frac{1}{1+\frac{N}{R}}.$$

Hence,

$$\sum_{0\le n\le N_1-1}\frac{R^n}{n!}<\frac{R^{N_1}}{N_1!}\,\frac{1}{1-\big(1-\frac{N}{R}\big)}=\frac{R}{N}\cdot\frac{R^{N_1}}{N_1!}$$

and

$$\sum_{n\ge N_2+1}\frac{R^n}{n!}<\frac{R^{N_2}}{N_2!}\,\frac{1}{1-\frac{1}{1+\frac{N}{R}}}<2\,\frac{R}{N}\cdot\frac{R^{N_2}}{N_2!}\,.$$

It remains to observe that neither of the quantities $R^{N_1}/N_1!$ and $R^{N_2}/N_2!$ exceeds $Ce^{-cR^{-1}N^2}\,\mu(R)$, provided that $\sqrt{R}\ll N\ll R$. Therefore,

$$F_\xi(Re(\theta))=\sum_{|n-R|\le N}\xi(n)e(n\theta)\,\frac{R^n}{n!}+O(1)\mu(R),$$

provided that $\sqrt{R\log R}\ll N\ll R$.

Now, we turn to the central group of terms of the series. By Stirling's formula, we have

$$\sum_{|n-R|\le N}\xi(n)e(n\theta)\frac{R^n}{n!}=\mu(R)\sum_{|n-R|\le N}\xi(n)e(n\theta)\,\frac{R^n}{n!}\cdot\frac{\sqrt{2\pi R}}{e^R}$$

$$=\mu(R)\sum_{|n-R|\le N}\xi(n)e(n\theta)\left(1+O(R^{-1})\right)\left(\frac{R}{n}\right)^{n+\frac12}e^{n-R}. \tag{4.1.1}$$

Put $t = n - R$. Then $|t| \le N$, and

$$\begin{aligned}\Big(\frac{R}{n}\Big)^{n+\frac12} e^{n-R} &= \exp\Big(\big(R+t+\tfrac12\big)\log\big(1-\frac{t}{R+t}\big)+t\Big)\\ &= \exp\Big(-\frac{t^2}{2(R+t)}-\frac{t}{2(R+t)}+O\Big(\frac{|t|^3}{R^2}\Big)\Big)\\ &= \exp\Big(-\frac{t^2}{2R}+O\Big(\frac{|t|}{R}\Big)+O\Big(\frac{|t|^3}{R^2}\Big)\Big)\\ &= \exp\Big(-\frac{t^2}{2R}+O\Big(\frac{N^3}{R^2}\Big)\Big)\\ &= \Big(1+O\big(R^{-\frac12+3\varepsilon}\big)\Big)e^{-\frac{t^2}{2R}}\,.\end{aligned}$$

Hence, the sum on the right-hand side of (4.1.1) equals

$$\mu(R)\sum_{|n-R|\le N}\xi(n)e(n\theta)\,e^{-\frac{1}{2R}(n-R)^2}+\Omega\mu(R)$$

with $|\Omega| \le O(1)\,N\cdot R^{-\frac12+3\varepsilon} = O\big(R^{4\varepsilon}\big)$. This completes the proof of Lemma 4.1.1.□

4.2. Combining Lemmas 3.2.1 and 4.1.1 we arrive at the following lemma.

Lemma 4.2.1. *Let*

$$F_\xi(z)=\sum_{n\ge0}\xi(n)\frac{z^n}{n!},$$

where $\xi\colon \mathbb{Z}_+\to\mathbb{C}$ *is a bounded sequence. Suppose that for every* $a\in[0,1]$*, there exist a thick sequence* $R_j\uparrow\infty$*, a sequence* $\theta_j\to a$*, and* $\delta>0$ *such that*

$$\big|W_{R_j}(\theta_j)\big|\ge R_j^\delta. \tag{4.2.1}$$

Then ξ *is an L-sequence.*

4.3. In many instances, it is easier to produce a lower bound for an average of $|W_R|^2$ over short intervals of θ. The following lemma is a straightforward corollary to the previous lemma.

From now on, we fix a non-negative even function $g\in C_0^2[-\frac12,\frac12]$ with $\int g(\theta)\,\mathrm{d}\theta=1$.

Lemma 4.3.1. *Let*

$$F_\xi(z)=\sum_{n\ge0}\xi(n)\frac{z^n}{n!},$$

where $\xi\colon \mathbb{Z}_+ \to \mathbb{C}$ is a bounded sequence. Suppose that for every $a \in [0,1]$ and for every $m \in \mathbb{N}$, there exist a thick sequence $R_j \uparrow \infty$ and $\delta > 0$ such that

$$\int_{a-\frac{1}{2m}}^{a+\frac{1}{2m}} \left|W_{R_j}(\theta)\right|^2 g(m(\theta-a))\mathrm{d}\theta \ge R_j^{\delta}, \quad j \ge j_0(a,m). \tag{4.3.1}$$

Then ξ is an L-sequence.

Curiously enough, the assumptions of Lemmas 4.2.1 and 4.3.1 impose restrictions only on relatively short blocks $\bigcup_j \left[R_j - R_j^{\frac{1}{2}+\varepsilon}, R_j + R_j^{\frac{1}{2}+\varepsilon}\right]$ of elements of the sequence ξ. The values attained by ξ off these blocks do not matter.

5 Proof of Theorems 1 and 2 ($\beta > 3/2$)

In this section, we put $\xi(n) = e(f(n))$ for some real-valued f. Then

$$W_R(\theta) = \sum_{|n|\le N} e(f(n+R)+n\theta)\, e^{-\frac{n^2}{2R}}, \quad \sqrt{R\log R} \ll N \ll R^{\frac{1}{2}+\varepsilon},$$

and we are looking for a lower bound for

$$X_R = \int_{a-\frac{1}{2m}}^{a+\frac{1}{2m}} \left|W_R(\theta)\right|^2 g(m(\theta-a))\,\mathrm{d}\theta, \quad a \in [0,1],\ m \in \mathbb{N}.$$

The upper bound $X_R \le C\sqrt{R}$ as well as the matching lower bound in the case when $m = 1$ follow from Parseval's theorem. There are some reasons to expect that if there are no unreasonable cancellations, then a similar lower bound holds in all scales, that is, $X_R \ge c(a,m)\sqrt{R}$ for every $m \in \mathbb{N}$ and every $a \in [0,1]$. In the next subsections, we justify these expectations.

5.1. The following lemma reduces the lower bound for X_R to upper bounds for certain Weyl sums. Put

$$S_T(M_1,M_2) = \sum_{M_1\le n<M_2} e\big(f(n+R)-f(n+R-T)\big).$$

Lemma 5.1.1. *There exist positive numerical constants c and C such that*

$$X_R \ge \frac{c\sqrt{R}}{m} - Cm\sum_{T=1}^{2N}\frac{1}{T^2} \max_{\substack{0<M_2-M_1\le\sqrt{R},\\ |M_1|,|M_2|\le N}} \left|S_T(M_1,M_2)\right|.$$

Proof. We have

$$X_R = \frac{1}{m}\int_{-1/2}^{1/2}\Big|\sum_{|n|\le N} e\Big(f(n+R)+na+\frac{n\theta}{m}\Big)e^{-n^2/(2R)}\Big|^2 g(\theta)\,\mathrm{d}\theta$$
$$= \frac{1}{m}\sum_{|n|,|n'|\le N} e\big(f(n+R)-f(n'+R)+(n-n')a\big)e^{-(n^2+n'^2)/(2R)}\,\widehat{g}\Big(\frac{n'-n}{m}\Big),$$

where $\widehat{g}$ denotes the Fourier transform of g extended by 0 to $\mathbb{R}\setminus[-\frac{1}{2},\frac{1}{2}]$. The diagonal sum ($n=n'$) contributes

$$\frac{1}{m}\sum_{|n|\le N} e^{-n^2/R} \overset{N\ge\sqrt{R}}{\ge} \frac{c\sqrt{R}}{m}.$$

We need to estimate from above the contribution of non-diagonal terms

$$\frac{2}{m}\Big|\sum_{\substack{|n|,|n'|\le N\\ n'<n}} e\big(f(n+R)-f(n'+R)+(n-n')a\big)\,e^{-(n^2+n'^2)/(2R)}\,\widehat{g}\Big(\frac{n'-n}{m}\Big)\Big|.$$

Letting $T=n-n'$ and using the fact that $\widehat{g}\big((n'-n)/m\big)=O\big(m^2/(n-n')^2\big)$, we see that the contribution of non-diagonal terms is

$$(5.1.1)\quad \le Cm\sum_{T=1}^{2N}\frac{1}{T^2}\Big|\sum_{-N+T\le n\le N} e\big(f(n+R)-f(n+R-T)\big)\,e^{-(n^2+(n-T)^2)/(2R)}\Big|.$$

