VARIATIONAL RESOLUTION FOR SOME GENERAL CLASSES OF NONLINEAR EVOLUTIONS. PART II

By

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Abstract. Using our results in [11], we provide existence theorems for general classes of nonlinear evolutions. Then we give examples of applications of our results to parabolic, hyperbolic, Schrödinger, Navier-Stokes and other time-dependent systems of equations.

1 Introduction

Let X be a reflexive Banach space. Consider the following evolutional initial value problem:

$$\begin{cases} \frac{d}{dt}\big\{I \cdot u(t)\big\} + \Lambda_t\big(u(t)\big) = 0 \quad \text{in } (0, T_0), \\ I \cdot u(0) = v_0. \end{cases} \tag{1.1}$$

Here, $I : X \to X^*$ (X^* is the space dual to X) is a fixed bounded linear inclusion operator, assumed to be self-adjoint and strictly positive, $u(t) \in L^q\big((0, T_0); X\big)$ is an unknown function such that $I \cdot u(t) \in W^{1,p}\big((0, T_0); X^*\big)$ (where $I \cdot h \in X^*$ is the value of the operator I at the point $h \in X$), $\Lambda_t(x) : X \to X^*$ is a fixed nonlinear mapping, considered for every fixed $t \in (0, T_0)$, and $v_0 \in X^*$ is a fixed initial value. The most trivial variational principle related to (1.1) is the following one. Consider some convex function $\Gamma(y) : X^* \to [0, +\infty)$ satisfying $\Gamma(y) = 0$ if and only if $y = 0$. Next define the energy functional

$$E_0\big(u(\cdot)\big) := \int_0^{T_0} \Gamma\left(\frac{d}{dt}\big\{I \cdot u(t)\big\} + \Lambda_t\big(u(t)\big)\right)dt \text{ for all } u(t) \in L^q\big((0, T_0); X\big)$$
$$\text{such that } I \cdot u(t) \in W^{1,p}\big((0, T_0); X^*\big) \text{ and } I \cdot u(0) = v_0. \tag{1.2}$$

Then it is obvious that $u(t)$ is a solution of (1.1) if and only if $E_0\big(u(\cdot)\big) = 0$. Moreover, the solution of (1.1) exists if and only if there exists a minimizer $u_0(t)$ of the energy $E_0(\cdot)$, which satisfies $E_0\big(u_0(\cdot)\big) = 0$.

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We have the following generalization of this variational principle. Let $\Psi_t(x) : X \to [0, +\infty)$ be some convex Gâteaux differentiable function, considered for every fixed $t \in (0, T_0)$ and satisfying $\Psi_t(0) = 0$. Next define the Legendre transform of Ψ_t by

$$\Psi_t^*(y) := \sup\left\{\langle z, y\rangle_{X\times X^*} - \Psi_t(z) :\ z \in X\right\} \quad \text{for all } y \in X^*. \tag{1.3}$$

It is well known that $\Psi_t^*(y) : X^* \to \mathbb{R}$ is a convex function and that

$$\Psi_t(x) + \Psi_t^*(y) \geq \langle x, y\rangle_{X\times X^*} \quad \text{for all } x \in X,\ y \in X^*, \tag{1.4}$$

with equality if and only if $y = D\Psi_t(x)$. Next, for $\lambda \in \{0, 1\}$, define the energy functional

$$E_\lambda\big(u\big) := \int_0^{T_0} \left\{\Psi_t\Big(\lambda u(t)\Big) + \Psi_t^*\bigg(-\frac{d}{dt}\{I \cdot u(t)\} - \Lambda_t\big(u(t)\big)\bigg) + \lambda\left\langle u(t), \frac{d}{dt}\{I \cdot u(t)\} + \Lambda_t\big(u(t)\big)\right\rangle_{X\times X^*}\right\}dt \tag{1.5}$$

for all $u(t) \in L^q\big((0, T_0); X\big)$ such that $I \cdot u(t) \in W^{1,p}\big((0, T_0); X^*\big)$ and $I \cdot u(0) = v_0$.

Then, by (1.4), we have $E_\lambda\big(\cdot\big) \geq 0$; and, moreover, $E_\lambda\big(u(\cdot)\big) = 0$ if and only if $u(t)$ is a solution of

$$\begin{cases} \frac{d}{dt}\{I \cdot u(t)\} + \Lambda_t\big(u(t)\big) + D\Psi_t\big(\lambda u(t)\big) = 0 & \text{in } (0, T_0), \\ I \cdot u(0) = v_0. \end{cases} \tag{1.6}$$

(Note here that since $\Psi_t(0) = 0$, in the case $\lambda = 0$, (1.6) coincides with (1.1). Moreover, if $\lambda = 0$, then the energy defined in (1.2) is a particular case of the energy in (1.5), where we take $\Gamma(x) := \Psi^*(-x)$). So, as before, a solution of (1.6) exists if and only if there exists a minimizer $u_0(t)$ of the energy $E_\lambda(\cdot)$ that satisfies $E_\lambda\big(u_0(\cdot)\big) = 0$. Consequently, in order to establish the existence of a solution of (1.6), we need to answer the following questions.

(a) Does a minimizer to the energy in (1.5) exist?

(b) Does the minimizer $u_0(t)$ of the corresponding energy $E_\lambda(\cdot)$ satisfy $E_\lambda\big(u_0(\cdot)\big) = 0$?

To the best of our knowledge, the energy in (1.5) with $\lambda = 1$, related to (1.6), was first considered for the heat equation and other types of evolutions by Brezis and Ekeland in [1]. In that work, they also first asked question (**b**): how, without a priori knowledge of the existence of a solution of (1.6), to prove that the minimum of the corresponding energy is 0. This question was asked even for very simple PDE's like the heat equation. A detailed investigation of the energy of type (1.5) with $\lambda = 1$ was done in a series of works by N. Ghoussoub and his coauthors; see [4] and also [5], [6], [7], [8]. In those works, they considered a similar variational principle, not only for evolutions, but also for some other classes of equations. They proved some theoretical results about general self-dual variational principles which, in many cases, can provide the existence of a zero energy state (answering questions (**a**) and (**b**) together) and, consequently, the existence of solutions of the related equations; see [4] for details.

In [11], we provided an alternative approach to questions (**a**) and (**b**). We treated them separately; and, in particular, for question (**b**), we derived the main information by studying the Euler-Lagrange equations for the corresponding energy. To our knowledge, such an approach was first considered in [10], where an alternative proof of existence of solutions for initial value problems for some parabolic systems is provided. Generalizing these results, we provided in [11] the answer to questions (**a**) and **(b)** for a wide class of evolution equations. In particular, regarding question **(b)**, we were able to prove that in some general cases, not only the minimizer, but also any critical point $u_0(t)$ (i.e., any solution of the corresponding Euler-Lagrange equation), satisfies $E_\lambda\big(u_0(\cdot)\big) = 0$, i.e., is a solution of (1.6).

The approach of Ghoussoub in [4] is more general than ours, as he considered a more abstract setting. The main advantages of our method are as follows.

- We are able to prove that under some growth and coercivity conditions *every critical point* of the energy (1.5) is actually a minimizer and a solution of (1.6).
- Our result, giving the answer for question (**b**), does not require any assumption of compactness or weak continuity of Λ_t (these assumptions are needed only for the proof of existence of minimizer, i.e., in connection with question (**a**)).
- Our method for answering question (**b**) uses only elementary arguments.

In particular, in order to answer question (**b**), we get the main information directly from the Euler-Lagrange equation for energy (1.5). Although the Euler-Lagrange equation of that energy differs from equation (1.6), we are able to show that the sets of solutions for these equations coincide in some general cases. We note here that the above property holds in the case of energy (1.5) related to evolutional equations and does not hold in many cases of stationary (time independent) problems.

We can rewrite the definition of E_λ in (1.5) as follows. Since I is a self-adjoint and strictly positive operator, there exist a Hilbert space H and an injective bounded linear operator $T : X \to H$ whose image is dense in H such that for the linear operator $\widetilde{T} : H \to X^*$ defined by the formula

$$\langle x, \widetilde{T} \cdot y\rangle_{X\times X^*} := \langle T \cdot x, y\rangle_{H\times H} \quad \text{for every } y \in H \text{ and } x \in X, \tag{1.7}$$

$\widetilde{T} \circ T \equiv I$; see [11, Lemma 2.7] for details. We call $\{X, H, X^*\}$ an **evolution triple with the corresponding inclusion operators** $T : X \to H$ **and** $\widetilde{T} : H \to X^*$. Thus, if $v_0 = \widetilde{T} \cdot w_0$ for some $w_0 \in H$ and $p = q^* := q/(q-1)$, where $q > 1$, then

$$\int_0^{T_0} \left\langle u(t), \frac{d}{dt}\{I \cdot u(t)\} \right\rangle_{X\times X^*} dt = \frac{1}{2}\|T \cdot u(T_0)\|_H^2 - \frac{1}{2}\|w_0\|_H^2$$

(see Lemma 2.3 for details), and therefore

$$\begin{aligned} E_\lambda\big(u\big) = J\big(u\big) := \\ \int_0^{T_0} \left\{ \Psi_t\Big(\lambda u(t)\Big) + \Psi_t^*\Big(-\frac{d}{dt}\{I \cdot u(t)\} - \Lambda_t\big(u(t)\big)\Big) + \lambda\Big\langle u(t), \Lambda_t\big(u(t)\big)\Big\rangle_{X\times X^*} \right\} dt \\ + \frac{\lambda}{2}\|T \cdot u(T_0)\|_H^2 - \frac{\lambda}{2}\|w_0\|_H^2 \end{aligned} \tag{1.8}$$

for all $u(t) \in L^q\big((0, T_0); X\big)$ such that $I \cdot u(t) \in W^{1,q^*}\big((0, T_0); X^*\big)$ and $I \cdot u(0) = \widetilde{T} \cdot w_0$.

Our first main result in [11] provides the answer to question (**b**) under some coercivity and growth conditions on Ψ_t and Λ_t.

Theorem 1.1. *Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion linear operators $T : X \to H$, assumed to be bounded, injective and to have dense image in H, $\widetilde{T} : H \to X^*$ defined by* (1.7). *Let $I := \widetilde{T} \circ T : X \to X^*$. Next, let $\lambda \in \{0, 1\}$, $q \geq 2$, $p = q^* := q/(q-1)$ and $w_0 \in H$. Furthermore, for every $t \in [0, T_0]$, let $\Psi_t(x) : X \to [0, +\infty)$ be a strictly convex function that is Gâteaux differentiable at every $x \in X$ and satisfies $\Psi_t(0) = 0$ and the condition*

$$\frac{1}{C_0}\|x\|_X^q - C_0 \leq \Psi_t(x) \leq C_0\,\|x\|_X^q + C_0 \quad \textit{for all } x \in X \textit{ for all } t \in [0, T_0] \tag{1.9}$$

for some $C_0 > 0$. Also assume that $\Psi_t(x)$ is a Borel function of its variables (x, t). Next, for each $t \in [0, T_0]$, let $\Lambda_t(x) : X \to X^*$ be a function that is Gâteaux differentiable at every $x \in X$ and such that $\Lambda_t(0) \in L^{q^*}\big((0, T_0); X^*\big)$ and the derivative $D\Lambda_t$ of Λ_t satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)} \leq g\big(\|T \cdot x\|_H\big)\left(\|x\|_X^{q-2} + \mu^{(q-2)/q}(t)\right) \tag{1.10}$$

for all $x \in X$ *for all* $t \in [0, T_0]$ *and some non-decreasing function* $g(s) : [0+\infty) \to (0, +\infty)$ *and some non-negative function* $\mu(t) \in L^1\big((0, T_0); \mathbb{R}\big)$. *Also assume that* $\Lambda_t(x)$ *is strongly Borel (see Definition 2.1) on the pair of variables* (x, t). *Assume also that* Ψ_t *and* Λ_t *satisfy the monotonicity condition*

$$\text{(1.11)}\quad \Big\langle h, \lambda\Big\{D\Psi_t\big(\lambda x + h\big) - D\Psi_t(\lambda x)\Big\} + D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*} \geq -\hat{g}\big(\|T\cdot x\|_H\big)\Big(\|x\|_X^q + \hat{\mu}(t)\Big)\,\|T\cdot h\|_H^2 \quad \text{for all } x, h \in X,\ \text{and } t \in [0, T_0]$$

for some non-decreasing function $\hat{g}(s) : [0+\infty) \to (0, +\infty)$ *and some nonnegative function* $\hat{\mu}(t) \in L^1\big((0, T_0); \mathbb{R}\big)$. *Consider the set*

$$\text{(1.12)}\qquad \mathcal{R}_q := \Big\{u(t) \in L^q\big((0, T_0); X\big) : I\cdot u(t) \in W^{1,q^*}\big((0, T_0); X^*\big)\Big\},$$

and the minimization problem

$$\text{(1.13)}\qquad \inf\Big\{J(u) :\ u(t) \in \mathcal{R}_q \text{ such that } I\cdot u(0) = \widetilde{T}\cdot w_0\Big\},$$

where $J(u)$ *is defined by* (1.8). *Then for every* $u \in \mathcal{R}_q$ *such that* $I\cdot u(0) = \widetilde{T}\cdot w_0$ *and for arbitrary function* $h(t) \in \mathcal{R}_q$, *such that* $I\cdot h(0) = 0$, $\lim_{s\to 0}\big(J(u+sh) - J(u)\big)/s$ *exists and is finite. Moreover, for every such u, the following four statements are equivalent:*

(1) *u is a critical point of* (1.13), *i.e., for any function* $h(t) \in \mathcal{R}_q$ *such that* $I\cdot h(0) = 0$,

$$\text{(1.14)}\qquad \lim_{s\to 0}\frac{J(u+sh) - J(u)}{s} = 0;$$

(2) *u is a minimizer of* (1.13)*;*

(3) $J(u) = 0$*;*

(4) *u is a solution of*

$$\text{(1.15)}\qquad \begin{cases} \frac{d}{dt}\big\{I\cdot u(t)\big\} + \Lambda_t\big(u(t)\big) + D\Psi_t\big(\lambda u(t)\big) = 0 & \text{in } (0, T_0),\\ I\cdot u(0) = \widetilde{T}\cdot w_0. \end{cases}$$

Finally, there exists at most one function $u \in \mathcal{R}_q$ *that satisfies* (1.15).

Remark 1.1. Assume that, instead of (1.11), one requires that Ψ_t and Λ_t satisfy the inequality

$$\text{(1.16)}\quad \Big\langle h,\lambda\Big\{D\Psi_t\big(\lambda x + h\big) - D\Psi_t(\lambda x)\Big\} + D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*} \geq \frac{\big|f(h,t)\big|^2}{\tilde{g}(\|T\cdot x\|_H)} - \tilde{g}\big(\|T\cdot x\|_H\big)\Big(\|x\|_X^q + \hat{\mu}(t)\Big)^{(2-r)/2}\big|f(h,t)\big|^r\,\|T\cdot h\|_H^{(2-r)}$$

for all $x, h \in X$ for all $t \in [0, T_0]$ for some non-decreasing function $\tilde{g}(s) : [0 + \infty) \to (0, +\infty)$, some function $\hat{\mu}(t) \in L^1\big((0, T_0); \mathbb{R}\big)$, some function $f(x, t) : X \times [0, T_0] \to \mathbb{R}$, and some constant $r \in (0, 2)$. Then (1.11) follows by the trivial inequality $(r/2)\, a^2 + \big((2 - r)/2\big) b^2 \geq a^r\, b^{2-r}$.

Our first result in [11] about the existence of minimizers for $J(u)$ is the following proposition.

Proposition 1.1. *Assume that* $\{X, H, X^*\}$, $T, \widetilde{T}, I, q, p, \Psi_t$ *and* Λ_t *satisfy all the conditions of Theorem 1.1 with* $\lambda = 1$. *Moreover, assume that* Ψ_t *and* Λ_t *satisfy the positivity condition*

$$\Psi_t(x) + \Big\langle x, \Lambda_t(x) \Big\rangle_{X \times X^*} \geq \frac{1}{\tilde{C}} \|x\|_X^q - \bar{\mu}(t)\Big(\|T \cdot x\|_H^2 + 1\Big) \tag{1.17}$$

for all $x \in X$ *and* $t \in [0, T_0]$, *where* $\tilde{C} > 0$ *is some constant and* $\bar{\mu}(t) \in L^1\big((0, T_0); \mathbb{R}\big)$ *is some non-negative function. Furthermore, assume that*

$$\Lambda_t(x) = A_t\big(S \cdot x\big) + \Theta_t(x) \quad \textit{for all } x \in X \textit{ for all } t \in [0, T_0], \tag{1.18}$$

where Z *is a Banach space,* $S : X \to Z$ *is a compact operator and for every* $t \in [0, T_0]$, $A_t(z) : Z \to X^*$ *is a function which is strongly Borel on the pair of variables* (z, t) *and Gâteaux differentiable at every* $z \in Z$, $\Theta_t(x) : X \to X^*$ *is strongly Borel on the pair of variables* (x, t) *and Gâteaux differentiable at every* $x \in X$, $\Theta_t(0), A_t(0) \in L^{q^*}\big((0, T_0); X^*\big)$ *and the derivative* DA_t *of* A_t *and the derivative* $D\Theta_t$ *of* Θ_t *satisfy the growth condition*

$$\|D\Theta_t(x)\|_{\mathcal{L}(X;X^*)} + \|DA_t(S \cdot x)\|_{\mathcal{L}(Z;X^*)} \leq g\big(\|T \cdot x\|\big) \Big(\|x\|_X^{q-2} + \mu^{(q-2)/q}(t)\Big) \tag{1.19}$$

for all $x \in X$ *and* $t \in [0, T_0]$ *for some nondecreasing function* $g(s) : [0, +\infty) \to (0 + \infty)$ *and some nonnegative function* $\mu(t) \in L^1\big((0, T_0); \mathbb{R}\big)$. *Next assume that for every sequence* $\big\{x_n(t)\big\}_{n=1}^{+\infty} \subset L^q\big((0, T_0); X\big)$ *such that the sequence* $\big\{I \cdot x_n(t)\big\}$ *is bounded in* $W^{1,q^*}\big((0, T_0); X^*\big)$ *and* $x_n(t) \rightharpoonup x(t)$ *weakly in* $L^q\big((0, T_0); X\big)$,

- $\Theta_t\big(x_n(t)\big) \rightharpoonup \Theta_t\big(x(t)\big)$ *weakly in* $L^{q^*}\big((0, T_0); X^*\big)$,
- $$\varliminf_{n \to +\infty} \int_0^{T_0} \Big\langle x_n(t), \Theta_t\big(x_n(t)\big) \Big\rangle_{X \times X^*} dt \geq \int_0^{T_0} \Big\langle x(t), \Theta_t\big(x(t)\big) \Big\rangle_{X \times X^*} dt.$$

Finally, let $w_0 \in H$ *be such that* $w_0 = T \cdot u_0$ *for some* $u_0 \in X$ *or, more generally, let* $w_0 \in H$ *be such that* $\mathcal{A}_{w_0} := \big\{u \in \mathcal{R}_q : I \cdot u(0) = \widetilde{T} \cdot w_0\big\} \neq \varnothing$. *Then there exists a minimizer of* (1.13).

As a consequence of Theorem 1.1 and Proposition 1.1, we have the following corollary.

Corollary 1.1. *Under the assumptions of Proposition 1.1, there exists a unique solution $u(t) \in \mathcal{R}_q$ of*

$$\begin{cases} \frac{d}{dt}\big\{I \cdot u(t)\big\} + \Lambda_t\big(u(t)\big) + D\Psi_t\big(u(t)\big) = 0 \quad in \ (0, T_0), \\ I \cdot u(0) = \widetilde{T} \cdot w_0. \end{cases} \tag{1.20}$$

As an important particular case of Corollary 1.1, we recover the following theorem in [11].

Theorem 1.2. *Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion linear operators $T : X \to H$, assumed bounded, injective and having dense image in H, and $\widetilde{T} : H \to X^*$ defined by* (1.7)*, and $I := \widetilde{T} \circ T : X \to X^*$. Next, let $q \geq 2$. Furthermore, for each $t \in [0, T_0]$, let $\Psi_t(x) : X \to [0, +\infty)$ be a strictly convex function that is Gâteaux differentiable at every $x \in X$, satisfies $\Psi_t(0) = 0$, and satisfies the growth condition*

$$\frac{1}{C_0}\|x\|_X^q - C_0 \leq \Psi_t(x) \leq C_0\,\|x\|_X^q + C_0 \tag{1.21}$$

for all $x \in X$ and $t \in [0, T_0]$ and the uniform convexity condition

$$\Big\langle h, D\Psi_t(x+h) - D\Psi_t(x)\Big\rangle_{X \times X^*} \geq \frac{1}{C_0}\Big(\big\|x\big\|_X^{q-2} + 1\Big) \cdot \|h\|_X^2 \tag{1.22}$$

for all $x, h \in X$ and $t \in [0, T_0]$ for some $C_0 > 0$. Also assume that $\Psi_t(x)$ is Borel on the pair of variables (x, t) Next let Z be a Banach space, $S : X \to Z$ be a compact operator; and, for every $t \in [0, T_0]$, let $F_t(z) : Z \to X^*$ be a function such that F_t is strongly Borel on the pair of variables (z, t) and Gâteaux differentiable at every $z \in Z$, $F_t(0) \in L^{q^*}\big((0, T_0); X^*\big)$ and the derivative DF_t of of F_t satisfies the growth condition

$$\big\|DF_t(S \cdot x)\big\|_{\mathcal{L}(Z;X^*)} \leq g\big(\|T \cdot x\|\big)\Big(\|x\|_X^{q-2} + 1\Big) \tag{1.23}$$

for all $x \in X$ and $t \in [0, T_0]$ for some non-decreasing function $g(s) : [0, +\infty) \to (0, +\infty)$. Moreover, assume that Ψ_t and F_t satisfy the positivity condition

$$\Psi_t(x) + \Big\langle x, F_t(S \cdot x)\Big\rangle_{X \times X^*} \geq \frac{1}{C}\|x\|_X^q - \bar{C}\|S \cdot x\|_Z^2 - \bar{\mu}(t)\Big(\|T \cdot x\|_H^2 + 1\Big) \tag{1.24}$$

for all $x \in X$ and $t \in [0, T_0]$ for some constant $\bar{C} > 0$ and some non-negative function $\bar{\mu}(t) \in L^1\big((0, T_0); \mathbb{R}\big)$. Furthermore, let $w_0 \in H$ be such that $w_0 = T \cdot u_0$ for some $u_0 \in X$, or more generally, $w_0 \in H$ be such that

$$\mathcal{A}_{w_0} := \big\{u \in \mathcal{R}_q : I \cdot u(0) = \widetilde{T} \cdot w_0\big\} \neq \varnothing.$$

Then there exists a unique solution $u(t) \in \mathcal{R}_q$ *of the equation*

$$\begin{cases} \frac{d}{dt}\big\{I \cdot u(t)\big\} + F_t\big(S \cdot u(t)\big) + D\Psi_t\big(u(t)\big) = 0 & \text{for a.e. } t \in (0, T_0), \\ I \cdot u(0) = \widetilde{T} \cdot w_0. \end{cases} \tag{1.25}$$

In this paper, using Theorem 1.2 as a basis, by the appropriate approximation, we obtain further existence theorems under much weaker assumptions on coercivity and compactness. The first theorem improves the existence part of Corollary 1.1 (Theorem 3.1 is an equivalent formulation and Theorem 3.2 is an important particular case).

