NON-REAL ZEROS OF DERIVATIVES OF MEROMORPHIC FUNCTIONS

By

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Abstract. A number of results are proved concerning non-real zeros of derivatives of real and strictly non-real meromorphic functions in the plane.

1 Introduction

If f is a non-constant meromorphic function in the plane, then so is the function $g(z) = \widetilde{f}(z) = \overline{f(\bar{z})}$. Here f is called real if $g = f$ and strictly non-real if g/f is non-constant. If f and $g = \widetilde{f}$ have zeros and poles at the same points with the same multiplicities, which certainly is the case if all zeros and poles of f are real, then g/f has no zeros and poles and has modulus 1 on $\mathbb{R}$, and so $\widetilde{f} = e^{ih} f$, where h is a real entire function.

There has been extensive research into the existence of non-real zeros of derivatives of real entire or meromorphic functions [2, 3, 5, 19, 20, 28, 32, 33, 37, 46, 48], but rather less in the strictly non-real case. Meromorphic functions which, together with all their derivatives, have only real zeros were classified in [24, 25, 26]. The only other general result treating the strictly non-real case appears to be the following theorem from [18].

Theorem 1.1 ([18, Theorem 1])**.** *Let f be a strictly non-real meromorphic function in the plane with only real poles, such that f, f' and f'' have only real zeros. Then f has one of the following forms:*

(I) $f(z) = Ae^{Bz}$;

(II) $f(z) = A\left(e^{i(cz+d)} - 1\right)$;

(III) $f(z) = A\exp(\exp(i(cz+d)))$;

(IV) $f(z) = A\exp\left[K(i(cz+d) - \exp(i(cz+d)))\right]$;

(V) $f(z) = \left(A\exp[-2i(cz+d) - 2\exp(2i(cz+d))]\right)/\sin^2(cz+d)$;

(VI) $f(z) = A/[e^{i(cz+d)} - 1]$.

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Here $A, B \in \mathbb{C}$, while c, d and K are real with $K \leq -1/4$.

In the last example (VI), it is easy to verify that f is strictly non-real but f' is not, while f and $g = \widetilde{f}$ have no zeros and the same poles, and $f^{(m)}$ and $g^{(m)}$ have the same zeros for all $m \geq 1$. Moreover, f' has no zeros, and f'' has only real zeros; but if $m \geq 3$, then $f^{(m)}$ has infinitely many non-real zeros by [33, Lemma 3.1]. We prove the following theorem and use standard terminology from [14].

Theorem 1.2. *Let f be a strictly non-real meromorphic function in the plane, and assume that*

(i) *f has finitely many zeros,*
(ii) *f has finitely many non-real poles,*
(iii) *$f^{(m)}$ has finitely many non-real zeros for some $m \geq 2$.*

Then the Nevanlinna characteristic of f'/f satisfies

$$T(r, f'/f) = O(r \log r) \quad \text{as } r \to \infty. \tag{1.1}$$

If, in addition, f has finite order, then one of the following two conclusions holds:

$$f = R_1 e^{P_1} \quad \text{for some rational function } R_1 \text{ and polynomial } P_1; \tag{1.2}$$

$$m = 2 \text{ and } f(z) = A(A_1 z + 1)/[U_1(z) e^{i(B_1 z + B_2)} - 1], \tag{1.3}$$

where $A \in \mathbb{C}$, while U_1 is a rational function with $|U_1(x)| = 1$ for all $x \in \mathbb{R}$, and A_1, B_1, B_2 are real numbers with $B_1 \neq 0$.

Conversely, if f is as in (1.3)*, then f satisfies* (i), (ii) and (iii) with $m = 2$.

For example, if $g(z) = z/(e^{iz} - 1)$, then all but finitely many zeros of g'' are real by Theorem 1.2 (see also Lemma 2.5(II) below), but it is easy to check that g' has infinitely many non-real zeros. Obviously, if f is transcendental and is given by (1.2), then every derivative of f has finitely many zeros. Examples (III), (IV), and (V) arising from Theorem 1.1 show that (1.1) is not far from being sharp and that, at least for $m = 2$, the hypothesis that f has finite order is not redundant in the second assertion of Theorem 1.2. Note that the analogous problem when f is real was treated, but again not fully solved, in [20, 35, 37, 46].

The next result deals with strictly non-real meromorphic functions f with only real zeros and poles and such that f''/f is real. Such functions do exist, but the following theorem shows that, except in one trivial case, the second derivative has at least one non-real zero.

Theorem 1.3. *Let f be a strictly non-real transcendental meromorphic function in the plane with finitely many zeros and poles in $\mathbb{C} \setminus \mathbb{R}$, and assume that f''/f is real. Then*

$$\frac{f'}{f} = -\frac{\beta'}{2\beta} + i\beta, \quad \frac{f'}{f} + \frac{g'}{g} = -\frac{\beta'}{\beta}, \tag{1.4}$$

where $g = \widetilde{f}$ and β is real and meromorphic in the plane, with finitely many poles, none of them real, and finitely many non-real zeros. Furthermore, f has finitely many zeros.

If, in addition, f'' has finitely many non-real zeros, then f satisfies (1.2)*: in particular, if all zeros and poles of f and f'' are real, then $f(z) = Ae^{iBz}$, where $A, B \in \mathbb{C}$ and B is real.*

It follows from (1.4) that a zero of β is a pole of f and hence of f''/f, while a pole of β is a zero of f or $\widetilde{f}$; thus, if f has only real zeros and f''/f is entire, then β has neither zeros nor poles, and so Theorem 1.3 contains [19, Theorem 5]. Observe further that if β is a real entire function with real zeros, all of even multiplicity, then (1.4) defines a strictly non-real meromorphic function f with real poles and no zeros, such that f''/f is real.

Corollary 1.1. *Let H be a non-constant real meromorphic function in the plane with only real zeros and poles. Then any strictly non-real meromorphic solution in the plane of the equation $w'' + Hw = 0$ has at least one non-real zero.*

Corollary 1.1 follows at once from the last part of Theorem 1.3, since any pole of a meromorphic solution of $w'' + Hw = 0$ is automatically a pole of H. The assertion of Corollary 1.1 is not valid for real solutions, as the example $w = \tan z$, $H(z) = -2\sec^2 z$ immediately shows.

The next two main results of this paper deal with the case of real functions. It is known [3, 48] that if f is a real transcendental entire function, then f and f'' have only real zeros if and only if f belongs to the Laguerre-Pólya class LP, consisting of all entire functions which are locally uniform limits of real polynomials with real zeros, in which case all derivatives of f have only real zeros. For the real meromorphic case, the following was conjectured in [19].

Conjecture 1.1 ([19])**.** Let f be a real transcendental meromorphic function in the plane with at least one pole, and assume that all zeros and poles of f, f' and f'' are real, and that all poles of f are simple. Then

$$f(z) = C\tan(az+b) + Dz + E, \quad a, b, C, D, E \in \mathbb{R}. \tag{1.5}$$

Without the condition that f has only simple poles, there are further examples for which f, f' and f'' have only real zeros and poles, such as $(2 + \tan z)^2$ (see [42]), as well as a substantial collection whose existence is established by [23, Theorem 5]. While Conjecture 1.1 appears to be difficult to resolve in general, results proved in [19, 27], and refined further in [33, 34, 44], show in particular that the conjecture is true, subject to the additional hypothesis that f' omits some finite value, as is the case for the functions in (1.5).

Theorems 1.4 and 1.5 below will resolve two further special cases of Conjecture 1.1, each of them linked to functions of the form (1.5). Consider first a real transcendental meromorphic function f in the plane which maps the open upper half-plane H^+ into itself; of course, f maps the open lower half-plane H^- into itself. Such a function f has only real zeros and poles, all necessarily simple, and by a theorem of Chebotarev [39, Ch. VII, p.310, Theorem 2], has a representation

$$\begin{gathered} f(z) = Az + B - \frac{d}{z} + \sum A_k \left(\frac{1}{a_k - z} - \frac{1}{a_k} \right), \\ B \in \mathbb{R}, \quad a_k \in \mathbb{R} \setminus \{0\}, \quad A, d, A_k \in [0, \infty), \quad \sum \frac{A_k}{a_k^2} < \infty. \end{gathered} \tag{1.6}$$

A well known example is $f(z) = \tan z$. Conversely, any function f given by an expansion (1.6) is real and maps H^+ into itself. This class is closely linked to the Laguerre-Pólya class because if $g \in LP$, then $f = -g'/g$ either is constant or satisfies (1.6); see [39, 48].

Theorem 1.4. *Let f be a transcendental meromorphic function in the plane given by a series expansion* (1.6). *If $m \geq 3$, then $f^{(m)}$ has infinitely many non-real zeros. If f'' has only real zeros, then f satisfies* (1.5).

If f'' has finitely many non-real zeros, then

$$f(z) = Az + B + \frac{R(z)e^{icz} - 1}{A_1 R(z) e^{icz} - \overline{A_1}}, \tag{1.7}$$

where $A \geq 0$, $B \in \mathbb{R}$, $c \in (0, \infty)$, $A_1 \in H^+$, and R is a rational function with all its zeros in H^+ and all its poles in H^-, and with $|R(x)| = 1$ for all $x \in \mathbb{R}$.

Conversely, if f is given by (1.7) with R and the coefficients as in the last conclusion of Theorem 1.4, then f maps H^+ into itself, and all but finitely many zeros of f'' are real by [33, Lemma 3.2]. The next result in the direction of Conjecture 1.1 concerns the case where zeros of f'' are zeros of f', as holds, for example, when $f(z) = z - \tan z$.

Theorem 1.5. *Let f be a real transcendental meromorphic function in the plane such that*

(a) *all but finitely many zeros and poles of f and f' are real,*

(b) *all but finitely many zeros of f'' are zeros of f',*

(c) *the poles of f have bounded multiplicities,*

(d) *either f has finitely many multiple poles or f has finitely many simple poles.*

Then f satisfies either (1.2) *or* (1.5).

It would clearly be preferable to know whether Theorem 1.5 holds without hypotheses (c) and (d), but the present method does not deliver this; and in particular, it seems difficult to exclude the possibility that f has simple poles interspersed with double poles. Of course, hypothesis (d) automatically holds if f is as in Conjecture 1.1, or is itself the derivative of a meromorphic function in the plane. Note that [23, Theorem 5] gives rise to the example

$$f(z) = \frac{1}{3}\tan^3 z - \tan z, \quad f'(z) = \tan^4 z - 1, \quad f''(z) = 4\tan^3 z \sec^2 z,$$

for which f, f', and f'' have only real zeros and poles. Here zeros of f'' are zeros of $f' + 1$, rather than of f', and f does not satisfy (1.5). A key ingredient in the proof of Theorem 1.5 is showing that f has finite order, so that the following result from [38] becomes relevant.

Theorem 1.6 ([38, Theorem 3])**.** *Let f be a meromorphic function in the plane with the following properties:*

(i) *f has finite lower order;*

(ii) *the zeros of f' have bounded multiplicities;*

(iii) *all but finitely many zeros of f'' are zeros of f';*

(iv) *there exists $M \in (0, +\infty)$ such that if ζ is a pole of f of multiplicity m_ζ, then $m_\zeta \leq M + |\zeta|^M$;*

(v) *there exist positive real numbers κ and R_0 such that if z is a zero of f'' with $|z| \geq R_0$, then $|f(z) - \alpha z| \geq \kappa |z|$ for all finite non-zero asymptotic values α of f'.*

Then $f'' = Re^P$ with R a rational function and P a polynomial.

Hypotheses (i) and (v) are not redundant in Theorem 1.6, as shown by $f(z) = z - \tan z$ and examples given in [31]. The proof of Theorem 1.5 also relies heavily on the next result.

Theorem 1.7. *Let $n \geq 2$ be an integer, and let f be a meromorphic function of finite lower order in the plane, with infinitely many poles, such that*

(i) *all but finitely many zeros and poles of f' have multiplicity n,*

(ii) *all but finitely many zeros of f'' are zeros of f'.*
Then there exist $a, b, C, \lambda \in \mathbb{C}$ such that

$$f'(z) = C\left(\frac{\lambda e^{az+b} - 1}{e^{az+b} - 1}\right)^n, \quad aC \neq 0, \quad \lambda^n = 1, \quad \lambda \neq 1. \tag{1.8}$$

Furthermore, there does not exist a meromorphic function h in the plane with $h' = f$.

In the converse direction, it follows from Lemma 8.1 below that the function in (1.8) is indeed the derivative of a meromorphic function of finite order in the plane.

It is worth noting that Theorem 1.7 fails completely for infinite lower order, as shown by the following example based on the Mittag-Leffler theorem, which is similar to Shen's construction of Bank-Laine functions with prescribed zeros [49]. Let $n \geq 2$ be an integer, let (a_k) be any complex sequence which tends to ∞ without repetition, and for each k let $b_k = \pm n$. Let G be an entire function with a simple zero at each a_k and no other zeros. Applying the Mittag-Leffler theorem then gives an entire function H such that, for each k,

$$G(z)e^{H(z)} = \frac{z - a_k}{b_k} + O(|z - a_k|^{n+1}) \quad \text{as } z \to a_k.$$

Next, a meromorphic function g in the plane is determined by the formula $g/g' = Ge^H$. This gives, for each k, as $z \to a_k$,

$$\frac{g'(z)}{g(z)} = \frac{b_k}{z - a_k} + O(|z - a_k|^{n-1}), \quad g(z) = (z - a_k)^{b_k}(C_k + O(|z - a_k|^n)), \quad C_k \in \mathbb{C} \setminus \{0\}.$$

Since g'/g has no zeros by construction, the formula $f' = g$ now defines a meromorphic function f in the plane satisfying all the hypotheses of Theorem 1.7 except for that of finite lower order, and each a_k is a zero or pole of f', depending on the sign of b_k. Moreover, g is in fact the $(n-1)$'th derivative of a meromorphic function in the plane.

2 Preliminaries

We require the following theorem.

Theorem 2.1 ([9, 29])**.** *Let f be a meromorphic function in the plane with finitely many zeros but not of the form* (1.2)*. Then $f^{(m)}$ has infinitely many zeros for every $m \geq 2$.*

Lemma 2.1. *Let f be a non-constant meromorphic function in the plane which satisfies at least one of the following two conditions:*

(a) *f and f'' have finitely many non-real zeros and poles;*
(b) *f and $f^{(m)}$ have finitely many non-real zeros, for some $m \geq 3$.*
Then the Tsuji characteristic $T_0(r, f'/f)$ in the upper half-plane satisfies

$$T_0(r, f'/f) = O(\log r) \quad \text{as } r \to \infty. \tag{2.1}$$

Proof. For details of the Tsuji characteristic, see [12, 52].

Case (a) is proved exactly as in [3, Lemma 2.3], by writing

$$F = \frac{f}{f'}, \quad F' = 1 - \frac{ff''}{(f')^2},$$

so that F and $F' - 1$ have finitely many non-real zeros, and (2.1) follows from the method of Hayman's alternative [14, Theorem 3.5, p.60].