The function $n\mapsto e^{-(n^2+(n-T)^2)/(2R)}$ increases for $-\infty<n\le\frac{1}{2}T$ and decreases for $\frac{1}{2}T\le n<+\infty$. We consider these two ranges separately. Then the expression in (5.1.1) is

$$\le Cm\sum_{T=1}^{2N}\frac{1}{T^2}\Big[\Big|\sum_{-N+T\le n<\frac{1}{2}T}\cdots\Big|+\Big|\sum_{\frac{1}{2}T\le n\le N}\cdots\Big|\Big].$$

Next, we split the sums in n into blocks of length $\sqrt{R}$ (and several blocks of smaller length that are treated similarly). We set $J_k=[\frac{T}{2}+(k-1)\sqrt{R},\ \frac{T}{2}+k\sqrt{R})$ and put

$$Y_{k,T}=\sum_{n\in J_k\cap[-N+T,N]} e(f(n+R)-f(n+R-T))e^{-(n^2+(n-T)^2)/(2R)},$$

with $|k|\le\frac{1}{\sqrt{R}}\big(N-\frac{T}{2}\big)+1$. Note that for $n=\frac{T}{2}+\lambda\sqrt{R}$ with $k-1\le\lambda<k$, we have

$$-\frac{1}{2R}\big(n^2+(n-T)^2\big)=-\frac{1}{2R}\Big(\big(\lambda\sqrt{R}+\frac{T}{2}\big)^2+\big(\lambda\sqrt{R}-\frac{T}{2}\big)^2\Big)$$
$$=-\frac{1}{2R}\Big(2\lambda^2R+\frac{1}{2}T^2\Big)<-\lambda^2\le -ck^2+c_1\,.$$

Then, applying the Abel summation formula to the sum $Y_{k,T}$, we see that

$$\left|Y_{k,T}\right| \le Ce^{-ck^2} \max_{\substack{0<M_2-M_1\le\sqrt{R}\\ |M_1|,|M_2|\le N}} \left|S_T(M_1,M_2)\right|,$$

and the sum of non-diagonal terms we are estimating is

$$\le Cm\sum_{T=1}^{2N}\frac{1}{T^2}\sum_{|k|\le\frac{1}{\sqrt{R}}(N-\frac{1}{2}T)+1}\left|Y_{k,T}\right| \le Cm\sum_{T=1}^{2N}\frac{1}{T^2}\max_{\substack{0<M_2-M_1\le\sqrt{R}\\ |M_1|,|M_2|\le N}}\left|S_T(M_1,M_2)\right|.$$

This completes the proof of Lemma 5.1.1. □

Now, Theorems 1 and 2 (for $\beta > 3/2$) follow readily from the classical Weyl and van der Corput estimates of exponential sums.

5.2 Proof of Theorem 1. First, we fix $T_0 = T_0(m)$ so large that

$$Cm\sum_{T>T_0}\frac{1}{T^2} < \frac{1}{2}\frac{c}{m},$$

where the positive numerical constants C and c are the same as in the assertion of Lemma 5.1.1. Then, using the trivial bound $\left|S_T(M_1,M_2)\right| \le \sqrt{R}$, we get

$$X_R > \frac{1}{2}\frac{c}{m}\sqrt{R} - CmT_0 \max_{1\le T\le T_0}\max_{0<M_2-M_1\le\sqrt{R}}\left|S_T(M_1,M_2)\right|.$$

Define $P_T(x) = Q(x) - Q(x-T)$, set $\sum_{k=1}^{d-1} p_k x^k = P_T(x)$, and observe that at least one of the coefficients p_k is irrational (if ℓ is the maximal index such that the coefficient q_ℓ of Q is irrational, then $p_{\ell-1}$ must be irrational too). Then, by Weyl's theorem [20, Section 3] (see also the argument in [17, pp. 17–18]), we have

$$\max_{0<M_2-M_1\le M}\left|\sum_{M_1\le n<M_2} e(P_T(n))\right| = o(M), \quad \text{as} \quad M\to\infty.$$

Hence, for each $T \in \{1,\dots,T_0\}$,

$$\max_{0\le M_2-M_1\le\sqrt{R}}\left|S_T(M_1,M_2)\right| = o(\sqrt{R}), \quad \text{as} \quad R\to\infty,$$

and, for $R > R_0(m)$, we have $X_R > c(m)\sqrt{R}$ with $c(m) > 0$. An application of Lemma 4.3.1 completes the proof of Theorem 1.

5.3 Proof of Theorem 2. Here, we prove Theorem 2 for $\beta > 3/2$; the case $\beta = \frac{3}{2}$ is treated in Section 6. Put $f_{T,R}(x) = (R+x)^\beta - (R+x-T)^\beta$.

5.3.1 $3/2 < \beta < 2$**.** In this case, we apply the classical summation formula (see, e.g., [19, (2.1.2)])

$$\sum_{M_1 \le n < M_2} \varphi(n) = \int_{M_1}^{M_2} \varphi(x)\mathrm{d}x + \int_{M_1}^{M_2} \Big(x - [x] - \frac{1}{2}\Big)\varphi'(x)\mathrm{d}x + \frac{1}{2}\varphi(M_1) - \frac{1}{2}\varphi(M_2)$$

with $\varphi(x) = e(f_{T,R}(x))$ and with integers M_1 and M_2, $|M_1|, |M_2| \le N$. We get

$$S_T(M_1, M_2) = \int_{M_1}^{M_2} e\big(f_{T,R}(x)\big)\mathrm{d}x + 2\pi\mathrm{i}\int_{M_1}^{M_2}\Big(x - [x] - \frac{1}{2}\Big) f'_{T,R}(x) e\big(f_{T,R}(x)\big)\mathrm{d}x + O(1).$$

Since the function $f'_{T,R}$ is monotonically decreasing, applying a classical estimate on integrals of oscillating functions (see, e.g., [19, Lemma 4.2]) and recalling that $M_2 \le N$, we get

$$\Big|\int_{M_1}^{M_2} e(f_{T,R}(x))\mathrm{d}x\Big| \le \frac{4}{f'_{T,R}(N)}. \tag{5.3.1}$$

For $R \ge R_0(\beta)$, $|x| \le N$, and $1 \le T \le 2N$, we have

$$f'_{T,R}(x) = \beta\big((R+x)^{\beta-1} - (R+x-T)^{\beta-1}\big) = \big(\beta(\beta-1) + o(1)\big)\,\frac{T}{R^{2-\beta}},$$

uniformly in $x \in [-N, N]$. Therefore, the LHS of (5.3.1) is $\le c(\beta)T^{-1}R^{2-\beta}$.

Next,

$$\Big|\int_{M_1}^{M_2} \Big(x - [x] - \frac{1}{2}\Big) f'_{T,R}(x)\, e\big(f_{T,R}(x)\big)\mathrm{d}x\Big| \le (M_2 - M_1) \max_{|x|\le N}\big|f'_{T,R}(x)\big| \le c(\beta)N\frac{T}{R^{2-\beta}},$$

whence, by Lemma 5.1.1,

$$X_R \ge \frac{c\sqrt{R}}{m} - C(\beta)m\big(R^{2-\beta} + NR^{\beta-2}\log N\big) \ge c(m,\beta)\sqrt{R},$$

provided that $R \ge R_0(m, \beta)$. In view of Lemma 4.3.1, this proves Theorem 2 in the case $3/2 < \beta < 2$.

5.3.2 $\beta > 2$**.** Suppose that $k < \beta < k+1$ with an integer $k \ge 2$. To estimate

$$\max_{\substack{M_2 - M_1 \le N \\ |M_1|, |M_2| \le N}} \big|S_T(M_1, M_2)\big|,$$

we apply a van der Corput bound [19, Theorem 5.13]. Using the fact that

$$f^{(k)}_{T,R}(x) \simeq_{\beta,k} \frac{T}{R^{k+1-\beta}} \quad \text{uniformly in} \quad |x| \le N, \quad 1 \le T \le 2N,$$

we get

$$\left|S_T(M_1, M_2)\right| \lesssim_{\beta,k} (M_2 - M_1)\Big(\frac{T}{R^{k+1-\beta}}\Big)^{\frac{1}{2K-2}} + (M_2 - M_1)^{1-\frac{2}{K}}\Big(\frac{R^{k+1-\beta}}{T}\Big)^{\frac{1}{2K-2}}$$

with $K = 2^{k-1}$ ($x \simeq_a y$ and $x \lesssim_a y$ mean, correspondingly, $c_1(a)y \le x \le c_2(a)y$ and $x \le c(a)y$). Since $M_2 - M_1 \le N$, the right-hand side is

$$\lesssim N\left(T^{1/2}R^{-\delta} + R^{-\frac{1}{K}+\frac{k+1-\beta}{2K-2}}\right),$$

with some $\delta > 0$. Since $K \ge 2$, we have

$$k + 1 - \beta < 1 \le 2 - \frac{2}{K} = \frac{2K-2}{K}.$$

Therefore,

$$\max_{\substack{M_2 - M_1 \le N \\ |M_1|, |M_2| \le N}} \left|S_T(M_1, M_2)\right| \lesssim_{\beta,\delta} T^{1/2}R^{1/2-\delta/2};$$

and, by Lemma 5.1.1, $X_R \ge c(m)\sqrt{R}$, provided that $R \ge R_0(m, \beta)$.