Theorem 1.3. *Let* $q \geq 2$ *and* $\{X, H, X^*\}$ *be an evolution triple with the corresponding inclusion linear operators* $T : X \to H$*, assumed injective and having dense image in* H*, and* $\widetilde{T} : H \to X^*$ *defined by* (1.7)*, and* $I := \widetilde{T} \circ T : X \to X^*$*. Assume also that the Banach space* X *is separable. Furthermore, for every* $t \in [0, T_0]$*, let* $\Psi_t(x) : X \to [0, +\infty)$ *be a convex function that is Gâteaux differentiable at every* $x \in X$*, satisfies* $\Psi_t(0) = 0$ *and satisfies the growth condition*

$$0 \leq \Psi_t(x) \leq C\,\|x\|_X^q + C \quad \text{for all } x \in X, \quad \text{for all } t \in [0, T_0], \tag{1.26}$$

for some $C > 0$*. Assume also that* $\Psi_t(x)$ *is Borel on the pair of variables* (x, t)*. Furthermore, for each* $t \in [0, T_0]$*, let* $\Lambda_t(x) : X \to X^*$ *be a function that is Gâteaux differentiable at every* $x \in X$*,* $\Lambda_t(0) \in L^{q^*}\big((0, T_0); X^*\big)$ *and the derivative* $D\Lambda_t$ *of* Λ_t *satisfies the growth condition*

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)} \leq g\big(\|T \cdot x\|_H\big)\left(\|x\|_X^{q-2} + 1\right) \quad \forall x \in X \quad \text{for all } t \in [0, T_0] \tag{1.27}$$

for some nondecreasing function $g(s) : [0, +\infty) \to (0, +\infty)$*. Also assume that* $\Lambda_t(x)$ *is Borel on the pair of variables* (x, t)*. Assume also that* Λ_t *and* Ψ_t *satisfy the monotonicity condition*

$$\Big\langle x, D\Psi_t(x) + \Lambda_t(x)\Big\rangle_{X \times X^*} \geq \frac{1}{\hat{C}}\,\|x\|_X^q - \hat{C}\|L \cdot x\|_V^2 - \mu(t)\Big(\|T \cdot x\|_H^2 + 1\Big) \tag{1.28}$$

for all $x \in X$ *and* $t \in [0, T_0]$*, where* V *is a given Banach space,* $L \in \mathcal{L}(X, V)$ *is a given compact operator,* $\mu(t) \in L^1\big((0, T_0); \mathbb{R}\big)$ *is some non-negative function and* $\hat{C} > 0$ *is some constant. Finally, assume that for every* $t \in [0, T_0]$*,* $\big(D\Psi_t + \Lambda_t\big)(x) : X \to X^*$ *satisfies the following compactness property:*

- *if* $x_n \rightharpoonup x$ *weakly in* X*, then* $\underline{\lim}_{n\to+\infty} \langle x_n - x, D\Psi_t(x_n) + \Lambda_t(x_n)\rangle_{X\times X^*} \geq 0$*;*
- *if* $x_n \rightharpoonup x$ *weakly in* X *and* $\lim_{n\to+\infty} \langle x_n - x, D\Psi_t(x_n) + \Lambda_t(x_n)\rangle_{X\times X^*} = 0$*, then* $D\Psi_t(x_n) + \Lambda_t(x_n) \rightharpoonup D\Psi_t(x) + \Lambda_t(x)$ *weakly in* X^**.*

Then for each $w_0 \in H$ and each $\lambda \in \mathbb{R}$, there exists $u(t) \in L^q\big((0,T_0);X\big)$ such that $I \cdot \big(u(t)\big) \in W^{1,q^*}\big((0,T_0);X^*\big)$, where $q^* := q/(q-1)$, and $u(t)$ is a solution of

$$\begin{cases} \frac{d}{dt}\big\{I \cdot u(t)\big\} + \lambda I \cdot u(t) + \Lambda_t\big(u(t)\big) + D\Psi_t\big(u(t)\big) = 0 \quad in \ (0,T_0), \\ I \cdot u(0) = \widetilde{T} \cdot w_0. \end{cases} \tag{1.29}$$

The second existence result is useful in the study of parabolic, hyperbolic, parabolic-hyperbolic, Schrödinger, Navier-Stokes and other types of equations (Theorem 3.3 is an equivalent formulation and Theorem 3.4 and Corollary 3.2 are important particular cases).

Theorem 1.4. *Let $q \geq 2$, X and Z be reflexive Banach spaces and X^* and Z^* be the corresponding dual spaces. Let H be a Hilbert space. Suppose that $Q : X \to Z$ is an injective bounded linear operator whose image is dense on Z. Furthermore, suppose that $P : Z \to H$ is an injective bounded linear operator whose image is dense on H. Let $T : X \to H$ be defined by $T := P \circ Q$, so that $\{X, H, X^*\}$ is an evolution triple with the corresponding inclusion operators $T : X \to H$, $\widetilde{T} : H \to X^*$ defined by* (1.7) *and $I := \widetilde{T} \circ T$. Assume also that the Banach space X is separable. Furthermore, for each $t \in [0,T_0]$, let $\Lambda_t(z) : Z \to X^*$ and $A_t(z) : Z \to X^*$ be functions that are Gâteaux differentiable at every $z \in Z$ and $A_t(0), \Lambda_t(0) \in L^{q^*}\big((0,T_0);X^*\big)$. Assume that for every $t \in [0,T]$,*

$$\big\|D\Lambda_t(z)\big\|_{\mathcal{L}(Z;X^*)} \leq g\big(\|P \cdot z\|_H\big) \cdot \Big(\|z\|_Z^{q-2} + 1\Big) \tag{1.30}$$

for all $z \in Z$ and $t \in [0,T_0]$,

$$\big\|\Lambda_t(z)\big\|_{X^*} \leq g\big(\|P \cdot z\|_H\big) \cdot \Big(\|L_0 \cdot z\|_{V_0}^{q-1} + \tilde{\mu}^{\frac{q-1}{q}}(t)\Big) \tag{1.31}$$

for all $z \in Z$ and $t \in [0,T_0]$, and

$$\big\|DA_t(z)\big\|_{\mathcal{L}(Z;X^*)} \leq g\big(\|P \cdot z\|_H\big) \cdot \Big(\|L_0 \cdot z\|_{V_0}^{q-2} + 1\Big) \tag{1.32}$$

for all $z \in Z$ and $t \in [0,T_0]$, where $\tilde{\mu}(t) \in L^1\big((0,T_0);\mathbb{R}\big)$ is some non-negative function, $g(s) : [0,+\infty) \to (0,+\infty)$ is some non-decreasing function, V_0 is some Banach space and $L_0 : Z \to V_0$ is some compact linear operator. Moreover, assume that Λ_t and A_t satisfy the monotonicity condition

$$\begin{aligned} &\Big\langle h, A_t(Q \cdot h) + \Lambda_t\big(Q \cdot h\big)\Big\rangle_{X \times X^*} \\ &\qquad \geq \big(1/\bar{C}\big)\big\|Q \cdot h\big\|_Z^q - \bar{C}\big|L \cdot (Q \cdot h)\big|_V^2 - \mu(t)\Big(\big\|T \cdot h\big\|_H^2 + 1\Big) \end{aligned} \tag{1.33}$$

for all $h \in X$ and $t \in [0,T_0]$, where V is a given Banach space, $L \in \mathcal{L}(Z,V)$ is a given compact operator, $\mu(t) \in L^1\big((0,T_0);\mathbb{R}\big)$ is some non-negative function and

$\bar{C} > 0$ *is some constant. Also assume that* $\Lambda_t(z)$ $A_t(z)$ *are Borel on the pair of variables* (z, t). *Finally, assume that there exist a family of Banach spaces* $\{V_j\}_{j=1}^{+\infty}$ *and a family of compact bounded linear operators* $\{L_j\}_{j=1}^{+\infty}$, *where* $L_j : Z \to V_j$, *which satisfy the condition*

- *if* $\{h_n\}_{n=1}^{+\infty} \subset Z$ *is a sequence and* $h_0 \in Z$ *are such that for every fixed* j, $\lim_{n\to+\infty} L_j \cdot h_n = L_j \cdot h_0$ *strongly in* V_j *and* $P \cdot h_n \rightharpoonup P \cdot h_0$ *weakly in* H, *then for every fixed* $t \in (0, T_0)$, $\Lambda_t(h_n) \rightharpoonup \Lambda_t(h_0)$ *weakly in* X^* *and* $DA_t(h_n) \to DA_t(h_0)$ *strongly in* $\mathcal{L}(Z, X^*)$.

Then, for every $w_0 \in H$, *there exists* $z(t) \in L^q\big((0, T_0); Z\big)$ *such that* $w(t) := P \cdot z(t) \in L^\infty\big((0, T_0); H\big)$, $v(t) := \widetilde{T} \cdot \big(w(t)\big) \in W^{1,q^*}\big((0, T_0); X^*\big)$, *and* $z(t)$ *satisfies the equation*

$$
\begin{cases} \frac{dv}{dt}(t) + A_t\big(z(t)\big) + \Lambda_t\big(z(t)\big) = 0 & \text{for a.e. } t \in (0, T_0), \\ v(a) = \widetilde{T} \cdot w_0. \end{cases} \tag{1.34}
$$

In Section 4, we give some applications of Theorems 1.3 and 1.4, providing the existence results for various classes of time dependent partial differential equations including parabolic, hyperbolic, Schrödinger and Navier-Stokes systems.

2 Notation and statement of preliminary results

Throughout the paper, by a linear space we mean a real linear space.

- Given a Banach space X, denote by X^* the corresponding dual space.
- Given a Banach space X, $h \in X$ and $x^* \in X^*$, denote by $\langle h, x^*\rangle_{X\times X^*}$ the value in $\mathbb{R}$ of the functional x^* at the vector h.
- Given two Banach spaces X and Y, denote by $\mathcal{L}(X; Y)$ the linear space of bounded linear operators from X to Y.
- Given Banach spaces X and Y, $A \in \mathcal{L}(X; Y)$ and $h \in X$, denote by $A \cdot h \in Y$ the value of the operator A at the point h.
- Set $\|A\|_{\mathcal{L}(X;Y)} = \sup\{\|A\cdot h\|_Y : h \in X, \|h\|_X \le 1\}$, making $\mathcal{L}(X; Y)$ a Banach space.
- Given two Banach spaces X and Y and a Gâteaux differentiable mapping $F : X \to Y$, denote by $DF(x) \in \mathcal{L}(X; Y)$ the Gâteaux derivative of F at the point $x \in X$.

Next we recall some definitions and lemmas from [11]. Many of them are well known.

Definition 2.1. Let X and Y be Banach spaces, and $U \subset X$ be a Borel subset. We say that the Borel mapping $F(x) : U \to Y$ is **strongly Borel** if for every separable subspace $X' \subset X$, the set $\{y \in Y : y = F(x),\ x \in U \cap X'\}$ is also contained in some separable subspace of Y.

Definition 2.2. For a Banach space X and an interval $(a, b) \subset \mathbb{R}$, we define $L^q(a, b; X)$ to be the linear space of (equivalence classes of) strongly measurable (i.e., equivalent to some strongly Borel mapping) functions $f : (a, b) \to X$ such that

$$\|f\|_{L^q(a,b;X)} := \begin{cases} \left(\int_a^b \|f(t)\|_X^q dt \right)^{1/q} & \text{if } 1 \le q < \infty, \\ \operatorname{ess\,sup}_{t \in (a,b)} \|f(t)\|_X & \text{if } q = \infty \end{cases} \tag{2.1}$$

is finite. It is known that $L^q(a, b; X)$ with the norm defined by (2.1) is a Banach space. Moreover, if X is reflexive and $1 < q < \infty$, then $L^q(a, b; X)$ is also reflexive with the corresponding dual space $L^{q^*}(a, b; X^*)$, where $q^* = q/(q-1)$.

Definition 2.3. Let Z be a Banach space and Z^* be the corresponding dual space. We say that a mapping $\Lambda(z) : Z \to Z^*$ is **monotone** if for all $y, z \in Z$,

$$\langle y - z, \Lambda(y) - \Lambda(z) \rangle_{Z \times Z^*} \ge 0.$$

Definition 2.4. Let Z be a Banach space and Z^* be the corresponding dual space. We say that a mapping $\Lambda(z) : Z \to Z^*$ is **pseudo-monotone** if it satisfies the following conditions:

(i) for every sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$ such that $z_n \rightharpoonup z$ weakly in Z,

$$\varliminf_{n \to +\infty} \langle z_n - z, \Lambda(z_n) \rangle_{Z \times Z^*} \ge 0;$$

(ii) $\Lambda(z_n) \rightharpoonup \Lambda(z)$ weakly* in Z^* for every sequence $\{z_n\}_{n=1}^{+\infty} \subset Z$ such that $z_n \rightharpoonup z$ weakly in Z and $\lim_{n \to +\infty} \langle z_n - z, \Lambda(z_n) \rangle_{Z \times Z^*} = 0$,

Lemma 2.1. *Let Z be a Banach space, Z^* be the corresponding dual space and $\Lambda(z) : Z \to Z^*$ be a monotone and strong-to-weak continuous mapping. Then $\Lambda(z)$ is a pseudo-monotone mapping.*

Definition 2.5. Let X be a reflexive Banach space, and let $(a, b) \subset \mathbb{R}$. We say that $v(t) \in L^q(a, b; X)$ belongs to $W^{1,q}(a, b; X)$ if there exists $f(t) \in L^q(a, b; X)$ such that for every $\delta(t) \in C_c^1\big((a, b); X^*\big)$,

$$\int_a^b \langle f(t), \delta(t) \rangle_{X \times X^*} dt = - \int_a^b \langle v(t), \frac{d\delta}{dt}(t) \rangle_{X \times X^*} dt.$$

We then denote $f(t)$ by $v'(t)$ or by $\frac{dv}{dt}(t)$.

Definition 2.6. Let X be a reflexive Banach space and X^* be the corresponding dual space. Next, let H be a Hilbert space and $T \in \mathcal{L}(X, H)$ be an injective inclusion operator whose image is dense in H. We call the triple $\{X, H, X^*\}$ an **evolution triple with the corresponding inclusion operator** T. Furthermore, we define the injective operator $\widetilde{T} \in \mathcal{L}(H; X^*)$ by the formula

$$\langle x, \widetilde{T} \cdot y\rangle_{X\times X^*} := \langle T \cdot x, y\rangle_{H\times H} \qquad \text{for every } y \in H \text{ and } x \in X\,. \tag{2.2}$$

Lemma 2.2. *Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion operators $T \in \mathcal{L}(X; H)$ and $\widetilde{T} \in \mathcal{L}(H; X^*)$, and let $a < b \in \mathbb{R}$. Let $w(t) \in L^\infty(a, b; H)$ be such that the function $v : [a, b] \to X^*$ defined by $v(t) := \widetilde{T} \cdot \big(w(t)\big)$ belongs to $W^{1,q}(a, b; X^*)$ for some $q \geq 1$. Then w can be redefined on a subset of $[a, b]$ of Lebesgue measure zero so that $w(t)$ is H-weakly continuous in t on $[a, b]$. Moreover, for every $a \leq \alpha < \beta \leq b$ and for every $\delta(t) \in C^1\big([a, b]; X\big)$, we then have*

$$\int_\alpha^\beta \left\{\left\langle \delta(t), \frac{dv}{dt}(t)\right\rangle_{X\times X^*} + \left\langle \frac{d\delta}{dt}(t), v(t)\right\rangle_{X\times X^*}\right\}dt$$
$$= \langle T \cdot \delta(\beta), w(\beta)\rangle_{H\times H} - \langle T \cdot \delta(\alpha), w(\alpha)\rangle_{H\times H}.$$

Lemma 2.3. *Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion operators $T \in \mathcal{L}(X; H)$ and $\widetilde{T} \in \mathcal{L}(H; X^*)$, and let $a < b \in \mathbb{R}$. Let $u(t) \in L^q(a, b; X)$ with $q > 1$ be such that the function $v(t) : [a, b] \to X^*$ defined by $v(t) := (\widetilde{T} \circ T) \cdot \big(u(t)\big)$ belongs to $W^{1,q^*}(a, b; X^*)$ with $q^* := q/(q-1)$. Then the function $w(t) : [a, b] \to H$ defined by $w(t) := T \cdot \big(u(t)\big)$ belongs to $L^\infty(a, b; H)$ and for every subinterval $[\alpha, \beta] \subset [a, b]$, up to a redefinition of $w(t)$ on a subset of $[a, b]$ of Lebesgue measure zero making w to be H-weakly continuous (see Lemma 2.2),*

$$\int_\alpha^\beta \left\langle u(t), \frac{dv}{dt}(t)\right\rangle_{X\times X^*} dt = \frac{1}{2}\Big(\|w(\beta)\|_H^2 - \|w(\alpha)\|_H^2\Big).$$

Lemma 2.4. *Let X be a reflexive Banach space and Y and Z Banach spaces. Let $T \in \mathcal{L}(X; Y)$ and $S \in \mathcal{L}(X; Z)$ be bounded linear operators. Assume that S is an injective operator and T is a compact operator. Assume that $a < b \in \mathbb{R}$, $1 \leq q < +\infty$ and $\{u_n(t)\}_{n=1}^{+\infty} \subset L^q(a, b; X)$ is a bounded sequence of functions in $L^q(a, b; X)$ such that the functions $v_n(t) : (a, b) \to Z$, defined by $v_n(t) := S \cdot \big(u_n(t)\big)$, belong to $L^\infty(a, b; Z)$, the sequence $\{v_n(t)\}_{n=1}^{+\infty}$ is bounded in $L^\infty(a, b; Z)$ and for a.e. $t \in (a, b)$, $v_n(t) \rightharpoonup v(t)$ weakly in Z as $n \to +\infty$. Then $\big\{T \cdot \big(u_n(t)\big)\big\}_{n=1}^{+\infty}$ converges strongly in $L^q(a, b; Y)$.*