In case (b), the result is proved via Frank's method [4, 9] coupled with the Tsuji characteristic. □

Lemma 2.2. *Let H be a non-constant meromorphic function in the plane, and let $G(z) = \overline{H(\bar{z})}$.*

(a) *If the Tsuji characteristics of H and G have growth given by*

$$T_0(r, H) + T_0(r, G) = O(\log r) \quad \text{as } r \to \infty,$$

then the Nevanlinna proximity function $m(r, H)$ satisfies

$$\int_R^\infty \frac{m(r, H)}{r^3} dr = O\left(\frac{\log R}{R}\right) \quad \text{as } R \to \infty. \tag{2.2}$$

(b) *If H satisfies* (2.2) *and $N(r, H) = O(r \log r)$ as $r \to \infty$, then $T(r, H) = O(r \log r)$ as $r \to \infty$.*
(c) *If $H = e^k$, where k is an entire function, and* (2.2) *holds, then k is a polynomial of degree* 1.

Proof. Applying a lemma of Levin-Ostrovskii [3, 12, 40] to H and G gives, as $R \to \infty$,

$$\begin{aligned} \int_R^\infty \frac{m(r, H)}{r^3} dr &= \int_R^\infty \int_0^\pi \frac{\log^+ |H(re^{i\theta})| + \log^+ |G(re^{i\theta})|}{r^3} d\theta dr \\ &\leq \int_R^\infty \frac{T_0(r, H) + T_0(r, G)}{r^2} dr = O\left(\frac{\log R}{R}\right), \end{aligned}$$

which proves (2.2). If H is as in (b) then (2.2) holds with $m(r, H)$ replaced by $T(r, H)$, and the remaining assertions follow from the monotonicity of $T(r, H)$. □

Lemma 2.3. *Let S be a rational function with $|S(x)| = 1$ for all real x, and let a and b be real numbers, with $a \neq 0$. Then all but finitely many solutions of $S(z)e^{i(az+b)} = 1$ are real.*

Proof. This can be deduced from [44, Lemma 6], but the proof is included for completeness. Assume that $S(\infty) = 1 = a$ and $b = 0$, and write $g(z) = S(z)e^{iz} = e^{iz+i\phi(z)}$, in which the principal logarithm $\log S(z) = i\phi(z)$ tends to 0 as $z \to \infty$, and $\phi(x) \in \mathbb{R}$ for real x with $|x|$ large. Denote by N_{NR} the counting function of the non-real 1-points of g. If $m \in \mathbb{Z}$ with $|m|$ large, then the Intermediate Value Theorem gives a solution of the equation $x+\phi(x) = 2m\pi$ in $((2m-1)\pi, (2m+1)\pi)$. Applying Nevanlinna's first fundamental theorem now yields, as $r \to \infty$,

$$\frac{r}{\pi} - O(\log r) + N_{NR}(r) \leq N(r, 1, g) \leq T(r, g) + O(1) \leq \frac{r}{\pi} + O(\log r). \qquad \square$$

Lemma 2.4. *Let f and g be meromorphic functions in the plane such that f, g, and $W = g/f$ are all non-constant. Assume further that*

$$\frac{f^{(m)}}{f} = \frac{g^{(m)}}{g} \tag{2.3}$$

for some integer $m \geq 2$. If m is odd, then every pole of f is a zero or pole of W. If m is even, then at a pole of f of multiplicity p which is neither a zero nor a pole of W, the function W' has a zero of multiplicity $2p + m - 2$, and

$$2N_0(r, f) + (m-2)\overline{N}_0(r, f) \leq N(r, W/W'), \tag{2.4}$$

in which N_0 and $\overline{N}_0$ count only those poles of f which are neither zeros nor poles of W.

Proof. Take a pole z_0 of f of multiplicity p which is neither a zero nor a pole of W; it may be assumed without loss of generality that $z_0 = 0$. Then there exist α and β in $\mathbb{C} \setminus \{0\}$ and a positive integer q such that, as $z \to 0$,

$$f(z) \sim \alpha z^{-p}, \quad V(z) = W(z) - W(0) \sim \beta z^q.$$

The coefficient of z^{-p+q-m} in the Laurent series of $U = (fV)^{(m)} - f^{(m)}V$ near 0 is

$$\alpha\beta\left[(-p+q)\cdots(-p+q-m+1) - (-p)\cdots(-p-m+1)\right].$$

But (2.3) implies that U vanishes identically, so that

$$p\cdots(p+m-1) = r\cdots(r+m-1),$$

where $r = p - q$. Now $r \geq 0$ is impossible, since $r < p$, while $r < 0 \leq r + m - 1$ makes the right-hand side vanish. Thus $s = -(r + m - 1) > 0$, and

$$(-1)^m s \cdots (s + m - 1) = p \cdots (p + m - 1),$$

which forces m to be even and $p = s = -(r + m - 1) = -(p - q + m - 1)$, so that $q = 2p + m - 1$. □

Lemma 2.4 may be applied, in particular, if f is a strictly non-real meromorphic function in the plane with finitely many non-real zeros and poles and such that $f^{(m)}/f$ is real for some integer $m \geq 2$: to see this, take $g(z) = \widetilde{f}(z) = \overline{f(\overline{z})}$. If m is odd, it follows that f has finitely many poles, while if m is even, then (2.4) yields

$$2N(r, f) + (m - 2)\overline{N}(r, f) \leq T(r, W'/W) + O(\log r) \leq 2m(r, f'/f) + O(\log r),$$

as is the case for $m = 2$ and the examples mentioned following Theorem 1.3.

Lemma 2.5. *Let T be a rational function with $|T(x)| = 1$ for all real x, let $K \not\equiv 0$ be a polynomial, and let a and b be real numbers with $a \neq 0$. Let*

$$f(z) = K(z)F(z) = \frac{K(z)}{T(z)e^{i(az+b)} - 1}.$$

(I) *For each $m \geq 3$, the function $f^{(m)}$ has infinitely many non-real zeros.*
(II) *If, in addition, K has degree at most 1, then all but finitely many zeros of f'' are real if and only if $L = K'/K$ is real.*

Proof. It may be assumed that $a = 1$, $b = 0$, and $T(\infty) = 1$. For $|z|$ large and $\zeta \in \mathbb{C}$, write

$$U(z) = iz + \log T(z), \quad f(z) = \frac{K(z)}{e^{U(z)} - 1}, \quad H(\zeta) = \frac{1}{e^{i\zeta} - 1}.$$

using the principal branch of the logarithm.

Part (I) is similar to [33, Lemma 3.2]. Let $m \geq 3$, denote positive constants by c_j, and let w be a non-real zero of $H^{(m)}$, the existence of which is assured by [33, Lemma 3.1]. Take a small positive t such that $|H^{(m)}(z)| \geq c_1$ and $|H^{(j)}(z)| \leq c_2$ for $0 \leq j \leq m$ and $t \leq |z - w| \leq 3t$. Now let n be a large positive integer, and let $t \leq |z - w - 2\pi n| \leq 3t$. Then $c_3 \leq |e^{iz} - 1| \leq c_4$ and

$$F(z) = \frac{1}{e^{U(z)} - 1} = \frac{1}{e^{iz}(1 + o(1)) - 1} = \frac{1}{e^{iz} - 1 + o(1)} = \frac{1 + o(1)}{e^{iz} - 1} = H(z) + o(1).$$

For $|z - w - 2\pi n| = 2t$, applying Cauchy's estimate for derivatives yields

$$\begin{aligned} F^{(j)}(z) &= H^{(j)}(z) + o(1) = O(1) \quad \text{for } 0 \leq j \leq m, \\ f^{(m)}(z) &= K(z)F^{(m)}(z) + \cdots + K^{(m)}(z)F(z) \\ &= K(z)F^{(m)}(z) + o(1)K(z) = K(z)H^{(m)}(z)(1 + o(1)). \end{aligned}$$

Since $w + 2\pi n$ is a zero of $H^{(m)}$, the assertion of part (I) now follows at once from Rouché's theorem.

To prove part (II), assume that K' is constant, and write $f'' = 2K'F' + KF''$ and

$$\begin{aligned} f''(e^U - 1)^3 &= e^{2U}(K(U')^2 - KU'' - 2K'U') + e^U(K(U')^2 + KU'' + 2K'U') \\ &= e^U(K(U')^2 + KU'' + 2K'U')(1 - Qe^U), \qquad (2.5) \\ Q &= \frac{U'' + 2LU' - (U')^2}{U'' + 2LU' + (U')^2}, \quad L = \frac{K'}{K}. \end{aligned}$$

Here Q is rational but not identically zero, since f'' has infinitely many zeros by Theorem 2.1. Moreover, if x is real with $|x|$ large, then $U'(x)$ and $U''(x)$ have zero real part, and $U'(x)^2$ is real. If all but finitely many zeros of f'' are real, then there exist $x \in \mathbb{R}$ with $|x|$ arbitrarily large such that $Q(x)e^{U(x)} = 1$; and so $|Q(x)| = 1$, which implies that x is a zero of $Q\widetilde{Q} - 1$, from which it follows that $Q\widetilde{Q} \equiv 1$ and $L \equiv \widetilde{L}$ as asserted. On the other hand, if L is real, then $|Q(x)| = 1$ on $\mathbb{R}$, so that all but finitely many zeros of f'' are real by (2.5) and Lemma 2.3. □

Lemma 2.6. *Let S, M and V be rational functions with $S(\infty) = 1$, $M \not\equiv 0$ and $V(\infty) \neq 0$, and let a and b be complex numbers with $a \neq 0$. For $|z|$ large, write $U(z) = az + b + \log S(z)$, using the principal branch of the logarithm. Assume that the function $f(z)$ is meromorphic for $|z|$ large and satisfies*

$$\frac{f'}{f} = \frac{M'}{M} + \frac{V}{e^U - 1}.$$

Then, for each $n \in \mathbb{N}$,

$$\frac{f^{(n)}}{f} = \frac{M^{(n)}}{M} + \frac{V_n}{(e^U - 1)^n}, \quad V_n = \sum_{j=0}^{n-1} R_{j,n} e^{jU}, \qquad (2.6)$$

in which the coefficients $R_{j,n}$ are rational functions and satisfy, as $z \to \infty$,

$$R_{0,n}(z) \sim V(z)^n \quad \text{and} \quad R_{n-1,n}(z) \sim V(z)(-U'(z))^{n-1}. \qquad (2.7)$$

Proof. Proceeding by induction on n, assume that $n \in \mathbb{N}$ and that (2.6) and (2.7) both hold, as is evidently the case for $n = 1$, with $V = V_1 = R_{0,1}$. Then (2.6) yields

$$\frac{f^{(n+1)}}{f} = \frac{M^{(n+1)}}{M} - \frac{M^{(n)}M'}{M^2} + \frac{V_n'}{(e^U-1)^n} - \frac{nV_nU'e^U}{(e^U-1)^{n+1}} + \frac{M^{(n)}M'}{M^2} + \frac{V_nM'/M}{(e^U-1)^n} + \frac{M^{(n)}V/M}{e^U-1} + \frac{V_nV}{(e^U-1)^{n+1}}.$$

This leads to (2.6), with n replaced by $n+1$ and

$$V_{n+1} = V_n'(e^U-1) - nV_nU'e^U + (V_nM'/M)(e^U-1) + (M^{(n)}V/M)(e^U-1)^n + V_nV.$$

Since $V_n' = \sum_{j=0}^{n-1}(R_{j,n}' + jU'R_{j,n})e^{jU}$, it follows that

$$R_{0,n+1} = -R_{0,n}' - R_{0,n}M'/M + (-1)^n M^{(n)}V/M + R_{0,n}V$$

and

$$R_{n,n+1} = R_{n-1,n}' + (n-1)U'R_{n-1,n} - nU'R_{n-1,n} + R_{n-1,n}M'/M + M^{(n)}V/M.$$

In view of (2.7) and the fact that $V(\infty) \neq 0$, this gives $R_{0,n+1}(z) \sim R_{0,n}(z)V(z)$ and

$$R_{n,n+1}(z) = -U'(z)R_{n-1,n}(z)(1+o(1)) + o(|V(z)|) \sim -U'(z)R_{n-1,n}(z),$$

as $z \to \infty$, and the induction is complete. □

Lemma 2.7 ([36], Lemma 4.7). *Let f be a transcendental meromorphic function in the plane, and let $k \in \mathbb{N}$. Let E be an unbounded subset of $[1,\infty)$ with the following property. For each $r \in E$ there exist real $\theta_1(r) < \theta_2(r) \le \theta_1(r) + 2\pi$ and an arc $\Omega_r = \{re^{i\theta} : \theta_1(r) \le \theta \le \theta_2(r)\}$ such that*

$$\lim_{r\to\infty, r\in E} \max\{|z^{2k} f^{(k)}(z)/f(z)| : z \in \Omega_r\} = 0.$$

Let $N = N(r)$ satisfy $0 \le \log N(r) \le o(\log r)$ as $r \to \infty$ in E. Then f satisfies, for all sufficiently large $r \in E$,

$$\left|\frac{zf'(z)}{f(z)}\right| \le kN(r)$$

for all $z \in \Omega_r$ outside a union $U(r)$ of open discs having sum of radii at most $r(k-1)/N(r)$.

Lemma 2.8. *Let $k \ge 2$ and $\rho, \sigma \in (0, \pi/2)$, and let $K_0 \in (0,\infty)$. Then there exists $K_1 \in (0,\infty)$, depending only on k, ρ, σ, and K_0, with the following property. If g is an analytic function on the domain*

$$D = \{z \in \mathbb{C} : 1/2 < |z| < 2,\ 0 < \arg z < \pi\}$$

such that g and $g^{(k)}$ have no zeros in D, and if

$$\min\{\,|g'(e^{i\theta})/g(e^{i\theta})| : \rho \le \theta \le \pi - \rho\} \le K_0,$$

then $|g'(e^{i\theta})/g(e^{i\theta})| \le K_1$ for all $\theta \in [\sigma, \pi - \sigma]$.

Lemma 2.8 is standard, and follows from the fact that if $\mathcal{G}$ is the family of analytic functions on D such that g and $g^{(k)}$ have no zeros in D, then the logarithmic derivatives g'/g, $g \in \mathcal{G}$, form a normal family on D [4, 47, 55]. The next lemma involves the Laguerre-Pólya class LP already mentioned in the introduction [39].