6 Proof of Theorem 2 ($\beta = 3/2$)

In [5], Chen and Littlewood showed that the zeroes of the function F_ξ with $\xi(n) = e(n^\beta)$, $1 < \beta < \frac{3}{2}$, are asymptotically very close to a sequence of points that are regularly distributed on the spiral given in polar coordinates by $\theta = -\pi + C(\beta)r^{\beta-1}$. Their analysis yields that this ξ is an L-sequence. In fact, they gave a detailed proof for another sequence $\xi(n) = e(n(\log n)^\beta)$ with $\beta > 1$, and mentioned that their arguments work with minor changes in the case we consider here. Apparently, it is an intriguing open question which part of their analysis can be extended to the case $3/2 \le \beta \le 2$ (or, even to $\beta = 3/2$). Nevertheless, as we show in this section, a certain combination of their method with our techniques is strong enough to show that the sequence $\xi(n) = e(n^{3/2})$ is an L-sequence.

Throughout this section,

$$W_R(\theta) = \sum_{|n| \le N} e\big((n+R)^{3/2} + (n+R)\theta\big)e^{-n^2/(2R)}$$

with $N = R^{1/2}\log R + O(1)$; this differs from our definition in Subsection 4.1 by a unimodular factor $e(R\theta)$.

6.1. Here, we give an asymptotic estimate of W_R which yields Theorem 2 in the case $\beta = 3/2$.

Lemma 6.1.1. *For* $R \to \infty$,

$$W_R(\theta) = \frac{2e(1/8 + MR)R^{1/4}}{\sqrt{3}} \sum_{|m| \le \frac{1}{2}\log R} e\big(mR - \tfrac{4}{27}(M + m - \theta)^3\big) e^{-\frac{8}{9}(m-\theta)^2} + O\big((\log R)^3\big)$$

with $M = \frac{3}{2}R^{1/2}$, *uniformly in* θ.

It is worth mentioning that in the case $1 < \beta < 3/2$ considered by Chen and Littlewood, at most two terms contribute to the corresponding sum on the right-hand side. This was crucial for finding the asymptotic locations of zeroes of F.

We split the proof of Lemma 6.1.1 into several parts.

6.1.1. Take $\chi \in C_0^\infty[0, +\infty)$ with $\chi \ge 0$,

$$\chi(t) = \begin{cases} 1, & 0 \le x \le N, \\ 0, & x \ge N + 1, \end{cases}$$

and set $\chi(z) = \chi(|z|)$, and

$$u(t) = \chi(t - R)e(t^{3/2} + t\theta)e^{-(t-R)^2/(2R)}, \quad t \in \mathbb{R}.$$

Then

$$W_R(\theta) = \sum_{n \in \mathbb{Z}} u(n) = \sum_{m \in \mathbb{Z}} \widehat{u}(m) \quad \text{(the Poisson summation),}$$

where

$$\begin{aligned} \widehat{u}(m) &= \int_{\mathbb{R}} u(t)e(-mt)\,\mathrm{d}t \\ &= e(mR) \int_{\mathbb{R}} \chi(t)e((t + R)^{3/2} - (m - \theta)(t + R))e^{-t^2/(2R)}\,\mathrm{d}t \\ &= e(mR) \int_{\mathbb{R}} \chi(t)e(\psi_m(t))e^{-t^2/(2R)}\,\mathrm{d}t\,, \end{aligned}$$

where $\psi_m(t) = (t + R)^{3/2} - \mu(t + R)$ is “a phase function”, and $\mu = m - \theta$ is “a distorted m”. Put

$$I_m = \int_{\mathbb{R}} \chi(t)e(\psi_m(t))e^{-t^2/(2R)}\,\mathrm{d}t\,.$$

Estimating the integrals I_m, we set $M = \frac{3}{2}R^{1/2}$ and consider separately three cases: $|m - M| > \log R$, $\frac{1}{2}\log R < |m - M| \le \log R$, and $|m - M| \le \frac{1}{2}\log R$. In what follows, we extend ψ_m to an analytic function in $\{z : \operatorname{Re} z > -R\}$ and use the Taylor approximation of ψ_m in the disk $\{z : |z| \le 10N\}$:

$$\psi_m(z) = -\tfrac{1}{2}R^{3/2} - \sigma R - \sigma z + \tfrac{3}{8}z^2R^{-1/2} - \tfrac{1}{16}z^3R^{-3/2} + O\big(R^{-1/2}(\log R)^4\big), \tag{6.1.1}$$

where $\sigma = \mu - M$.

6.1.2. We start with the case $|m-M| > \log R$. Then the derivative of the phase ψ_m is large on the support of χ; see (6.1.3) below. We show that for $R \ge R_0$,

$$|I_m| \le \frac{e^{-c(\log R)^2}}{(m-M)^2}. \tag{6.1.2}$$

Integrating twice by parts, we obtain

$$\begin{aligned} I_m &= \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{R}}\Big[\frac{1}{\psi_m'}\Big(\frac{\chi e^{-t^2/(2R)}}{\psi_m'}\Big)'\Big]'(t)e(\psi_m(t))\,\mathrm{d}t \\ &= \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{R}}\frac{\lambda(t)}{\psi_m'^2(t)}e(\psi_m(t))e^{-t^2/(2R)}\,\mathrm{d}t, \end{aligned}$$

where

$$\lambda = \chi'' - \frac{3\chi'\psi_m''}{\psi_m'} - \frac{2\chi' t}{R} - \frac{\chi}{R} + \frac{\chi t^2}{R^2} + \frac{3\chi\psi_m'' t}{\psi_m' R} - \frac{\chi\psi_m'''}{\psi_m'} + 3\Big(\frac{\psi_m''}{\psi_m'}\Big)^2\chi.$$

For $|z| \le N+1$ and $R > R_0$, we have

$$\begin{aligned} |\psi_m'(z)| &= \Big|\tfrac{3}{2}(R+z)^{1/2} - \tfrac{3}{2}R^{1/2} - \sigma\Big| \ge |\sigma| - \tfrac{3}{2}R^{1/2}\Big[\sqrt{1+\frac{N+1}{R}} - 1\Big] \\ &\ge |\sigma| - (\tfrac{3}{4}+o(1))\log R \ge \tfrac{1}{5}|\sigma|. \end{aligned} \tag{6.1.3}$$

Since the functions ψ_m'', ψ_m''' are bounded on the disk $\{z\colon |z| \le N+1\}$, we conclude that λ is bounded on the same disk.

Next, we set

$$H(z) = 2\pi \mathrm{i}\psi_m(z) - \frac{z^2}{2R} = 2\pi\mathrm{i}(R+z)^{3/2} - 2\pi\mathrm{i}\Big(\frac{3}{2}R^{1/2}+\sigma\Big)(z+R) - \frac{z^2}{2R}.$$

Then

$$I_m = -\frac{1}{4\pi^2}\int_{\mathbb{R}}\frac{\lambda(t)}{\psi_m'^2(t)}\,e^{H(t)}\mathrm{d}t.$$

Using the Taylor expansion (6.1.1), we get

$$\begin{aligned} &|\exp H(x+\mathrm{i}y)| \\ &\le C\exp\Big(2\pi\sigma y - \frac{3\pi}{2}xyR^{-1/2} + \frac{3\pi}{8}x^2yR^{-3/2} - \frac{\pi}{8}y^3R^{-3/2} - \frac{x^2}{2R} + \frac{y^2}{2R}\Big) \\ &= C\exp\Big[\Big(2\pi\sigma - \frac{3\pi}{2}xR^{-1/2} + o(1)\Big)y - \frac{x^2}{2R} + \frac{y^2}{2R}\Big], \quad |x+iy| \le 3N. \end{aligned} \tag{6.1.4}$$

Now,

$$4\pi^2|I_m| \le \Big|\int_{\frac{N}{2}\le|x|\le N+1}\Big| + \Big|\int_{|x|\le N/2}\Big|.$$

For $|x| \geq \frac{1}{2}N$, $|\exp H(x)| \leq C\exp[-x^2/(2R)] \leq \exp[-c(\log R)^2]$. Thus, the first integral does not exceed

$$CN|\sigma|^{-2}e^{-c(\log R)^2} \leq \frac{e^{-c_1(\log R)^2}}{(m-M)^2}\,.$$

In the second integral, instead of integrating over the interval $[-\frac{1}{2}N, \frac{1}{2}N]$, we integrate over the contour Γ_σ shown in Figure 1.