3 The existence results

Lemma 3.1. *Let X and Z be reflexive Banach spaces and X^* and Z^* be the corresponding dual spaces. Let H be a Hilbert space. Suppose that $Q \in \mathcal{L}(X, Z)$ is an injective inclusion operator (i.e., satisfies* $\ker Q = \{0\}$*) whose image is dense on Z. Furthermore, suppose that $P \in \mathcal{L}(Z, H)$ is an injective inclusion operator whose image is dense on H. Let $T \in \mathcal{L}(X, H)$ be defined by $T := P \circ Q$, so that $\{X, H, X^*\}$ is an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X; H)$ as defined in Definition 2.6 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H; X^*)$ defined as in (2.2). Next let $a, b \in \mathbb{R}$ be such that $a < b$ and $q \geq 2$. For every $t \in [a, b]$, let $\Psi_t(x) : X \to [0, +\infty)$ be a convex function that is Gâteaux differentiable at every $x \in X$, satisfies $\Psi_t(0) = 0$ and satisfies the growth condition*

$$\frac{1}{C}\|x\|_X^q - C \leq 0 \leq \Psi_t(x) \leq C\,\|x\|_X^q + C \quad \text{for all } x \in X \text{ and } t \in [a, b] \tag{3.1}$$

for some $C > 0$. Also assume that $\Psi_t(x)$ is Borel on the pair of variables (x, t). Next, for every $t \in [a, b]$, let $\Lambda_t(z) : Z \to X^*$ be a function that is Gâteaux differentiable at every $z \in Z$ and that satisfies the bound

$$\big\|\Lambda_t(z)\big\|_{X^*} \leq g\big(\|P \cdot z\|_H\big) \cdot \Big(\|z\|_Z^{q-1} + \mu^{\frac{q-1}{q}}(t)\Big) \tag{3.2}$$

for all $z \in Z$ and $t \in [a, b]$, where $g(s) : [0, +\infty) \to (0, +\infty)$ is some non-decreasing function and $\mu(t) \in L^1(a, b; \mathbb{R})$ is some non-negative function. Moreover, assume that Λ_t satisfies the positivity condition

$$\Big\langle h, \Lambda_t\big(Q \cdot h\big)\Big\rangle_{X \times X^*} \geq \big(1/\bar{C}\big)\big\|Q \cdot h\big\|_Z^q - \bar{C}\big\|L \cdot (Q \cdot h)\big\|_V^2 - \tilde{\mu}(t)\Big(\big\|T \cdot h\big\|_H^2 + 1\Big) \tag{3.3}$$

for all $h \in X$ and $t \in [a, b]$, where V is a given Banach space, $L \in \mathcal{L}(Z, V)$ is a given compact linear operator, $\bar{C} > 0$ is some constant and $\tilde{\mu}(t) \in L^1(a, b; \mathbb{R})$ is some non-negative function. Also assume that $\Lambda_t(z)$ is strongly Borel on the pair of variables (z, t). Furthermore, let $\{w_n^{(0)}\}_{n=1}^{\infty} \subset H$ be such that $w_n^{(0)} \to w_0$ strongly in H, and let $\varepsilon_n > 0$ be such that $\varepsilon_n \to 0$ as $n \to +\infty$. Moreover, assume that $u_n(t) \in L^q(a, b; X)$ is such that $v_n(t) := (\widetilde{T} \circ T) \cdot u_n(t) \in W^{1,q^*}(a, b; X^*)$, where $q^* = q/(q-1)$, and $u_n(t)$ is a solution of

$$\begin{cases} \frac{dv_n}{dt}(t) + \Lambda_t\big(z_n(t)\big) + \varepsilon_n D\Psi_t\big(u_n(t)\big) = 0 & \text{for a.e. } t \in (a, b) \\ w_n(a) = w_n^{(0)}, \end{cases} \tag{3.4}$$

where $w_n(t) := T \cdot u_n(t)$, $z_n(t) := Q \cdot u_n(t)$, and $w_n(t)$ is H-weakly continuous on $[a, b]$; see Lemma 2.2. Then there exist $z(t) \in L^q(a, b; Z)$ and $\bar{\Lambda}(t) \in L^{q^*}(a, b; X^*)$

such that $w(t) := P \cdot z(t) \in L^\infty(a,b;H)$, $v(t) := \widetilde{T} \cdot w(t) \in W^{1,q^*}(a,b;X^*)$, $w(t)$ *is* H*-weakly continuous on* $[a,b]$*, and up to a subsequence, we have*

$$\begin{cases} z_n(t) \rightharpoonup z(t) \text{ weakly in } L^q(a,b;Z) \\ \frac{dv_n}{dt}(t) \rightharpoonup \frac{dv}{dt}(t) \text{ weakly in } L^{q^*}(a,b;X^*) \\ \Lambda_t\big(z_n(t)\big) \rightharpoonup \overline{\Lambda}(t) \text{ weakly in } L^{q^*}(a,b;X^*) \\ w_n(t) \rightharpoonup w(t) \text{ weakly in } H \text{ for every fixed } t \in [a,b], \\ \big\{w_n(t)\big\}_{n=1}^{+\infty} \text{ is bounded in } L^\infty(a,b;H), \end{cases} \tag{3.5}$$

and $w(t)$ *satisfies the equation*

$$\begin{cases} \frac{dv}{dt}(t) + \overline{\Lambda}(t) = 0 \text{ for a.e. } t \in (a,b), \\ w(a) = w_0. \end{cases} \tag{3.6}$$

Moreover,

$$\frac{1}{2}\big\|w(t)\big\|_H^2 + \overline{\lim_{n\to+\infty}} \left(\int_a^t \Big\langle u_n(s), \Lambda_s\big(z_n(s)\big)\Big\rangle_{X\times X^*} ds \right) \le \frac{1}{2}\big\|w_0\big\|_H^2 \tag{3.7}$$

for all $t \in [a,b]$.

Proof. By a well-known embedding result (see [11, Appendix, Lemma A.1]), there exists a constant $K > 0$ such that

$$\big\|L \cdot z\big\|_V^2 \le \frac{1}{2C^2}\,\|z\|_Z^2 + K\big\|P \cdot z\big\|_H^2$$

for all $z \in Z$. Plugging this inequality into (3.3), we obtain

$$\begin{aligned} \Big\langle h, \Lambda_t\big(Q \cdot h\big)\Big\rangle_{X\times X^*} &\ge \frac{1}{2C}\Big(2\|Q\cdot h\|_Z^q - \|Q\cdot h\|_Z^2\Big) - \big(\tilde{\mu}(t) + \bar{C}K\big)\Big(\|T\cdot h\|_H^2 + 1\Big) \\ &\ge \frac{1}{2C}\,\|Q\cdot h\|_Z^q - \big(\tilde{\mu}(t) + \tilde{K}\big)\Big(\|T\cdot h\|_H^2 + 1\Big) \end{aligned} \tag{3.8}$$

for all $h \in X$ and $t \in [a,b]$, where $\tilde{K} > 0$ is a constant. Thus, setting $\bar{\mu}(t) := \big(\tilde{\mu}(t) + \tilde{K}\big) \in L^1(a,b;\mathbb{R})$, we obtain

$$\Big\langle h, \Lambda_t\big(Q\cdot h\big)\Big\rangle_{X\times X^*} \ge \big(1/2\bar{C}\big)\big\|Q\cdot h\big\|_Z^q - \bar{\mu}(t)\Big(\big\|T\cdot h\big\|_H^2 + 1\Big) \tag{3.9}$$

for all $h \in X$ for all $t \in [a,b]$. On the other hand, by (3.4), we deduce

$$\begin{aligned} \int_a^t \Big\langle u_n(s), \frac{dv_n}{dt}(s)\Big\rangle_{X\times X^*} ds + \int_a^t \Big\langle u_n(s), \Lambda_s\big(z_n(s)\big)\Big\rangle_{X\times X^*} ds \\ + \varepsilon_n \int_a^t \Big\langle u_n(s), D\Psi_t\big(u_n(s)\big)\Big\rangle_{X\times X^*} ds = 0 \end{aligned} \tag{3.10}$$

for all $t \in [a, b]$. However, since by Lemma 2.3 we have

$$\int_a^t \left\langle u_n(s), \frac{dv_n}{dt}(s) \right\rangle_{X \times X^*} ds = \frac{1}{2}\Big(\|w_n(t)\|_H^2 - \|w_n^{(0)}\|_H^2 \Big),$$

using (3.10), we obtain

$$\begin{aligned} \frac{1}{2}\|w_n(t)\|_H^2 + \int_a^t \Big\langle u_n(s), \Lambda_s\big(z_n(s)\big) \Big\rangle_{X \times X^*} ds \\ + \varepsilon_n \int_a^t \Big\langle u_n(s), D\Psi_t\big(u_n(s)\big) \Big\rangle_{X \times X^*} ds = \frac{1}{2}\|w_n^{(0)}\|_H^2 \end{aligned} \tag{3.11}$$

for all $t \in [a, b]$. However, since $\Psi_t(\cdot)$ is convex and since $\Psi_t(\cdot) \geq 0$, $\Psi_t(0) = 0$, and then also $D\Psi_t(0) = 0$, we have

$$\Big\langle u_n(t), D\Psi_t\big(u_n(t)\big) \Big\rangle_{X \times X^*} \geq \Psi_t\big(u_n(t)\big) \geq 0 \tag{3.12}$$

for all $t \in (a, b)$. Therefore, using (3.12), from (3.11) we deduce that

$$\varepsilon_n \int_a^t \Psi_s\big(u_n(s)\big) ds + \frac{1}{2}\|w_n(t)\|_H^2 + \int_a^t \Big\langle u_n(s), \Lambda_s\big(z_n(s)\big) \Big\rangle_{X \times X^*} ds \leq \frac{1}{2}\|w_n^{(0)}\|_H^2 \tag{3.13}$$

for all $t \in [a, b]$; and, in particular,

$$\frac{1}{2}\|w_n(t)\|_H^2 + \int_a^t \Big\langle u_n(s), \Lambda_s\big(z_n(s)\big) \Big\rangle_{X \times X^*} ds \leq \frac{1}{2}\|w_n^{(0)}\|_H^2 \tag{3.14}$$

for all $t \in [a, b]$. Thus, inserting (3.9) into (3.13), we deduce that

$$\|w_n(t)\|_H^2 + \varepsilon_n \int_a^t \Psi_s\big(u_n(s)\big) ds + \int_a^t \|z_n(s)\|_Z^q ds \leq C_2 \int_a^t \bar{\mu}(s)\|w_n(s)\|_H^2 ds + C_2 \tag{3.15}$$

for all $t \in [a, b]$, where $C_2 > 0$ is a constant. In particular,

$$\|w_n(t)\|_H^2 \leq C_2 \int_a^t \bar{\mu}(s)\|w_n(s)\|_H^2 ds + C_2 \tag{3.16}$$

for all $t \in [a, b]$. Thus

$$\begin{aligned} \frac{d}{dt}\left\{ \exp\left(-C_2 \int_a^t \bar{\mu}(s) ds \right) \int_a^t \bar{\mu}(s)\|w_n(s)\|_H^2 ds \right\} \\ \leq C_2 \bar{\mu}(t) \exp\left(-C_2 \int_a^t \bar{\mu}(s) ds \right) \leq C_2 \bar{\mu}(t) \end{aligned} \tag{3.17}$$

for a.e. $t \in [a, b]$ and $n \in \mathbb{N}$, and thus

$$\begin{aligned} \int_a^t \bar{\mu}(s)\|w_n(s)\|_H^2 ds &\leq C_2 \exp\left(C_2 \int_a^t \bar{\mu}(s) ds \right) \cdot \int_a^t \bar{\mu}(s) ds \\ &\leq C_2 \exp\left(C_2 \int_a^b \bar{\mu}(s) ds \right) \cdot \int_a^b \bar{\mu}(s) ds \end{aligned} \tag{3.18}$$

for all $t \in [a,b]$ and $n \in \mathbb{N}$. Then, by (3.18), from (3.16) we obtain that the sequence $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$. Then, by (3.15), we deduce that that sequence $\{z_n(t)\}$ is bounded in $L^q(a,b;Z)$. Moreover, by (3.2), we obtain that $\Lambda_t\big(z_n(t)\big)$ is bounded in $L^{q^*}(a,b;X^*)$. Therefore, in particular, up to a subsequence, we have

$$\begin{cases} z_n(t) \rightharpoonup z(t) \text{ weakly in } L^q(a,b;Z),\\ w_n(t) \rightharpoonup w(t) \text{ weakly in } L^q(a,b;H),\\ v_n(t) \rightharpoonup v(t) \text{ weakly in } L^q(a,b;X^*),\\ \Lambda_t\big(z_n(t)\big) \rightharpoonup \overline{\Lambda}(t) \text{ weakly in } L^{q^*}(a,b;X^*), \end{cases} \tag{3.19}$$

where $w(t) := P\cdot z(t)$, $v(t) := \widetilde{T}\cdot w(t)$. Next plugging (3.19) into (3.15) and using the fact that $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$, we deduce

$$\varepsilon_n \int_a^t \Psi_s\big(u_n(s)\big)ds \le C_4\,, \tag{3.20}$$

where C_4 is a constant. Then using (3.1) we deduce from (3.20),

$$\varepsilon_n \int_a^b \big\|u_n(s)\big\|_X^q ds \le C_5\,. \tag{3.21}$$

Next, since Ψ_t is a convex function satisfying (3.1), using [11, Lemma 2.3], we obtain that

$$\Big\|D\Psi_t\big(u_n(t)\big)\Big\|_{X^*} \le \bar{C}\big\|u_n(t)\big\|_X^{q-1} + \bar{C} \tag{3.22}$$

for all $t \in (a,b)$, for some constant $\bar{C} > 0$. Then

$$\Big\|D\Psi_t\big(u_n(t)\big)\Big\|_{X^*}^{q^*} \le \bar{C}_0\big\|u_n(t)\big\|_X^{q} + \bar{C}_0 \quad \text{for all } t \in (a,b). \tag{3.23}$$

Thus, plugging (3.23) into (3.21), we deduce

$$\int_a^b \Big\|\varepsilon_n D\Psi_t\big(u_n(s)\big)\Big\|_{X^*}^{q^*} ds \le \hat{C}\varepsilon_n^{1/(q-1)}. \tag{3.24}$$

So

$$\lim_{n\to+\infty} \Big\|\varepsilon_n D\Psi_t\big(u_n(t)\big)\Big\|_{L^{q^*}(a,b;X^*)} = 0. \tag{3.25}$$

On the other hand, by (3.4) and Lemma 2.2, for any $\beta \in [a,b]$ and every $\delta(t) \in C^1\big([a,b];X\big)$,

$$\text{(3.26)}\quad \Big\langle T\cdot\delta(\beta), w_n(\beta)\Big\rangle_{H\times H} - \Big\langle T\cdot\delta(a), w_n^{(0)}\Big\rangle_{H\times H} - \int_a^\beta \Big\langle \frac{d\delta}{dt}(t), v_n(t)\Big\rangle_{X\times X^*} dt + \int_a^\beta \Big\langle \delta(t), \varepsilon_n D\Psi_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt + \int_a^\beta \Big\langle \delta(t), \Lambda_t\big(z_n(t)\big)\Big\rangle_{X\times X^*} dt = 0.$$

Letting $n \to +\infty$ in (3.26) and using (3.19), (3.25) and the fact that $w_n^{(0)} \to w_0$ in H, we obtain

$$\text{(3.27)}\quad \lim_{n\to+\infty} \Big\langle T\cdot\delta(\beta), w_n(\beta)\Big\rangle_{H\times H} - \Big\langle T\cdot\delta(a), w_0\Big\rangle_{H\times H} - \int_a^\beta \Big\langle \frac{d\delta}{dt}(t), v(t)\Big\rangle_{X\times X^*} dt + \int_a^\beta \Big\langle \delta(t), \overline{\Lambda}(t)\Big\rangle_{X\times X^*} dt = 0$$

for every $\delta(t) \in C^1\big([a,b];X\big)$. In particular, for every $\delta(t) \in C^1\big([a,b];X\big)$ such that $\delta(b) = 0$ we have

$$\text{(3.28)}\quad -\Big\langle T\cdot\delta(a), w_0\Big\rangle_{H\times H} - \int_a^b \Big\langle \frac{d\delta}{dt}(t), v(t)\Big\rangle_{X\times X^*} dt + \int_a^b \Big\langle \delta(t), \overline{\Lambda}(t)\Big\rangle_{X\times X^*} dt = 0\,.$$

Thus, in particular, $\frac{dv}{dt}(t) = -\overline{\Lambda}(t) \in L^{q^*}(a,b;X^*)$; and so $v(t) \in W^{1,q^*}(a,b;X^*)$. Then, since $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$, we have $w(t) \in L^\infty(a,b;H)$ and thus, as before, we can redefine w on a subset of $[a,b]$ of Lebesgue measure zero, so that $w(t)$ is H-weakly continuous in t on $[a,b]$; and by (3.28), we then have $w(a) = w_0$. So $w(t)$ is a solution of the equation

$$\text{(3.29)}\quad \begin{cases} \frac{dv}{dt}(t) + \overline{\Lambda}(t) = 0 & \text{for a.e. } t \in (a,b),\\ w(a) = w_0. \end{cases}$$

Thus, in particular, for any $\beta \in [a,b]$ and every $\delta(t) \in C^1\big([a,b];X\big)$, we have

$$\text{(3.30)}\quad \Big\langle T\cdot\delta(\beta), w(\beta)\Big\rangle_{H\times H} - \Big\langle T\cdot\delta(a), w_0\Big\rangle_{H\times H} - \int_a^\beta \Big\langle \frac{d\delta}{dt}(t), v(t)\Big\rangle_{X\times X^*} dt + \int_a^\beta \Big\langle \delta(t), \overline{\Lambda}(t)\Big\rangle_{X\times X^*} dt = 0.$$

Plugging (3.30) into (3.27), we deduce

$$\text{(3.31)}\quad \lim_{n\to+\infty} \Big\langle T\cdot x, w_n(\beta)\Big\rangle_{H\times H} = \Big\langle T\cdot x, w(\beta)\Big\rangle_{H\times H} X$$

for all $x \in X$ and $\beta \in [a,b]$. Therefore, since the image of T has dense range in H and $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$, we deduce that

$$\text{(3.32)}\quad w_n(t) \rightharpoonup w(t) \text{ weakly in } H \quad \text{for all } t \in [a,b].$$

Next, by (3.19), (3.25), (3.4) and (3.29), we obtain

$$\frac{dv_n}{dt}(t) \rightharpoonup \frac{dv}{dt}(t) \quad \text{weakly in } L^{q^*}(a,b;X^*). \tag{3.33}$$

So we have established (3.5) and (3.6). Finally, since $w_n^{(0)} \to w_0$ strongly in H, plugging (3.32) into (3.14), we obtain (3.7). □

As a consequence of Lemma 3.1 in a particular case we have the following corollary.

Corollary 3.1. *Let X and Z be reflexive Banach spaces and X^* and Z^* be their corresponding dual spaces. Let H be a Hilbert space. Suppose that $Q \in \mathcal{L}(X,Z)$ is an injective inclusion operator (i.e. it satisfies $\ker Q = \{0\}$) whose image is dense on Z. Furthermore, suppose that $P \in \mathcal{L}(Z,H)$ is an injective inclusion operator whose image is dense on H. Let $T \in \mathcal{L}(X,H)$ be defined by $T := P \circ Q$, and let $\widetilde{P} \in \mathcal{L}(H;Z^*)$ be defined by*

$$\langle z, \widetilde{P} \cdot y \rangle_{Z\times Z^*} := \langle P \cdot z, y \rangle_{H\times H} \quad \textit{for every } y \in H \textit{ and } z \in Z, \tag{3.34}$$

so that $\{X,H,X^*\}$ is an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X;H)$ as defined in Definition 2.6 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$ defined as in (2.2). Moreover, $\{Z,H,Z^*\}$ is another evolution triple with the corresponding inclusion operator $P \in \mathcal{L}(Z;H)$, together with the corresponding operator $\widetilde{P} \in \mathcal{L}(H;Z^*)$. Next let $a, b \in \mathbb{R}$ be such that $a < b$ and $q \geq 2$. Furthermore, for every $t \in [a,b]$, let $\Psi_t(x) : X \to [0,+\infty)$ be a convex function that is Gâteaux differentiable at every $x \in X$, satisfies $\Psi_t(0) = 0$ and satisfies the growth condition

$$\frac{1}{C}\|x\|_X^q - C \leq \Psi_t(x) \leq C\,\|x\|_X^q + C \tag{3.35}$$

for all $x \in X$ and $t \in [a,b]$ for some $C > 0$. Also assume that $\Psi_t(x)$ is Borel on the pair of variables (x,t). Furthermore, for each $t \in [a,b]$, let $\Lambda_t(z) : Z \to Z^*$ be a function that is Gâteaux differentiable at every $z \in Z$ and satisfies the bound

$$\big\|\Lambda_t(z)\big\|_{Z^*} \leq g\big(\|P \cdot z\|_H\big) \cdot \Big(\|z\|_Z^{q-1} + \mu^{\frac{q-1}{q}}(t)\Big) \tag{3.36}$$

for all $z \in Z$ and $t \in [a,b]$, where $g(s) : [0,+\infty) \to (0,+\infty)$ is some non-decreasing function and $\mu(t) \in L^1(a,b;\mathbb{R})$ is some non-negative function. Moreover, assume that Λ_t satisfies the positivity condition

$$\Big\langle h, \Lambda_t\big(h\big)\Big\rangle_{Z\times Z^*} \;\geq\; \big(1/\bar{C}\big)\big\|h\big\|_Z^q - \bar{C}\big\|L \cdot h\big\|_V^2 - \bar{\mu}(t)\Big(\big\|P \cdot h\big\|_H^2 + 1\Big) \tag{3.37}$$

for all $h \in Z$ *and* $t \in [a,b]$, *where* V *is a given Banach space,* $L \in \mathcal{L}(Z,V)$ *is a given compact linear operator,* $\overline{C} > 0$ *is some constant and* $\overline{\mu}(t) \in L^1(a,b;\mathbb{R})$ *is some non-negative function. Also assume that* $\Lambda_t(z)$ *is strongly Borel on the pair of variables* (z,t). *Moreover, assume the following compactness property: for every sequence* $\{\sigma_n(t)\}_{n=1}^{+\infty} \subset L^q(a,b;Z)$ *such that* $\{P \cdot \sigma_n(t)\}_{n=1}^{+\infty} \subset L^\infty(a,b;H)$, $\sigma_n(t) \rightharpoonup \sigma(t)$ *weakly in* $L^q(a,b;Z)$, $\{P \cdot \sigma_n(t)\}_{n=1}^{+\infty}$ *is bounded in* $L^\infty(a,b;H)$ *and* $P \cdot \sigma_n(t) \rightharpoonup P \cdot \sigma(t)$ *weakly in* H *for a.e.* $t \in (a,b)$, *the inequality*

$$\underline{\lim}_{n\to+\infty} \int_a^b \Big\langle \sigma_n(t) - \sigma(t), \Lambda_t\big(\sigma_n(t)\big)\Big\rangle_{Z\times Z^*} dt \le 0, \tag{3.38}$$

implies that, up to a subsequence, $\Lambda_t\big(\sigma_n(t)\big) \rightharpoonup \Lambda_t\big(\sigma(t)\big)$ *weakly in* $L^{q^*}(a,b;Z^*)$. *Next, let* $\{w_n^{(0)}\}_{n=1}^\infty \subset H$ *be such that* $w_n^{(0)} \to w_0$ *strongly in* H, *and let* $\varepsilon_n > 0$ *be such that* $\varepsilon_n \to 0$ *as* $n \to +\infty$. *Moreover, assume that* $u_n(t) \in L^q(a,b;X)$ *is such that* $v_n(t) := (\widetilde{T} \circ T) \cdot u_n(t) \in W^{1,q^*}(a,b;X^*)$, *where* $q^* = q/(q-1)$, *and* $u_n(t)$ *is a solution of*

$$\begin{cases} \frac{dv_n}{dt}(t) + Q^* \cdot \Lambda_t\big(z_n(t)\big) + \varepsilon_n D\Psi_t\big(u_n(t)\big) \quad \text{for a.e. } t \in (a,b) \\ w_n(a) = w_n^{(0)}, \end{cases} \tag{3.39}$$

where $Q^* \in \mathcal{L}(Z^*;X^*)$ *is the adjoint operator to* Q, $w_n(t) := T \cdot u_n(t)$, $z_n(t) := Q \cdot u_n(t)$, *and* $w_n(t)$ *is* H*-weakly continuous on* $[a,b]$, *as stated in Lemma* 2.2. *Then there exists* $z(t) \in L^q(a,b;Z)$ *such that* $w(t) := P \cdot z(t) \in L^\infty(a,b;H)$, $\zeta(t) := \widetilde{P} \cdot w(t) \in W^{1,q^*}(a,b;Z^*)$, $v(t) := \widetilde{T} \cdot w(t) \in W^{1,q^*}(a,b;X^*)$, $w(t)$ *is* H*-weakly continuous on* $[a,b]$, *up to a subsequence, we have*

$$\begin{cases} z_n(t) \rightharpoonup z(t) \text{ weakly in } L^q(a,b;Z) \\ \frac{dv_n}{dt}(t) \rightharpoonup \frac{dv}{dt}(t) \text{ weakly in } L^{q^*}(a,b;X^*) \\ \Lambda_t\big(z_n(t)\big) \rightharpoonup \Lambda_t\big(z(t)\big) \text{weakly in } L^{q^*}(a,b;Z^*) \\ w_n(t) \rightharpoonup w(t) \text{ weakly in } H \quad \text{for every fixed } t \in [a,b], \\ \{w_n(t)\}_{n=1}^{+\infty} \text{ is bounded in } L^\infty(a,b;H), \end{cases} \tag{3.40}$$

and $z(t)$ *satisfies the equation*

$$\begin{cases} \frac{d\zeta}{dt}(t) + \Lambda_t\big(z(t)\big) = 0 \, \text{for a.e. } t \in (a,b), \\ w(a) = w_0. \end{cases} \tag{3.41}$$

Moreover,

$$\frac{1}{2}\|w(t)\|_H^2 + \int_a^t \Big\langle z(s), \Lambda_s\big(z(s)\big)\Big\rangle_{Z\times Z^*} ds = \frac{1}{2}\|w_0\|_H^2 \tag{3.42}$$

for all $t \in [a,b]$.