Lemma 2.9. *Let $g \not\equiv 0$ belong to LP, let M be a meromorphic function in the plane, and write*

$$Q = 4M^3 + 6MM' + M'', \quad Q' = 12M^2M' + 6(M')^2 + 6MM'' + M'''. \tag{2.8}$$

(A) *If $M = R - g'/g$, in which g has infinitely many zeros and R is a real rational function with $R(\infty)$ finite, then $Q'(x)$ is positive or infinite for all x in $\mathbb{R}$ with $|x|$ sufficiently large.*

(B) *If $M = -g'/g$ is non-constant, then $Q'(x)$ is positive or infinite for all $x \in \mathbb{R}$.*

Proof. Assume first that M is as in (A) and that $x \in \mathbb{R}$ with $|x|$ large. Then the standard representation [39] (see also (1.6)) for the logarithmic derivative of a function in LP leads to

$$M' = R' - \left(\frac{g'}{g}\right)', \quad M'(x) = R'(x) + C_0 + \sum \frac{1}{(a_k - x)^2} \ge \sum_{|a_k| \le |x|} \frac{1}{4|x|^2} - O(|x|^{-2}),$$

in which $C_0 \ge 0$ and the a_k are the zeros of g, repeated according to multiplicity, as well as

$$M'''(x) = \sum \frac{6}{(a_k - x)^4} + O(|x|^{-4}) \ge \sum_{|a_k| \le |x|} \frac{3}{8|x|^4} - O(|x|^{-4}).$$

This gives

$$M'(x) \sim C_0 + \sum \frac{1}{(a_k - x)^2}, \quad M'''(x) \sim \sum \frac{6}{(a_k - x)^4}. \tag{2.9}$$

Write

$$A = |M(x)|, \quad B = M'(x) > 0, \quad C = |M''(x)|, \quad D = M'''(x) > 0.$$

Then the Cauchy-Schwarz inequality and (2.9) deliver

$$\begin{aligned} C &\le O(|x|^{-3}) + 2\sum \left(\frac{1}{|a_k - x|} \cdot \frac{1}{|a_k - x|^2}\right) \\ &\le o(\sqrt{BD}) + 2\sqrt{\sum \frac{1}{|a_k - x|^2} \sum \frac{1}{|a_k - x|^4}} \le (1 + o(1))\sqrt{\frac{2}{3}BD}. \end{aligned}$$

Assuming that the assertion of the lemma fails at x gives, by (2.8),

$$12A^2B + 6B^2 + D \leq 6AC,$$

and squaring both sides produces

$$\begin{aligned} E &= 144A^4B^2 + 36B^4 + D^2 + 144A^2B^3 + 12B^2D + 24A^2BD \\ &\leq 36A^2C^2 \leq (24 + o(1))A^2BD, \end{aligned}$$

which implies at once that

$$144A^4B^2 + 36B^4 + D^2 + 144A^2B^3 + 12B^2D \leq o(A^2BD). \tag{2.10}$$

But (2.10) yields $A^4B^2 = o(A^2BD)$ and hence $A^2B = o(D)$, as well as

$$0 < D^2 = o(A^2BD) = o(D^2),$$

a contradiction which completes the proof of part (A).

Assume now that M is as in part (B), and let $x \in \mathbb{R}$. If g has at least one zero, then

$$M'(x) = C_0 + \sum \frac{1}{(a_k - x)^2} > 0, \quad M'''(x) = \sum \frac{6}{(a_k - x)^4} > 0,$$

in which $C_0 \geq 0$ and $a_k \in \mathbb{R}$; this time the Cauchy-Schwarz inequality gives $C^2 \leq 2BD/3$. If the assertion of the lemma fails at x, then the left-hand side of (2.10) is non-positive, which is impossible since $D > 0$.

Suppose finally that M is as in (B) but g has no zeros. Since M is assumed non-constant, this forces $M' = C_0 > 0$ and $M'' = M''' = 0$, and the conclusion of the lemma follows trivially. □

Lemma 2.10. *Let L be a real transcendental meromorphic function in the plane with upper half-plane Tsuji characteristic satisfying $T_0(r, L) = O(\log r)$ as $r \to \infty$, such that at least one of L and $1/L$ has finitely many poles in H^+. Assume further that $F(z) = z - 1/L(z)$ has no asymptotic values $w \in H^+$, and that F' has finitely many zeros in H^+. Then there exists a positive integer N with the following property: if $w \in H^+$ and C is a component of the set $W^+ = \{z \in H^+ : F(z) \in H^+\}$, then each of L and F takes the value w at most N times in C, counting multiplicity.*

Proof. Let C be a component of W^+. The assertion concerning the valency of F on C is fairly standard [3, Lemma 4.2]: choose a Jordan arc γ^+ which, apart from its initial point, lies in H^+, and is such that every critical value $w \in H^+$ of F lies on γ^+. Suppose that $D \subseteq C$ is a component of $Y^+ = F^{-1}(H^+ \setminus \gamma^+)$ with no non-real zero of F' in ∂D: then the branch of F^{-1} mapping $H^+ \setminus \gamma^+$ to D may be analytically continued along $\gamma^+ \cap H^+$, giving a domain D_1 with $D \subseteq D_1 \subseteq C$, mapped univalently onto H^+ by F, which forces $D_1 = C$. Thus the number of components of Y^+ which lie in C is bounded, independent of C, as is the valency of F on C.

Controlling the number of w-points of L in C, for $w \in H^+$, requires a refinement of arguments from [36, 37]. By [37, Lemma 2.2], there exist at most finitely many $\alpha \in \mathbb{C}$ such that $F(z)$ or $L(z)$ tends to α as $z \to \infty$ along a path in H^+. This makes it possible to choose $\theta \in (0, \pi)$ such that the two rays $P^{\pm}$, given respectively by $w = te^{\pm i\theta}$, $0 < t < \infty$, contain no critical values of L and no values α such that $L(z)$ tends to α as $z \to \infty$ along a path in H^+.

Let $\Gamma \subseteq H^+$ be a component of ∂C. If Γ is bounded, then F has a pole on Γ. On the other hand, if Γ is unbounded, then Γ contains a level curve of F tending to ∞ in H^+, on which $F(z)$ must tend to some asymptotic value belonging to $\mathbb{R} \cup \{\infty\}$, because F is finite-valent on C. It follows that the number of components $\Gamma \subseteq H^+$ of ∂C is bounded, independent of C.

Now take $w^* = t^* e^{i\theta} \in P^+$, and distinct $z_1, \ldots, z_n \in C$ with $L(z_j) = w^*$. For each j, continue the branch of L^{-1} mapping w^* to z_j along P^+ in the direction of decreasing t. This gives pairwise disjoint paths σ_j, which remain in C since $\theta \in (0, \pi)$. Each σ_j must tend either to ∞ or to a pole of F on ∂C, of which only finitely many are available. Assume, after re-labelling if necessary, that σ_j tends to ∞ for $j = 1, \ldots, m$.

Each σ_j, for $j = 1, \ldots, m$, may be extended to a simple path $\tau_j = \sigma_j \cup \mu_j$ in C, where μ_j is bounded, so that the τ_j are pairwise disjoint apart from a common starting point $z^* \in C$. After re-labelling if necessary, this gives $m - 1$ pairwise disjoint domains $\Omega_j \subseteq H^+$, each bounded by τ_j and τ_{j+1}. Because of the bound on the number of components $\Gamma \subseteq H^+$ of ∂C, the number of Ω_j for which $\Omega_j \not\subseteq C$ is also bounded, independent of C.

Suppose now that $1 \leq k < k' \leq m - 1$ and that Ω_k and $\Omega_{k'}$ are contained in C: then so are their closures. Because F has no poles in C, the function $|L(z)|$ has a positive lower bound on the union of the μ_j. Choose q small and positive such that the circle $|w| = q$ contains no critical values of L and no α such that $L(z) \to \alpha$ as $z \to \infty$ along a path in H^+. Take $u_k \in \sigma_k$ with $L(u_k) = qe^{i\theta}$, and continue $z = L^{-1}(w)$, starting from $qe^{i\theta}$ and along the circle $|w| = q$, so that the continuation takes z into Ω_k. Since q is small and because of the choice of θ, this gives $v_k \in \Omega_k$ with $L(v_k) = qe^{-i\theta}$ and a simple path ν_k in Ω_k which is mapped by L onto the set $\{w = te^{-i\theta} : 0 < t \leq q\}$. The fact that L has no zeros in C implies that ν_k must tend to ∞, and so there exists an unbounded component V_k of the set

$\{z \in \mathbb{C} : \text{Im}\,(1/L(z)) > 2/q\}$, such that $V_k \cup \partial V_k \subseteq \Omega_k$. Furthermore, the function

$$U_k(z) = \text{Im}\,\frac{1}{L(z)} \quad (z \in V_k), \qquad U_k(z) = \frac{2}{q} \quad (z \notin V_k),$$

is non-constant and subharmonic in $\mathbb{C}$. But the same argument applied to $\Omega_{k'}$ gives a corresponding component $V_{k'}$ and subharmonic function $U_{k'}$. A standard application of the Phragmén-Lindelöf principle [17] yields z in V_k or $V_{k'}$, with $|z|$ large and $\text{Im}\,(1/L(z)) \geq |z|^{3/2}$, so that $\text{Im}\,F(z) < 0$, which contradicts the fact that $z \in C$.

Therefore, at most one of the Ω_j is contained in C, and this gives an upper bound, independent of C, for the number n of pre-images z_j in C of $w^* \in P^+$ under L. The open mapping theorem and analytic continuation of L^{-1} extend this same upper bound to the number of w-points of L in C, counting multiplicities, for any $w \in H^+$. □

Lemma 2.11. *Let Q be a transcendental meromorphic function in the plane such that the Nevanlinna deficiency $\delta(\infty, Q)$ is positive. Let $C > 1$ and let $E_C \subseteq [1, \infty)$ be unbounded, such that $T(2r, Q) \leq CT(r, Q)$ for $r \in E_C$. Let $H_r = \{\theta \in [0, 2\pi] : 2\log|Q(re^{i\theta})| > \delta(\infty, Q)T(r, Q)\}$. Then, for large $r \in E_C$, the linear measure m_r of H_r satisfies $m_r \geq d > 0$, where d depends only on C and $\delta(\infty, Q)$.*

Proof. This is standard. An inequality of Edrei and Fuchs [6, p.322] yields, for large $r \in E_C$,

$$\begin{aligned}\frac{3\delta(\infty, Q)}{4}T(r, Q) \leq m(r, Q) &\leq \frac{\delta(\infty, Q)}{2}T(r, Q) + \frac{1}{2\pi}\int_{H_r} \log^+|Q(re^{i\theta})|d\theta \\ &\leq \frac{\delta(\infty, Q)}{2}T(r, Q) + 11\left(\frac{2r}{2r - r}\right)m_r\left(1 + \log^+\frac{1}{m_r}\right)T(2r, Q) \\ &\leq \frac{\delta(\infty, Q)}{2}T(r, Q) + 22Cm_r\left(1 + \log^+\frac{1}{m_r}\right)T(r, Q).\end{aligned}$$ □

3 An auxiliary result

The following proposition plays a fundamental role in the proof of Theorem 1.2 and, in particular, proves the first assertion (1.1).

Proposition 3.1. *Let f be a function satisfying hypotheses* (i), (ii) *and* (iii) *of Theorem 1.2. Then*

$$g = \tilde{f} = Re^{ih}f = Wf, \qquad \frac{g^{(m)}}{g} = Se^{ik}\frac{f^{(m)}}{f}, \tag{3.1}$$

in which $\widetilde{f}(z) = \overline{f(\overline{z})}$*, while R and S are rational functions, h is an entire function with*

$$T(r, h') = O(r \log r) \quad \textit{as } r \to \infty, \tag{3.2}$$

and k is a polynomial of degree at most 1*. Furthermore,* f'/f *satisfies* (1.1)*. If, in addition, k is constant in* (3.1)*, then*

$$N(r, f) = O(T(r, h') + \log r) \quad \textit{as } r \to \infty. \tag{3.3}$$

Proof. It is clear that f and $g = \widetilde{f}$ satisfy (3.1) with R and S rational functions and h and k entire. Now Lemma 2.1 implies that, with T_0 the Tsuji characteristic,

$$T_0(r, f'/f) + T_0(r, g'/g) = O(\log r) \quad \text{as } r \to \infty. \tag{3.4}$$

Hence h' and e^{ik} satisfy the hypotheses of Lemma 2.2, from which it follows that (3.2) holds, and that k is a polynomial of degree at most 1.

Now (3.4) also implies that (2.2) holds with $H = f'/f$. But f has finitely many zeros, and so (1.1) follows, provided it can be shown that

$$\overline{N}(r, f) = O(r \log r) \quad \text{as } r \to \infty. \tag{3.5}$$

If k is non-constant, then (3.5) clearly holds, since all but finitely many poles of f are real 1-points of Se^{ik} by (3.1). In view of (3.2), it therefore remains only to prove that (3.3) holds when k is constant: if $Se^{ik} \not\equiv 1$, this follows again from (3.1). Suppose finally that $Se^{ik} \equiv 1$: then Lemma 2.4 may be applied, and (2.4) yields

$$N(r, f) \leq O(N(r, W/W') + \log r) \leq O(T(r, h') + \log r) \quad \text{as } r \to \infty.$$

□

4 Proof of Theorem 1.2

Let f be as in the hypotheses. Since (1.1) has already been proved in Proposition 3.1, it suffices to consider the case where f has finite order but (1.2) does not hold. Then (i) and Theorem 2.1 imply that f has infinitely many poles and $f^{(m)}$ has infinitely many zeros, all but finitely many of which are real, by (ii) and (iii). Moreover, f satisfies (3.1), in which R and S are rational functions, while h and k are polynomials.

Lemma 4.1. *It may be assumed that h and k are real, and that* $|R(x)| = |S(x)| = 1$ *for all* $x \in \mathbb{R}$.

Proof. Write $h(x) = a(x) + ib(x)$ with a and b real polynomials. If x is real but not a zero or pole of f, then $|f(x)| = |g(x)|$ and, by (3.1),

$$1 = |R(x)e^{ih(x)}|^2 = R(x)\overline{R(x)}\exp\left(ih(x) - i\overline{h(x)}\right) = |R(x)|^2\exp(-2b(x)).$$

Therefore, $b(x) = O(\log|x|)$ as $|x| \to \infty$ with x real. Thus b is constant, and it may be assumed that $b = 0$.

A similar argument may be applied to Se^{ik}. □

If k is constant in (3.1), then (3.3) shows that f has finitely many poles, giving an immediate contradiction. Assume henceforth that k is non-constant in (3.1), and observe that if x is a real pole of f, then $S(x)e^{ik(x)} = 1$. Since k has degree at most 1 by Proposition 3.1, it may be assumed by employing a linear change of variables that $S(\infty) = 1$ and $k(z) = 2\pi z$ which, on combination with (1.1), gives the following lemma.