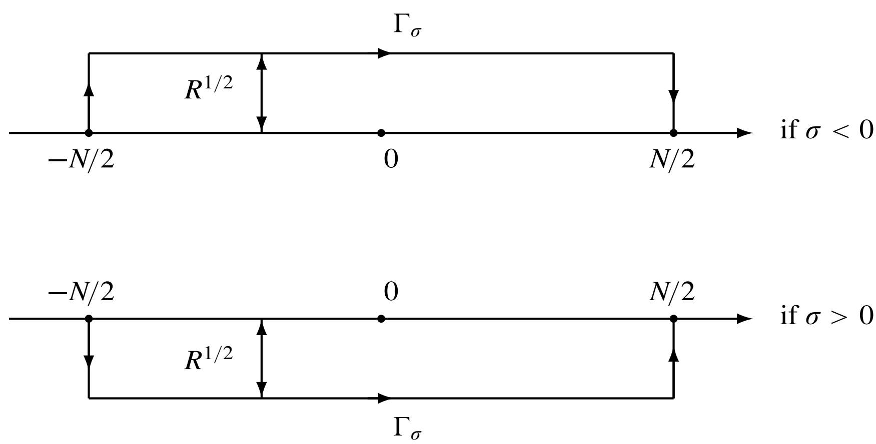

Figure 1

Estimate (6.1.4) shows that

$$|e^{H(x+iy)}| \leq \begin{cases} C\,e^{-cR^{1/2}\log R}, & z \in \Gamma_\sigma,\ |y| = R^{1/2}, \\ C\,e^{-c(\log R)^2}, & z \in \Gamma_\sigma,\ |x| = N/2. \end{cases}$$

Therefore, the integral over the contour Γ_σ is also bounded by

$$(m-M)^{-2}\,e^{-c(\log R)^2},$$

and estimate (6.1.2) follows.

6.1.3. Next suppose $\frac{1}{2}\log R < |m-M| \leq \log R$. This case is similar to the previous one but the proof is somewhat shorter since there is no need to integrate by parts (instead of (6.1.2) we check a simpler estimate (6.1.5)). We again split the integral into two parts:

$$
\begin{aligned}
|I_m| &= \Big|\int_{\mathbb{R}} \chi(t) e(\psi_m(t)) e^{-t^2/(2R)}\,\mathrm{d}t\Big| \le \Big|\int_{\frac{N}{2}\le|x|\le N+1}\Big| + \Big|\int_{|x|\le N/2}\Big| \\
&\le Ce^{-c(\log R)^2} + \Big|\int_{\Gamma_\sigma} e^{H(z)}\,\mathrm{d}z\Big|.
\end{aligned}
$$

Arguing as above, we obtain

$$
|I_m| \le e^{-c(\log R)^2}. \tag{6.1.5}
$$

6.1.4. Finally, we deal with I_m such that $|m-M| \le \frac{1}{2}\log R$. This case requires a saddle point approximation. Set

$$
z_0 = \frac{4}{3}\sigma R^{1/2}, \quad A_0 = -\frac{8}{9}\sigma^2 - \frac{8\pi\mathrm{i}}{27}\mu^3.
$$

Then, using the Taylor approximations (6.1.1), we get

$$
\begin{aligned}
H(z_0) &= A_0 + O\big(R^{-1/2}(\log R)^4\big), \\
H'(z_0) &= O\big(R^{-1/2}(\log R)^2\big), \\
H''(z) &= \frac{3\pi\mathrm{i}}{2}R^{-1/2} + O(R^{-1}\log R), \quad |z| < 5N\,.
\end{aligned}
$$

Now,

$$
I_m = \int_{|x|\le N} + \int_{N\le|x|\le N+1} = \int_{\Lambda_\sigma} e^{H(z)}\mathrm{d}z + O\big(e^{-c(\log R)^2}\big),
$$

where the Fresnel-type contour Λ_σ is as in Figure 2.

Let Λ_σ^0 be the vertical part of Λ_σ, and Λ_σ^1 be the rest. Then by estimate (6.1.4) we have

$$
\Big|\int_{\Lambda_\sigma^0} e^{H(z)}\mathrm{d}z\Big| \le e^{-c(\log R)^2}.
$$

Hence

$$
I_m = \int_{\Lambda_\sigma^1} e^{H(z)}\mathrm{d}z + O\big(e^{-c(\log R)^2}\big).
$$

Furthermore,

$$
\int_{\Lambda_\sigma^1} e^{H(z)}\mathrm{d}z = e^{\mathrm{i}\pi/4}\int_{-R^{1/2}(NR^{-1/2}+4\sigma/3)}^{R^{1/2}(NR^{-1/2}-4\sigma/3)} e^{H(te^{\mathrm{i}\pi/4}+z_0)}\mathrm{d}t.
$$

For $|t| \le 2N$, we have

$$
H(te^{\mathrm{i}\pi/4}+z_0) = A_0 + O\Big(\frac{(\log R)^4}{R^{1/2}}\Big) + O\Big(t\frac{(\log R)^2}{R^{1/2}}\Big) - \frac{3\pi}{4}t^2R^{-1/2} + O\Big(t^2\frac{\log R}{R}\Big),
$$

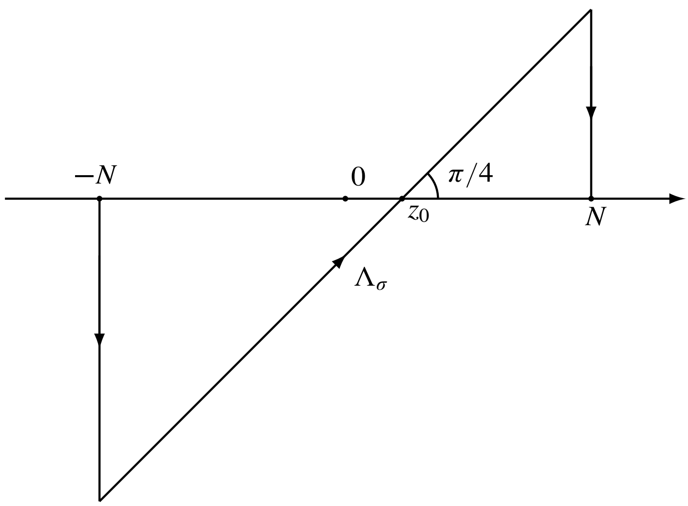

Figure 2

and hence

$$\int_{\Lambda_\sigma^1} e^{H(z)}\mathrm{d}z = e^{\mathrm{i}\pi/4+A_0}\, R^{1/4} \int_{R^{1/4}(-\log R-4\sigma/3+o(1))}^{R^{1/4}(\log R-4\sigma/3+o(1))} \exp\Big[-\frac{3\pi}{4}t^2 + O\Big(\frac{(\log R)^4}{R^{1/2}} + t\frac{(\log R)^2}{R^{1/4}} + t^2\frac{\log R}{R^{1/2}}\Big)\Big]\,\mathrm{d}t$$
$$= e^{\mathrm{i}\pi/4+A_0}\, R^{1/4}\Big(\int_{\mathbb{R}} \exp\big[-\frac{3\pi}{4}t^2\big]dt + O\Big(\frac{(\log R)^3}{R^{1/4}}\Big)\Big)$$
$$= \frac{2}{\sqrt{3}}R^{1/4}e\big(\tfrac{1}{8} - \tfrac{4}{27}\mu^3\big)e^{-\frac{8}{9}\sigma^2} + O\big((\log R)^3\big).$$

Finally, for $|m-M| \le \frac{1}{2}\log R$, we get

$$I_m = \frac{2}{\sqrt{3}}R^{1/4}e\big(\tfrac{1}{8} - \tfrac{4}{27}\mu^3\big)e^{-\frac{8}{9}\sigma^2} + O\big((\log R)^3\big) + O\big(e^{-c(\log R)^2}\big)$$
$$= \frac{2}{\sqrt{3}}R^{1/4}e\big(\tfrac{1}{8} - \tfrac{4}{27}(m-\theta)^3\big)e^{-\frac{8}{9}(m-M-\theta)^2} + O\big((\log R)^3\big).$$

6.1.5. Thus,

$$\begin{aligned} W_R(\theta) &= \sum_{m\in\mathbb{Z}} \widehat{u}(m+M) = \Big[\sum_{|m|\le \frac12 \log R} + \sum_{\frac12 \log R < m \le \log R} + \sum_{|m|>\log R} \Big] e((M+m)R) I_{m+M} \\ &= \frac{2e(1/8+MR)R^{1/4}}{\sqrt{3}} \sum_{|m|\le\frac12\log R} e\big(mR - \tfrac{4}{27}(M+m-\theta)^3\big) e^{-\frac{8}{9}(m-\theta)^2} \\ &\qquad + O\big((\log R)^3\big), \end{aligned}$$

which proves Lemma 6.1.1.