Proof. By Lemma 3.1, there exist $z(t) \in L^q(a,b;Z)$ and $\overline{\Lambda}(t) \in L^{q^*}(a,b;Z^*)$ such that $w(t) := P \cdot z(t) \in L^\infty(a,b;H)$, $v(t) := \widetilde{T} \cdot w(t) \in W^{1,q^*}(a,b;X^*)$, $w(t)$ is H-weakly continuous on $[a,b]$, up to a subsequence, we have

$$\begin{cases} z_n(t) \rightharpoonup z(t) \quad \text{weakly in } L^q(a,b;Z), \\ \frac{dv_n}{dt}(t) \rightharpoonup \frac{dv}{dt}(t) \text{ weakly in } L^{q^*}(a,b;X^*), \\ \Lambda_t\big(z_n(t)\big) \rightharpoonup \overline{\Lambda}(t) \text{ weakly in } L^{q^*}(a,b;Z^*), \\ w_n(t) \rightharpoonup w(t) \text{ weakly in } H \text{ for every fixed } t \in [a,b], \\ \big\{w_n(t)\big\}_{n=1}^{+\infty} \text{ is bounded in } L^\infty(a,b;H), \end{cases} \tag{3.43}$$

and $z(t)$ satisfies the equation

$$\begin{cases} \frac{dv}{dt}(t) + Q^* \cdot \overline{\Lambda}(t) = 0 \quad \text{for a.e. } t \in (a,b), \\ w(a) = w_0. \end{cases} \tag{3.44}$$

Moreover,

$$\frac{1}{2}\|w(t)\|_H^2 + \varlimsup_{n\to+\infty} \left(\int_a^t \Big\langle z_n(s), \Lambda_s\big(z_n(s)\big) \Big\rangle_{Z\times Z^*} ds \right) \le \frac{1}{2}\|w_0\|_H^2 \tag{3.45}$$

for all $t \in [a,b]$. Next, using (3.44) with [11, Lemma 2.2], we deduce that $\zeta(t) := \widetilde{P} \cdot w(t) \in W^{1,q^*}(a,b;Z^*)$. Moreover, by Lemma 2.3, we have

$$\frac{1}{2}\|w(t)\|_H^2 + \int_a^t \Big\langle z(s), \overline{\Lambda}(s) \Big\rangle_{Z\times Z^*} ds = \frac{1}{2}\|w_0\|_H^2 \quad \text{for all } t \in [a,b]. \tag{3.46}$$

Thus, plugging (3.46) into (3.45) and using (3.43) gives

$$\begin{aligned} \varlimsup_{n\to+\infty} \left(\int_a^b \Big\langle z_n(t), \Lambda_t\big(z_n(t)\big) \Big\rangle_{Z\times Z^*} dt \right) &\le \int_a^b \Big\langle z(t), \overline{\Lambda}(t) \Big\rangle_{Z\times Z^*} dt \\ &= \lim_{n\to+\infty} \left(\int_a^b \Big\langle z(t), \Lambda_t\big(z_n(t)\big) \Big\rangle_{Z\times Z^*} dt \right). \end{aligned} \tag{3.47}$$

So

$$\varlimsup_{n\to+\infty} \int_a^b \Big\langle z_n(t) - z(t), \Lambda_t\big(z_n(t)\big) \Big\rangle_{Z\times Z^*} dt \le 0,$$

which implies $\overline{\Lambda}(t) = \Lambda_t\big(z(t)\big)$. This completes the proof. □

Definition 3.1. Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X;H)$ as defined in Definition 2.6. Furthermore, let (a,b) be a real interval, $q > 1$ and $q^* := q/(q-1)$. We say that the mapping $\Gamma(u) : \big\{u \in L^q(a,b;X) : T \cdot u \in L^\infty(a,b;H)\big\} \to L^{q^*}(a,b;X^*) \equiv \big\{L^q(a,b;X)\big\}^*$ is **weakly pseudo-monotone** if for every sequence $u_n(t) \rightharpoonup u(t)$ weakly in

$L^q(a,b;X)$ such that $\{T \cdot u_n(t)\}_{n=1}^{+\infty}$ is bounded in $L^\infty(a,b;H)$ and such that $T \cdot u_n(t) \rightharpoonup T \cdot u(t)$ weakly in H for a.e. $t \in (a,b)$, the following conditions are satisfied:

-
$$\underline{\lim}_{n\to+\infty} \Big\langle u_n - u, \Gamma(u_n)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)} \geq 0; \tag{3.48}$$

- if
$$\lim_{n\to+\infty} \Big\langle u_n - u, \Gamma(u_n)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)} = 0, \tag{3.49}$$
then $\Gamma(u_n) \rightharpoonup \Gamma(u)$ weakly in $L^{q^*}(a,b;X^*)$.

Remark 3.1. It follows immediately from Definition 2.4 that if the mapping $\Gamma(u) : L^q(a,b;X) \to L^{q^*}(a,b;X^*)$ is pseudo-monotone, then $\Gamma(u)$ is weakly pseudo-monotone.

Remark 3.2. It is trivially follows from the definition of a weakly pseudo-montone mapping that if

$$\Gamma_1(u), \Gamma_2(u) : \big\{u \in L^q(a,b;X) :\ T \cdot u \in L^\infty(a,b;H)\big\} \to L^{q^*}(a,b;X^*)$$

are weakly pseudo-monotone mappings, then $\Gamma_1(u)+\Gamma_2(u)$ is also a weakly pseudo-monotone mapping.

Lemma 3.2. *Let $\{X,H,X^*\}$ be an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X;H)$ as defined in Definition 2.6 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$ defined as in (2.2). Furthermore, let $q \geq 2$ and, for every $t \in [a,b]$, let $\Theta_t(x) : X \to X^*$ be a function that satisfies the growth condition*

$$\|\Theta_t(x)\|_{X^*} \leq g\big(\|T \cdot x\|_H\big) \left(\|x\|_X^{q-1} + \mu^{\frac{q-1}{q}}(t)\right) \tag{3.50}$$

for all $x \in X$ and $t \in [a,b]$, for some non-decreasing function $g(s) : [0,+\infty) \to (0,+\infty)$ and some non-negative function $\mu(t) \in L^1\big(a,b;\mathbb{R}\big)$. Also assume that $\Theta_t(x)$ is strongly Borel on the pair of variables (x,t) and satisfies the monotonicity condition

$$\Big\langle x, \Theta_t(x)\Big\rangle_{X\times X^*} \geq \frac{1}{\hat{C}}\|x\|_X^q - \Big(\|x\|_X^p + \tilde{\mu}^{\frac{p}{2}}(t)\Big) \cdot \tilde{\mu}^{\frac{2-p}{2}}(t)\Big(\|T\cdot x\|_H^{(2-p)} + 1\Big) \tag{3.51}$$

for all $x \in X$ and $t \in [a,b]$, where $p \in [0,2)$, $\hat{C} > 0$ are some constants and $\tilde{\mu}(t) \in L^1\big(a,b;\mathbb{R}\big)$ is some non-negative function. Finally, assume that for a.e.

fixed $t \in (a,b)$, the function $\Theta_t(x) : X \to X^*$ is pseudo-monotone; see Definition 2.4. Then, the mapping $\Gamma(u) : \{u \in L^q(a,b;X) : \; T \cdot u \in L^\infty(a,b;H)\} \to L^{q^*}(a,b;X^*)$, defined by

$$\Big\langle h(t), \Gamma\big(u(t)\big)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)} := \int_a^b \Big\langle h(t), \Theta_t\big(u(t)\big)\Big\rangle_{X\times X^*} dt \tag{3.52}$$

for all $u(t) \in \{\bar{u}(t) \in L^q(a,b;X) : \; T\cdot\bar{u}(t) \in L^\infty(a,b;H)\}$ and all $h(t) \in L^q(a,b;X)$ is weakly pseudo-monotone; see Definition 3.1.

Proof. Consider a sequence $\{u_n(t)\}_{n=1}^{+\infty} \subset L^q(a,b;X)$ such that $u_n(t) \rightharpoonup u(t)$ weakly in $L^q(a,b;X)$, $\{T \cdot u_n(t)\}_{n=1}^{+\infty}$ is bounded in $L^\infty(a,b;H)$ and $T \cdot u_n(t) \rightharpoonup T \cdot u(t)$ weakly in H for a.e. $t \in (a,b)$. Then, by (3.50) and (3.51), for every $h(t) \in L^q(a,b;X)$, there exists $\eta_h(t) \in L^1\big(a,b;\mathbb{R}\big)$ such that

$$\Big\langle u_n(t) - h(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} \geq \frac{1}{2\hat{C}}\,\|u_n(t)\|_X^q + \eta_h(t) \tag{3.53}$$

for all $t \in [a,b]$ Therefore, by Fatou's Lemma,

$$\begin{aligned} \varliminf_{n\to+\infty} \int_a^b \Big\langle u_n(t) - h(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt \\ \geq \int_a^b \left(\varliminf_{n\to+\infty} \Big\langle u_n(t) - h(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}\right) dt \end{aligned} \tag{3.54}$$

for all $h(t) \in L^q(a,b;X)$. Then, assuming

$$\varliminf_{n\to+\infty} \int_a^b \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt < +\infty$$

and taking $h(t) = u(t)$ in (3.54), we deduce

$$\begin{aligned} \int_a^b \left(\varliminf_{n\to+\infty} \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}\right) dt \\ \leq \varliminf_{n\to+\infty} \int_a^b \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt < +\infty. \end{aligned} \tag{3.55}$$

In particular, for a.e. $t \in (a,b)$, there exists a strictly increasing subsequence $\{n_k^{(t)}\}_{k=1}^{+\infty} \subset \mathbb{N}$ such that

$$\lim_{k\to+\infty} \Big\langle u_{n_k^{(t)}}(t) - u(t), \Theta_t\big(u_{n_k^{(t)}}(t)\big)\Big\rangle_{X\times X^*} = \varliminf_{n\to+\infty} \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} < +\infty. \tag{3.56}$$

Therefore, by (3.53), for a.e. fixed $t \in (a,b)$, the sequence $\{u_{n_k^{(t)}}(t)\}_{k=1}^{+\infty}$ is bounded in X. On the other hand, $T \cdot u_n(t) \rightharpoonup T \cdot u(t)$ weakly in H for a.e. $t \in (a,b)$.

Thus, since T is an injective operator, we obtain that for a.e. fixed $t \in (a,b)$, $u_{n_k^{(t)}}(t) \rightharpoonup u(t)$ weakly in X. Therefore, since for a.e. fixed $t \in (a,b)$ the function $\Theta_t(x) : X \to X^*$ is pseudo-monotone, using (3.56) and Definition 2.4, for a.e. $t \in (a,b)$, we deduce

$$\underline{\lim}_{n\to+\infty} \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} = \lim_{k\to+\infty} \Big\langle u_{n_k^{(t)}}(t) - u(t), \Theta_t\big(u_{n_k^{(t)}}(t)\big)\Big\rangle_{X\times X^*} \geq 0. \tag{3.57}$$

Plugging this into (3.55) yields

$$\begin{aligned} \underline{\lim}_{n\to+\infty} \int_a^b \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt \\ \geq \int_a^b \left(\underline{\lim}_{n\to+\infty} \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}\right) dt \geq 0. \end{aligned} \tag{3.58}$$

Moreover, obviously in the case that

$$\underline{\lim}_{n\to+\infty} \int_a^b \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt = +\infty,$$

the first inequality in (3.58) still holds. So

$$\underline{\lim}_{n\to+\infty} \Big\langle u_n - u, \Gamma(u_n)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)} \geq 0. \tag{3.59}$$

Next assume that

$$\lim_{n\to+\infty} \Big\langle u_n - u, \Gamma(u_n)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)} = 0\,.$$

Plugging this into (3.58), we deduce

$$\begin{aligned} \int_a^b \left(\underline{\lim}_{n\to+\infty} \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}\right) dt \\ = \lim_{n\to+\infty} \int_a^b \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} dt = 0. \end{aligned} \tag{3.60}$$

On the other hand, plugging (3.60) into (3.57), we deduce

$$\underline{\lim}_{n\to+\infty} \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} = 0 \tag{3.61}$$

for a.e. $t \in (a,b)$. Therefore,

$$\lim_{n\to+\infty} \left(\min\left\{ 0\,,\ \Big\langle u_n(t) - u(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} \right\}\right) = 0 \tag{3.62}$$

for a.e. $t \in (a, b)$. Then, using (3.53) and the dominated convergence theorem, by (3.62) we deduce

$$\lim_{n\to+\infty}\int_a^b \left(\min\left\{0\,,\,\Big\langle u_n(t)-u(t),\,\Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}\right\}\right)dt=0. \tag{3.63}$$

Thus plugging (3.63) into (3.60), we obtain

$$\lim_{n\to+\infty}\int_a^b \left(\max\left\{0\,,\,\Big\langle u_n(t)-u(t),\,\Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}\right\}\right)dt=0. \tag{3.64}$$

So by (3.64) and (3.63), we deduce

$$\lim_{n\to+\infty}\int_a^b \left|\Big\langle u_n(t)-u(t),\,\Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}\right|dt=0. \tag{3.65}$$

Therefore, up to a subsequence, we have

$$\lim_{n\to+\infty}\Big\langle u_n(t)-u(t),\,\Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}=0\quad\text{for a.e. } t\in(a,b). \tag{3.66}$$

Furthermore, using the fact that $u_n(t) \rightharpoonup u(t)$ weakly in $L^q(a, b; X)$ and (3.50), we obtain that there exists $\widetilde{\Theta}(t) \in L^{q^*}\big(a, b; X^*\big)$ such that up to a further subsequence, $\Theta_t\big(u_n(t)\big) \rightharpoonup \widetilde{\Theta}(t)$ weakly in $L^{q^*}\big(a, b; X^*\big)$. Using this fact and (3.65), we deduce that for every $h(t) \in L^q(a, b; X)$, which we now fix,

$$\int_a^b \Big\langle h(t),\,\widetilde{\Theta}(t)\Big\rangle_{X\times X^*}dt=\lim_{n\to+\infty}\int_a^b\Big\langle u_n(t)-u(t)+h(t),\,\Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}dt. \tag{3.67}$$

Thus, using (3.53) and Fatou's Lemma, by (3.67) and (3.66), we infer

$$\begin{aligned}\int_a^b \Big\langle h(t),\,\widetilde{\Theta}(t)\Big\rangle_{X\times X^*}dt &\geq \int_a^b\left(\varliminf_{n\to+\infty}\Big\langle u_n(t)-u(t)+h(t),\,\Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}\right)dt\\ &=\int_a^b\left(\varliminf_{n\to+\infty}\Big\langle h(t),\,\Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}\right)dt.\end{aligned} \tag{3.68}$$

On the other hand, by (3.66), for a.e. $t \in (a, b)$, there exists a strictly increasing subsequence $\{\bar{n}_k^{(t)}\}_{k=1}^{+\infty} \subset \mathbb{N}$ such that

$$\begin{aligned}&\lim_{k\to+\infty}\Big\langle u_{\bar{n}_k^{(t)}}(t)-u(t)+h(t),\,\Theta_t\big(u_{\bar{n}_k^{(t)}}(t)\big)\Big\rangle_{X\times X^*}\\ &\quad=\lim_{k\to+\infty}\Big\langle h(t),\,\Theta_t\big(u_{\bar{n}_k^{(t)}}(t)\big)\Big\rangle_{X\times X^*}=\varliminf_{n\to+\infty}\Big\langle h(t),\,\Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*}<+\infty.\end{aligned} \tag{3.69}$$

Therefore, by (3.53), for a.e. fixed $t \in (a, b)$ the sequence $\{u_{\bar{n}_k^{(t)}}(t)\}_{k=1}^{+\infty}$ is bounded in X. On the other hand, $T \cdot u_n(t) \rightharpoonup T \cdot u(t)$ weakly in H for a.e. $t \in (a, b)$. Thus, since T is injective, we obtain that for a.e. fixed $t \in (a,b)$ $u_{\bar{n}_k^{(t)}}(t) \rightharpoonup u(t)$ weakly in X. Therefore, since for a.e. fixed $t \in (a,b)$ the function $\Theta_t(x) : X \to X^*$ is pseudo-monotone, using (3.66) and Definition 2.4, for a.e. $t \in (a,b)$, we deduce

$$\Theta_t\big(u_{\bar{n}_k^{(t)}}(t)\big) \rightharpoonup \Theta_t\big(u(t)\big) \quad \text{weakly in } X^*. \tag{3.70}$$

Plugging this into (3.69), we deduce

$$\varliminf_{n\to+\infty} \Big\langle h(t), \Theta_t\big(u_n(t)\big)\Big\rangle_{X\times X^*} = \Big\langle h(t), \Theta_t\big(u(t)\big)\Big\rangle_{X\times X^*} \quad \text{for a.e. } t \in (a,b). \tag{3.71}$$

Thus, plugging (3.71) into (3.68) gives

$$\int_a^b \Big\langle h(t), \widetilde{\Theta}(t)\Big\rangle_{X\times X^*} dt \geq \int_a^b \Big\langle h(t), \Theta_t\big(u(t)\big)\Big\rangle_{X\times X^*} dt. \tag{3.72}$$

Thus, since $h(t) \in L^q(a,b;X)$ was arbitrary, interchanging the roles of $h(t)$ and $-h(t)$ gives

$$\int_a^b \Big\langle h(t), \Theta_t\big(u(t)\big)\Big\rangle_{X\times X^*} dt \leq \int_a^b \Big\langle h(t), \widetilde{\Theta}(t)\Big\rangle_{X\times X^*} dt. \tag{3.73}$$

Together, (3.72) and (3.73) give

$$\int_a^b \Big\langle h(t), \Theta_t\big(u(t)\big)\Big\rangle_{X\times X^*} dt = \int_a^b \Big\langle h(t), \widetilde{\Theta}(t)\Big\rangle_{X\times X^*} dt; \tag{3.74}$$

and, since $h(t) \in L^q(a,b;X)$ was arbitrarily chosen, we deduce $\Theta_t\big(u(t)\big) = \widetilde{\Theta}(t)$ for a.e. $t \in (a,b)$. So $\Theta_t\big(u_n(t)\big) \rightharpoonup \Theta\big(u(t)\big)$ weakly in $L^{q^*}\big(a,b;X^*\big)$. This completes the proof. □