Lemma 4.2. *The function*

$$H(z) = \frac{f'(z)}{f(z)}\left(S(z)e^{\pi iz} - e^{-\pi iz}\right) \tag{4.1}$$

is meromorphic of order at most 1 *in the plane, and has finitely many poles.*

Lemma 4.3. *Let ε and M be positive real numbers, with ε small and M large. For $j = 1, 2$, let S_j be the sector given by $|z| \geq M$, $\varepsilon \leq (-1)^{j+1}\arg z \leq \pi - \varepsilon$. Then g and f satisfy, on S_1,*

$$\frac{g'(z)}{g(z)} = T_1(z)+E(z)e^{2\pi iz}, \quad \frac{f'(z)}{f(z)} = W_1(z) + E(z)e^{2\pi iz}, \tag{4.2}$$
$$W_1(z) = -\frac{R'(z)}{R(z)} - ih'(z) + T_1(z).$$

Moreover, f satisfies, on S_2,

$$\frac{f'(z)}{f(z)} = T_2(z) + E(z)e^{-2\pi iz}. \tag{4.3}$$

Here each T_j is k_j'/k_j for some polynomial $k_j \not\equiv 0$ of degree at most $m - 1$, and $\chi(z) = E(z)$ on S_j means that $\log^+|\chi(z)| = o(|z|)$ *as $z \to \infty$ in S_j.*

Proof. It suffices to give the proof of (4.2), that of (4.3) requiring only trivial modifications. The function f has finitely many zeros and non-real poles, and $k(z) = 2\pi z$. Hence (3.1) and standard estimates for logarithmic derivatives [13] show that

$$\frac{f^{(m)}(z)}{f(z)} = E(z), \quad g^{(m)}(z) = \delta(z)g(z), \quad \delta(z) = E(z)e^{2\pi iz}$$

on the sector S_1. Fix a large positive A and, for $z \neq 0$ let L_z be the path consisting of the shorter circular arc from iA to $z^* = Az/|z|$, followed by the straight line segment from z^* to z. If A is large enough, then

$$2^m \int_{L_z} |\delta(t)t^{m-1}||dt| \leq 1 \tag{4.4}$$

for all $z \in S_1$ with $|z| \geq A$. Now there exist constants a_j, independent of z for $z \in S_1$, such that

$$g(z) = a_{m-1}z^{m-1} + \cdots + a_0 + \int_{iA}^{z} \frac{(z-t)^{m-1}}{(m-1)!}\delta(t)g(t)dt,$$

which can be written in the form

$$q(z) = \frac{g(z)}{z^{m-1}} = a_{m-1} + \cdots + \frac{a_0}{z^{m-1}} + \int_{iA}^{z} \frac{(1-t/z)^{m-1}}{(m-1)!}\delta(t)t^{m-1}q(t)dt.$$

The first step is to show that q is bounded for $z \in S_1$ with $|z| \geq A$. If this is not the case, then it is possible to choose $z \in S_1$ with $|z| \geq A$ and $q(z) = N$ large such that $|q(t)| \leq |N|$ on L_z. Since $|t| \leq |z|$ on L_z, this gives, with use of (4.4),

$$|N| \leq |a_{m-1}| + \cdots + |a_0| + 2^{m-1}|N| \int_{L_z} |\delta(t)t^{m-1}|\,|dt| \leq |a_{m-1}| + \cdots + |a_0| + \frac{|N|}{2},$$

which is obviously a contradiction if N is large enough. It follows that, for z in S_1,

$$g(z) = k_1(z) + \int_{i\infty}^{z} \frac{(z-t)^{m-1}}{(m-1)!}\delta(t)g(t)dt = k_1(z) + \int_{i\infty}^{z} \frac{(z-t)^{m-1}}{(m-1)!}E(t)e^{2\pi i t}dt,$$
$$g'(z) = k_1'(z) + \int_{i\infty}^{z} \frac{(z-t)^{m-2}}{(m-2)!}\delta(t)g(t)dt,$$

in which the path of integration Λ_z is along the positive imaginary axis from $i\infty$ to $i|z|$ followed by the shorter arc of the circle $|t| = |z|$ from $i|z|$ to z, while k_1 is a polynomial of degree at most $m-1$. Since $|z| \leq |t|$ on Λ_z, this implies that

$$|g(z) - k_1(z)| \leq 2^{m-1} \int_{\Lambda_z} |t^{m-1}\delta(t)g(t)|\,|dt| \leq \int_{\Lambda_z} |E(t)|e^{-2\pi \operatorname{Im} t}|dt| \quad \text{on } S_1.$$

The next step is to show that $k_1 \not\equiv 0$. If k_1 vanishes identically, then obviously $g(z)$ tends to 0 on the positive imaginary axis. So take a large positive y such that $|g(is)| \leq |g(iy)|$ for all real $s \geq y$, which gives

$$|g(iy) \leq |g(iy)|\, 2^{m-1} \int_{y}^{\infty} s^{m-1}|\delta(is)|ds,$$

an evident contradiction if y is large enough.

Splitting the path Λ_z into the part from $i\infty$ to $4i|z|$ and the part Λ_z^* from $4i|z|$ to z now yields, for large z in S_1,

$$\int_{\Lambda_z} |E(t)|e^{-2\pi \operatorname{Im} t}|dt| \le |e^{2\pi i z}| \int_{\Lambda_z^*} |E(t)||dt| + e^{-4\pi|z|} \int_{4|z|}^{\infty} |E(is)|e^{-\pi s} ds,$$

and hence

$$g(z) = k_1(z) + E(z)e^{2\pi i z}, \quad g'(z) = k_1'(z) + E(z)e^{2\pi i z},$$

which leads to (4.2) with $T_1 = k_1'/k_1$ and completes the proof of the lemma. □

It now follows from (4.1) and (4.2) that

$$H(z) = \frac{f'(z)}{f(z)} \left(S(z)e^{\pi i z} - e^{-\pi i z}\right) = -W_1(z)e^{-\pi i z} + E(z)e^{\pi i z} \quad \text{on } S_1, \tag{4.5}$$

and from (4.1) and (4.3) that

$$H(z) = T_2(z)S(z)e^{\pi i z} + E(z)e^{-\pi i z} \quad \text{on } S_2. \tag{4.6}$$

Since H has finite order and finitely many poles, and ε may be chosen arbitrarily small, the Phragmén-Lindelöf principle gives

$$H(z) = T_2(z)S(z)e^{\pi i z} - W_1(z)e^{-\pi i z} = T_2(z)\left(S(z)e^{\pi i z} - e^{-\pi i z}\right) + V(z)e^{-\pi i z},$$

in which $V = T_2 - W_1 = T_2 - T_1 + \frac{R'}{R} + ih'$ is a rational function. With (4.1) again, this leads to

$$\frac{f'(z)}{f(z)} = T_2(z) + \frac{V(z)}{S(z)e^{2\pi i z} - 1}. \tag{4.7}$$

Recalling that $S(\infty) = 1$ and using the principal logarithm, we write, for $|z|$ large,

$$U(z) = 2\pi i z + \log S(z), \quad \frac{f'}{f} = T_2 + \frac{V}{e^U - 1}. \tag{4.8}$$

Lemma 4.4. $V = -U'$.

Proof. Observe first that (4.8) shows that f has infinitely many real poles x with multiplicity

$$m_x = -\frac{V(x)}{U'(x)} \sim -\frac{V(x)}{2\pi i}, \tag{4.9}$$

and so $V(\infty) \neq 0$. Furthermore, $T_2 = k_2'/k_2$, where $k_2 \not\equiv 0$ has degree at most $m - 1$. Thus f satisfies the hypotheses of Lemma 2.6, with $M = k_2$, by (4.8). It follows from (2.6) and (2.7) that, as $z \to \infty$ in the sector S_2, on which e^U is large,

$$\frac{f^{(m)}(z)}{f(z)} = \frac{V_m(z)}{(e^{U(z)} - 1)^m} \sim R_{m-1,m}(z)e^{-U(z)} \sim V(z)(-U'(z))^{m-1}e^{-U(z)}. \tag{4.10}$$

On the other hand, since $|S| = 1$ on $\mathbb{R}$, which implies that $\widetilde{U}' = -U'$, formula (4.7) leads to

$$\frac{g'}{g} = \widetilde{T}_2 + \frac{\widetilde{V}}{e^{-U} - 1}, \tag{4.11}$$

in which $\widetilde{T}_2 = \widetilde{k}_2'/\widetilde{k}_2$. Since x and m_x are real in (4.9), it must be the case that $\widetilde{V} = -V$. Combining Lemma 2.6 with (4.11) now yields, as $z \to \infty$ in S_2,

$$\frac{g'}{g} = \widetilde{T}_2 - \frac{V}{e^{-U} - 1}, \quad e^{U(z)} \frac{f^{(m)}(z)}{f(z)} = \frac{g^{(m)}(z)}{g(z)} \sim V(z)^m,$$

in light of (3.1) and the fact that e^{-U} is small on S_2. On comparison with (4.10), this shows that $V(\infty)/U'(\infty)$ has modulus 1, so that m_x has to be 1 in (4.9), and the rational function V/U' must be identically -1. □

It now follows from (4.8), Lemma 4.4, and the fact that $T_2 = k_2'/k_2$ for some polynomial $k_2 \not\equiv 0$, that f satisfies the hypotheses of Lemma 2.5, with $T(z)e^{i(az+b)} = e^{-U(z)}$ and K/k_2 constant. Applying Lemma 2.5, part (I) shows that m must be 2. Furthermore, when $m = 2$, the degree of k_2 is at most $m - 1 = 1$, and part (II) of the same lemma implies that k_2'/k_2 is real, so that any zero of k_2 must also be real. Conversely, if f is as in conclusion (1.3) of the theorem, then all but finitely many zeros of f'' are real, again by Lemma 2.5(II). This completes the proof of Theorem 1.2.

5 Proof of Theorem 1.3

To prove Theorem 1.3, assume that f is a strictly non-real transcendental meromorphic function in the plane, with finitely many zeros and poles in $\mathbb{C} \setminus \mathbb{R}$, such that f''/f is real. Write

$$g = \widetilde{f}, \quad \frac{f'}{f} = L = \alpha + i\beta, \quad \frac{g'}{g} = \widetilde{L} = \alpha - i\beta, \quad L - \widetilde{L} = 2i\beta, \tag{5.1}$$

where α and β are real meromorphic functions, and $\beta \not\equiv 0$, since g/f is non-constant. Then

$$\frac{f''}{f} = \alpha' + i\beta' + \alpha^2 - \beta^2 + 2i\alpha\beta = \frac{g''}{g} = \alpha' - i\beta' + \alpha^2 - \beta^2 - 2i\alpha\beta,$$

from which it follows that

$$\beta' + 2\alpha\beta = 0, \quad L = \frac{f'}{f} = -\frac{\beta'}{2\beta} + i\beta, \quad \widetilde{L} = \frac{g'}{g} = -\frac{\beta'}{2\beta} - i\beta,$$

and so f'/f and β are related as in (1.4).

Now the last equation of (5.1) implies that all poles of β are simple and that β has finitely many non-real poles. Moreover, a real pole of β would give rise to real residues for β, β'/β and f'/f, which is impossible by the first equation of (1.4). Thus β has finitely many poles, all non-real. It is also evident from (1.4) that all zeros of β have even multiplicity and are poles of f, and that β has finitely many non-real zeros, and finally that f has finitely many zeros, as asserted. Obviously, if β is constant, then $f(z) = Ae^{i\beta z}$, with A constant.

Assume henceforth that β is non-constant and that all but finitely many zeros of f'' are real. Then it is convenient to write, using (1.4),

$$(5.2) \qquad \beta = S\gamma^2, \quad P = \beta^{-1/2}, \quad \frac{f'}{f} = \frac{P'}{P} + \frac{i}{P^2}, \quad M = \frac{P'}{P} = -\frac{S'}{2S} - \frac{\gamma'}{\gamma},$$

where S is a real rational function and γ is a real entire function with only real zeros. Here M is single-valued in the plane, and $P(z)$ is single-valued for $|z|$ large, since the zeros of β have even multiplicity and the finitely many poles occur in non-real conjugate pairs.

Lemma 5.1. *The function γ belongs to the Laguerre-Pólya class LP.*

Proof. Formula (1.4) and Lemma 2.1 give as $r \to \infty$, using Tsuji functionals as before,

$$m_0(r, f'/f) \leq T_0(r, f'/f) = O(\log r), \quad T_0(r, \beta) \leq O(\log r) + m_0(r, \beta'/\beta),$$

and hence $T_0(r, \beta) = O(\log r)$, by the lemma of the logarithmic derivative for the Tsuji characteristic [12]. Now β has order of growth at most 1, by Lemma 2.2. Thus γ is a real entire function of order at most 1 with only real zeros, and so belongs to LP [39]. □

Lemma 5.2. (a) *Assume that γ has infinitely many zeros and x_0 is a large positive real number. If $I \subseteq \mathbb{R} \setminus [-x_0, x_0]$ is an open interval containing no poles of P, then f''/f has at most two zeros, counting multiplicity, in I.*

(b) *Assume that $S = 1$ in (5.2) and that M is non-constant. Then f''/f has at most two zeros, counting multiplicity, in any open real interval I which contains no poles of P.*

Proof. Observe that (5.2) gives

$$(5.3) \qquad \frac{f''}{f} = \frac{P''}{P} - \frac{1}{P^4} = \frac{P''}{P} - \beta^2 = \frac{P''}{P} - S^2\gamma^4 = \frac{P^3P'' - 1}{P^4}.$$

Here P''/P and P^3P'' are singled-valued in $\mathbb{C}$, since P^2 and P'/P are.

Suppose first that γ and I are as in (a). Then $M = P'/P$ satisfies the hypotheses of part (A) of Lemma 2.9, by (5.2), and so the function Q in (2.8) has at most one zero in I, counting multiplicity. Hence the same is true of

$$(P^3P'')' = P^4\left(\frac{P'''}{P} + 3\frac{P'}{P}\frac{P''}{P}\right) = P^4\left(M^3 + 3MM' + M'' + 3M(M^2 + M')\right) = P^4Q.$$

This implies that $P^3P'' - 1$ has at most two zeros in I, counting multiplicity, and so has f''/f, by (5.3).

Part (b) is proved the same way, since if $S = 1$ and M is non-constant, then M satisfies the hypotheses of Lemma 2.9(B). □

Lemma 5.3. *The function β is rational, and f satisfies* (1.2).

Proof. Assume that β is transcendental. If β has finitely many zeros, then $\beta(z) = R_1(z)e^{b_1 z}$, with R_1 a rational function and $b_1 \in \mathbb{R} \setminus \{0\}$; and (5.3) shows that f''/f has infinitely many non-real zeros, which is a contradiction.

Assume henceforth that β has infinitely many zeros; then so has γ. Since f''/f has a double pole at each real pole x of P with $|x|$ large, and has finitely many non-real zeros, Lemma 5.2(a) implies that the following estimates hold as $r \to \infty$. First,

$$n(r, f/f'') \le n(r, f''/f) + O(1), \quad N(r, f/f'') \le N(r, f''/f) + O(\log r),$$

from which applying Jensen's formula yields, in view of (5.2), (5.3), and the fact that β has finite order,

$$\begin{aligned} 2m(r, \beta) &\le m(r, f''/f) + O(\log r) \le m(r, f/f'') + O(\log r) \\ &\le T(r, f''/f) + O(\log r) = O(T(r, \beta)). \end{aligned}$$

Thus the zeros of f''/f have positive Nevanlinna deficiency $\delta(0, f''/f)$.

A contradiction can now be obtained using a method similar to the proof of [36, Lemma 5.4]. Since β and f''/f have finite order, a well known result of Hayman [15, Lemma 4] gives $C_1 > 0$ and a set $E_1 \subseteq [1, \infty)$, of positive lower logarithmic density, such that

$$(5.4) \qquad T(4s, \beta) \le C_1 T(s, \beta) \quad \text{and} \quad T(4s, f''/f) \le C_1 T(s, f''/f)$$

for $s \in E_1$. By estimates from [13], the function β also satisfies

$$(5.5) \qquad \left|\frac{\beta'(z)}{\beta(z)}\right| \le r^{M_0} \quad \text{for } |z| = r \notin F_2,$$

where M_0 is a positive constant and F_2 has finite logarithmic measure.