6.2. At last, we are able to prove Theorem 2 for $\beta = \frac{3}{2}$. Consider the shifts $W_R(\theta+t)$ with $0 \le t \le \frac{9}{8}M^{-1}$. We have

$$\begin{aligned} W_R(\theta+t) &= \frac{2e(1/8+MR)R^{1/4}}{\sqrt{3}} \sum_{|m|\le\frac12\log R} e\big(mR - \tfrac{4}{27}(M+m-\theta-t)^3\big) e^{-\frac{8}{9}(m-\theta-t)^2} \\ &\qquad + O\big((\log R)^3\big). \end{aligned}$$

Furthermore, since $|t| = O\big(M^{-1}\big)$ with $M = \frac{3}{2}R^{1/2}$, we have

$$\begin{aligned} &e\big(-\tfrac{4}{27}(M+m-\theta-t)^3\big)e^{-\frac{8}{9}(m-\theta-t)^2} \\ &\quad = e\big(-\tfrac{4}{27}(M+m-\theta)^3 + \tfrac{4}{9}(M^2-2M\theta)t + \tfrac{8}{9}Mmt\big)e^{-\frac{8}{9}(m-\theta)^2} \\ &\qquad\qquad + O\Big(\frac{(\log R)^2}{R^{1/2}}\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} W_R(\theta+t) &= KR^{1/4} \sum_{|m|\le\frac12\log R} e\big(\tfrac{8}{9}Mtm + mR - \tfrac{4}{27}(M+m-\theta)^3\big)e^{-\frac{8}{9}(m-\theta)^2} \\ &\qquad + O\big((\log R)^3\big) \end{aligned}$$

with

$$K = K(M,\theta,t) = \frac{2e\big(\frac18 + MR + \frac49(M^2-2M\theta)t\big)}{\sqrt{3}}.$$

Now, notice that the sum on the right-hand side is a Fourier series in the variable $\frac{8}{9}Mt$. Hence, by Parseval's theorem, there exists $t \in [0, \frac{9}{8}M^{-1}]$ such that

$$|W_R(\theta+t)| \ge \frac{2R^{1/4}}{\sqrt{3}} \Big(\sum_{|m|\le\frac12\log R} e^{-\frac{16}{9}(m-\theta)^2} \Big)^{1/2} - O\big((\log R)^3\big) \ge CR^{1/4}$$

with a positive numerical constant C. Applying Lemma 4.2.1, we finish off the proof.

7 Wide-sense stationary sequences

Here, we prove several simple lemmas pertaining to the case when $\xi\colon \mathbb{Z}_+ \to \mathbb{C}$ is a **wide-sense stationary sequence**, that is, $\mathbb{E}|\xi(n)|^2 < \infty$ for every n, and $\mathbb{E}\xi(n)$ and $\mathbb{E}\{\xi(n)\overline{\xi(n+m)}\}$ do not depend on n. We also always assume that ξ is not the zero sequence. By ρ we denote the spectral measure of such a sequence ξ, i.e., ρ is a finite non-negative measure on the unit circle $\mathbb{T}$ such that

$$\mathbb{E}\left[\xi(n_1)\overline{\xi(n_2)}\right] = \widehat{\rho}(n_2 - n_1).$$

By $\sigma(\xi)$, we denote the spectrum of ξ, i.e., the closed support of the spectral measure ρ. In what follows, by σ^* we always mean the reflection of the spectrum σ in the real axis.

Observe that if ξ is a wide-sense stationary sequence then, almost surely, F_ξ is an entire function of exponential type at most 1. Indeed, for every $\varepsilon > 0$,

$$\mathbb{P}\left\{|\xi(n)| > (1+\varepsilon)^n\right\} \le (1+\varepsilon)^{-2n}\,\mathbb{E}|\xi(n)|^2,$$

whence, by the Borel–Cantelli Lemma, $\limsup_{n\to\infty} |\xi(n)|^{1/n} \le 1$ almost surely, which is equivalent to the inequality $|F_\xi(z)| \le C(\varepsilon)e^{(1+\varepsilon)|z|}$ being valid for every $z \in \mathbb{C}$ and every $\varepsilon > 0$.

7.1. First, we compute the variance of F_ξ in terms of the spectral measure ρ.

Lemma 7.1.1. *Suppose ξ is a wide-sense stationary sequence. Then*

$$\mathbb{E}\left|F_\xi(re^{\mathrm{i}\theta})\right|^2 = \int_{-\pi}^{\pi} e^{2r\cos(\theta+t)}\,\mathrm{d}\rho(t)\,, \tag{7.1.1}$$

and

$$\log \mathbb{E}\left|F_\xi(re^{\mathrm{i}\theta})\right|^2 = 2rh_{\sigma^*}(\theta) + o(r)\,, \quad r\to\infty\,. \tag{7.1.2}$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left|F_\xi(re^{\mathrm{i}\theta})\right|^2 &= \sum_{n_1,n_2\ge 0} \mathbb{E}\left[\xi(n_1)\overline{\xi(n_2)}\right] e^{\mathrm{i}(n_1-n_2)\theta}\,\frac{r^{n_1+n_2}}{n_1!n_2!} \\
&= \sum_{n_1,n_2\ge 0}\left[\int_{-\pi}^{\pi} e^{-\mathrm{i}(n_2-n_1)t}\,\mathrm{d}\rho(t)\right] e^{\mathrm{i}(n_1-n_2)\theta}\,\frac{r^{n_1+n_2}}{n_1!n_2!} \\
&= \int_{-\pi}^{\pi}\Big[\sum_{n_1,n_2\ge 0} e^{\mathrm{i}n_1(\theta+t)}\,\frac{r^{n_1}}{n_1!}\cdot e^{-\mathrm{i}n_2(\theta+t)}\,\frac{r^{n_2}}{n_2!}\Big]\,\mathrm{d}\rho(t) \\
&= \int_{-\pi}^{\pi} e^{r\left[e^{\mathrm{i}(\theta+t)}+e^{-\mathrm{i}(\theta+t)}\right]}\,\mathrm{d}\rho(t) \\
&= \int_{-\pi}^{\pi} e^{2r\cos(\theta+t)}\mathrm{d}\rho(t),
\end{aligned}$$

which proves (7.1.1). Now, recalling the definition of the supporting function

$$h_{\sigma^*}(\theta) = \max_{t\in spt(\rho)} \cos(\theta + t),$$

we readily get asymptotics (7.1.2). □

7.2. As a straightforward consequence of the previous lemma, we get the following result.

Lemma 7.2.1. *Suppose ξ is a wide-sense stationary sequence. Then, almost surely,*

$$h^{F_\xi}(\theta) \le h_{\sigma^*}(\theta)\,, \quad \theta \in [-\pi, \pi]\,.$$

In other words, the indicator diagram I^{F_ξ} of F_ξ is contained in the closed convex hull of the spectrum $\sigma(\xi)$ reflected in the real axis.

Proof. Using (7.1.2) and Chebyshev's inequality, we see that for every $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\big\{\log|F_\xi(re^{\mathrm{i}\theta})| > (h_{\sigma^*}(\theta)+\varepsilon)r\big\} &= \mathbb{P}\big\{|F_\xi(re^{\mathrm{i}\theta})|^2 > e^{2(h_{\sigma^*}(\theta)+\varepsilon)r}\big\} \\ &\le \mathbb{E}\big\{|F_\xi(re^{\mathrm{i}\theta})|^2\big\}\, e^{-2(h_{\sigma^*}(\theta)+\varepsilon)r} = e^{-2\varepsilon r + o(r)}\,, \quad r \to \infty\,.\end{aligned}$$

Hence, by the Borel–Cantelli Lemma, for every $\kappa > 0$ and every $\theta \in [-\pi, \pi]$,

$$\limsup_{n\to\infty} \frac{\log|F_\xi(\kappa n e^{\mathrm{i}\theta})|}{\kappa n} \le h_{\sigma^*}(\theta), \quad \text{almost surely.}$$

Since the exponential type of the entire function F_ξ does not exceed 1, for any $\kappa < \pi$, we have

$$\limsup_{r\to\infty} \frac{\log|F_\xi(re^{\mathrm{i}\theta})|}{r} = \limsup_{n\to\infty} \frac{\log|F_\xi(\kappa n e^{\mathrm{i}\theta})|}{\kappa n}.$$

This is a special instance of a classical result that goes back to Pólya and to Vl. Bernstein; for a simple proof of this result see, e.g., [2, Theorem 1.3.5]. Therefore, given $\theta \in [-\pi, \pi]$, almost surely, we have $h^{F_\xi}(\theta) \le h_{\sigma^*}(\theta)$. Since both functions in this inequality are continuous on $[-\pi, \pi]$, we immediately conclude that, almost surely, the inequality holds for all $\theta \in [-\pi, \pi]$. □

We will use Lemmas 7.1.1 and 7.2.1 in the Gaussian case (Theorem 4).