Theorem 3.1. *Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X;H)$ as defined in Definition 2.6 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$ defined in* (2.2)*. Assume also that the Banach space X is separable. Furthermore, let $a, b, q \in \mathbb{R}$ be such that $a < b$ and $q \geq 2$. Next, for each $t \in [a,b]$, let $\Phi_t(x) : X \to [0,+\infty)$ be a convex function that is Gâteaux differentiable at every $x \in X$, satisfies $\Phi_t(0) = 0$ and satisfies the growth condition*

$$0 \leq \Phi_t(x) \leq C\,\|x\|_X^q + C \tag{3.75}$$

for all $x \in X$ and $t \in [a,b]$ for some $C > 0$. Also assume that $\Phi_t(x)$ is Borel on the pair of variables (x,t). Furthermore, for every $t \in [a,b]$, let $\Lambda_t(x) : X \to X^*$ be a function that is Gâteaux differentiable at every $x \in X$, $\Lambda_t(0) \in L^{q^*}(a,b;X^*)$ and the derivative of $D\Lambda_t$ of Λ_t satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)} \leq g\big(\|T\cdot x\|_H\big)\left(\|x\|_X^{q-2} + 1\right) \tag{3.76}$$

for all $x \in X$ *and* $t \in [a, b]$ *for some non-decreasing function* $g(s) : [0, +\infty) \to (0, +\infty)$*. Also assume that* $\Lambda_t(x)$ *is Borel on the pair of variables* (x, t) *and that* Λ_t *and* Φ_t *satisfy the monotonicity condition*

$$\begin{aligned}(3.77)\quad &\Big\langle x, D\Phi_t(x) + \Lambda_t(x)\Big\rangle_{X\times X^*}\\ &\geq \frac{1}{\hat{C}}\|x\|_X^q - \Big(\|x\|_X^p + \mu^{\frac{p}{2}}(t)\Big)\Big(\hat{C}\|L\cdot x\|_V^{(2-p)} + \mu^{\frac{2-p}{2}}(t)\Big(\|T\cdot x\|_H^{(2-p)} + 1\Big)\Big)\end{aligned}$$

for all $x \in X$ *for all* $t \in [a, b]$*, where* V *is a given Banach space,* $L \in \mathcal{L}(X, V)$ *is a given compact operator,* $p \in [0, 2)$*,* $\mu(t) \in L^1(a, b; \mathbb{R})$ *is a non-negative function and* $\hat{C} > 0$ *is a constant. Finally, assume that for each* $t \in [a, b]$*, the mapping* $\big(D\Phi_t + \Lambda_t\big)(x) : X \to X^*$ *is pseudo-monotone; see Definition 2.4. Then for every* $w_0 \in H$ *and every* $\lambda \in \mathbb{R}$*, there exists* $u(t) \in L^q(a, b; X)$*, such that* $w(t) := T\cdot\big(u(t)\big) \in L^\infty(a, b; H)$*,* $v(t) := \widetilde{T}\cdot\big(w(t)\big) = (\widetilde{T}\circ T)\cdot\big(u(t)\big) \in W^{1,q^*}(a, b; X^*)$ *and* $u(t)$ *is a solution of*

$$(3.78)\qquad \begin{cases}\frac{dv}{dt}(t) + \lambda v(t) + \Lambda_t\big(u(t)\big) + D\Phi_t\big(u(t)\big) = 0 & \text{for a.e. } t \in (a, b),\\ w(a) = w_0,\end{cases}$$

where $w(t)$ *is* H*-weakly continuous on* $[a, b]$*; see Lemma 2.2. Moreover, if* Λ_t *and* Φ_t *satisfy the monotonicity condition*

$$\begin{aligned}(3.79)\quad &\Big\langle h, \big\{D\Phi_t(x+h) - D\Phi_t(x)\big\} + D\Lambda_t(x)\cdot h\Big\rangle_{X\times X^*} \geq\\ &\frac{k_0\big|f(h,t)\big|^2}{\hat{g}(\|T\cdot x\|_H)} - \hat{g}\big(\|T\cdot x\|_H\big)\cdot\Big(\|x\|_X^q + \mu(t)\Big)^{(2-p)/2}\cdot\big|f(h,t)\big|^p\cdot\|T\cdot h\|_H^{(2-p)}\end{aligned}$$

for all $x, h \in X$ *and* $t \in [a, b]$ *for some constant* $k_0 \geq 0$ *such that* $k_0 \neq 0$ *if* $p > 0$*, some function* $f(h, t) : X \times [a, b] \to \mathbb{R}$ *and some non-decreasing function* $\hat{g}(s) : [0, +\infty) \to (0, +\infty)$*, then such a solution of* (3.78) *is unique.*

Proof. *Step 1: Existence of the solution.* Assume first that $\lambda = 0$. Since the Banach space is X separable, using [11, Lemma A.2], we deduce that there exists a separable Hilbert space Y and a bounded linear inclusion operator $S \in \mathcal{L}(Y; X)$ such that S is injective, the image of S is dense in X and, moreover, S is a compact operator. Let $S^* \in \mathcal{L}(X^*; Y^*)$ be the corresponding adjoint operator, which satisfies

$$(3.80)\qquad \langle y, S^*\cdot x^*\rangle_{Y\times Y^*} := \langle S\cdot y, x^*\rangle_{X\times X^*}$$

for all $x^* \in X^*$ and $y \in Y$. Define $P \in \mathcal{L}(Y; H)$ by $P := T\circ S$ and $\widetilde{P} \in \mathcal{L}(H; Y^*)$ by $\widetilde{P} := S^*\circ\widetilde{T}$. Then it is clear that $\{Y, H, Y^*\}$ is another evolution triple with the corresponding inclusion operator $P \in \mathcal{L}(Y;H)$ as defined in Definition 2.6 together with the corresponding adjoint operator $\widetilde{P} \in \mathcal{L}(H;Y^*)$ defined as in (2.2).

Furthermore, let $\psi(t) \in L^q(a,b;Y)$ be such that the function $\varphi(t) : (a,b) \to X^*$ defined by $\varphi(t) := I_Y \cdot \big(\psi(t)\big)$ belongs to $W^{1,q^*}(a,b;Y^*)$, where $I_Y := \widetilde{P} \circ P : Y \to Y^*$. Denote the set of all such functions ψ by $\mathcal{R}_{Y,q}(a,b)$. As before, by Lemma 2.3, for each $\psi(t) \in \mathcal{R}_q(a,b)$ the function $w(t) : [a,b] \to H$ defined by $w(t) := P \cdot \big(\psi(t)\big)$ belongs to $L^\infty(a,b;H)$ and, up to a redefinition of $w(t)$ on a subset of $[a,b]$ of Lebesgue measure zero, w is H-weakly continuous, as stated in Lemma 2.2.

Next, for all $y \in Y$, let $\Psi(y) : Y \to [0,+\infty)$ be a function defined by

$$\Psi(y) := \|y\|_Y^q + \|y\|_Y^2. \tag{3.81}$$

Then $\Psi(y)$ is a convex function that is Gâteaux differentiable on every $y \in Y$, satisfies $\Psi(0) = 0$ and satisfies the growth condition

$$\frac{1}{C_0}\|y\|_Y^q - C_0 \le \Psi(y) \le C_0\,\|y\|_Y^q + C_0 \tag{3.82}$$

for all $y \in Y$ and the uniform convexity condition

$$\Big\langle h, D\Psi(y+h) - D\Psi(y)\Big\rangle_{Y\times Y^*} \ge \frac{1}{C_0}\big(\|y\|_Y^{q-2} + 1\big)\cdot\|h\|_Y^2 \tag{3.83}$$

for all $y, h \in Y$, for some $C_0 > 0$.

Next let $w_0 \in H$. Then, since the image of the operator $T \circ S$ is dense in H, there exists a sequence $\{\psi_n^{(0)}\} \subset Y$ such that $w_n^{(0)} := (T \circ S)\cdot \psi_n^{(0)} \to w_0$ strongly in H as $n \to +\infty$. Furthermore, let $\varepsilon_n \to 0^+$ as $n \to +\infty$. By Theorem 1.2, for every n there exists $\psi_n(t) \in \mathcal{R}_{Y,q}(a,b)$ such that

$$\frac{d\varphi_n}{dt}(t) + S^*\cdot\Big(\Lambda_t\big(u_n(t)\big) + D\Phi_t\big(u_n(t)\big)\Big) + \varepsilon_n D\Psi\big(\psi_n(t)\big) = 0 \tag{3.84}$$

for a.e. $t \in (a,b)$ and $w_n(a) = w_n^{(0)}$, where $u_n(t) := S\cdot\big(\psi_n(t)\big)$, $w_n(t) := P\cdot\big(\psi_n(t)\big)$, $\varphi_n(t) := \widetilde{P}\cdot\big(w_n(t)\big)$ and we assume that $w_n(t)$ is H-weakly continuous on $[a,b]$, as stated in Lemma 2.2.

On the other hand, by the trivial inequality

$$\frac{p}{2}a^2 + \frac{2-p}{2}b^2 \ge a^p b^{2-p},$$

using (3.77), we deduce

$$\Big\langle x, D\Phi_t(x) + \Lambda_t(x)\Big\rangle_{X\times X^*} \ge \frac{1}{C_1}\,\|x\|_X^q - C_1\|L\cdot x\|_V^2 - C_1\mu(t)\Big(\|T\cdot x\|_H^2 + 1\Big) \tag{3.85}$$

for all $x \in X$ and $t \in [a, b]$ for some constant $C_1 > 0$. Then, as before in (3.9), we obtain

$$\Big\langle x, D\Phi_t(x) + \Lambda_t(x)\Big\rangle_{X\times X^*} \geq \frac{1}{K}\,\|x\|_X^q - \tilde{\mu}(t)\Big(\|T\cdot x\|_H^2 + 1\Big) \tag{3.86}$$

for all $x \in X$ and $t \in [a, b]$, for some constant $K > 0$ and $\tilde{\mu}(t) \in L^1(a, b; \mathbb{R})$. Thus, since for every $t \in [a, b]$ the mapping $\big(D\Phi_t + \Lambda_t\big)(x) : X \to X^*$ is pseudo-monotone, Lemma 3.2 implies that the mapping

$$\Gamma\big(x(t)\big) : \big\{\bar{x}(t) \in L^q(a, b; X) : \; T\cdot \bar{x}(t) \in L^\infty(a, b; H)\big\} \to L^{q^*}(a, b; X^*)$$

defined by

$$\Big\langle h(t), \Gamma\big(x(t)\big)\Big\rangle_{L^q(a,b;X)\times L^{q^*}(a,b;X^*)} := \int_a^b \Big\langle h(t), \Lambda_t\big(x(t)\big) + D\Phi_t\big(x(t)\big)\Big\rangle_{X\times X^*} dt \tag{3.87}$$

for all $x(t) \in \big\{\bar{x}(t) \in L^q(a, b; X) : T\cdot \bar{x}(t) \in L^\infty(a, b; H)\big\}$ and $h(t) \in L^q(a, b; X)$ is weakly pseudo-monotone with respect to the evolution triple $\{X, H, X^*\}$; see Definition 3.1.

So all the conditions of Corollary 3.1 satisfied; and therefore, by that corollary, up to a subsequence, $u_n(t) \rightharpoonup u(t)$ weakly in $L^q(a, b; X)$, where $u(t) \in L^q(a, b; X)$ is such that

$$w(t) := T\cdot\big(u(t)\big) \in L^\infty(a, b; H), \quad v(t) := \widetilde{T}\cdot\big(w(t)\big) = (\widetilde{T}\circ T)\cdot\big(u(t)\big) \in W^{1,q^*}(a, b; X^*)$$

and $u(t)$ is a solution of (3.78) with $\lambda = 0$, where $w(t)$ is H-weakly continuous on $[a, b]$, as stated in Lemma 2.2.

Step 2: Assume that $\lambda \neq 0$. Then by the above, for every $w_0 \in H$ and every $\lambda \in \mathbb{R}$, there exists $u_\lambda(t) \in L^q(a, b; X)$, such that $w_\lambda(t) := T\cdot\big(u_\lambda(t)\big) \in L^\infty(a, b; H)$, $v_\lambda(t) := \widetilde{T}\cdot\big(w_\lambda(t)\big) = (\widetilde{T}\circ T)\cdot\big(u_\lambda(t)\big) \in W^{1,q^*}(a, b; X^*)$ and $u_\lambda(t)$ is a solution of

$$\begin{cases} \frac{dv_\lambda}{dt}(t) + e^{\lambda(t-a)}\Lambda_t\big(e^{-\lambda(t-a)}u_\lambda(t)\big) + e^{\lambda(t-a)}D\Phi_t\big(e^{-\lambda(t-a)}u_\lambda(t)\big) = 0 \\ \qquad\qquad \text{for a.e. } t \in (a, b), \\ w_\lambda(a) = w_0, \end{cases} \tag{3.88}$$

where we assume that $w(t)$ is H-weakly continuous on $[a, b]$, as stated in Lemma 2.2. Then, defining $u(t) := e^{-\lambda(t-a)}u_\lambda(t)$, we obtain that $u(t) \in L^q(a, b; X)$ is such that $w(t) := T\cdot\big(u(t)\big) \in L^\infty(a, b; H)$, $v(t) := \widetilde{T}\cdot\big(w(t)\big) = (\widetilde{T}\circ T)\cdot\big(u(t)\big) \in W^{1,q^*}(a, b; X^*)$ and $u(t)$ is a solution of (3.78).

Step 3: Uniqueness of the solution. Assume that Φ_t satisfies (3.79). Then applying Theorem 1.1 completes the proof. □

Remark 3.3. By Lemma 2.3, the solution of (3.78) from Theorem 3.1 satisfies the energy equality

$$\frac{\|w(t)\|_H^2}{2} + \int_a^t \left(\lambda \|w(s)\|_H^2 + \left\langle u(s), \Lambda_s\big(u(s)\big) + D\Phi_s\big(u(s)\big)\right\rangle_{X\times X^*}\right) ds = \frac{\|w_0\|_H^2}{2} \tag{3.89}$$

for all $t \in [a, b]$.

As a particular case of Theorem 3.1 we have the following theorem.

Theorem 3.2. *Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X; H)$ as defined in Definition 2.6 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H; X^*)$, defined in* (2.2). *Assume also that the Banach space X is separable. Furthermore, let $a, b, q \in \mathbb{R}$ be such that $a < b$ and $q \geq 2$. Next, for each $t \in [a, b]$ let $\Phi_t(x) : X \to [0, +\infty)$ be a convex function that is Gâteaux differentiable at every $x \in X$, satisfies $\Phi_t(0) = 0$ and satisfies the growth condition*

$$0 \leq \Phi_t(x) \leq C\,\|x\|_X^q + C \tag{3.90}$$

for all $x \in X$ and $t \in [a, b]$, for some $C > 0$. Also assume that $\Phi_t(x)$ is Borel on the pair of variables (x, t). Furthermore, for every $t \in [a, b]$ let $\Lambda_t(x) : X \to X^*$ be a function which is Gâteaux differentiable at every $x \in X$, $\Lambda_t(0) \in L^{q^*}(a, b; X^*)$ and the derivative of Λ_t satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)} \leq g\big(\|T \cdot x\|_H\big)\left(\|x\|_X^{q-2} + 1\right) \tag{3.91}$$

for all $x \in X$ and $t \in [a, b]$, for some non-decreasing function $g(s) : [0 + \infty) \to (0 + \infty)$. Also assume that $\Lambda_t(x)$ is Borel on the pair of variables (x, t); see Definition 2.1. Assume also that Λ_t satisfies the monotonicity conditions

$$\left\langle h, D\Lambda_t(x) \cdot h\right\rangle_{X\times X^*} \geq 0 \tag{3.92}$$

for all $x, h \in X$ and $t \in [a, b]$. Finally, let $F_t(x) : X \to X^*$ be a function that is Gâteaux differentiable at every $x \in X$, $F_t(0) \in L^{q^*}(a, b; X^*)$ and such that the derivative DF_t of F_t satisfies the condition

$$\|DF_t(x)\|_{\mathcal{L}(X;X^*)} \leq g\big(\|T \cdot x\|_H\big)\big(\|x\|_X^{q-2} + 1\big) \tag{3.93}$$

for all $x \in X$ and $t \in [a, b]$, for some non-decreasing function $g(s) : [0 + \infty) \to (0 + \infty)$. Also assume that $F_t(x)$ is Borel on the pair of variables (x, t). Next assume *that*

$$\Big\langle x, D\Phi_t(x) + \Lambda_t(x) + F_t(x)\Big\rangle_{X\times X^*} \geq \frac{1}{\hat{C}}\|x\|_X^q - \hat{C}\big(\|x\|_X + 1\big)\Big(\|L\cdot x\|_V + \|T\cdot x\|_H + 1\Big) - \mu(t) \tag{3.94}$$

for all $x \in X$ and $t \in [a,b]$, where V is a given Banach space, $L \in \mathcal{L}(X,V)$ is a given compact operator, $\hat{C} > 0$ is some constant and $\mu(t) \in L^1(a,b;\mathbb{R})$ is some non-negative function. Finally, assume that $F_t(x)$ is weak-to-strong continuous, i.e., for every fixed $t \in [a,b]$ and every sequence $\{x_n\}$ such that $x_n \rightharpoonup x$ weakly in X, $F_t(x_n) \to F_t(x)$ strongly in X^*. Then, for every $w_0 \in H$ and every $\lambda \in \mathbb{R}$, there exists $u(t) \in L^q(a,b;X)$, such that $w(t) := T\cdot\big(u(t)\big) \in L^\infty(a,b;H)$, $v(t) := \widetilde{T}\cdot\big(w(t)\big) = (\widetilde{T}\circ T)\cdot\big(u(t)\big) \in W^{1,q^*}(a,b;X^*)$ and $u(t)$ is a solution of

$$\begin{cases} \frac{dv}{dt}(t) + \lambda v(t) + F_t\big(u(t)\big) + \Lambda_t\big(u(t)\big) + D\Phi_t\big(u(t)\big) = 0 & \text{for a.e. } t \in (a,b),\\ w(a) = w_0, \end{cases} \tag{3.95}$$

where we assume that $w(t)$ is H-weakly continuous on $[a,b]$, as stated in Lemma 2.2.

Proof. Since $F_t(x) : X \to X^*$ is weak to strong continuous, it is pseudo-monotone on X. Moreover, for every $t \in [a,b]$, the mappings $D\Phi_t(x) : X \to X^*$ and $\Lambda_t(x) : X \to X^*$ are monotone. Therefore, since Λ_t is Gâteaux differentiable and Φ_t is convex, using Lemma 2.1 and Definition 2.4, we deduce that the mapping $(D\Phi_t + \Lambda_t + F_t)(x) : X \to X^*$ is pseudo-monotone. Thus, applying Theorem 3.1 with $\Lambda_t + F_t$ instead of Λ_t, gives the desired result. □

Theorem 3.3. *Let X and Z be reflexive Banach spaces and X^* and Z^* be their corresponding dual spaces. Furthermore, let H be a Hilbert space. Suppose that $Q \in \mathcal{L}(X,Z)$ is an injective inclusion operator whose image is dense on Z. Furthermore, suppose that $P \in \mathcal{L}(Z,H)$ is an injective inclusion operator whose image is dense on H. Let $T \in \mathcal{L}(X,H)$ be defined by $T := P\circ Q$. So $\{X,H,X^*\}$ is an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X;H)$ as defined in Definition 2.6 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$ defined as in (2.2). Assume also that the Banach space X is separable. Next let $a,b \in \mathbb{R}$ be such that $a < b$ and $q \geq 2$. Furthermore, for every $t \in [a,b]$, let $\Lambda_t(z) : Z \to X^*$ and $A_t(z) : Z \to X^*$ be functions that are Gâteaux differentiable at every $z \in Z$ and such that $\Lambda_t(0), A_t(0) \in L^{q^*}\big(a,b;X^*\big)$. Assume that for every $t \in [a,b]$, Λ_t and A_t satisfy the bounds*

$$\big\|D\Lambda_t(z)\big\|_{\mathcal{L}(Z;X^*)} \leq g\big(\|P\cdot z\|_H\big)\cdot\Big(\|z\|_Z^{q-2} + 1\Big) \tag{3.96}$$

for all $z \in Z$ and $t \in [a,b]$,

$$\big\|\Lambda_t(z)\big\|_{X^*} \leq g\big(\|P\cdot z\|_H\big)\cdot\Big(\|L_0\cdot z\|_{V_0}^{q-1} + \tilde{\mu}^{\frac{q-1}{q}}(t)\Big) \tag{3.97}$$

for all $z \in Z$ and $t \in [a,b]$ and

$$\big\|DA_t(z)\big\|_{\mathcal{L}(Z;X^*)} \leq g\big(\|P\cdot z\|_H\big)\cdot\Big(\|L_0\cdot z\|_{V_0}^{q-2} + 1\Big) \tag{3.98}$$

for all $z \in Z$ for all $t \in [a,b]$, where $\tilde{\mu}(t) \in L^1(a,b;\mathbb{R})$ is some non-negative function, $g(s) : [0,+\infty) \to (0,+\infty)$ is some non-decreasing function, V_0 is some Banach space and $L_0 \in \mathcal{L}(Z;V_0)$ is some compact linear operator. Moreover, assume that Λ_t and A_t satisfy the monotonicity condition

$$\begin{aligned}\Big\langle h, A_t(Q\cdot h) + \Lambda_t\big(Q\cdot h\big)\Big\rangle_{X\times X^*} &\geq \big(1/\bar{C}\big)\big\|Q\cdot h\big\|_Z^q \\ &- \Big(\big\|Q\cdot h\big\|_Z^p + \mu^{\frac{p}{2}}(t)\Big)\bigg(\bar{C}\big\|L\cdot(Q\cdot h)\big\|_V^{(2-p)} + \mu^{\frac{2-p}{2}}(t)\Big(\big\|T\cdot h\big\|_H^{(2-p)} + 1\Big)\bigg)\end{aligned} \tag{3.99}$$

for all $h \in X$ and $t \in [a,b]$, where V is a given Banach space, $L \in \mathcal{L}(Z,V)$ is a given compact operator, $p \in [0,2)$, $\mu(t) \in L^1(a,b;\mathbb{R})$ is some non-negative function and $\bar{C} > 0$ is some constant. Also assume that $\Lambda_t(z)$ $A_t(z)$ are Borel on the pair of variables (z,t). Finally, assume that there exist a family of Banach spaces $\{V_j\}_{j=1}^{+\infty}$ and a family of compact bounded linear operators $\{L_j\}_{j=1}^{+\infty}$, where $L_j \in \mathcal{L}(Z,V_j)$, which satisfy the following condition:

- *if $\{h_n\}_{n=1}^{+\infty} \subset Z$ is a sequence and $h_0 \in Z$, are such that for every fixed j $\lim_{n\to+\infty} L_j\cdot h_n = L_j\cdot h_0$ strongly in V_j and $P\cdot h_n \rightharpoonup P\cdot h_0$ weakly in H, then for every fixed $t \in (a,b)$, $\Lambda_t(h_n) \rightharpoonup \Lambda_t(h_0)$ weakly in X^* and $DA_t(h_n) \to DA_t(h_0)$ strongly in $\mathcal{L}(Z,X^*)$.*

Then for every $w_0 \in H$, there exists a function $z(t) \in L^q(a,b;Z)$ such that $w(t) := P\cdot z(t) \in L^\infty(a,b;H)$, $v(t) := \widetilde{T}\cdot\big(w(t)\big) \in W^{1,q^*}(a,b;X^*)$ and $z(t)$ satisfies the following equation:

$$\begin{cases}\frac{dv}{dt}(t) + A_t\big(z(t)\big) + \Lambda_t\big(z(t)\big) = 0 \quad \text{for a.e. } t \in (a,b)\,,\\ w(a) = w_0\,,\end{cases} \tag{3.100}$$

where we assume that $w(t)$ is H-weakly continuous on $[a,b]$, as stated in Lemma 2.2. Moreover, if in addition, there exist a Banach space V, a compact operator $L \in \mathcal{L}(Z,V)$, a non-decreasing function $\widetilde{g}(s) : [0,+\infty) \to (0,+\infty)$ and for every $t \in [a,b]$ a convex Gâteaux differentiable functions $\Phi_t : Z \to \mathbb{R}$, Borel measurable on (z,t), and a Gâteaux differentiable mapping $F_t(\sigma) : V \to Z^*$, Borel measurable on (σ,t), satisfying $F_t(0) \in L^{q^*}(a,b;Z^*)$ and such that

$$0 \leq \Phi_t(z) \leq \widetilde{g}\big(\|P\cdot z\|_H\big)\cdot\big(\|z\|_Z^q + 1\big) \tag{3.101}$$

for all $z \in Z$ and $t \in [a,b]$,

$$\big\|DF_t(L \cdot z)\big\|_{\mathcal{L}(V;Z^*)} \leq \widetilde{g}\big(\|P \cdot z\|_H\big) \cdot \big(\|L \cdot z\|_V^{q-2} + 1\big) \tag{3.102}$$

for all $z \in Z$ and $t \in [a,b]$, and

$$\Big\langle h, A_t(Q \cdot h) + \Lambda_t\big(Q \cdot h\big)\Big\rangle_{X \times X^*} \geq \Phi_t\big(Q \cdot h\big) + \Big\langle Q \cdot h, F_t\big((L \circ Q) \cdot h\big)\Big\rangle_{Z \times Z^*} \tag{3.103}$$

for all $h \in X$ and $t \in [a,b]$, then the function $z(t)$, as above, satisfies the energy inequality

$$\frac{1}{2}\big\|w(t)\big\|_H^2 + \int_a^t \Big(\Phi_s\big(z(s)\big) + \Big\langle z(s), F_s\big(L \cdot z(s)\big)\Big\rangle_{Z \times Z^*}\Big) ds \leq \frac{1}{2}\big\|w_0\big\|_H^2 \tag{3.104}$$

for all $t \in [a,b]$.