Now let σ, K_0, K_1, and K_2 be positive constants, with K_0, K_1/K_0 and K_2/K_1 large, and σ small. Let $s \in E_1$ be large. Since f''/f is transcendental and $\delta(0, f''/f) > 0$, a standard application of (5.4) and Fuchs' small arcs lemma [17, p.721] give $r \in [s, 2s] \setminus F_2$ and an arc of the circle $|z| = r$, of angular measure 6σ, on which $|f''(z)/f(z)| \leq r^{-5}$. The fact that f''/f is real then implies that $|f''(z)/f(z)| \leq r^{-5}$ on a subarc I_r of $\{z \in \mathbb{C} : |z| = r, \sigma \leq \arg z \leq \pi - \sigma\}$ of angular measure at least σ. Next, applying Lemma 2.7 with $k = 2$ and $N(r) = K_0$ shows that there exists $z \in I_r$ with $|zf'(z)/f(z)| \leq K_1$. Now Lemma 2.8, applied to the function $f(rz)$, delivers $|zf'(z)/f(z)| \leq K_2$ for all z with $|z| = r, \sigma \leq \arg z \leq \pi - \sigma$. Because β is real, combining this estimate with (1.4), (5.4), and (5.5) yields an unbounded set of positive r such that $T(2r, \beta) \leq T(4s, \beta) \leq C_1 T(s, \beta) \leq C_1 T(r, \beta)$ and such that $|\beta(z)| \leq r^{M_0}$ for all z with $|z| = r$, apart from a set J_r of angular measure at most 4σ, where σ may be chosen arbitrarily small, independent of C_1. Since β has finitely many poles, this contradicts Lemma 2.11.

Thus β is rational, as asserted, and so is f'/f by (1.4), which implies (1.2) and completes the proof of the lemma. □

To finish the proof of the theorem, assume henceforth that all zeros and poles of f and f'' are real. Then β has no poles, by (5.1), and so it may be assumed that $S = 1$ in (5.2). Since zeros of β have even multiplicity, and the case where β is constant has already been disposed of, it can now be assumed that β is a polynomial with real zeros, of even positive degree, and M is non-constant in (5.2). Thus (1.4) and (5.3) show that f''/f is a rational function with double poles at the zeros of β, which are real poles of P. Moreover f''/f has only real zeros, and by Lemma 5.2(b), the number of zeros of f''/f exceeds the number of poles by at most 2. Hence f''/f has at most a double pole at ∞, and so β has degree at most 1, by (5.3) again, which is a contradiction.

6 Some applications of harmonic measure

Lemma 6.1 ([7, 43]). *Let G be a domain bounded by a Jordan curve C consisting of a Jordan arc B and its complement $A = C \setminus B$. Let L be a rectifiable curve in G joining $a \in A$ to $b \in B$, and for $z \in L$, let $\rho(z)$ be the distance from z to A. Then the harmonic measure $\omega(z)$ of B with respect to G satisfies, for z on L,*

$$\omega(z) \geq \frac{1}{2\pi} \exp\left(-4 \int_z^b \frac{|du|}{\rho(u)}\right),$$

in which the integration is from z to b along L.

Lemma 6.2. *Let Q be a transcendental meromorphic function of finite order in the plane such that the zeros of Q have positive Nevanlinna deficiency $\delta(0, Q)$. Assume that for each $\delta > 0$, there exists $N(\delta) > 0$ such that*

$$\log |Q(z)| \leq N(\delta) \log |z| \tag{6.1}$$

for all z with $|z|$ large and $\delta \leq |\arg z| \leq \pi - \delta$.

Let η and ε be positive. Then, for all sufficiently large r, the function Q satisfies

$$\log |Q(z)| \leq 2N(\varepsilon/2) \log r - r^{-\eta} T(r, Q) \tag{6.2}$$

for all z in at least one of the arcs

$$I^{+}(r, \varepsilon) = \{re^{i\theta} : \varepsilon \leq \theta \leq \pi - \varepsilon\}, \quad I^{-}(r, \varepsilon) = \{re^{-i\theta} : \varepsilon \leq \theta \leq \pi - \varepsilon\}.$$

Proof. The initial steps are standard. Choose $\delta > 0$ small compared to η. By the same result of Hayman [15, Lemma 4] as used in the proof of Lemma 5.3, there exists $C_1 > 0$, depending on δ and the order of Q, as well as a set $E_\delta \subseteq [1, \infty)$, of lower logarithmic density at least $1 - \delta/2$, such that if $s \in E_\delta$, then $T(4s, Q) \leq C_1 T(s, Q)$. Let $s \in E_\delta$ be large,

$$H_s = \{\theta \in [0, 2\pi] : 2 \log |Q(2se^{i\theta})| < -\delta(0, Q) T(2s, Q)\},$$

and m_s be the linear measure of H_s. Then Lemma 2.11 yields $m_s \geq 16\delta_1 > 0$, where δ_1 is small but independent of s.

Now let r be large and positive: then there exists $s \in E_\delta$ with

$$2r \leq s \leq r^{1+\delta} \leq r^2. \tag{6.3}$$

Since H_s has measure $m_s \geq 16\delta_1$, it may be assumed without loss of generality that Q satisfies $2 \log |Q(z)| < -\delta(0, Q) T(2s, Q)$ for all z in a subset I_s of $I^{+}(2s, 2\delta_1)$, of angular measure at least $4\delta_1$. Let D_s be the domain

$$\{z \in \mathbb{C} : s/2 < |z| < 2s, \ \delta_1 < \arg z < \pi - \delta_1\},$$

and let $w \in I^{+}(s, \pi/4)$. Then the harmonic measure $\omega(w, I_s, D_s)$ of I_s with respect to D_s is bounded below by a positive constant δ_2 which is independent of s and r. Thus (6.1) and the two constants theorem [43] yield, since Q is transcendental and r and s are large,

$$\log |Q(w)| \leq N(\delta_1) \log 2s - \frac{\delta_2 \delta(0, Q)}{2} T(2s, Q) \leq -\frac{\delta_2 \delta(0, Q)}{4} T(2s, Q) \tag{6.4}$$

for all $w \in I^+(s, \pi/4)$. Next, let Ω be the domain

$$\{z \in \mathbb{C} : r/2 < |z| < s,\ \varepsilon/2 < \arg z < \pi - \varepsilon/2\},$$

and let $z_0 \in I^+(r, \varepsilon)$. Join z_0 to is by the simple path γ consisting of the shorter arc of the circle $|z| = r$ from z_0 to ir, followed by the radial segment $z = ix$, $r \leq x \leq s$. Let $B = I^+(s, \pi/4)$ and $A = \partial\Omega \setminus B$. Denoting by $\rho(u)$ the distance from u to A then gives, on integrating with respect to arc length and using (6.3),

$$\int_\gamma \frac{|du|}{\rho(u)} \leq d_1 \left(\frac{1}{\varepsilon} + \int_r^s \frac{dt}{t} \right) \leq d_1 \left(\frac{1}{\varepsilon} + \delta \log r \right),$$

where $d_1 > 0$ is independent of ε, δ and r. This time the two constants theorem delivers, in view of (6.1), (6.4), and Lemma 6.1,

$$\log |Q(z_0)| \leq 2N(\varepsilon/2) \log r - \frac{\delta_2 \delta(0, Q)}{8\pi} T(2s, Q) \exp \left(-4d_1 \left(\frac{1}{\varepsilon} + \delta \log r \right) \right).$$

Since r is large and δ/η is small, (6.2) follows for $z = z_0$, and the proof is complete. □

Lemma 6.3. *Let u be a non-constant continuous subharmonic function in the plane of finite order ρ, and let $\varepsilon > 0$. Let F be the set of $r \in [1, \infty)$ for which there exists an arc of the circle $|z| = r$ of length at least εr, on which $u(z) > 0$. Then F has lower logarithmic density at least $1 - \varepsilon\rho/\pi$.*

Proof. This is a standard application of a well-known estimate for harmonic measure [53]. For $r > 0$, let $B(r, u) = \max\{u(z) : |z| = r\}$, and let $r\,\theta(r)$ be the length of the longest open arc of the circle $|z| = r$ on which $u(z) > 0$, except that $\theta(r) = \infty$ if $u(z) > 0$ on the whole circle. Then, as $r \to \infty$, by [53, p.116],

$$\int_{[1,r]\setminus F} \frac{dt}{t} \leq \frac{\varepsilon}{\pi} \int_1^r \frac{\pi\, dt}{t\theta(t)} \leq \frac{\varepsilon}{\pi} \log B(2r, u) + O(1) \leq \frac{\varepsilon}{\pi}(\rho + o(1)) \log r.$$ □

Lemma 6.4. *Let G be a transcendental meromorphic function of finite order in the plane, and assume that there exist $\alpha_1, \alpha_2 \in \mathbb{C}$, not necessarily distinct, with the following property: for each $\varepsilon > 0$, the function G satisfies $G(z) \to \alpha_j$ as $z \to \infty$ with $\varepsilon < (-1)^j \arg z < \pi - \varepsilon$. If $\beta \in (\mathbb{C} \cup \{\infty\}) \setminus \{\alpha_1, \alpha_2\}$, then the inverse function of G cannot have a direct transcendental singularity over β.*

Proof. This is again standard: for the terminology, see [1, 43]. Assuming without loss of generality that G^{-1} has a direct transcendental singularity over

$\beta \in \mathbb{C}\setminus\{\alpha_1, \alpha_2\}$ gives a small $\delta > 0$, a component of U of $\{z \in \mathbb{C} : |G(z) - \beta| < \delta\}$, and a non-constant continuous subharmonic function u of finite order in the plane which satisfies $u(z) = \log(\delta/|G(z) - \beta|)$ on U and vanishes outside U. Here δ may be chosen arbitrarily small, as may ε. But then the intersection of U with the set $\{z \in \mathbb{C} : \varepsilon < |\arg z| < \pi - \varepsilon\}$ is bounded, which contradicts Lemma 6.3. □

7 Proof of Theorem 1.4

Let f be a transcendental meromorphic function given by (1.6).

Lemma 7.1. *Let n be a non-negative integer, and let $N_R(r, 1/f^{(n)})$ count the real zeros of $f^{(n)}$, with respect to multiplicity. If n is odd, then $N_R(r, 1/f^{(n)}) = 0$. If n is even, then $f^{(n)}$ has at most one zero in any open interval of the real axis which contains no poles of f, and $N_R(r, 1/f^{(n)}) \leq N(r, f) + O(\log r)$ as $r \to \infty$. Furthermore, if a_k and a_{k+1} are poles of f, with $a_k < a_{k+1}$ and no poles of f in $I_k = (a_k, a_{k+1})$, then I_k contains precisely one zero of f''. Finally, $m(r, f) = O(\log r)$ as $r \to \infty$.*

Proof. The first three assertions follow from differentiating (1.6), which shows that if m is an odd positive integer, then $f^{(m)}(x)$ is positive or infinite for every real x. Next, the fact that all residues of f are negative while all poles of f'' have multiplicity 3 forces f'' to change sign on I_k. Hence f'' has precisely one zero in I_k, since f''' has none there. The bound on $m(r, f)$ holds, since f is real and maps the upper half-plane H^+ into itself, so that [39, Ch. I.6, Thm. 8′]

$$(7.1) \qquad \frac{1}{5}|f(i)|\frac{\sin\theta}{r} < |f(re^{i\theta})| < 5|f(i)|\frac{r}{\sin\theta} \quad \text{for} \quad r \geq 1,\ \theta \in (0, \pi). \qquad □$$

Lemma 7.2. *Let $m \geq 3$, let ε be small and positive, and let $N_{NR}(r, 1/f^{m)})$ count the non-real zeros of $f^{(m)}$. Then f satisfies $(m-2-\varepsilon)T(r, f) \leq N_{NR}(r, 1/f^{(m)})$ as $r \to \infty$ outside a set of finite measure. In particular, $f^{(m)}$ has infinitely many non-real zeros.*

Proof. Since f is transcendental with only real poles, all of which are simple, Lemma 7.1 and an inequality of Frank, Steinmetz and Weissenborn [8] (see also [10, 11, 50]) yield, for large r outside a set of finite measure,

$$\begin{aligned}(m+1)T(r, f) &= (m+1)N(r, f) + O(\log r) = N(r, f^{(m)}) + o(T(r, f)) \\ &\leq N(r, 1/f^{(m)}) + (2 + \varepsilon/2)N(r, f) + o(T(r, f)) \\ &\leq N_{NR}(r, 1/f^{(m)}) + (3 + \varepsilon/2)N(r, f) + o(T(r, f)).\end{aligned} \qquad □$$

Lemma 7.2 proves the first assertion of Theorem 1.4. Assume henceforth that f'' has finitely many non-real zeros. Clearly all zeros of f' are non-real by Lemma 7.1. Let

(7.2)
$$F(z) = z - \frac{f(z)}{f'(z)}, \quad W^+ = \{z \in H^+ : F(z) \in H^+\}, \quad W^- = \{z \in H^+ : F(z) \in H^-\}.$$

It may be assumed that $A = B = 0$ in (1.6), since $f(z) - Az - B$ has the same second derivative as f.

Lemma 7.3. *Let $\varepsilon > 0$. Then $f(z)/z \to 0$ as $z \to \infty$ with $\varepsilon < |\arg z| < \pi - \varepsilon$.*

Proof. This is standard. Fix $\delta > 0$ and let $R \geq 1$. Then (1.6) gives a rational function T_R, with $T_R(\infty) = 0$, such that, for $\varepsilon < |\arg z| < \pi - \varepsilon$,

$$\frac{f(z)}{z} = T_R(z) + \sum_{|a_k|>R} \frac{A_k}{a_k(a_k - z)}, \quad \left|\frac{f(z)}{z}\right| \leq |T_R(z)| + \sum_{|a_k|>R} \frac{A_k}{a_k^2 \sin \varepsilon} = |T_R(z)| + S.$$

Now choose R so large that (1.6) gives $S < \delta$, and $|z|$ so large that $|T_R(z)| < \delta$. □

Lemma 7.4. *All poles of F are non-real, while all but finitely many zeros of F' are real. In any open interval of the real axis which contains no poles of f, the function F' has at most two zeros, counting multiplicity.*

Proof. These assertions all follow from Lemma 7.1 and the formula $F' = (ff'')/(f')^2$. □

Lemma 7.5. *The Tsuji characteristic of f'/f satisfies* (2.1)*, and f has order of growth at most* 1 *in the plane.*

Proof. The first assertion follows from Lemma 2.1. Alternatively, it may be observed that the function $(f - i)/(f + i)$ has modulus less than 1 on H^+.

To prove that f has order at most 1, write f''/f as follows. Assume that the a_k in (1.6) are ordered so that $a_k < a_{k+1}$ for each k. If $|k| \geq k_0$, where k_0 is large, then a_k and a_{k+1} have the same sign and, by Lemma 7.1, there is precisely one zero b_k of f'' in (a_k, a_{k+1}), counting multiplicity. For $z \in H^+$, write

$$\psi(z) = \prod_{|k|\geq k_0} \frac{1 - z/b_k}{1 - z/a_k}, \quad 0 < \sum_{|k|\geq k_0} \arg \frac{1 - z/b_k}{1 - z/a_k} = \sum_{|k|\geq k_0} \arg \frac{b_k - z}{a_k - z} < \pi.$$

The product ψ converges by the alternating series test, and $\psi(H^+) \subseteq H^+$. Next, write $f''/f = \psi/g$, where $g = \psi f/f''$ has finitely many poles, using Lemma 7.1, and all but finitely many poles of f are simple zeros of g. It follows from (2.1) and standard properties of the Tsuji characteristic that the hypotheses of Lemma 2.2(a) are satisfied with $H = f/f''$ (and so $\widetilde{H} = H$). This gives (2.2) with $H = f/f''$.