7.3. The next lemma is needed for the mixing case (Theorem 3). As above, we use the notation

$$W_R(\theta) = \sum_{|n|\le N} \xi(n+R)e(n\theta)\, e^{-\frac{n^2}{2R}}, \quad N = R^{1/2}\log R + O(1),$$

fix a non-negative even function $g \in C_0^2[-\frac{1}{2}, \frac{1}{2}]$ with $\int g(\theta)\,\mathrm{d}\theta = 1$, and set

$$\begin{aligned} X_R &= \int_{a-\frac{1}{2m}}^{a+\frac{1}{2m}} \big|W_R(\theta)\big|^2 g(m(\theta - a))\mathrm{d}\theta \\ &= \frac{1}{m} \sum_{|n_1|,|n_2|\le N} \xi(n_1+R)\overline{\xi(n_2+R)} e\big((n_1-n_2)a\big) e^{-(n_1^2+n_2^2)/(2R)} \widehat{g}\Big(\frac{n_2-n_1}{m}\Big). \end{aligned}$$

Lemma 7.3.1. *Suppose ξ is a wide-sense stationary sequence whose spectral measure ρ has no gaps in its support. Then for every $a \in [0, 1]$ and every $m \in \mathbb{N}$, there exists a positive limit*

$$\lim_{R\to\infty} R^{-1/2}\, \mathbb{E}X_R = c(a, m) > 0. \tag{7.3.1}$$

Proof. We have

$$\mathbb{E}X_R = \frac{1}{m} \sum_{|n_1|,|n_2|\le N} \widehat{\rho}(n_2-n_1)\, \widehat{g}\,\Big(\frac{n_2-n_1}{m}\Big)\, e((n_1-n_2)a) e^{-(n_1^2+n_2^2)/2R}.$$

Put $k = n_2 - n_1$, $\ell = n_2 + n_1$. Then

$$|k| \le 2N, \quad |\ell| \le 2N - k, \quad \ell \equiv k \bmod 2,$$

and $n_1^2 + n_2^2 = \frac{1}{2}(k^2 + \ell^2)$. Hence,

$$\mathbb{E}X_R = \frac{1}{m} \sum_{|k|\le 2N} \widehat{\rho}(k)\, \widehat{g}\Big(\frac{k}{m}\Big) e(-ka)\, e^{-k^2/(4R)} \sum_{\substack{|\ell|\le 2N-k \\ \ell\equiv k \bmod 2}} e^{-\ell^2/(4R)}.$$

Because of the cut-off $e^{-k^2/(4R)}$, we discard the sum over $N \le |k| \le 2N$ (recall that $N = R^{1/2}\log R + O(1)$) and consider only the range $|k| \le N$. Then the inner "ℓ-sum" equals $\sqrt{\pi R} + O(R^{-1/2})$, and we get

$$\mathbb{E}X_R = \frac{\sqrt{\pi R}}{m} \sum_{|k|\le N} \widehat{\rho}(k)\, \widehat{g}\Big(\frac{k}{m}\Big) e(-ka) e^{-k^2/(4R)} + O(\log R).$$

Since $\widehat{g} \in l^1(\mathbb{Z})$, by the dominated convergence on $\mathbb{Z}$, we have

$$\lim_{R\to\infty} R^{-1/2}\mathbb{E}X_R = \frac{\sqrt{\pi}}{m} \sum_{k\in\mathbb{Z}} \widehat{\rho}(k)\, \widehat{g}\Big(\frac{k}{m}\Big) e(-ka).$$

The sum on the right-hand side is the density of the convolution $\rho * g_m$ at the point $-a$, where

$$g_m(\theta) = \begin{cases} mg(m\theta), & |\theta| \le 1/(2m) \\ 0, & \text{otherwise.} \end{cases}$$

Since the support of ρ is the whole circle $\mathbb{T}$ and the function g is non-negative, this value is positive. This proves the lemma. □

8 Proof of Theorem 6

8.1. First, we assume that $\xi\colon \mathbb{Z} \to \mathbb{Z}$ is an integer-valued stationary sequence with the spectral measure ρ. Let K_ξ be the convex hull of $\mathrm{spt}(\rho)$, and let K_ξ^* be its reflection in the real axis. Suppose that $\mathrm{spt}(\rho) \neq \mathbb{T}$, i.e., $K_\xi^* \neq \overline{\mathbb{D}}$. By Pólya's theorem (see [14, Theorem 33, Chapter I] or [2, Theorem 1.1.5]), the series

$$f_\xi(w) = \sum_{n\geq 0} \frac{\xi(n)}{w^{n+1}}$$

is analytic on $\widehat{\mathbb{C}} \setminus K_\xi^*$. Since ξ attains only integer values, another theorem of Pólya [2, Theorem 6.2.1] yields that for every fixed ξ, the function f_ξ is rational with poles at roots of 1, $f_\xi = P/Q$, with mutually prime $P, Q \in \mathbb{Z}[w]$ and monic Q.

Next, we use simple algebra. Noting that P is a product of irreducible polynomials, and recalling that if a polynomial is irreducible in $\mathbb{Z}[w]$ then it is also irreducible in $\mathbb{Q}[w]$ ("Gauss lemma"), and that two different irreducible polynomials in $\mathbb{Q}[w]$ are mutually prime, we conclude that P has no common zeroes with Q.

Since every polynomial in $\mathbb{Z}[w]$ is a product of irreducible polynomials, and since the cyclotomic polynomials $\Phi_n(w) = \prod_{\gcd(k,n)=1}(w - e(k/n))$ belong to $\mathbb{Z}[w]$ and are irreducible therein, we see that $Q = \prod_{1\leq k\leq u} \Phi_{n(k)}$. Since f_ξ is analytic on a fixed arc of the unit circle, we obtain that $n(k) \leq M$ for some M independent of ξ. Thus, the set of poles of f_ξ is contained in $\{w\colon w^N = 1\}$ for some $N \geq 1$ independent of ξ.

Furthermore, since $\mathbb{E}|\xi(n)|^2$ is finite (and does not depend on n), applying Chebyshev's inequality and the Borel-Cantelli Lemma, we see that, for any $\lambda > 1/2$, almost surely, $|\xi(n)| = o(n^\lambda)$, whence

$$\max_{|w|=r} |f_\xi(w)| = o((r-1)^{-2}), \quad r \downarrow 1$$

follows. Therefore, all poles of f_ξ are simple and f_ξ can be written in the form $f_\xi(w) = (w^N - 1)^{-1} S(w)$, where S is a polynomial (depending on ξ) and $N \in \mathbb{N}$ does not depend on ξ. Hence, the coefficients $\xi(n)$ of f_ξ are eventually periodic with period N.

Since the sequence ξ is stationary, we conclude that it is periodic with period N. Indeed, given $M < \infty$, we consider the bounded sequence ξ_M given by

$$\xi_M(n) = \begin{cases} M, & \xi(n) > M, \\ \xi(n), & -M \leq \xi_M(n) \leq M, \\ -M, & \xi(n) < -M. \end{cases}$$

Then ξ_M is also stationary, and the values $\mathbb{E}(\xi_M(k+N) - \xi_M(k))^2$ do not depend on k. On the other hand, almost surely, the sequence $\xi_M(n)$ is eventually periodic with period N, and by the bounded convergence theorem, the values $\mathbb{E}(\xi_M(k+N) - \xi_M(k))^2$ converge to 0 for $k \to \infty$. Hence they are equal to 0, and the sequences $\xi_M(n)$ are periodic with period N for every M. Thus, ξ is periodic with period N. This completes the proof of the first part of Theorem 6.

Note that we used stationarity of ξ only in the last step of the proof. The rest is valid for wide-sense stationary integer-valued sequences. Also note that this last step can be made for wide-sense stationary sequences ξ satisfying the condition $\sup_n \mathbb{E}|\xi(n)|^\kappa < \infty$ for some $\kappa > 2$. Hence, the first statement in Theorem 6 is valid for wide-sense stationary sequences satisfying this moment condition.

8.2. To prove the second part of Theorem 6, we use the following result of Hausdorff [2, Theorem 4.2.4].

If the set A is uniformly discrete, then there exist at most countably many sequences ξ such that the series $f_\xi(w)$ can be analytically continued across an arc in $\mathbb{T}$.

Let μ be a translation invariant probability measure in the space of sequences $A^{\mathbb{Z}}$ corresponding to the stationary sequence ξ. Suppose that there exists a lacuna in the support of the spectral measure ρ. Then, as above, by Lemma 7.2.1 combined with Pólya's theorem, almost surely, the function f_ξ has an analytic continuation across an arc in $\mathbb{T}$; and by the theorem of Hausdorff, the measure μ has at most countable support. Since μ is translation invariant, we conclude that, almost surely, the sequence $\xi(n)$ is periodic. Since μ is ergodic, the period is not random and the stationary sequence ξ is periodic.