Proof. Since the Banach space X is separable, as before, by [11, Lemma A.2], we deduce that there exists a separable Hilbert space Y and a bounded linear inclusion operator $S \in \mathcal{L}(Y;X)$ such that S is injective, the image of S is dense in X and S is a compact operator. Moreover, let $S^* \in \mathcal{L}(X^*;Y^*)$ be the corresponding adjoint operator, which satisfies

$$\langle y, S^* \cdot x^* \rangle_{Y \times Y^*} := \langle S \cdot y, x^* \rangle_{X \times X^*} \tag{3.105}$$

for all $z^* \in X^*$ and $y \in Y$. Define $P_0 \in \mathcal{L}(Y;H)$ by $P_0 := T \circ S$ and $\widetilde{P}_0 \in \mathcal{L}(H;Y^*)$ by $\widetilde{P}_0 := S^* \circ \widetilde{T}$. Then it is clear that $\{Y, H, Y^*\}$ is another evolution triple with the corresponding inclusion operator $P_0 \in \mathcal{L}(Y;H)$ as defined in Definition 2.6 together with the corresponding adjoint operator $\widetilde{P}_0 \in \mathcal{L}(H;Y^*)$ defined as in (2.2).

Furthermore, let $\psi(t) \in L^q(a,b;Y)$ be such that the function $\varphi(t) : (a,b) \to Y^*$ defined by $\varphi(t) := I_Y \cdot \big(\psi(t)\big)$ belongs to $W^{1,q^*}(a,b;Y^*)$, where $I_Y := \widetilde{P}_0 \circ P_0 : Y \to Y^*$. Denote the set of all such functions ψ by $\mathcal{R}_{Y,q}(a,b)$. As before, by Lemma 2.3, for every $\psi(t) \in \mathcal{R}_q(a,b)$, the function $w(t) : [a,b] \to H$ defined by $w(t) := P_0 \cdot \big(\psi(t)\big)$ belongs to $L^\infty(a,b;H)$ and, up to a redefinition of $w(t)$ on a subset of $[a,b]$ of Lebesgue measure zero, w is H-weakly continuous, as stated in Lemma 2.2.

Next define the function $\Psi(y) : Y \to [0,+\infty)$ by

$$\Psi(y) := \|y\|_Y^q + \|y\|_Y^2 \quad \text{for all } y \in Y. \tag{3.106}$$

Then $\Psi(y)$ is a convex function that is Gâteaux differentiable at every $y \in Y$, satisfies $\Psi(0) = 0$ and satisfies the growth condition

$$\frac{1}{C_0}\|y\|_Y^q - C_0 \leq \Psi(y) \leq C_0\,\|y\|_Y^q + C_0 \tag{3.107}$$

for all $y \in Y$ and the uniform convexity condition

$$\Big\langle h, D\Psi(y+h) - D\Psi(y)\Big\rangle_{Y\times Y^*} \geq \frac{1}{C_0}\big(\|y\|_Y^{q-2}+1\big)\cdot\|h\|_Y^2$$

for all $y, h \in Y$, for some $C_0 > 0$.

Next let $w_0 \in H$. Then, since the image of the operator $T \circ S$ is dense in H, there exists a sequence $\{\psi_n^{(0)}\} \subset Y$ such that $w_n^{(0)} := (T\circ S)\cdot \psi_n^{(0)} \to w_0$ strongly in H as $n \to +\infty$. Furthermore, let $\varepsilon_n \to 0^+$ as $n \to +\infty$. By Theorem 1.2, for every n, there exists $\psi_n(t) \in \mathcal{R}_{Y,q}(a,b)$ such that

$$\begin{cases} \frac{d\varphi_n}{dt}(t) + S^*\cdot\Big(A_t\big(z_n(t)\big) + \Lambda_t\big(z_n(t)\big)\Big) + \varepsilon_n D\Psi\big(\psi_n(t)\big) = 0 \\ \qquad\qquad \text{for } t\in(a,b), \\ w_n(a) = (T\circ S)\cdot\psi_n^{(0)}, \end{cases} \tag{3.108}$$

where

$$\begin{aligned} u_n(t) &:= S\cdot\big(\psi_n(t)\big), \\ z_n(t) &:= (Q\circ S)\cdot\big(\psi_n(t)\big) = Q\cdot\big(u_n(t)\big), \\ w_n(t) &:= (T\circ S)\cdot\big(\psi_n(t)\big) = P\cdot\big(z_n(t)\big), \\ \varphi_n(t) &:= (S^*\circ\widetilde{T}\circ T\circ S)\cdot\big(\psi_n(t)\big) = (S^*\circ\widetilde{T})\cdot\big(w_n(t)\big), \end{aligned}$$

and we assume that $w_n(t)$ is H-weakly continuous on $[a,b]$. Thus all the conditions of Lemma 3.1 satisfied; and, by Lemma 3.1, using [11, Lemma 2.2], we deduce that there exist $z(t) \in L^q(a,b;Z)$ and $\overline{\Lambda}(t), \overline{A}(t) \in L^{q^*}(a,b;X^*)$ such that $w(t) := P\cdot z(t) \in L^\infty(a,b;H)$, $v(t) := \widetilde{T}\cdot w(t) \in W^{1,q^*}(a,b;X^*)$, $w(t)$ is H-weakly continuous on $[a,b]$, up to a subsequence, we have

$$\begin{cases} z_n(t) \rightharpoonup z(t) \quad \text{weakly in } L^q(a,b;Z), \\ \frac{d\varphi_n}{dt}(t) \rightharpoonup \frac{d\varphi}{dt}(t) \text{ weakly in } L^{q^*}(a,b;Y^*), \\ \Lambda_t\big(z_n(t)\big) \rightharpoonup \overline{\Lambda}(t) \text{ weakly in } L^{q^*}(a,b;X^*), \\ A_t\big(z_n(t)\big) \rightharpoonup \overline{A}(t) \text{ weakly in } L^{q^*}(a,b;X^*), \\ w_n(t) \rightharpoonup w(t) \text{ weakly in } H \quad \text{for every fixed } t\in[a,b], \\ \big\{w_n(t)\big\}_{n=1}^{+\infty} \text{ is bounded in } L^\infty(a,b;H), \end{cases} \tag{3.109}$$

where $\varphi(t) = S^*\cdot v(t)$, and $z(t)$ satisfies the equation

$$\begin{cases} \frac{dv}{dt}(t) + \overline{A}(t) + \overline{\Lambda}(t) = 0 \quad \text{for a.e. } t\in(a,b), \\ w(a) = w_0. \end{cases} \tag{3.110}$$

Moreover,

$$\frac{1}{2}\|w(t)\|_H^2 + \overline{\lim_{n\to+\infty}}\left(\int_a^t \Big\langle u_n(s), A_s\big(z_n(s)\big) + \Lambda_s\big(z_n(s)\big)\Big\rangle_{X\times X^*} ds\right) \leq \frac{1}{2}\|w_0\|_H^2 \tag{3.111}$$

for all $t \in [a, b]$. Next there exists a family of reflexive Banach spaces $\{V_j\}_{j=1}^{+\infty}$ and a family of compact bounded linear operators $\{L_j\}_{j=1}^{+\infty}$, where $L_j \in \mathcal{L}(Z, V_j)$, which satisfy the following condition:

- if $\{h_n\}_{n=1}^{+\infty} \subset Z$ is a sequence and $h_0 \in Z$, are such that for every fixed j, $\lim_{n\to+\infty} L_j \cdot h_n = L_j \cdot h_0$ strongly in V_j and $P \cdot h_n \rightharpoonup P \cdot h_0$ weakly in H, then for every fixed $t \in (a, b)$, $\Lambda_t(h_n) \rightharpoonup \Lambda_t(h_0)$ weakly in X^* and $DA_t(h_n) \to DA_t(h_0)$ strongly in $\mathcal{L}(Z, X^*)$.

On the other hand, using (3.109) and Lemma 2.4, we deduce that for every j, $L_j \cdot z_n(t) \to L_j \cdot z(t)$ strongly in $L^q(a, b; V_j)$ as $n \to +\infty$. In the same way, we obtain $L_0 \cdot z_n(t) \to L_0 \cdot z(t)$ strongly in $L^q(a, b; V_0)$ as $n \to +\infty$. Thus, up to a further subsequence, we have $L_j \cdot z_n(t) \to L_j \cdot z(t)$ strongly in V_j for a.e. $t \in (a, b)$ and every j. Therefore, by (3.109) and the above condition, we must have $\Lambda_t\big(z_n(t)\big) \rightharpoonup \Lambda_t\big(z(t)\big)$ weakly in X^* and $DA_t\big(sz_n(t) + (1-s)z(t)\big) \to DA_t\big(z(t)\big)$ strongly in $\mathcal{L}(Z, X^*)$ for a.e. $t \in (a, b)$ and for every $s \in [0, 1]$. Therefore, using (3.97), the facts that $\{w_n(t)\}$ is bounded in $L^\infty(a, b; H)$ and that $L_0 \cdot z_n(t) \to L_0 \cdot z(t)$ strongly in $L^q(a, b; V_0)$, we deduce that

$$\int_a^b \Big\langle h(t), \Lambda_t\big(z_n(t)\big)\Big\rangle_{X\times X^*} dt \to \int_a^b \Big\langle h(t), \Lambda_t\big(z(t)\big)\Big\rangle_{X\times X^*} dt$$

for all $h \in L^q(a, b; X)$. Thus

$$\Lambda_t\big(z_n(t)\big) \rightharpoonup \Lambda_t\big(z(t)\big) \text{ weakly in } L^{q^*}(a, b; X^*). \tag{3.112}$$

In a similar way, by (3.98), the fact that $\{w_n(t)\}$ is bounded in $L^\infty(a, b; H)$ and the fact that $L_0 \cdot z_n(t) \to L_0 \cdot z(t)$ strongly in $L^q(a, b; V_0)$, we deduce that, for $q = 2$,

$$\begin{aligned} &DA_t\big(sz_n(t) + (1-s)z(t)\big) \to DA_t\big(z(t)\big) \text{ strongly in } \mathcal{L}(Z, X^*) \\ &\qquad\qquad \text{for a.e. } t \in (a, b) \text{ for all } s \in [0, 1], \text{ and} \\ &DA_t\big(sz_n(t) + (1-s)z(t)\big) \text{ is bounded in } L^\infty\big(a, b; \mathcal{L}(Z, X^*)\big) \text{ uniformly in } s; \end{aligned} \tag{3.113}$$

and, for $q > 2$,

$$DA_t\big(sz_n(t) + (1-s)z(t)\big) \to DA_t\big(z(t)\big) \text{ strongly in } L^{q/(q-2)}\big(a, b; \mathcal{L}(Z, X^*)\big) \tag{3.114}$$

for all $s \in [0, 1]$. In both cases,

$$\left\{DA_t\big(sz_n(t)+(1-s)z(t)\big)\right\}^* \cdot h(t) \to \left\{DA_t\big(z(t)\big)\right\}^* \cdot h(t) \text{ strongly in } L^{q^*}(a,b,Z) \tag{3.115}$$

for all $h(t) \in L^q(a,b;X)$ and all $s \in [0,1]$, where $\{DA_t(\cdot)\}^* \in \mathcal{L}(X,Z^*)$ is the adjoint operator to $DA_t(\cdot) \in \mathcal{L}(Z,X^*)$. Thus, by (3.98), the fact that $\{w_n(t)\}$ is bounded in $L^\infty(a,b;H)$ and the fact that $L_0\cdot z_n(t) \to L_0\cdot z(t)$ strongly in $L^q(a,b;V_0)$, together with (3.115) and (3.109), we obtain

$$\begin{aligned}\int_a^b \Big\langle h(t), A_t\big(z_n(t)\big) - A_t\big(z(t)\big)\Big\rangle_{X\times X^*} dt \\ = \int_0^1\int_a^b \Big\langle h(t), DA_t\Big(sz_n(t)+(1-s)z(t)\Big)\cdot\Big(z_n(t)-z(t)\Big)\Big\rangle_{X\times X^*} dtds \\ = \int_0^1\int_a^b \Big\langle \Big(z_n(t)-z(t)\Big), \Big\{DA_t\big(sz_n(t)+(1-s)z(t)\big)\Big\}^*\cdot h(t)\Big\rangle_{Z\times Z^*} dtds \to 0\end{aligned}$$

for all $h(t) \in L^q(a,b;X)$. So, by (3.109) and (3.112), we have $\overline{\Lambda}(t) = \Lambda_t\big(z(t)\big)$ and $\overline{A}(t) = A_t\big(z(t)\big)$; and thus using (3.110), we finally deduce that $z(t)$ is a solution of (3.100).

Finally, assume that there exist a reflexive Banach space V, a compact operator $L \in \mathcal{L}(Z,V)$, and for every $t \in [a,b]$ a convex Gâteaux differentiable function $\Phi_t : Z \to \mathbb{R}$ and a Gâteaux differentiable mapping $F_t(\sigma) : V \to Z^*$ satisfying (3.101), (3.102) and (3.103). Then, since, as before, $L\cdot z_n(t) \to L\cdot z(t)$ strongly in $L^q(a,b;V)$, we deduce that, up to a subsequence, $F_t\big(L\cdot z_n(t)\big) \to F_t\big(L\cdot z(t)\big)$ strongly in $L^{q^*}(a,b;Z^*)$. On the other hand, by (3.103) and (3.111), we infer

$$\frac{1}{2}\|w(t)\|_H^2 + \overline{\lim_{n\to+\infty}}\left\{\int_a^t\left(\Phi_s\big(z_n(s)\big) + \Big\langle z_n(s), F_s\big(L\cdot z_n(s)\big)\Big\rangle_{Z\times Z^*}\right)ds\right\} \le \frac{1}{2}\|w_0\|_H^2 \tag{3.116}$$

for all $t \in [a,b]$. Therefore, letting $n \to +\infty$ in (3.116) and using (3.109) and the convexity of Φ_t, we finally obtain (3.104). □

As a particular case of Theorem 3.3, we have the following theorem.

Theorem 3.4. *Let X and Z be reflexive Banach spaces and X^* and Z^* be their corresponding dual spaces. Furthermore, let H be a Hilbert space. Suppose that $Q \in \mathcal{L}(X,Z)$ is an injective inclusion operator whose image is dense in Z. Furthermore, suppose that $P \in \mathcal{L}(Z,H)$ is an injective inclusion operator whose image is dense in H. Let $T \in \mathcal{L}(X,H)$ be defined by $T := P\circ Q$, so that $\{X,H,X^*\}$ is an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X;H)$ as defined in Definition 2.6 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$ defined* *as in* (2.2)*. Assume also that the Banach space X is separable. Next let $a, b \in \mathbb{R}$ be such that $a < b$. Furthermore, for each $t \in [a,b]$, let $\Lambda_t \in L^\infty\big(a,b;\mathcal{L}(Z,X^*)\big)$. Next let $F_t(z) : Z \to X^*$ be a function that is Gâteaux differentiable at every $z \in Z$ for every $t \in [a,b]$ and satisfies $F_t(0) \in L^2(a,b;X^*)$ and the Lipschitz condition*

$$\big\|DF_t(z)\big\|_{\mathcal{L}(Z;X^*)} \leq g\big(\|P\cdot z\|_H\big) \tag{3.117}$$

for all $z \in Z$ and $t \in [a,b]$, for some non-decreasing function $g(s) : [0,+\infty) \to (0,+\infty)$. Also assume that $F_t(z)$ is Borel on the pair of variables (z,t). Moreover, suppose that Λ_t and F_t satisfy the lower bound condition

$$\Big\langle h, \Lambda_t\cdot\big(Q\cdot h\big) + F_t\big(Q\cdot h\big)\Big\rangle_{X\times X^*} \geq \frac{1}{C}\big\|Q\cdot h\big\|_Z^2 - \Big(\big\|Q\cdot h\big\|_Z^p + \mu^{\frac{p}{2}}(t)\Big)\Big(\bar{C}\big\|L\cdot(Q\cdot h)\big\|_V^{(2-p)} + \mu^{\frac{2-p}{2}}(t)\Big(\big\|T\cdot h\big\|_H^{(2-p)}+1\Big)\Big) \tag{3.118}$$

for all $h \in X$ and $t \in [a,b]$, where V is a given Banach space, $L \in \mathcal{L}(Z,V)$ is a given compact operator, $p \in [0,2)$ and $\bar{C} > 0$ are some constants and $\mu(t) \in L^1(a,b;\mathbb{R})$ is a non-negative function. Finally assume that there exist a family of reflexive Banach spaces $\{V_j\}_{j=1}^{+\infty}$ and a family of compact bounded linear operators $\{L_j\}_{j=1}^{+\infty}$, where $L_j \in \mathcal{L}(Z,V_j)$, which satisfy the following condition:

- *if $\{h_n\}_{n=1}^{+\infty} \subset Z$ is a sequence such that for all fixed j $\lim_{n\to+\infty} L_j\cdot h_n = L_j\cdot h_0$ strongly in V_j and $P\cdot h_n \rightharpoonup P\cdot h_0$ weakly in H, then for every fixed $t \in (a,b)$, $F_t(h_n) \rightharpoonup F_t(h_0)$ weakly in X^*.*

Then, for each $w_0 \in H$, there exists $z(t) \in L^2(a,b;Z)$ such that $w(t) := P\cdot z(t)$ belongs to $L^\infty(a,b;H)$, $v(t) := \widetilde{T}\cdot\big(w(t)\big)$ belongs to $W^{1,2}(a,b;X^*)$ and $z(t)$ satisfies the equation

$$\begin{cases} \frac{dv}{dt}(t) + \Lambda_t\cdot\big(z(t)\big) + F_t\big(z(t)\big) = 0 & \text{for a.e. } t \in (a,b),\\ w(a) = w_0, \end{cases} \tag{3.119}$$

where we assume that $w(t)$ is H-weakly continuous on $[a,b]$, as stated in Lemma 2.2. Moreover, if, in addition, there exist a reflexive Banach space E, a compact operator $L_0 \in \mathcal{L}(Z,E)$, and for every $t \in [a,b]$ a Gâteaux differentiable mapping $H_t(\zeta) : E \to Z^*$, measurable on (ζ,t), such that $H_t(0) \in L^2(a,b;Z^*)$ and satisfying

$$\big\|DH_t(L_0\cdot z)\big\|_{\mathcal{L}(E;Z^*)} \leq \widetilde{g}\big(\|P\cdot z\|_H\big) \tag{3.120}$$

for all $z \in Z$ and $t \in [a,b]$, for some non-decreasing function $\widetilde{g}(s) : [0,+\infty) \to (0,+\infty)$, and satisfying

$$\Big\langle h, \Lambda_t\cdot(Q\cdot h) + F_t(Q\cdot h)\Big\rangle_{X\times X^*} \geq \Big\langle Q\cdot h, A_t\cdot(Q\cdot h) + H_t\big((L_0\circ Q)\cdot h\big)\Big\rangle_{Z\times Z^*} \tag{3.121}$$

for all $h \in X$ and all $t \in [a,b]$, where $A_t \in L^\infty\big(a,b;\mathcal{L}(Z,Z^*)\big)$ is such that $\langle z, A_t \cdot z\rangle_{Z\times Z^*} \geq 0$ for all $z \in Z$, then the function $z(t)$, as above, satisfies the energy inequality

$$\frac{1}{2}\|w(t)\|_H^2 + \int_a^t \Big\langle z(s), A_s\cdot\big(z(s)\big) + H_s\big(L_0\cdot z(s)\big)\Big\rangle_{Z\times Z^*} ds \leq \frac{1}{2}\|w_0\|_H^2 \tag{3.122}$$

for all $t \in [a,b]$.