Now $m(r, f) = O(\log r)$ by Lemma 7.1, and the same is true with f replaced by ψ, because $\psi(H^+) \subseteq H^+$. Therefore, (2.2) also holds with $H = g$. Thus Lemma 2.2(b) shows that $T(r, g)$ has order of growth at most 1, and hence so have $N(r, f)$ and $T(r, f)$. □

Lemma 7.6. *There does not exist $\beta \in \mathbb{C} \setminus \{0\}$ such that $f(z)/z \to \beta$ as z tends to ∞ on a path in $\mathbb{C} \setminus \mathbb{R}$.*

Proof. If such an asymptotic value β exists, the inverse function of $f(z)/z$ has a direct transcendental singularity over ∞, by Lemma 7.3. But this is impossible, by Lemmas 6.4 and 7.3 and the fact that f has finite order of growth. □

Lemma 7.7. *Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Then the inverse function F^{-1} has no direct transcendental singularities over α.*

Proof. Assume that F^{-1} has a direct transcendental singularity over $\alpha \in \mathbb{C}\setminus\mathbb{R}$. Then, without loss of generality, there exist $\delta > 0$ and a component $U \subseteq H^+$ of the set $\{z \in \mathbb{C} : |F(z) - \alpha| < \delta\}$, such that the function

$$u(z) = \log \frac{\delta}{|F(z) - \alpha|} \quad (z \in U), \quad u(z) = 0 \quad (z \in \mathbb{C} \setminus U), \tag{7.3}$$

is subharmonic and non-constant in the plane. By a result of Lewis, Rossi, and Weitsman [41], there exists a path Γ tending to ∞ in U on which $u(z) \to +\infty$ with

$$\int_\Gamma e^{-u(z)} \, |dz| < \infty. \tag{7.4}$$

For $z \in \Gamma$ with $|z|$ large, write

$$z - \frac{f(z)}{f'(z)} = F(z) = \alpha + p(z), \quad \frac{f'(z)}{f(z)} = \frac{1}{z - \alpha} + q(z), \quad |q(z)| \leq |p(z)| = \delta e^{-u(z)}.$$

Hence (7.4) shows that there exists a non-zero complex number β such that $f(z) \sim \beta(z - \alpha)$ as $z \to \infty$ on Γ, contradicting Lemma 7.6. □

Lemma 7.8. *The function F has finitely many critical values, and no asymptotic values, in $\mathbb{C} \setminus \mathbb{R}$.*

Proof. The fact that all but finitely many critical values of F are real is an immediate consequence of Lemma 7.4. Since all poles of f'/f are real, it follows from Lemma 7.5 and [37, Lemma 2.2] that F has finitely many asymptotic values in $\mathbb{C} \setminus \mathbb{R}$. Because F has finite order, any non-real finite asymptotic value of F must give rise to a direct singularity of F^{-1}, by [1], contradicting Lemma 7.7. □

Lemma 7.9. *There exists a positive integer M such that if C is a component of W^+ or W^-, then F takes each value at most M times in C, counting multiplicity. Furthermore, a component of W^+ (respectively, W^-) which contains no zeros of f'' is simply connected and conformally equivalent to H^+ (respectively, H^-) under F, and this is true for all but finitely many components of W^+ (respectively, W^-).*

Proof. The first assertion is proved as in Lemma 2.10, using 7.8, and the second is standard. □

Lemma 7.10. *Let C be a component of W^+ or W^- which contains no zeros of f'', and let $\alpha \in \mathbb{R}$. Then there exists z in the finite boundary ∂C with $F(z) = \alpha$.*

Proof. Let C and α be as in the hypotheses, and assume that $\alpha \notin F(\partial C)$. Let $G(z) = 1/(\alpha - F(z))$, so that G is univalent on C and $G(C)$ is H^+ or H^-. Let $g : G(C) \to C$ be the inverse function of G, and let Γ be the path in $G(C)$ given by

$$w = it, \quad t \in \mathbb{R}, \quad 1 \leq |t| < \infty.$$

Then $\gamma = g(\Gamma)$ is a curve in C on which iG is real, and γ tends either to ∞ or to an α-point of F on ∂C. Hence γ must tend to ∞ in C. For $z \in \gamma$ with $|z|$ large, write

$$z - \frac{f(z)}{f'(z)} = F(z) = \alpha - \frac{1}{G(z)} = \alpha + o(1),$$

which leads to

$$\frac{f'(z)}{f(z)} = \frac{1}{z - \alpha + 1/G(z)} = \frac{1}{z - \alpha} + h(z), \quad \text{where} \quad h(z) = O\left(\frac{1}{|z|^2 |G(z)|}\right).$$

But Koebe's 1/4 theorem applied to $\log g$ gives $g'(w)/g(w) = O(1/|w|)$ on Γ, and so

$$\int_\gamma |h(z)||dz| = \int_\Gamma O\left(\frac{|g'(w)|}{|g(w)|^2 |w|}\right)|dw| = \int_\Gamma O\left(\frac{1}{|w|^2 |g(w)|}\right)|dw| < \infty.$$

It follows that there exists a non-zero complex number β such that $f(z) \sim \beta(z - \alpha)$ as $z \to \infty$ on γ, and this contradicts Lemma 7.6. □

Lemma 7.11. *Let $a \in \mathbb{R}$ be a zero of f''. Then f has at least one pole in each of $(-\infty, a)$ and (a, ∞).*

Proof. Suppose that f has no poles in $(-\infty, a)$. Then $(X - a)^3 > 0$ for every pole X of f, and the series expansion for f'' obtained from (1.6) shows that a cannot be a zero of f''. □

Lemma 7.12. *Every pole of f lies on the boundary of a component of W^+, but not in the closure of W^-.*

Proof. This holds because every pole X of f is a real fixed point of F with $F'(X) > 1$. □

Lemma 7.13. *Let $a \in \mathbb{R}$ be a multiple zero of F'. Then $F'''(a) > 0$.*

Proof. Lemma 7.1 shows that a must be a common zero of f and f'', and a triple zero of $F - F(a)$. Assume that $F'''(a)$ is negative, and let δ be small and positive: then $a-\delta$ and $a+\delta$ both lie in ∂W^-. Let A and B be the nearest poles of f to a in $(-\infty, a)$ and (a, ∞) respectively; these exist by Lemma 7.11, and Lemma 7.12 ensures that each lies on the boundary of a component of W^+. It follows that F must have critical points in (A, a) and (a, B), contradicting Lemma 7.1. □

Lemma 7.14. *The function f' has finitely many zeros, and none at all if f'' has only real zeros.*

Proof. Let w be a zero of f'. Then w is non-real by Lemma 7.1, and it may be assumed that $w \in H^+$. Thus w is a pole of F: with finitely many exceptions, and none if f'' has only real zeros, the pole of F at w is simple.

Assume henceforth that $w \in H^+$ is a zero of f' and a simple pole of F: then w lies on the boundary of a uniquely determined component C_w of W^-. Consider those w for which the component C_w either is multiply connected, or has a non-real zero of f'' in its closure. There are finitely many of these, by Lemma 7.9, and none if f'' has only real zeros.

Attention may thus be restricted to those w for which $C = C_w$ is simply connected, with no non-real zero of f'' in its closure. Then F maps C univalently onto H^-, and $F(\partial C) = \mathbb{R} \cup \{\infty\}$, by Lemmas 7.9 and 7.10 and the fact that $F(w) = \infty$. Thus C is bounded; otherwise, there exist $\zeta_n \in C$ with $\zeta_n \to \infty$ and $F(\zeta_n) \to \zeta^* \in F(C \cup \partial C)$, contradicting the univalence of F on C.

Suppose that ∂C has a component $\Gamma \subseteq H^+$. Then Γ is a Jordan curve, and $\Gamma = \partial C$, because C is simply connected. Moreover, Γ forms part of the boundary of a multiply connected component E of W^+. But F has a pole on ∂C, and F is finite-valent on each such E, and so there are finitely many components C of this type, and none at all if f'' has only real zeros.

Assume henceforth that every component of ∂C meets $\mathbb{R}$, and take $z_0 \in \partial C$ with the property that $\operatorname{Im} z_0 = \max\{\operatorname{Im} z : z \in C \cup \partial C\}$. Follow ∂C in each direction, starting from z_0, until the first encounter with $\mathbb{R}$. This gives a Jordan arc or curve γ in $\partial C \cap (H^+ \cup \mathbb{R})$ such that $\gamma \cap \mathbb{R} = \{a, b\}$, where a and b are real zeros of F' with $a \leq b$. Here it is necessary to allow for the possibility that $a = b$, in which case a is a multiple zero of F' and so of ff''. Now $\lambda = \gamma \cup [a, b]$ is a Jordan curve, and since $F'(z_0) \neq 0$, local considerations show that there are points in C which lie in the interior domain of λ, and hence so does all of C.

Let $c = \sup\{x \in \mathbb{R} : [a, x] \subseteq \partial C\}$. Then $[a, c] \subseteq \partial C$, and a and c are zeros of ff'' (again, in principle, a and c might coincide, and so might b and c). Lemmas 7.1 and 7.12 show that f has no poles in ∂C, each of f and f'' has one simple zero in the set $\{a, c\}$, and $c \leq b$.

Now f has at least one pole in $(-\infty, a)$, since otherwise neither a nor c can be a zero of f'', by Lemma 7.11. Let A be the nearest pole of f to a in $(-\infty, a)$. Then A lies on the boundary of a component D of W^+. Because F has no multiple points in $[A, a)$ by Lemma 7.1, the interval $[A, a]$ is a subset of ∂D. Furthermore, γ meets ∂D: if a is a simple zero of F', then this is clear, while if a is a multiple zero of F', then $F'''(a) > 0$ by Lemma 7.13, in which case γ meets ∂D because C lies in the interior domain of $\lambda = \gamma \cup [a, b]$. Since f'' has no non-real zeros in the closure of C, it follows that $\gamma \subseteq \partial D$. A similar argument shows that there exists a pole B of f with $B > b$ such that the interval $[b, B]$ lies in the boundary of a component D' of W^+, and so does γ, from which it follows that $D = D' = D_w$.

In the case where f'' has only real zeros, F must be univalent on D, and the branch g of the inverse function F^{-1} which maps H^+ to D has at least two attracting fixed points on the boundary of H^+, at A and B, contradicting the Denjoy-Wolff theorem [51, Chapter 2]. Indeed, the iterates g^n form a normal family on H^+, since $g(H^+) = D \subseteq H^+$, but g extends to be analytic on a neighbourhood U_A of A such that $g(U_A) \subseteq U_A$ and the g^n converge to A on U_A, and in the same way they converge to B on a neighbourhood of B.

In the general case where f'' has finitely many non-real zeros, suppose that there exist infinitely many zeros $w \in H^+$ of f'. This gives infinitely many distinct components C_w of W^- as above, each with a corresponding component D_w of W^+. The D_w need not be distinct, but Lemma 2.10 implies that L has finitely many poles on the boundary of any component of W^+, and therefore so has f. Hence there must exist at least one D_w which is mapped univalently onto H^+ by F, and the Denjoy-Wolff theorem supplies a contradiction as before. □

To complete the proof of Theorem 1.4, observe that it now follows from Lemma 7.14 and the fact that all but finitely many zeros of f and f'' are real that f satisfies the hypotheses of [34, Theorem 6.4] (see also [33, Theorem 1.5]), subject to the assumption made earlier that $A = B = 0$ in (1.6). Then

$$f(z) = \frac{R(z)e^{icz} - 1}{A_1 R(z)e^{icz} - \overline{A_1}},$$

with $c \in (0, \infty)$, $A_1 \in \mathbb{C} \setminus \mathbb{R}$, and R a rational function satisfying $|R(x)| = 1$ for all $x \in \mathbb{R}$, by [34, Theorem 6.4]. Since all residues of f have to be negative, it follows easily that $A_1 \in H^+$, and the fact that $f(H^+) \subseteq H^+$ shows that all zeros of R lie in H^+, and all poles in H^-.

Finally, suppose that all zeros of f'' are real. Then the Schwarzian derivative S_f is entire, because f' has no zeros and all poles of f are simple [21, 22]. Since f is transcendental of order at most 1, it must be the case that S_f is a non-zero constant, so there exist $a \in \mathbb{C}$ and a Möbius transformation T such that $f(z) = T(e^{i2az})$. Because f is real with only real zeros and poles, a must be real, and $f(z) = C\tan(az + b) + E$, with b, C and E also real.

8 A special case of Theorem 1.7

The following special case illustrates Theorem 1.7 and plays a key role in its proof.

Lemma 8.1. *Let $a, b, D, E \in \mathbb{C}$ with $a \neq 0$ and $D \neq E$, and let $2 \leq n \in \mathbb{Z}$. Let*

$$F(z) = \left(\frac{De^{az+b} - E}{e^{az+b} - 1} \right)^n . \tag{8.1}$$

(i) *There exists a meromorphic function G in the plane with $G' = F$ if and only if $D = \lambda E$ where $\lambda^n = 1$, $\lambda \neq 1$.*

(ii) *There does not exist a meromorphic function H in the plane with $H'' = F$.*

Proof. It may be assumed that $a = 1$ and $b = 0$. By periodicity, there exists a meromorphic function G with $G' = F$ if and only if $\mathrm{Res}\,(F, 0) = 0$. The function $w = e^z - 1$ is univalent on a neighbourhood of the origin and has local inverse

$$z = \phi(w) = \log(1 + w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \cdots . \tag{8.2}$$

Let ε be small and positive, and let γ describe the circle $|z| = \varepsilon$ once counter-clockwise. Let Γ be the image of γ under $w = e^z - 1$. Then $\mathrm{Res}\,(F, 0) = 0$ if and only if

$$0 = \int_\gamma F(z)\,dz = \int_\Gamma \psi(w)dw, \quad \psi(w) = \left(D + \frac{D - E}{w} \right)^n \phi'(w). \tag{8.3}$$

Now (8.2) and (8.3) give, as $w \to 0$,

$$\psi(w) = \left(D^n + nD^{n-1}\left(\frac{D - E}{w}\right) + \cdots + \left(\frac{D - E}{w}\right)^n \right) \times \left(1 - w + \cdots + (-1)^{n-1}w^{n-1} + \cdots \right),$$

and so (i) follows from the fact that

$$\begin{aligned}\mathrm{Res}\,(\psi, 0) &= nD^{n-1}(D-E) - \frac{n!}{2!(n-2)!}D^{n-2}(D-E)^2 + \cdots + (-1)^{n-1}(D-E)^n \\ &= -\left(nD^{n-1}(E-D) + \frac{n!}{2!(n-2)!}D^{n-2}(E-D)^2 + \cdots + (E-D)^n\right) \\ &= -((D+E-D)^n - D^n) = D^n - E^n.\end{aligned}$$

To establish (ii), suppose that there exists a meromorphic function H in the plane with $H'' = F$. Then $D = \lambda E$, with $\lambda^n = 1$ by (i), and it may be assumed that $E = 1$ and $D = \lambda \neq 1$. This time write

$$w = q(z) = \frac{e^z - 1}{\lambda e^z - 1}, \quad z = q^{-1}(w) = \sigma(w) = \log\left(\frac{1-w}{1-\lambda w}\right), \tag{8.4}$$

each of these being univalent near the origin. This forces, with γ as before and Λ the image of γ under $w = q(z)$,

$$0 = \int_\gamma zF(z)\,dz = \int_\gamma \frac{z}{w^n}dz = \int_\Lambda \frac{\tau(w)}{w^n}dw, \quad \tau(w) = \sigma(w)\sigma'(w). \tag{8.5}$$