9 Proof of Theorem 3

9.1. The proof of Theorem 3 needs in addition an estimate of the fourth order correlations.

Lemma 9.1.1. *Let ξ be a bounded stationary sequence of random variables, and let the maximal correlation coefficient of ξ satisfy*

$$r(t) = O\big((\log t)^{-\kappa}\big)\,, \quad t \to \infty\,,$$

with some $\kappa > 1$. Then, for every $a \in [0,1]$ and every $m \in \mathbb{N}$,

$$\mathbb{E}\big(X_R - \mathbb{E}X_R\big)^2 = O\Big(\frac{R}{(\log R)^{\kappa_1}}\Big), \quad R \to \infty,$$

with some $1 < \kappa_1 < \kappa$.

Proof. We have

$$
\begin{aligned}
&m^2\mathbb{E}\big(X_R-\mathbb{E}X_R\big)^2\\
&\quad=\mathbb{E}\Big[\sum_{|n_1|,|n_2|\le N}\big(\xi(n_1+R)\overline{\xi(n_2+R)}-\mathbb{E}\{\xi(n_1+R)\overline{\xi(n_2+R)}\}\big)\widehat{g}\big(\frac{n_2-n_1}{m}\big)\\
&\qquad\qquad\times e((n_1-n_2)a)\,e^{(n_1^2+n_2^2)/(2R)}\Big]^2\\
&\quad=\sum_{|n_1|,\dots,|n_4|\le N} C(n_1,n_2,n_3,n_4)\,\widehat{g}\big(\frac{n_2-n_1}{m}\big)\widehat{g}\big(\frac{n_4-n_3}{m}\big)\\
&\qquad\qquad\times e((n_1-n_2+n_3-n_4)a)\,e^{(n_1^2+n_2^2+n_3^2+n_4^2)/(2R)},
\end{aligned}
$$

where

$$C(n_1,n_2,n_3,n_4)=\mathbb{E}\big\{\eta(n_1,n_2)\cdot\eta(n_3,n_4)\big\}$$

with

$$\eta(n_i,n_j)=\xi(n_i+R)\overline{\xi(n_j+R)}-\mathbb{E}\big\{\xi(n_i+R)\overline{\xi(n_j+R)}\big\}.$$

Let I be the interval with endpoints n_1 and n_2, and let J be the interval with endpoints n_3 and n_4. Setting $t=\mathrm{dist}(I,J)$, we estimate C by the maximal correlation coefficient $r(t)$:

$$\big|C(n_1,n_2,n_3,n_4)\big|\le r(t)\sqrt{\mathbb{E}|\eta(n_1,n_2)|^2\cdot\mathbb{E}|\eta(n_3,n_4)|^2}\le 4r(t)\|\xi\|_\infty^4.$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\big(X_R-\mathbb{E}X_R\big)^2&=O(1)\|\xi\|_\infty^4\\
&\qquad\times\sum_{|n_1|,\dots,|n_4|\le N} r(t)\frac{m^2}{1+(n_1-n_2)^2}\,\frac{m^2}{1+(n_3-n_4)^2}e^{-(n_1^2+n_2^2+n_3^2+n_4^2)/(2R)}.
\end{aligned}
$$

To estimate the sum on the right-hand side, we put

$$k_1=n_1-n_2,\ \ \ell_1=n_1+n_2,\quad k_2=n_3-n_4,\quad \ell_2=n_3+n_4.$$

Then $t\ge\frac12\big(|\ell_1-\ell_2|-(|k_1|+|k_2|)\big)$, and we need to estimate the sum

$$\sum_{|k_1|,|k_2|,|\ell_1|,|\ell_2|\le 2N} r\big(\tfrac12\big(|\ell_1-\ell_2|-(|k_1|+|k_2|)\big)\big)\frac{e^{-(k_1^2+k_2^2+\ell_1^2+\ell_2^2)/(4R)}}{(1+k_1^2)(1+k_2^2)}.$$

Here and later on, $r(t)=r(\max([t],0))$, where $[t]$ is the maximal integer not exceeding t.

We split this sum into two parts: the first taken over $|\ell_1-\ell_2|\le 2(|k_1|+|k_2|)$, the second taken over $|\ell_1-\ell_2|>2(|k_1|+|k_2|)$.

The first sum does not exceed

$$\begin{aligned}
&\sum_{k_1,k_2\ge 0} \frac{e^{-(k_1^2+k_2^2)/(4R)}}{(1+k_1^2)(1+k_2^2)} \cdot O(1+k_1+k_2)\cdot O(N)\\
&= O(R^{1/2}\log R) \sum_{k_1,k_2\ge 0} \frac{1+k_1+k_2}{(1+k_1^2)(1+k_2^2)}\, e^{-(k_1^2+k_2^2)/(4R)}\\
&= O(R^{1/2}\log R)\Big[\sum_{k\ge 1} \frac{e^{-k^2/(4R)}}{k} + O(1)\Big] = O\big(R^{1/2}(\log R)^2\big)\,,
\end{aligned}$$

while the second sum is bounded by

$$O(1) \sum_{|\ell_1|,|\ell_2|\le 2N} r\big(\tfrac14|\ell_1-\ell_2|\big)e^{-(\ell_1^2+\ell_2^2)/(4R)} = O(\sqrt{R}) \sum_{\ell>1} r\big(\tfrac14\ell\big)e^{-\ell^2/(8R)}\,.$$

Recall that $r(t) = O\big(1/\log^{\kappa} R\big)$, and let $\kappa = 1+2\varepsilon$. Then

$$\begin{aligned}
\sum_{\ell>1} r\big(\tfrac14\ell\big)e^{-\ell^2/(8R)} &\le \sum_{1\le\ell\le\sqrt{R}\log^{\varepsilon} R} r\big(\tfrac14\ell\big) + \sum_{\ell>\sqrt{R}\log^{\varepsilon} R} e^{-\ell^2/(8R)}\\
&= O(1)\Big[\frac{\sqrt{R}\,\log^{\varepsilon} R}{\log^{1+2\varepsilon} R} + \sqrt{R}\,e^{-c\log^{2\varepsilon} R}\Big] = O(1)\,\frac{\sqrt{R}}{\log^{1+\varepsilon} R}.
\end{aligned}$$

This completes the proof of Lemma 9.1.1. □

9.2. Now, the proof of Theorem 3 is straightforward. Since the maximal correlation coefficient $r(m)$ decays to 0 as $m\to\infty$, the bounded stationary sequence ξ is **linearly regular**, i.e., $\bigcap_m L^2_{(-\infty,m]} = \{0\}$, where $L^2_{(-\infty,m]}$ is the Hilbert space which consists of the random variables measurable with respect to the σ-algebra generated by $\{\xi(n)\colon -\infty < n \le m\}$ that have a finite second moment. Then the spectral measure ρ has a density $|f|^2$, where f belongs to the Hardy space $H^2(\mathbb{T})$, see [10, Chapter XVII, §1], and therefore, $\operatorname{spt}(\rho) = \mathbb{T}$. Hence, we are in the assumptions of Lemma 7.3.1. Fix $a\in[0,1]$, $m\ge 1$. Then, combining Lemma 7.3.1 with Lemma 9.1.1 and using Chebyshev's inequality, we see that, for some $c = c(a,m) > 0$, $\kappa > 0$,

$$\mathbb{P}\{X_R < c\sqrt{R}\} = O(R^{-1})\,\mathbb{E}\big(X_R - \mathbb{E}X_R\big)^2 = O\big((\log R)^{-\kappa}\big).$$

Then we take any $\delta\in(\kappa^{-1},1)$, and put $R_j = e^{j^{\delta}}$. This is a thick sequence (i.e., $R_{j+1}/R_j \to 1$), while

$$\mathbb{P}\{X_{R_j} < c\sqrt{R_j}\} = O\big(j^{-\delta\kappa}\big)$$

with $\delta\kappa > 1$. Applying the Borel–Cantelli Lemma, we get estimate (4.3.1). Then Lemma 4.3.1 does the job.

10 Proof of Theorem 4

Given $z = re^{i\theta}$, $F_\xi(z)$ is a Gaussian random variable. As before, σ^* is the reflection of the spectrum $\sigma(\xi)$ in the real axis. By Lemma 7.1.1,

$$\mathbb{E}|F_\xi(re^{i\theta})|^2 = e^{2h_{\sigma^*}(\theta)r+o(r)}, \quad r \to \infty.$$

Then, for every $\varepsilon > 0$, every $r > r_\varepsilon$, and every $\theta \in [-\pi, \pi]$, we have

$$\mathbb{P}\big\{\log|F_\xi(re^{i\theta})| < (h_{\sigma^*}(\theta)-\varepsilon)r\big\} = \mathbb{P}\big\{|F_\xi(re^{i\theta})| < e^{-\varepsilon r+o(r)}\sqrt{\mathbb{E}|F_\xi(re^{i\theta})|^2}\big\} < e^{-\frac{1}{2}\varepsilon r}$$

(the last inequality is where we are using the Gaussianity of F_ξ). Applying this with $R = j$ and using the Borel–Cantelli Lemma, we see that, given $\theta \in [-\pi, \pi]$,

$$\liminf_{j\to\infty} \frac{1}{j} \log|F_\xi(je^{i\theta})| \ge h_{\sigma^*}(\theta), \quad \text{almost surely.}$$

By Lemma 7.2.1, $h^{F_\xi} \le h_{\sigma^*}$ everywhere on $[-\pi, \pi]$. Therefore, applying Lemma 3.2.1, we conclude that, almost surely, F_ξ has completely regular growth on the ray $\{\arg(z) = \theta\}$ with the indicator $h_{\sigma^*}(\theta)$. To complete the proof, we apply this argument to a dense countable set of θ's.