As a particular case of Theorem 3.4, where $Z = H$, we have the following statement, which is useful in the study of hyperbolic systems.

Corollary 3.2. *Let $\{X, H, X^*\}$ be an evolution triple with the corresponding inclusion operator $T \in \mathcal{L}(X;H)$ as defined in Definition 2.6 together with the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$ defined as in (2.2). Assume also that the Banach space X is separable. Next let $a,b \in \mathbb{R}$ be such that $a < b$. Furthermore, for every $t \in [a,b]$, let $\Lambda_t \in L^\infty\big(a,b;\mathcal{L}(H,X^*)\big)$. Next let $F_t(w) : H \to X^*$ be a function that is Gâteaux differentiable at every $w \in H$ for every $t \in [a,b]$, and satisfies $F_t(0) \in L^2(a,b;X^*)$ and the Lipschitz condition*

$$\big\|DF_t(w)\big\|_{\mathcal{L}(H;X^*)} \leq g\big(\|w\|_H\big) \tag{3.123}$$

for all $w \in H$ and all $t \in [a,b]$, for some non-decreasing function $g(s) : [0,+\infty) \to (0,+\infty)$. Also assume that $F_t(w)$ is Borel on the pair of variables (w,t); see Definition 2.1. Moreover, assume that F_t is weak to weak continuous from H to X^* for every fixed t, i.e., for every sequence $\{h_n\}_{n=1}^{+\infty} \subset H$ such that $h_n \rightharpoonup h_0$ weakly in H and for every $t \in [a,b]$, $F_t(h_n) \rightharpoonup F_t(h_0)$ weakly in X^*. Finally, suppose that Λ_t and F_t satisfy the lower bound condition

$$\Big\langle h, \Lambda_t\cdot(T\cdot h) + F_t(T\cdot h)\Big\rangle_{X\times X^*} \geq -\mu(t)\Big(\|T\cdot h\|_H^2 + 1\Big) \tag{3.124}$$

for all $h \in X$ for all $t \in [a,b]$, for some non-negative function $\mu(t) \in L^1(a,b;\mathbb{R})$. Then, for each $w_0 \in H$, there exists $w(t) \in L^\infty(a,b;H)$ such that $v(t) := \widetilde{T}\cdot\big(w(t)\big) \in W^{1,2}(a,b;X^*)$ and $w(t)$ satisfies the equation

$$\begin{cases} \frac{dv}{dt}(t) + \Lambda_t\cdot\big(w(t)\big) + F_t\big(w(t)\big) = 0 & \text{for a.e. } t \in (a,b),\\ w(a) = w_0, \end{cases} \tag{3.125}$$

where $w(t)$ is H-weakly continuous on $[a,b]$, as stated in Lemma 2.2.

4 Applications

4.1 Notation. For a $p\times q$ matrix A with ij-th entry a_{ij}, we denote by $|A| = \big(\Sigma_{i=1}^p \Sigma_{j=1}^q a_{ij}^2\big)^{1/2}$ the Frobenius norm of A.

For matrices $A, B \in \mathbb{R}^{p\times q}$ with ij-th entries a_{ij} and b_{ij} respectively, we write $A : B := \sum_{i=1}^{p}\sum_{j=1}^{q} a_{ij}b_{ij}$.

Given a vector-valued function $f(x) = \big(f_1(x), \dots, f_k(x)\big) : \Omega \to \mathbb{R}^k$ ($\Omega \subset \mathbb{R}^N$), we denote by $\nabla_x f$ the $k \times N$ matrix with ij-th entry $\frac{\partial f_i}{\partial x_j}$.

For a matrix-valued function $F(x) := \{F_{ij}(x)\} : \mathbb{R}^N \to \mathbb{R}^{k\times N}$, we denote by $div\, F$ the $\mathbb{R}^k$-valued vector field defined by $div\, F := (l_1, \dots, l_k)$, where $l_i = \sum_{j=1}^{N} \frac{\partial F_{ij}}{\partial x_j}$.

For $u = (u_1, \dots, u_p) \in \mathbb{R}^p$ and $v = (v_1, \dots, v_q) \in \mathbb{R}^q$ we denote by $\boldsymbol{u} \otimes \boldsymbol{v}$ the $p \times q$ matrix with ij-th entry $u_i v_j$.

4.2 A general parabolic system in divergence form. Suppose that $\Psi(A, x, t) : \mathbb{R}^{k\times N}_A \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}$ is a non-negative measurable function. Moreover, assume that $\Psi(A, x, t)$ is C^1 as a function of the first argument A when (x, t) are fixed, which satisfies $\Psi(0, x, t) = 0$ and is convex in the first argument A when (x, t) are fixed, i.e.,

$$\Psi\big(\alpha A_1 + (1-\alpha)A_2, x, t\big) \le \alpha\Psi(A_1, x, t) + (1-\alpha)\Psi(A_2, x, t)$$

for every $\alpha \in [0, 1]$, $A_1, A_2 \in \mathbb{R}^{k\times N}$, $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Moreover, assume that Ψ satisfies the growth condition

$$\frac{1}{C}|A|^q - |g_0(x)| \le \Psi(A, x, t) \le C|A|^q + |g_0(x)| \tag{4.1}$$

for all $A \in \mathbb{R}^{k\times N}$, $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$, where $C > 0$ is some constant, $g_0(x) \in L^1(\mathbb{R}^N, \mathbb{R})$ and $q \in [2, +\infty)$. Next let $\Gamma(A, x, t) : \mathbb{R}^{k\times N}_A \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^{k\times N}$ be a measurable function. Moreover, assume that $\Gamma(A, x, t)$ is C^1 as a function of the first argument A when (x, t) are fixed, which satisfies

$$\Gamma(0, x, t) \in L^{q^*}\big(\mathbb{R}; L^2(\mathbb{R}^N, \mathbb{R}^{k\times N})\big), \tag{4.2}$$

the monotonicity condition

$$\sum_{1\le j,n\le N}\ \sum_{1\le i,m\le k} H_{ij}H_{mn}\frac{\partial \Gamma_{mn}}{\partial A_{ij}}(A, x, t) \ge 0 \tag{4.3}$$

for all $H, A \in \mathbb{R}^{k\times N}$ and all $x \in \mathbb{R}^N$ for all $t \in \mathbb{R}$, and the growth condition

$$\left|\frac{\partial \Gamma}{\partial A_{ij}}(A, x, t)\right| \le C\,|A|^{q-2} + C \tag{4.4}$$

for all $A \in \mathbb{R}^{k\times N}$, all $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$ for all $i \in \{1, \dots, k\}$ and all $j \in \{1, \dots, N\}$, where $C > 0$ is some constant. Finally, suppose that

$\Xi(B,x,t) : \mathbb{R}^k_B \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^{k\times N}$ and $\Theta(B,x,t) : \mathbb{R}^k_B \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^k$ are measurable functions. Moreover, assume that $\Xi(B,x,t)$ and $\Theta(B,x,t)$ are C^1 as functions of the first argument B when (x,t) are fixed. Also assume that $\Xi(B,x,t)$ and $\Theta(B,x,t)$ are globally Lipschitz in the first argument B and satisfy

$$\Xi(0,x,t) \in L^{q^*}\big(\mathbb{R}; L^2(\mathbb{R}^N, \mathbb{R}^{k\times N})\big), \quad \Theta(0,x,t) \in L^{q^*}\big(\mathbb{R}; L^2(\mathbb{R}^N, \mathbb{R}^k)\big). \tag{4.5}$$

Proposition 4.1. *Let Ψ, Γ, Ξ, Θ be as above, and let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $2 \le q < +\infty$ and $T_0 > 0$. Then, for each $w_0(x) \in L^2(\Omega, \mathbb{R}^k)$, there exists $u(x,t) \in L^q\big(0,T_0; W_0^{1,q}(\Omega,\mathbb{R}^k)\big)$ such that $u(x,t) \in L^\infty\big(0,T_0; L^2(\Omega,\mathbb{R}^k)\big) \cap W^{1,q^*}\big(0,T_0; W^{-1,q^*}(\Omega,\mathbb{R}^k)\big)$, where $q^* := q/(q-1)$, $u(x,t)$ is $L^2(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $u(x,0) = w_0(x)$ and $u(x,t)$ is a solution of*

$$\frac{du}{dt}(x,t) = \Theta\big(u(x,t),x,t\big) + div_x\Big(\Xi\big(u(x,t),x,t\big)\Big) + \\ div_x\Big(\Gamma\big(\nabla_x u(x,t),x,t\big)\Big) + div_x\Big(D_A\Psi\big(\nabla_x u(x,t),x,t\big)\Big) \quad \text{in } \Omega\times(0,T_0), \tag{4.6}$$

where

$$D_A\Psi(A,x,t) := \left\{\frac{\partial\Psi}{\partial A_{ij}}(A,x,t)\right\}_{1\le i\le k, 1\le j\le N} \in \mathbb{R}^{k\times N}.$$

Moreover, if $\Psi(A,x,t)$ is a uniformly convex function in the first argument A, then such a solution u is unique.

Proof. Let $X := W_0^{1,q}(\Omega,\mathbb{R}^k)$ (a separable reflexive Banach space), $H := L^2(\Omega,\mathbb{R}^k)$ (a Hilbert space) and $T \in \mathcal{L}(X;H)$ be the usual embedding operator from $W_0^{1,q}(\Omega,\mathbb{R}^k)$ into $L^2(\Omega,\mathbb{R}^k)$. Then T is an injective inclusion with dense image. Furthermore, $X^* = W^{-1,q^*}(\Omega,\mathbb{R}^k)$ where $q^* = q/(q-1)$, and the corresponding operator $\widetilde{T} \in \mathcal{L}(H;X^*)$, defined as in (2.2), is the usual inclusion of $L^2(\Omega,\mathbb{R}^k)$ into $W^{-1,q^*}(\Omega,\mathbb{R}^k)$. Then $\{X,H,X^*\}$ is an evolution triple with the corresponding inclusion operators $T \in \mathcal{L}(X;H)$ and $\widetilde{T} \in \mathcal{L}(H;X^*)$, as defined in Definition 2.6. Moreover, by the theorem about the compact embedding in Sobolev spaces, it is well known that T is a compact operator.

Next, for each $t \in [0,T_0]$, define $\Phi_t(x) : X \to [0,+\infty)$ by

$$\Phi_t(u) := \int_\Omega \Psi\big(\nabla u(x),x,t\big)dx + \frac{k_\Omega}{2}\int_\Omega |u(x)|^2 dx \quad \forall u \in W^{1,q}(\Omega,\mathbb{R}^k) \equiv X,$$

where

$$k_\Omega := \begin{cases} 0 & \text{if } \Omega \text{ is bounded,} \\ 1 & \text{if } \Omega \text{ is unbounded.} \end{cases} \tag{4.7}$$

Then $\Phi_t(x)$ is Gâteaux differentiable at every $x \in X$, satisfies $\Phi_t(0) = 0$ and by (4.1) satisfies the growth condition

$$\frac{1}{C}\|x\|_X^q - C \le \Phi_t(x) \le C\,\|x\|_X^q + C$$

for all $x \in X$ and all $t \in [0, T]$. Furthermore, for each $t \in [0, T_0]$, define the mapping $\Lambda_t(x) : X \to X^*$ by

$$\Big\langle \delta, \Lambda_t(u) \Big\rangle_{X \times X^*} := \int_\Omega \Gamma\big(\nabla u(x), x, t\big) : \nabla\delta(x)\, dx$$

for all $u, \delta \in W^{1,q}(\Omega, \mathbb{R}^k) \equiv X$. Then $\Lambda_t(x) : X \to X^*$ is Gâteaux differentiable at every $x \in X$; and, by (4.4), its derivative $D\Lambda_t$ satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)} \le C\,\|x\|_X^{q-2} + C$$

for all $x \in X$ and all $t \in [0, T]$, for some $C > 0$. Moreover, by (4.3), Λ_t satisfies the monotonicity conditions

$$\Big\langle h, D\Lambda_t(x)\cdot h \Big\rangle_{X \times X^*} \ge 0$$

for all $x, h \in X$ and all $t \in [0, T_0]$. Finally, for each $t \in [0, T_0]$, define the mapping $F_t(w) : H \to X^*$ by

$$\Big\langle \delta, F_t(w) \Big\rangle_{X \times X^*} := \int_\Omega \Big\{ \Xi\big(w(x), x, t\big) : \nabla\delta(x) - \Big(k_\Omega w(x) + \Theta\big(w(x), x, t\big)\Big)\cdot \delta(x) \Big\} dx \tag{4.8}$$

for all $w \in L^2(\Omega, \mathbb{R}^k) \equiv H$ for all $\delta \in W^{1,q}(\Omega, \mathbb{R}^k) \equiv X$. Then $F_t(w)$ is Gâteaux differentiable at every $w \in H$; and, since Ξ and Θ are Lipschitz functions, the derivative DF_t of F_t satisfies the Lipschitz condition

$$\|DF_t(w)\|_{\mathcal{L}(H;X^*)} \le C \tag{4.9}$$

for all $w \in H$ and all $t \in [0, T_0]$, for some $C > 0$. Thus all the conditions of Theorem 3.2 are satisfied. Applying this theorem completes the proof. □

Remark 4.1. If, in the framework of Proposition 4.1, we suppose that $q = 2$ and that $D_A\Psi(A, x, t)$ and $\Gamma(A, x, t)$ are linear in their first argument A, but assume that Ω is unbounded, we obtain an existence result similar to Proposition 4.1 as a consequence of Theorem 3.4 with $Z = X$.

Indeed, in the case of unbounded Ω, let $V_j = L^2(\Omega \cap B_{R_j}(0), \mathbb{R}^k)$ for some sequence $R_j \to +\infty$ and define $L_j \in \mathcal{L}(H, V_j)$ by

$$L_j \cdot \big(h(x)\big) := h(x)\llcorner\big(\Omega \cap B_{R_j}(0)\big) \in L^2(\Omega \cap B_{R_j}(0), \mathbb{R}^k) = V_j$$

for all $h(x) \in L^2(\Omega, \mathbb{R}^k) = H$. Then, by standard embedding theorems on Sobolev spaces, the operator $L_j \circ T \in \mathcal{L}(X, V_j)$ is compact for every j. Moreover, if $\{h_n\} \subset H$ is a sequence such that $h_n \rightharpoonup h_0$ weakly in H and $L_j \cdot h_n \to L_j \cdot h_0$ strongly in V_j as $n \to +\infty$ for every j, then $h_n \to h_0$ strongly in $L^2_{loc}(\Omega, \mathbb{R}^k)$; and thus, by (4.8) and (4.9), we must have $F_t(h_n) \rightharpoonup F_t(h_0)$ weakly in X^*.

4.3 Parabolic systems in non-divergence form. Suppose that $\Psi(L, x, t) : \mathbb{R}^k_L \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}$ is a non-negative measurable function. Moreover, assume that $\Psi(L, x, t)$ is C^1 as a function of the first argument L when (x, t) are fixed, which satisfies $\Psi(0, x, t) = 0$ and is convex in the first argument L when (x, t) are fixed, i.e.,

$$\Psi\big(\alpha L_1 + (1-\alpha)L_2, x, t\big) \le \alpha\Psi(L_1, x, t) + (1-\alpha)\Psi(L_2, x, t)$$

for every $\alpha \in [0, 1]$, $L_1, L_2 \in \mathbb{R}^k$, $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Moreover, we assume that Ψ satisfies the growth condition

$$\frac{1}{C}|L|^q - C \le \Psi(L, x, t) \le C|L|^q + C \tag{4.10}$$

for all $L \in \mathbb{R}^k$ and all $x \in \mathbb{R}^N$ for all $t \in \mathbb{R}$, where $C > 0$ is some constant and $q \in [2, +\infty)$. Next let $\Gamma(L, x, t) : \mathbb{R}^k_L \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^k$ be a measurable function. Moreover, assume that $\Gamma(L, x, t)$ is C^1 as a function of the first argument L when (x, t) are fixed, which satisfies

$$\Gamma(0, x, t) \in L^{q^*}\big(\mathbb{R}; L^2(\mathbb{R}^N, \mathbb{R}^k)\big), \tag{4.11}$$

the monotonicity condition

$$\sum_{1 \le i,j \le k} h_i h_j \frac{\partial \Gamma_i}{\partial L_j}(L, x, t) \ge 0 \tag{4.12}$$

for all $h, L \in \mathbb{R}^k$, all $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$, and the growth condition

$$\left|\frac{\partial \Gamma}{\partial L_j}(L, x, t)\right| \le C\,|L|^{q-2} + C \tag{4.13}$$

for all $L \in \mathbb{R}^k$, all $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$ for all $j \in \{1, \dots, k\}$. Finally let $\Theta(A, L, x, t) : \mathbb{R}^{k \times N}_A \times \mathbb{R}^k_L \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^k$ be a measurable function. Moreover, assume that $\Theta(A, L, x, t)$ is C^1 as a function of the first two arguments A and L when (x, t) are fixed. We also assume that $\Theta(A, L, x, t)$ is globally Lipschitz in the first two arguments A and L and

$$\Theta(0, 0, x, t) \in L^{q^*}\big(\mathbb{R}; L^2(\mathbb{R}^N, \mathbb{R}^k)\big). \tag{4.14}$$

Proposition 4.2. *Let Ψ, Γ, Θ be as above, and let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $2 \le q < +\infty$ and $T_0 > 0$. Then, for every $w_0(x) \in W_0^{1,2}(\Omega, \mathbb{R}^k)$, there exists $u(x,t) \in L^q\big(0, T_0; W^{2,q}_{loc}(\Omega, \mathbb{R}^k)\big)$ such that $\Delta_x u(x,t) \in L^q\big(0, T_0; L^q(\Omega, \mathbb{R}^k)\big)$, $u(x,t) \in L^\infty\big(0, T_0; W_0^{1,2}(\Omega, \mathbb{R}^k)\big) \cap W^{1,q^*}\big(0, T_0; L^{q^*}(\Omega, \mathbb{R}^k)\big)$, where $q^* := q/(q-1)$, $u(x,t)$ is $W_0^{1,2}(\Omega, \mathbb{R}^k)$-weakly continuous on $[0, T_0]$, $u(x,0) = w_0(x)$ and $u(x,t)$ is a solution of*

$$\frac{du}{dt}(x,t) = \Theta\big(\nabla_x u(x,t), u(x,t), x, t\big) + \Gamma\big(\Delta_x u(x,t), x, t\big) + \nabla_L \Psi\big(\Delta_x u(x,t), x, t\big) \text{ in } \Omega \times (0, T_0), \tag{4.15}$$

where $\nabla_L \Psi(L, x, t)$ is the partial gradient in the first variable L. Moreover, if $\Psi(L, x, t)$ is uniformly convex in the first argument L, then such a solution u is unique.

Proof. Let

$$X := \Big\{u(x) \in W_0^{1,2}(\Omega, \mathbb{R}^k) : \Delta u(x) \in L^q(\Omega, \mathbb{R}^k)\Big\} \tag{4.16}$$

for $2 \le q < +\infty$, endowed with the norm

$$\|u\|_X := \|\Delta u\|_{L^q(\Omega, \mathbb{R}^k)} + \|\nabla u\|_{L^2(\Omega, \mathbb{R}^{k\times N})} \tag{4.17}$$

for all $u \in X \subset W_0^{1,2}(\Omega, \mathbb{R}^k)$. Then X is a separable reflexive Banach space. Next let $H := W_0^{1,2}(\Omega, \mathbb{R}^k)$, endowed with the standard scalar product

$$< \phi_1, \phi_2 >_{H\times H} = \int_\Omega \nabla\phi_1(x) : \nabla\phi_2(x)\, dx$$

(a Hilbert space), and $T \in \mathcal{L}(X; H)$ be the trivial embedding operator from $X \subset W_0^{1,2}(\Omega, \mathbb{R}^k)$ into $H = W_0^{1,2}(\Omega, \mathbb{R}^k)$. Then T is an injective inclusion with dense image. Moreover, T is a compact operator. In order to follow the definitions above, we identify the dual space H^* with H. So in our notation, $\big\{W_0^{1,2}(\Omega, \mathbb{R}^k)\big\}^* = W_0^{1,2}(\Omega, \mathbb{R}^k)$ (although, in the usual notation, $\big\{W_0^{1,2}(\Omega, \mathbb{R}^k)\big\}^*$ is identified with the isomorphic space $W^{-1,2}(\Omega, \mathbb{R}^k)$). Next define $S \in \mathcal{L}\big(L^{q*}(\Omega, \mathbb{R}^k), X^*\big)$ by the formula

$$\Big\langle \delta, S \cdot h \Big\rangle_{X\times X^*} = -\int_\Omega h(x) \cdot \Delta\delta(x)\, dx \tag{4.18}$$

for all $\delta \in X$ and all $h \in L^{q^*}(\Omega, \mathbb{R}^k)$. Then, since for every $\phi \in L^q(\Omega, \mathbb{R}^k)$ there exists unique $\delta_\phi \in X$ such that $\Delta\delta_\phi = \phi$, we deduce that S is an injective inclusion, i.e., $\ker S = 0$.