Now, as $w \to 0$, expanding (8.4) yields

$$\begin{aligned}\tau(w) &= \left(w(\lambda - 1) + \ldots + \frac{w^{n-1}}{n-1}(\lambda^{n-1} - 1) + \cdots\right) \\ &\qquad \times \left(\lambda - 1 + \cdots + w^{n-2}(\lambda^{n-1} - 1) + \cdots\right) \\ &= a_1 w + \cdots + a_{n-1}w^{n-1} + \cdots .\end{aligned}$$

Here the coefficient a_{n-1} of w^{n-1} must vanish by (8.5), which delivers

$$\begin{aligned}0 &= \frac{1}{n-1}(\lambda^{n-1} - 1)(\lambda - 1) + \cdots + (\lambda - 1)(\lambda^{n-1} - 1) \\ &= \sum_{j=1}^{n-1} \frac{1}{n-j}(\lambda^{n-j} - 1)(\lambda^j - 1).\end{aligned} \tag{8.6}$$

But $\lambda^n = 1$, and so $\lambda = \exp(2\pi ik/n)$ for some $k \in \{1, \ldots, n-1\}$. It follows that, for $1 \le j \le n-1$,

$$\mu_j = (\lambda^{n-j} - 1)(\lambda^j - 1) = 2 - (\lambda^j + \lambda^{-j}) = 2 - 2\cos(2\pi jk/n) \ge 0.$$

Since $\mu_1 > 0$, the sum in (8.6) is real and positive, and this contradiction completes the proof. □

9 Proof of Theorem 1.7

Let f be as in the hypotheses, let R be a large positive real number, and define g formally by

$$f' = g^n. \tag{9.1}$$

Then g admits unrestricted analytic continuation in $R < |z| < \infty$, these continuations having only simple poles and no critical points. Since g'/g is single-valued in the plane, so is the function A defined by

$$2A = S_g = \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2, \tag{9.2}$$

where S_g denotes the Schwarzian derivative [21, 22]. Moreover, A has finitely many poles, and none in $R < |z| < \infty$, because the continuations of g are free of multiple points there.

Lemma 9.1. *The function A is rational, but does not satisfy $A(z) = O(|z|^{-2})$ as $z \to \infty$.*

Proof. The first assertion follows from the lemma of the logarithmic derivative and the fact that f has finite lower order. Now suppose that $A(z) = O(|z|^{-2})$ as $z \to \infty$. Take $z_0 \in \mathbb{C}$ with $|z_0| > R$ such that z_0 is neither a pole nor a zero of f', and define the functions W and V in a simply connected open neighbourhood U of z_0 by

$$W^2 = \frac{1}{g'} = \frac{ng^{n-1}}{f''}, \quad V = W^{2n} = \frac{n^n(f')^{n-1}}{(f'')^n}. \tag{9.3}$$

It follows from (9.3), hypothesis (ii), and the fact that R is large that V extends to be analytic in $R < |z| < \infty$, with a zero of multiplicity $2n$ at each pole of f, and no other zeros. In particular, V has an essential singularity at ∞. By a result of Valiron [54, p.15], the function V may be written in the form

$$V(z) = z^q Y(z)(1 + o(1)) \quad \text{as } z \to \infty, \tag{9.4}$$

in which q is an integer and Y is a transcendental entire function.

A standard calculation starting from (9.2) and (9.3) shows that W is a solution on U of

$$w'' + A(z)w = 0. \tag{9.5}$$

On the other hand, (9.3) and (9.5) now yield, again on U,

$$W = V^{1/2n}, \quad -A = \frac{W''}{W}, \quad -A = \frac{1}{2n}\left(\frac{1}{2n} - 1\right)\left(\frac{V'}{V}\right)^2 + \frac{1}{2n}\frac{V''}{V}. \tag{9.6}$$

The last equation of (9.6) then holds by analytic continuation throughout the region $R < |z| < \infty$.

Now let $\nu(r)$ denote the central index of the transcendental entire function Y. By (9.4) and the Wiman-Valiron theory [16], if r is large and lies outside a set of finite logarithmic measure, and if $|z_1| = r$ and $|Y(z_1)| = M(r, Y)$, then $\nu(r)$ is large, and

$$\frac{V'(z_1)^2}{V(z_1)^2} \sim \frac{V''(z_1)}{V(z_1)} \sim \frac{\nu(r)^2}{z_1^2} \quad \text{and} \quad \frac{1}{4n^2}\frac{\nu(r)^2}{z_1^2} \sim -A(z_1) = O(r^{-2}),$$

which is a contradiction. □

Lemma 9.1 makes it possible to write, as $z \to \infty$,

$$A(z) \sim cz^m, \quad c \in \mathbb{C} \setminus \{0\}, \quad m \in \mathbb{Z}, \quad m \geq -1, \tag{9.7}$$

and so Hille's asymptotic method [21, 22] may now be applied to (9.5). The $m+2$ critical rays $\arg z = \theta_0$ for the equation (9.5) are determined by the formula

$$\arg c + (m+2)\theta_0 = 0 \quad (\bmod 2\pi). \tag{9.8}$$

Let ε and $1/R_1$ be small and positive: then (9.5) has linearly independent solutions u_1, u_2 satisfying

$$\begin{gathered} u_1(z) \sim A(z)^{-1/4}e^{-iZ}, \quad u_2(z) \sim A(z)^{-1/4}e^{iZ}, \\ Z = \int_{2R_1}^{z} A(t)^{1/2}dt \sim \frac{2c^{1/2}}{m+2}z^{(m+2)/2}, \end{gathered} \tag{9.9}$$

as $z \to \infty$ in the sectorial region

$$S(R_1, \varepsilon) = \left\{ z \in \mathbb{C} : |z| > R_1, \ |\arg z - \theta_0| < \frac{2\pi}{m+2} - \varepsilon \right\}.$$

If $m = -1$, there is only one critical ray given by (9.8), and $S(R_1, \varepsilon)$ should be understood as lying on the Riemann surface of $\log z$. It follows from (9.1), (9.2), and (9.5) that there exist complex numbers A_j and B_j such that f' satisfies, on $S(R_1, \varepsilon)$,

$$f' = g^n, \quad g = \frac{A_1u_1 - A_2u_2}{B_1u_1 - B_2u_2}, \tag{9.10}$$

and $A_1B_2 - A_2B_1 \neq 0$, since f' is non-constant.

It may be assumed that θ_0 is chosen so that f has infinitely many poles in the narrower sectorial region $S(R_1, 4\varepsilon)$, which forces $B_1B_2 \neq 0$ in (9.10) and makes it possible to write

$$f' = \left(\frac{De^{2\pi iL} - E}{e^{2\pi iL} - 1}\right)^n, \quad D, E \in \mathbb{C}, \quad D \neq E, \tag{9.11}$$

where

$$L(z) = \frac{1}{2\pi i} \log\left(\frac{B_2u_2(z)}{B_1u_1(z)}\right) \sim \frac{Z}{\pi} \sim \frac{2c^{1/2}}{\pi(m+2)} z^{(m+2)/2} \tag{9.12}$$

as $z \to \infty$ in $S(R_1, 2\varepsilon)$. In view of (9.8), it may be assumed that the branch of the square root in (9.9) is chosen so as to make $\operatorname{Re} L(z)$ positive as $z \to \infty$ on the critical ray, and the poles ζ_j of f in $S(R_1, 4\varepsilon)$ must have $\arg \zeta_j \to \theta_0$ as $\zeta_j \to \infty$.

The asymptotics (9.12) show that $w = L(z)$ maps a subdomain S^* of $S(R_1, 3\varepsilon)$ univalently onto a a sectorial region $\Omega = \{w \in \mathbb{C} : |w| > R_2, \ |\arg w| < \pi - \delta\}$, where R_2 is large and δ may be made arbitrarily small by choosing ε small enough. In particular, Ω contains a half-plane H given by $\operatorname{Re} w > q_0 > 0$. Let $z = \phi(w)$ be the inverse mapping from Ω to S^*, choose a large positive integer q, and let the contour γ in H describe once counter-clockwise the circle of centre q and radius $1/4$. Then f has no poles on $\phi(\gamma)$, and (9.11) gives

$$0 = \int_{\phi(\gamma)} f'(z)\, dz = \int_\gamma \psi(w)\, dw, \quad \psi(w) = \left(\frac{De^{2\pi iw} - E}{e^{2\pi iw} - 1}\right)^n \phi'(w). \tag{9.13}$$

As $w \to q$, periodicity yields

$$Q(w) = \left(\frac{De^{2\pi iw} - E}{e^{2\pi iw} - 1}\right)^n = \left(\frac{De^{2\pi i(w-q)} - E}{e^{2\pi i(w-q)} - 1}\right)^n = \frac{D_n}{(w-q)^n} + \cdots + \frac{D_1}{w-q} + O(1),$$

in which the D_j depend on n, D and E, but not on q. Moreover, Lemma 8.1 implies that the function $Q(w)$ is not the second derivative of a meromorphic function in the plane; and so, by periodicity again, at least one of D_1 and D_2 is non-zero. Now (9.13) delivers

$$0 = \operatorname{Res}(\psi, q) = \sigma(q), \quad \sigma(w) = D_1\phi'(w) + D_2\phi''(w) + \cdots + D_n \frac{\phi^{(n)}(w)}{(n-1)!},$$
$$|D_1| + |D_2| > 0.$$

Since $m + 2 \geq 1$ in (9.12), the function $\sigma(w)$ has at most polynomial growth in the half-plane $\operatorname{Re} w > q_0 + 1$. Now the fact that $\sigma(q) = 0$ for all sufficiently large positive integers q forces σ to vanish identically (apply, for example, [30, Lemma 5]). This implies that ϕ satisfies, in the domain Ω, a linear differential equation with constant coefficients, and so ϕ is an entire function of exponential type. Because ϕ has polynomial growth in Ω, by (9.12), while δ is small, applying the Phragmén-Lindelöf principle shows that ϕ is a polynomial. But then the condition $|D_1|+|D_2| > 0$ and the vanishing of σ together ensure that ϕ is a polynomial of degree 1, and so is its inverse function L. Thus (9.11) implies that Lemma 8.1 may be applied to f', which completes the proof.

10 Proof of Theorem 1.5

Let f be a real transcendental meromorphic function in the plane satisfying hypotheses (a), (b) and (c) of Theorem 1.5. It is not assumed at this stage that hypothesis (d) holds. The function

$$h = \frac{f'}{f''} \tag{10.1}$$

has finitely many poles and non-real zeros. If h is a rational function, then $f' = R_0 e^{P_0}$ with R_0 a real rational function and P_0 a real polynomial. Because f has finitely many non-real zeros, this forces (1.2). Assume for the remainder of the proof that h is transcendental.

Lemma 10.1. *The function $L = f'/f$ is transcendental and its Tsuji characteristic satisfies $T_0(r, L) = O(\log r)$ as $r \to \infty$.*

Proof. L must be transcendental, because $1/h = L + L'/L$. The second assertion holds by Lemma 2.1 and the fact that all but finitely many zeros and poles of f and f'' are real. □

Lemma 10.2. *The Nevanlinna characteristic of h satisfies $T(r, h) = O(r \log r)$ as $r \to \infty$, while*

$$\overline{N}(r, f) + \overline{N}(r, 1/f) + \overline{N}(r, 1/f') = O(r \log r) \quad \text{as } r \to \infty. \tag{10.2}$$

Furthermore, $T(r, L) = O(r \log r)$ as $r \to \infty$.

Proof. Lemma 10.1 and standard properties of the Tsuji characteristic give $T_0(r, h) = O(\log r)$ as $r \to \infty$, so that $T(r, h) = O(r \log r)$ as $r \to \infty$ by Lemma 2.2. It then follows from this and (10.1) that

$$\overline{n}(r, f) + \overline{n}(r, 1/f') \leq \overline{n}(r, 1/h) = O(r \log r) \quad \text{as } r \to \infty.$$

The corresponding result for $\overline{n}(r, 1/f)$ now follows from Rolle's theorem. This gives (10.2), which, with Lemmas 2.2 and 10.1, implies the estimate for $T(r, L)$. □

Lemma 10.3. *The function f admits a representation*

$$f = \frac{G}{H}, \quad \frac{G'}{G} = \phi\psi, \tag{10.3}$$

in which

(i) *G and H are real entire functions, and H has order at most* 1*;*

(ii) *ϕ and ψ are real meromorphic functions, and ϕ has finitely many poles and order at most* 1*;*

(iii) *either $\psi \equiv 1$ or ψ maps the upper half-plane H^+ into itself.*

Proof. Here H is the canonical product formed using the poles of f, all but finitely many of which are real, the rest occurring in conjugate pairs because f is real. Since the poles of f have bounded multiplicities, it follows from (10.2) that H has order at most 1. Now G is a real entire function with finitely many non-real zeros, and the formula $G'/G = \phi\psi$ is just the standard Levin-Ostrovskii factorisation [3, 32], in which ψ is formed as in the proof of Lemma 7.5, using real zeros a_k of G and b_k of G'. Finally, ϕ has order at most 1 because (7.1) holds with f replaced by ψ so that, as $r \to \infty$,

$$m(r, \phi) \leq m(r, G'/G) + m(r, 1/\psi) \leq m(r, G'/G) + O(\log r) \leq m(r, L) + O(\log r).$$ □

Lemma 10.4. *The function ϕ in* (10.3) *is rational, and G and f have finite order.*

Proof. Assume that ϕ is transcendental. Fix a small positive real number ε and a large positive integer N, and set

$$W_1(z) = \frac{h(z)}{z^N} = \frac{f'(z)}{z^N f''(z)}, \quad W_2(z) = \frac{\phi(z)}{z^N}. \tag{10.4}$$

Each W_j has finite order and finitely many poles, and so Lemma 6.3 gives an unbounded set $E_1 \subseteq [1, \infty)$ such that for $r \in E_1$ and $j = 1, 2$ there exists $\theta_j \in \mathbb{R}$ with

$$|W_j(re^{i\theta})| \geq 1 \quad \text{for} \quad |\theta - \theta_j| \leq 8\varepsilon. \tag{10.5}$$

For $r \in E_1$, integration gives $c_r \in \mathbb{C} \setminus \{0\}$ and $d_r \in \mathbb{C}$ such that

$$f'(re^{i\theta}) = c_r\left(1 + O\left(r^{1-N}\right)\right), \quad f(re^{i\theta}) = c_r\left(re^{i\theta} + O\left(r^{2-N}\right)\right) + d_r$$

for $|\theta - \theta_1| \leq 8\varepsilon$. This in turn gives, for θ in an interval of length 4ε,

$$P(re^{i\theta}) = re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})} = \frac{re^{i\theta}(1 + o(1))}{re^{i\theta} + d_r/c_r + o(1)} = O(1). \tag{10.6}$$

Because f is real, it may be assumed that (10.6) holds for at least one θ in the interval $[\varepsilon, \pi - \varepsilon]$, and so Lemma 2.8 yields $P(re^{i\theta}) = O(1)$ for $r \in E_1$ and all $\theta \in [\varepsilon, \pi - \varepsilon]$. Since H has order at most 1 and finitely many non-real zeros, (7.1), with f replaced by ψ, and (10.3) yield

$$\frac{G'(re^{i\theta})}{G(re^{i\theta})} = \frac{f'(re^{i\theta})}{f(re^{i\theta})} + \frac{H'(re^{i\theta})}{H(re^{i\theta})} = O(r) \quad \text{and} \quad \phi(re^{i\theta}) = \frac{G'(re^{i\theta})}{G(re^{i\theta})\psi(re^{i\theta})} = O(r^2)$$

for $r \in E_1$ and $|\theta| \in [\varepsilon, \pi - \varepsilon]$. By (10.4), this contradicts (10.5) for $j = 2$.