11 Proof of Theorem 5

Now, ξ is a uniformly almost-periodic sequence. By $\widehat{\xi}$ we denote the Fourier transform of ξ, $\widehat{\xi}\colon \mathbb{T} \to \mathbb{C}$. The spectrum of ξ is $\sigma(\xi) = \{e^{i\lambda} \in \mathbb{T}\colon \widehat{\xi}(e^{i\lambda}) \ne 0\}$; this is an at most countable subset of $\mathbb{T}$.

We use the following result of Bochner.

There exist an enumeration of the spectrum $\sigma(\xi) = \{e^{i\lambda_1}, e^{i\lambda_2}, \dots\}$ *and a sequence of multipliers* $\beta_k^{(m)}$, $k \in \{1, \dots, m\}$, *satisfying* $0 \le \beta_k^{(m)} \le 1$ *and* $\beta_k^{(m)} \to 1$ *as* $m \to \infty$ *and* k *stays fixed, such that the finite exponential sums*

$$\sum_{k=1}^{m} \beta_k^{(m)} \widehat{\xi}(e^{i\lambda_k})e^{i\lambda_k n}$$

converge to $\xi(n)$ *uniformly in* $n \in \mathbb{Z}$ *as* $m \to \infty$.

For the proof, see, for instance, [13, Chapter VI, § 5], where the proof is given for almost periodic functions. The proof for almost periodic sequences is almost the same.

As before, by σ^* we denote the reflection of $\sigma(\xi)$ in the real axis. First, we show that $h^F \le h_{\sigma^*}$ everywhere, and then that $|F(re^{i\theta})| \ge c(\theta)e^r$ with some $c(\theta) > 0$, whenever $\theta \in \sigma^*$ and $r \ge r_0(\theta)$. Then Lemma 3.3.1 does the job.

11.1. The following lemma is an old result of Bochner and Bohnenblust [3]. The proof given here follows that in [14, Chapter VI].

Lemma 11.1.1. *Everywhere,* $h^{F_\xi} \le h_{\sigma^*}$.

Proof. If the spectrum $\sigma(\xi)$ is dense on $\mathbb{T}$, then $h_{\sigma^*} \equiv 1$, and there is nothing to prove. So we assume that there is an open arc $J \subset \mathbb{T}$ such that $\sigma(\xi) \bigcap J = \varnothing$. Rotating the complex plane, $z \mapsto ze^{-\mathrm{i}t}$, we shift the spectrum $\sigma^*(\xi)$ and the indicator function h^{F_ξ} by t. Therefore, without loss of generality, we may assume that $\sigma(\xi)$ is contained in the arc $\{e^{\mathrm{i}\theta} \colon |\theta| \le \pi - \delta\}$ for some $\delta > 0$. We need to show that the indicator diagram I^F is contained in the closed convex hull of $\{e^{\mathrm{i}\theta} \colon |\theta| \le \pi - \delta\}$.

By our assumption, the functions

$$w \mapsto \Xi_m(w) = \sum_{k=1}^{m} \beta_k^{(m)} \widehat{\xi}(e^{\mathrm{i}\lambda_k}) e^{\mathrm{i}\lambda_k w}$$

are entire functions of exponential type at most $\pi - \delta$. By Bochner's theorem, given $\varepsilon > 0$, there exists M_ε such that, for all $m_1, m_2 > M_\varepsilon$,

$$\| \Xi_{m_1} - \Xi_{m_2} \|_{\ell^\infty(\mathbb{Z})} < \varepsilon.$$

Then, by Cartwright's theorem [14, Chapter IV, Theorem 15],

$$\| \Xi_{m_1} - \Xi_{m_2} \|_{L^\infty(\mathbb{R})} < C(\delta)\varepsilon;$$

and, invoking one of the Phragmén–Lindelöf theorems, we conclude that the sequence of entire functions Ξ_m converges to an entire function Ξ uniformly in any horizontal strip. Obviously, the entire function Ξ interpolates the sequence ξ at $\mathbb{Z}$, the exponential type of Ξ does not exceed $\pi - \delta$, and Ξ is bounded on $\mathbb{R}$. Thus, the indicator diagram of Ξ is contained in the interval $[(-\pi + \delta)\mathrm{i}, (\pi - \delta)\mathrm{i}]$ of the imaginary axis. It is worth noting that in what follows, we use only the fact that the exponential type of Ξ does not exceed $\pi - \delta$.

Now, consider the Taylor series

$$f(s) = \sum_{n \ge 0} \xi(n) s^n,$$

which is analytic in the unit disk. Since the coefficients $\xi(n)$ can be interpolated by an entire function of exponential type at most $\pi - \delta$, the function f can be analytically continued across the arc $\{e^{\mathrm{i}\theta} \colon |\theta - \pi| < \delta\}$ to $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. This is a classical result that goes back to Carlson and Pólya (see [14, Appendix 1, § 5] or

[2, Theorem 1.3.1]). On the other hand, the function $w^{-1}f(w^{-1})$ is nothing but the Laplace transform of the entire function F_ξ, and, as we have seen, this function is analytic outside the closed convex hull of the arc $\{e^{\mathrm{i}\theta}\colon |\theta|\le \pi-\delta\}$. Then, by Pólya's theorem (see [14, Chapter I, Theorem 33] or [2, Theorem 1.1.5]), the indicator diagram I^{F_ξ} is contained in the closed convex hull of $\{e^{\mathrm{i}\theta}\colon |\theta|\le\pi-\delta\}$. This completes the proof of the lemma. □

11.2. Here, we show that F_ξ grows as e^r on the rays corresponding to the set σ^*.

Lemma 11.2.1. *For every $\theta\in\sigma^*$, there exists $c(\theta)>0$ and $r(\theta)<\infty$ such that*

$$\left|F_\xi(re^{\mathrm{i}\theta})\right| \ge c(\theta)e^r, \quad r\ge r(\theta).$$

Proof. Once again, we use Bochner's theorem. We fix $e^{\mathrm{i}\lambda_j}\in\sigma(\xi)$, take $m\ge j$ and put

$$\xi_m(n) = \sum_{k=1}^m \beta_k^{(m)}\widehat{\xi}(e^{\mathrm{i}\lambda_k})e^{\mathrm{i}\lambda_k n}.$$

Then, uniformly in z,

$$\left|F_\xi(z) - F_{\xi_m}(z)\right| \le \varepsilon_m e^{|z|}, \quad \text{with } \varepsilon_m\to 0. \tag{11.2.1}$$

Furthermore, $F_{\xi_m}(z)$ is a finite sum of exponential functions

$$F_{\xi_m}(z) = \sum_{k=1}^m \beta_k^{(m)}\widehat{\xi}(e^{\mathrm{i}\lambda_k})e^{ze^{\mathrm{i}\lambda_k}},$$

whence

$$\begin{aligned}\left|F_{\xi_m}(re^{-\mathrm{i}\lambda_j})\right| &\ge \beta_j^{(m)}|\widehat{\xi}(e^{\mathrm{i}\lambda_j})|e^r - \sum_{\substack{k=1\\k\ne j}}^m \beta_k^{(m)}|\widehat{\xi}(e^{\mathrm{i}\lambda_k})|e^{r\cos(\lambda_k-\lambda_j)}\\ &\ge \beta_j^{(m)}|\widehat{\xi}(e^{\mathrm{i}\lambda_j})|e^r - C_m e^{(1-\delta_m)r},\end{aligned}$$

follows with some $\delta_m>0$. Therefore,

$$\liminf_{r\to\infty} e^{-r}\left|F_{\xi_m}(re^{-\mathrm{i}\lambda_j})\right| \ge \beta_j^{(m)}|\widehat{\xi}(e^{\mathrm{i}\lambda_j})| \ge \tfrac12|\widehat{\xi}(e^{\mathrm{i}\lambda_j})|, \tag{11.2.2}$$

provided that $m\ge m_0(j)$. Juxtaposing (11.2.1) and (11.2.2), we obtain Lemma 11.2.1. □

To finish off the proof of Theorem 5, we observe that, by Lemmas 11.1.1 and 11.2.1, the function F_ξ satisfies the assumptions of Lemma 3.3.1. Theorem 5 then follows readily.

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