Thus ϕ is rational, and the assertion that G has finite order, which in turn implies that so has f, follows from a standard argument [3, Lemma 5.1]. □

Lemma 10.5. *The function f' has finitely many asymptotic values, all transcendental singularities of the inverse function of f' are logarithmic, and f''/f' has lower order at least* $1/2$.

Proof. Since f''/f' has finitely many zeros, f' has finitely many critical values. Thus, because f' has finite order, all transcendental singularities of the inverse function are direct, by the main result of [1], and they are finite in number by the Denjoy-Carleman-Ahlfors theorem [17]. Hence all such singularities are in fact logarithmic.

The last assertion is proved as in [38, Lemma 11]. Since f''/f' has finitely many zeros, the same result of Lewis, Rossi and Weitsman [41] as used in Lemma 7.7 gives a path γ tending to ∞ on which f' tends to $\beta \in \mathbb{C} \setminus \{0\}$. If f''/f' has lower order less than $1/2$, the $\cos \pi\rho$ theorem [17] implies that f''/f' is small, and f' is close to β, on the union of a sequence of circles $|z| = r_n \to \infty$. This contradicts the fact that the singularity over β is logarithmic. □

Lemma 10.6. *Let $\delta_1 > 0$, and let $\rho < \infty$ be the order of growth of f. Then $\left|f''(z)/f'(z)\right| \leq |z|^\rho$ as $z \to \infty$ with $\delta_1 \leq |\arg z| \leq \pi - \delta_1$.*

Proof. This follows from standard estimates based on the differentiated Poisson-Jensen formula [14] and the fact that f' has order ρ and finitely many non-real zeros and poles. □

Lemma 10.7. *There exists $\alpha \in \mathbb{C} \setminus \{0\}$ with the following property. If $\varepsilon > 0$ then, as $z \to \infty$ with $\varepsilon \leq \arg z \leq \pi - \varepsilon$,*

$$\left| \frac{f''(z)}{f'(z)} \right| \leq \exp\left(-|z|^{1/4}\right), \tag{10.7}$$

and $f'(z) = \alpha + o(1)$.

Proof. To prove (10.7), apply Lemma 6.2 with $Q = f''/f'$ and $\eta = 1/16$, in conjunction with Lemmas 10.5 and 10.6. Integration then gives $f'(z) = \alpha + o(1)$ in the same sector, where $\alpha \in \mathbb{C} \setminus \{0\}$, and it is clear that α is independent of ε. □

Lemma 10.8. *The inverse function of f' has exactly one of the following:*

(I) *a logarithmic singularity over each of α and $\overline{\alpha}$, where $\alpha \in \mathbb{C} \setminus \mathbb{R}$, and no other transcendental singularities;*

(II) *one or two logarithmic singularities over $\alpha \in \mathbb{R} \setminus \{0\}$, and no other transcendental singularities.*

Proof. Lemma 10.7 gives $f'(z) = \overline{\alpha} + o(1)$ as $z \to \infty$ with $\varepsilon \leq -\arg z \leq \pi - \varepsilon$, where ε may be chosen arbitrarily small. The result now follows from Lemmas 6.3 and 6.4. □

Following [38], let J be a polygonal Jordan curve in $\mathbb{C} \setminus \{0\}$, symmetric with respect to the real axis, such that every finite non-zero critical or asymptotic value of f' lies on J but is not a vertex of J. Here J can be formed so that its complement in $\mathbb{C} \cup \{\infty\}$ consists of two simply connected domains B_1 and B_2, with $0 \in B_1$ and $\infty \in B_2$. Fix conformal mappings

$$h_m : B_m \to \{w \in \mathbb{C} : |w| < 1\}, \quad m = 1, 2, \quad h_1(0) = 0, \quad h_2(\infty) = 0. \tag{10.8}$$

The mapping h_1 may then be extended to be quasiconformal on the plane [45, Ch.5], fixing ∞, and there exist a meromorphic function G_1 and a quasiconformal mapping ψ_1 such that

$$h_1 \circ f' = G_1 \circ \psi_1 \quad \text{on } \mathbb{C}. \tag{10.9}$$

The following lemma is [38, Lemma 4], translated to the present setting in the light of Lemma 10.8.

Lemma 10.9. *For $j = 1, 2$, all components of $(f')^{-1}(B_j)$ are simply connected, and all but finitely many are unbounded. If C_0 is a component of $(f')^{-1}(B_1)$, then C_0 contains one zero of f', of multiplicity $m_1 \in \mathbb{N}$, and C_0 is mapped m_1 to 1 onto B_1 by f'. Furthermore, if a zero z_1 of f'' lies in a component C_1 of $(f')^{-1}(B_1)$, then z_1 is the only zero of f'' in C_1. Similarly, each component of $(f')^{-1}(B_2)$ contains exactly one pole of f, disregarding multiplicities.*

The next step is to combine [38, Lemma 5] with Lemma 10.8.

Lemma 10.10. *Arbitrarily small positive real numbers ε_1 and ε_2 may be chosen with the following properties. There exist one or two unbounded simply connected domains U_n, each a component of the set $\{z \in \mathbb{C} : |f'(z) - b_n| < \varepsilon_1\}$, such that U_n contains a path tending to ∞ on which $f'(z)$ tends to b_n. Here each b_n is α or $\overline{\alpha}$, and $f'(z) \neq b_n$ on U_n, while $|f(z) - b_n z| < \varepsilon_2|z|$ for all z in U_n with $|z|$ large enough. If Γ is a path tending to ∞ on which f' tends to an asymptotic value β, then there exists n such that $\beta = b_n$ and $\Gamma \setminus U_n$ is bounded.*

Lemma 10.11. *The function f' has infinitely many zeros x_j, all but finitely many of which satisfy the following. First, x_j is real and lies in a component C_j of $(f')^{-1}(B_1)$ which is unbounded, simply connected, and symmetric with respect to the real axis, and there are no zeros of f'' on the boundary ∂C_j. Furthermore, ∂C_j is $\Gamma_j^- \cup \Gamma_j^+$, where each $\Gamma_j^\pm$ is a simple curve tending to ∞ in both directions, symmetric with respect to $\mathbb{R}$, and meeting the real axis exactly once. Analogous considerations apply to poles of f'.*

Proof. There exist infinitely many zeros x_j of f' by Lemma 10.8. For $|x_j|$ large, let

(10.10)
$$-\infty < y_j^- = \inf\{x \in \mathbb{R} : [x, x_j] \subseteq C_j\} < y_j^+ = \sup\{x \in \mathbb{R} : [x_j, x] \subseteq C_j\} < \infty.$$

Each $y_j^\pm$ lies in a component $\Gamma_j^\pm$ of ∂C_j which is symmetric with respect to $\mathbb{R}$, and $\psi_1(\Gamma_j^\pm)$ is a level curve of the function G_1 in (10.9). Thus $\Gamma_j^\pm \cap \mathbb{R} = \{y_j^\pm\}$, because C_j is simply connected. Finally, observe that any component of ∂C_j other than the $\Gamma_j^\pm$ would have to lie in $\mathbb{C} \setminus \mathbb{R}$ and form part of the boundary of a component of $(f')^{-1}(B_2)$, that component having to contain a non-real pole of f. □

Lemma 10.12. *The zeros of f' have bounded multiplicities, and case* (II) *holds in Lemma 10.8.*

Proof. Each $\Gamma_j^\pm$ in Lemma 10.11 forms part of the boundary of a a component of $(f')^{-1}(B_2)$, and the poles of f have bounded multiplicities. Hence the variation of $\arg f'$ on $\Gamma_j^\pm$ has an upper bound which is independent of j, thus proving the first assertion.

Suppose now that case (I) holds in Lemma 10.8. If z_0 is large and is a zero of f'', then z_0 and $f(z_0)$ are real, so that

$$|f(z_0) - \alpha z_0| = |f(z_0) - \overline{\alpha} z_0| \geq |z_0 \operatorname{Im} \alpha|.$$

Theorem 1.6 may now be applied, yielding $f'' = R_2 e^{P_2}$ with R_2 a real rational function and P_2 a real polynomial. Thus f has finitely many poles, which contradicts Lemma 10.8. □

It may be assumed henceforth that case (II) holds in Lemma 10.8, with $\alpha = 1$.

Lemma 10.13. *Fix positive real numbers M_1 and M_2 with M_1 large and $M_1 < M_2$. Let $v_j \in \mathbb{R}$ with $|v_j|$ large be a pole of f of multiplicity m_j, and let D_j be the component of $(f')^{-1}(B_2)$ in which x_j lies. Then $|f(z) - z| \leq 2\varepsilon_2|z|$ for all $z \in D_j$ with $M_1 < |f'(z)| < M_2$, where ε_2 is as in Lemma 10.10. Moreover, f has at least m_j real simple zeros in D_j, and m_j is* 1 *or* 2.

Proof. The component D_j is simply connected and, as shown in Lemma 10.11, its boundary consists of two disjoint simple curves $\Lambda_j^{\pm}$. The function $v = (h_2 \circ f')^{1/m_j}$ maps D_j conformally onto the unit disc, and as z tends to ∞ in either direction along either of the $\Lambda_j^{\pm}$, the image $f'(z)$ tends to the unique asymptotic value 1 of f', since f' is finite-valent on D_j. This implies that D_j meets one of the components U_n of Lemma 10.10. It follows that there exist μ_j with $\mu_j^{m_j} = h_2(1)$ and a positive ε_3 such that if $z \in D_j$ and $|v(z) - \mu_j| \leq \varepsilon_3$, then $z \in U_n$. Here ε_3 may be chosen arbitrarily small and independent of j, since the m_j are bounded by hypothesis.

Let u be the inverse function of v, mapping the unit disc onto D_j. Then $u'(0) = o(|v_j|)$, by Koebe's 1/4 theorem and Lemma 10.7. Koebe's distortion theorem then yields $u'(w) = o(|v_j|)$ for $|w| \leq 1 - \varepsilon_3$. Now let $z_1 \in D_j$ be such that $w_1 = v(z_1)$ satisfies $\varepsilon_3 \leq |w_1| \leq 1 - \varepsilon_3$. Then w_1 can be joined to a point w_2 with $|w_2| < 1$, $|w_2 - \mu_j| \leq \varepsilon_3$ by a path Σ in $\varepsilon_3 \leq |w| \leq 1 - \varepsilon_3$, so that $\sigma = v(\Sigma)$ is a path in D_j of length $o(|v_j|)$ joining z_1 to $z_2 = u(w_2) \in U_n$. But then $|f(z_2) - z_2| \leq \varepsilon_2|z_2|$ by Lemma 10.10. Since f' is bounded on σ, integration of f' gives $|f(z_1) - z_1| \leq 2\varepsilon_2|z_1|$, proving the first assertion.

Next, let τ be the image under u of the circle $|w| = \varepsilon_3$. Then τ is a Jordan curve in D_j enclosing v_j, and symmetric with respect to the real axis. Furthermore, $|f(z) - z| < |z|$ on τ; thus Rouché's theorem implies that f has m_j zeros inside τ, and these zeros must be real. Since f' has no zeros in D_j, these zeros of f are also simple, and $m_j \in \{1, 2\}$ by Rolle's theorem. □

In view of Lemma 10.13, the hypothesis (d) may now be used for the first time, to separate the remainder of the proof into two cases.

Case A: *assume that all but finitely many poles of f have multiplicity* 2.

The first step in this case is the following.

Lemma 10.14. *All but finitely many zeros of f' have multiplicity* 3.

Proof. It suffices to take successive real zeros $x_{j-1} < x_j < x_{j+1}$ of f' with $|x_{j-1}|$ and $|x_{j+1}|$ large, and to show that the multiplicity n_j of x_j is 3. Since all but finitely many zeros of f'' are zeros of f', Rolle's theorem implies that there exist poles v_k, v_{k+1} of f' which satisfy $x_{j-1} < v_k < x_j < v_{k+1} < x_{j+1}$, and these may be assumed to be the nearest poles of f' to x_j and to have multiplicity 3 for f'. It then follows from Lemmas 10.11 and 10.12 and the argument principle that $2 \leq n_j \leq 4$. On the other hand, Lemma 10.13 and Rolle's theorem together show that v_k lies close to, and must lie between, a pair of real simple zeros of f, and the same is true of v_{k+1}. Thus x_j lies between zeros of f which are not separated by poles of f, and so x_j is a zero of f' of odd multiplicity, forcing $n_j = 3$. □

Now Theorem 1.7 can be applied with $n = 3$ and $\lambda^3 = 1, \lambda \neq 1$ in (1.8), and the constants a and b must have zero real part. Hence, without loss of generality, assume that

$$f'(z) = C\left(\frac{\lambda e^{iz} - 1}{e^{iz} - 1}\right)^3,$$

and $C = 1$ since 1 is the only asymptotic value of f'. If x is a pole of f then, as $z \to x$,

$$\begin{aligned} f'(z) &\sim \frac{\mu}{(z-x)^3}, \quad f(z) \sim \frac{-\mu}{2(z-x)^2}, \\ \mu &= \frac{(\lambda-1)^3}{i^3} = -6\,\mathrm{Im}\,\lambda \in \mathbb{R} \setminus \{0\}. \end{aligned} \tag{10.11}$$

Next, let ε_4 be small and positive, and let U be the union of the discs of centre $2\pi n$ and radius ε_4, for $n \in \mathbb{Z}$. Let m be an integer with $|m|$ large such that m has the same sign as $-\mu$. Then $2\pi m$ is a pole of f, and the real limit $\Lambda = \lim_{t\to 2\pi m} f(t)$ exists and is infinite, with the same sign as m. Since integration shows that $f(z) \sim z$ for z with $|z|$ large but $z \notin U$, it follows that Λ has the same sign as $f(2\pi m - \varepsilon_4)$ and $f(2\pi m + \varepsilon_4)$. Now Rolle's theorem and the fact that f' has no zeros near to $2\pi m$ together imply that f has no real zeros close to $2\pi m$. But Rouché's theorem gives, counting multiplicity, two zeros of f close to $2\pi m$, both necessarily real, and this contradiction excludes Case A.

Case B: *assume that all but finitely many poles of f have multiplicity* 1. In this case, all but finitely many zeros of f' have multiplicity 2, by the argument principle. This time, Theorem 1.7 may be applied with $n = 2$, and hence $\lambda = -1$, in (1.8). This yields $f'(z) = C\cot^2(Az + B)$, with A, B, C real, and the conclusion of the theorem follows easily.

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