For the corresponding operator $\widetilde{T} \in \mathcal{L}(H; X^*)$, by (2.2) and (4.18), we must have

$$\begin{aligned}\langle u, \widetilde{T} \cdot w \rangle_{X \times X^*} := \langle T \cdot u, w \rangle_{H \times H} &= \int_\Omega \nabla u(x) : \nabla w(x)\, dx \\ &= -\int_\Omega w(x) \cdot \Delta u(x)\, dx = \Big\langle u, S \cdot (L \cdot w) \Big\rangle_{X \times X^*}\end{aligned} \tag{4.19}$$

for all $w \in H$ and $u \in X$, where L is the trivial inclusion of $W_0^{1,2}(\Omega, \mathbb{R}^k)$ into $L^{q^*}(\Omega, \mathbb{R}^k)$ $(q^* \leq 2)$. So

$$\widetilde{T} = S \circ L\,. \tag{4.20}$$

Then $\{X, H, X^*\}$ is an evolution triple with the corresponding inclusion operators $T \in \mathcal{L}(X; H)$ and $\widetilde{T} \in \mathcal{L}(H; X^*)$ as defined in Definition 2.6.

Next, for each $t \in [0, T_0]$, define $\Phi_t(x) : X \to [0, +\infty)$ by

$$\Phi_t(u) := \int_\Omega \Big(\Psi\big(\Delta u(x), x, t\big) + \frac{1}{2}\big|\nabla u(x)\big|^2 \Big) dx$$

for all $u \in X$. Then $\Phi_t(x)$ is Gâteaux differentiable at every $x \in X$, satisfies $\Phi_t(0) = 0$ and satisfies the growth condition

$$\frac{1}{C}\|x\|_X^q - C \leq \Phi_t(x) \leq C\,\|x\|_X^q + C$$

for all $x \in X$ and all $t \in [0, T_0]$ Furthermore, for each $t \in [0, T_0]$, define the mapping $\Lambda_t(x) : X \to X^*$ by

$$\Big\langle \delta, \Lambda_t(u) \Big\rangle_{X \times X^*} := \int_\Omega \Gamma\big(\Delta u(x), x, t\big) \cdot \Delta\delta(x)\, dx$$

for all $u, \delta \in X$, i.e.,

$$\Lambda_t(u) = -S \cdot \Big(\Gamma\big(\Delta u(x), x, t\big)\Big) \tag{4.21}$$

for all $u \in X$. Then $\Lambda_t(x) : X \to X^*$ is Gâteaux differentiable at every $x \in X$; and, by (4.4), its derivative $D\Lambda_t$ satisfies the growth condition

$$\|D\Lambda_t(x)\|_{\mathcal{L}(X;X^*)} \leq C\,\|x\|_X^{q-2} + C$$

for all $x \in X$ and all $t \in [0, T_0]$, for some $C > 0$. Moreover, by (4.3), Λ_t satisfies the monotonicity condition

$$\Big\langle h, D\Lambda_t(x) \cdot h \Big\rangle_{X \times X^*} \geq 0$$

for all $x, h \in X$ and all $t \in [0, T_0]$. Finally, for each $t \in [0, T_0]$, define the mapping $F_t(w) : H \to X^*$ by

$$\Big\langle \delta, F_t(w) \Big\rangle_{X \times X^*} := \int_\Omega \Big(\Theta\big(\nabla w(x), w(x), x, t\big) + w(x)\Big) \cdot \Delta\delta(x) dx$$

for all $w \in W_0^{1,2}(\Omega, \mathbb{R}^k) \equiv H$ and all $\delta \in X$, i.e.,

$$F_t(w) = -S \cdot \Big(\Theta\big(\nabla w(x), w(x), x, t\big) + w(x)\Big) \tag{4.22}$$

for all $w \in H$. Then $F_t(w)$ is Gâteaux differentiable at every $w \in H$; and, since Θ is a Lipschitz function, the derivative DF_t of F_t satisfies a Lipschitz condition

$$\|DF_t(w)\|_{\mathcal{L}(H;X^*)} \le C \tag{4.23}$$

for all $w \in H$ and all $t \in [0, T_0]$.

Thus all the conditions of Theorem 3.2 are satisfied. Applying this theorem, together with (4.18), we obtain that for each $w_0(x) \in W_0^{1,2}(\Omega, \mathbb{R}^k)$, there exists $u(x,t) \in L^q\big(0, T_0; W_{loc}^{2,q}(\Omega, \mathbb{R}^k)\big)$ such that $u(x,t) \in L^\infty\big(0, T_0; W_0^{1,2}(\Omega, \mathbb{R}^k)\big)$, where $q^* := q/(q-1)$, $u(x,t)$ is $W_0^{1,2}(\Omega, \mathbb{R}^k)$-weakly continuous on $[0, T_0]$, $u(x,0) = w_0(x)$ and $u(x,t)$ is a solution of

$$\frac{dv}{dt}(t) + \Lambda_t\big(u(t)\big) + F_t\big(u(t)\big) + D\Phi_t\big(u(t)\big) = 0 \quad \text{for a.e. } t \in (0, T_0)\,. \tag{4.24}$$

Thus, by (4.24), (4.18), (4.20), (4.21), (4.22) and [11, Lemma 2.2], we infer that $u(x,t) \in W^{1,q^*}\big(0, T_0; L^{q^*}(\Omega, \mathbb{R}^k)\big)$ and

$$\int_\Omega \bigg\{ -\frac{du}{dt}(x,t) + \Theta\big(\nabla_x u(x,t), u(x,t), x, t\big) + \Gamma\big(\Delta_x u(x,t), x, t\big) + \nabla_L \Psi\big(\Delta_x u(x,t), x, t\big) \bigg\} \cdot \Delta\delta(x)\, dx = 0 \tag{4.25}$$

for all $t \in (0, T_0)$ for all $\delta \in X$. Therefore,

$$\frac{du}{dt}(x,t) = \Theta\big(\nabla_x u(x,t), u(x,t), x, t\big) + \Gamma\big(\Delta_x u(x,t), x, t\big) + \nabla_L \Psi\big(\Delta_x u(x,t), x, t\big) \tag{4.26}$$

for all $(x,t) \in \Omega \times (0, T_0)$, and the result follows. □

4.4 Hyperbolic systems of second order.

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $T_0 > 0$. Furthermore, let $\Xi(L,x,t) : \mathbb{R}^k_L \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^{k \times N}$, $\Upsilon(L,x,t) : \mathbb{R}^k_L \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^k$ and*

$\Theta(L,x,t):\mathbb{R}^k_L\times\mathbb{R}^N_x\times\mathbb{R}_t\to\mathbb{R}^k$ be measurable functions. Moreover, assume that $\Xi(L,x,t)$, $\Upsilon(L,x,t)$ and $\Theta(L,x,t)$ are C^1 as a functions of the first argument L when (x,t) are fixed. Also assume that $\Upsilon(L,x,t)$ $\nabla_x\Upsilon(L,x,t)$, $\Theta(L,x,t)$, $\Xi(L,x,t)$ and $\nabla_x\Xi(L,x,t)$ are globally Lipschitz in the first argument L, $\Upsilon(L,x,t)$ is globally Lipschitz in the last argument t, and that $\Theta(0,x,t)\in L^2\big(\mathbb{R};L^2(\mathbb{R}^N,\mathbb{R}^k)\big)$, $\Xi(0,x,t)\in L^2\big(\mathbb{R};W^{1,2}(\mathbb{R}^N,\mathbb{R}^{k\times N})\big)$ and that $\Upsilon(0,x,t)\in L^2\big(\mathbb{R};W^{1,2}_0(\Omega,\mathbb{R}^k)\big)$. Then, for every $w_0(x)\in W^{1,2}_0(\Omega,\mathbb{R}^k)$ and $h_0(x)\in L^2(\Omega,\mathbb{R}^k)$, there exists $u(x,t)\in L^\infty\big(0,T_0;W^{1,2}_0(\Omega,\mathbb{R}^k)\big)$ such that $\frac{du}{dt}(x,t)\in L^\infty\big(0,T_0;L^2(\Omega,\mathbb{R}^k)\big)\cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big)$, $u(x,t)$ is $W^{1,2}_0(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $\frac{du}{dt}(x,t)$ is $L^2(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $u(x,0)=w_0(x)$, $\frac{du}{dt}(x,0)=h_0(x)$ and $u(x,t)$ is a solution of

$$\frac{d^2u}{dt^2}(x,t)-\Delta_x u(x,t)+\partial_t\big\{\Upsilon\big(u(x,t),x,t\big)\big\}+div_x\big\{\Xi\big(u(x,t),x,t\big)\big\}\\+\Theta\big(u(x,t),x,t\big)=0\ in\ \Omega\times(0,T_0)\,. \tag{4.27}$$

Proof. Let $X_0:=\big\{\varphi\in W^{1,2}_0(\Omega,\mathbb{R}^k)\cap W^{2,2}_{loc}(\Omega,\mathbb{R}^k):\ \Delta\varphi\in L^2(\Omega,\mathbb{R}^k)\big\}$ endowed with the norm

$$\|\varphi\|_{X_0}:=\big(\|\Delta\varphi\|^2_{L^2(\Omega,\mathbb{R}^k)}+\|\nabla\varphi\|^2_{L^2(\Omega,\mathbb{R}^{k\times N})}+\|\varphi\|^2_{L^2(\Omega,\mathbb{R}^k)}\big)^{1/2} \tag{4.28}$$

for all $\varphi\in X_0\subset W^{2,2}_{loc}(\Omega,\mathbb{R}^k)\cap W^{1,2}_0(\Omega,\mathbb{R}^k)$. Then X_0 is a separable reflexive Banach space. Next endow $H_0:=W^{1,2}_0(\Omega,\mathbb{R}^k)$ with the standard scalar product

$$\langle\phi_1,\phi_2\rangle_{H\times H}=\int_\Omega\big(\nabla\phi_1(x):\nabla\phi_2(x)+\phi_1(x)\cdot\phi_2(x)\big)\,dx$$

(a Hilbert space) and let $\mathcal{T}_0\in\mathcal{L}(X_0;H_0)$ be the trivial embedding operator from $X_0\subset W^{1,2}_0(\Omega,\mathbb{R}^k)$ into $H_0=W^{1,2}_0(\Omega,\mathbb{R}^k)$. Then $\mathcal{T}_0$ is an injective inclusion with dense image. As before, in our notation, $\big\{W^{1,2}_0(\Omega,\mathbb{R}^k)\big\}^*=W^{1,2}_0(\Omega,\mathbb{R}^k)$ (although, in the usual notation, $\big\{W^{1,2}_0(\Omega,\mathbb{R}^k)\big\}^*$ identified with the isomorphic space $W^{-1,2}(\Omega,\mathbb{R}^k)$). Next, define $S_0\in\mathcal{L}\big(L^2(\Omega,\mathbb{R}^k),X_0^*\big)$ by

$$\langle\delta,S_0\cdot h\rangle_{X_0\times X_0^*}=\int_\Omega\Big(\delta(x)-\Delta\delta(x)\Big)\cdot h(x)\,dx \tag{4.29}$$

for all $\delta\in X_0$ and all $h\in L^2(\Omega,\mathbb{R}^k)$. Then, since for every $\phi\in L^2(\Omega,\mathbb{R}^k)$ there exists unique $\delta_\phi\in X_0$ such that $(\Delta\delta_\phi-\delta_\phi)=\phi$, we deduce that S_0 is an injective inclusion (i.e., $\ker S_0=0$). As before, $\{X_0,H_0,X_0^*\}$ is an evolution triple with the corresponding inclusion operators $\mathcal{T}_0\in\mathcal{L}(X_0;H_0)$ and $\widetilde{\mathcal{T}}_0\in\mathcal{L}(H_0;X_0^*)$ as defined in Definition 2.6 by

$$\langle\delta,\widetilde{\mathcal{T}}_0\cdot\varphi\rangle_{X_0\times X_0^*}:=\langle\mathcal{T}_0\cdot\delta,\varphi\rangle_{H_0\times H_0} \tag{4.30}$$

for all $\varphi \in H_0$ and $\delta \in X_0$. However,

$$\begin{aligned}\langle \mathfrak{T}_0 \cdot \delta, \varphi \rangle_{H_0 \times H_0} &= \int_\Omega \Big(\nabla \delta(x) : \nabla \varphi(x) + \delta(x) \cdot \varphi(x) \Big) dx \\ &= \int_\Omega \Big(\delta(x) - \Delta \delta(x) \Big) \cdot \varphi(x) dx = \langle \delta, (S_0 \circ L) \cdot \varphi \rangle_{X_0 \times X_0^*}\end{aligned} \tag{4.31}$$

for all $\varphi \in H_0$ and $\delta \in X_0$, where $L \in \mathcal{L}\big(W_0^{1,2}(\Omega, \mathbb{R}^k), L^2(\Omega, \mathbb{R}^k)\big)$ is a trivial inclusion of $W_0^{1,2}(\Omega, \mathbb{R}^k)$ into $L^2(\Omega, \mathbb{R}^k)$. Thus plugging (4.31) into (4.30), we obtain

$$\widetilde{\mathfrak{T}}_0 \cdot \varphi = S_0 \cdot (L \cdot \varphi) \tag{4.32}$$

for all $\varphi \in H_0$.

Next, as in the proof of Proposition 4.1, let $X_1 := W_0^{1,2}(\Omega, \mathbb{R}^k)$, $H_1 := L^2(\Omega, \mathbb{R}^k)$ and $T_1 \in \mathcal{L}(X_1; H_1)$ be the usual embedding operator from $W_0^{1,2}(\Omega, \mathbb{R}^k)$ into $L^2(\Omega, \mathbb{R}^k)$. Then T_1 is an injective inclusion with dense image. Furthermore, $X_1^* = W^{-1,2}(\Omega, \mathbb{R}^k)$, and the corresponding operator $\widetilde{T}_1 \in \mathcal{L}(H_1; X_1^*)$, defined as in (2.2), is the usual inclusion of $L^2(\Omega, \mathbb{R}^k)$ into $W^{-1,2}(\Omega, \mathbb{R}^k)$. Thus $\{X_1, H_1, X_1^*\}$ is another evolution triple with the corresponding inclusion operators $T_1 \in \mathcal{L}(X_1; H_1)$ and $\widetilde{T}_1 \in \mathcal{L}(H_1; X_1^*)$, as defined in Definition 2.6. Finally set

$$\begin{aligned}X := \Big\{ \big(u(x), v(x)\big) :\ & u(x) : \Omega \to \mathbb{R}^k,\ v(x) : \Omega \to \mathbb{R}^k \\ & u(x) \in X_0 \subset W^{2,2}_{loc}(\Omega, \mathbb{R}^k) \cap W_0^{1,2}(\Omega, \mathbb{R}^k),\ v(x) \in X_1 \equiv W_0^{1,2}(\Omega, \mathbb{R}^k) \Big\}.\end{aligned} \tag{4.33}$$

On X, we consider the norm

$$\begin{aligned}\|z\|_{X*} &:= \big(\|u\|^2_{X_0} + \|v\|^2_{X_1} \big)^{1/2} \\ &= \big(\|\Delta u\|^2_{L^2(\Omega, \mathbb{R}^k)} + \|u\|^2_{W_0^{1,2}(\Omega, \mathbb{R}^k)} + \|v\|^2_{W_0^{1,2}(\Omega, \mathbb{R}^k)} \big)^{1/2}\end{aligned} \tag{4.34}$$

for all $z = (u, v) \in X$. Thus X is a separable reflexive Banach space. Next set

$$\begin{aligned}H := \Big\{ \big(u(x), v(x)\big) :\ & u(x) : \Omega \to \mathbb{R}^k,\ v(x) : \Omega \to \mathbb{R}^k \\ & u(x) \in H_0 \equiv W_0^{1,2}(\Omega, \mathbb{R}^k),\ v(x) \in H_1 \equiv L^2(\Omega, \mathbb{R}^k) \Big\}.\end{aligned} \tag{4.35}$$

On H, we consider the scalar product

$$\begin{aligned}\langle z_1, z_2 \rangle_{H \times H} &:= \langle u_1, u_2 \rangle_{H_0 \times H_0} + < v_1, v_2 >_{H_1 \times H_1} \\ &= \int_\Omega \big\{ \nabla u_1(x) : \nabla u_2(x) + u_1(x) \cdot u_2(x) + v_1(x) \cdot v_2(x) \big\} dx\end{aligned} \tag{4.36}$$

for all $z_1 = (u_1, v_1), z_2 = (u_2, v_2) \in H$. Then H is a Hilbert space. Furthermore, consider $T \in \mathcal{L}(X, H)$ defined by

$$T \cdot z = \big(\mathfrak{T}_0 \cdot u, T_1 \cdot v\big) \tag{4.37}$$

for all $z = (u, v) \in X$. Thus T is an injective inclusion with dense image. Furthermore,

$$X^* := \Big\{\big(u, v\big) : u \in X_0^*,\ v \in X_1^* \equiv W^{-1,2}(\Omega, \mathbb{R}^k)\Big\}, \tag{4.38}$$

where

$$\langle \delta, h\rangle_{X\times X^*} = \langle \delta_0, h_0\rangle_{X_0\times X_0^*} + \langle \delta_1, h_1\rangle_{X_1\times X_1^*} \tag{4.39}$$

for all $\delta = (\delta_0, \delta_1) \in X$ and all $h = (h_0, h_1) \in X^*$, and

$$\|z\|_{X^*} := \Big(\|u\|^2_{X_0^*} + \|v\|^2_{X_1^*}\Big)^{1/2} \tag{4.40}$$

for all $z = (u, v) \in X^*$. Moreover, the corresponding operator $\widetilde{T} \in \mathcal{L}(H; X^*)$, defined as in (2.2), is defined by

$$\widetilde{T} \cdot z = \big(\widetilde{\mathfrak{T}_0} \cdot u, \widetilde{T}_1 \cdot v\big) \tag{4.41}$$

for all $z = (u, v) \in H$. Thus $\{X, H, X^*\}$ is an evolution triple with the corresponding inclusion operators $T \in \mathcal{L}(X; H)$ and $\widetilde{T} \in \mathcal{L}(H; X^*)$ as defined in Definition 2.6.

Next let $\Lambda \in \mathcal{L}(H, X^*)$ be defined by

$$\Lambda \cdot z := (S_0 \cdot v, \Delta u - u) \tag{4.42}$$

for all $z = (u, v) \in H$, i.e., $u \in W_0^{1,2}(\Omega, \mathbb{R}^k)$, $v \in L^2(\Omega, \mathbb{R}^k)$. Then, using (4.39) and (4.29), we deduce

$$\begin{aligned}\langle h, \Lambda \cdot (T \cdot h)\rangle_{X\times X^*} &= \langle u, S_0 \cdot (T_1 \cdot v)\rangle_{X_0\times X_0^*} + \langle v, \Delta(\mathfrak{T}_0 \cdot u) - \mathfrak{T}_0 \cdot u\rangle_{X_1\times X_1^*}\\ &= \int_\Omega v(x) \cdot \big(u(x) - \Delta u(x)\big) dx - \int_\Omega \big(\nabla v(x) : \nabla u(x) + v(x) \cdot u(x)\big)\, dx\\ &= 0\end{aligned} \tag{4.43}$$

for all $h = (u, v) \in X$.

Furthermore, for $t \in [0, T_0]$, define the function $F_t(z) : H \to H$ by

$$F_t(z) := \Big(\Upsilon\big(u(x), x, t\big), u(x) - \Theta\big(u(x), x, t\big) - div_x \Xi\big(u(x), x, t\big)\Big) \tag{4.44}$$

for all $z=(u,v)\in H$, (we have $\Upsilon(u(x),x,t)\in W_0^{1,2}(\Omega,\mathbb{R}^k)$ for a.e. t), i.e.,

$$\langle F_t(z),z_0\rangle_{H\times H}=\int_\Omega\Big(\nabla_x\big\{\Upsilon\big(u(x),x,t\big)\big\}:\nabla u_0(x)+\Upsilon\big(u(x),x,t\big)\cdot u_0(x)\Big)dx \\ +\int_\Omega\Big\{u(x)-\Theta\big(u(x),x,t\big)-div_x\,\Xi\big(u(x),x,t\big)\Big\}\cdot v_0(x)dx \tag{4.45}$$

for all $z=(u,v)\in H$ and all $z_0=(u_0,v_0)\in H$. Then F_t satisfies the conditions

$$\|F_t(z)\|_H\leq C\|z\|_H+f(t) \tag{4.46}$$

for all $z\in H$ and all $t\in[0,T_0]$, and

$$\|\widetilde{T}\circ DF_t(z)\|_{\mathcal{L}(H;X^*)}\leq C \tag{4.47}$$

for all $z\in H$ and all $t\in[0,T_0]$, for some $C>0$ and some $f(t)\in L^2(0,T_0;\mathbb{R})$. Moreover, for bounded Ω, since the embedding of $W_0^{1,2}(\Omega,\mathbb{R}^k)$ into $L^2(\Omega,\mathbb{R}^k)$ is compact, we obtain that F_t is weak to weak continuous on H. If we assume Ω to be unbounded then, for every $\Omega'\subset\subset\Omega$, F_t is weak to weak continuous, as a mapping defined on H with the valued functions, restricted to the smaller set Ω'. Therefore, since Ω' is arbitrary, using (4.46), we deduce that in all cases, F_t is weak to weak continuous on H. Then all the conditions of Corollary 3.2 satisfied; and by that corollary, for every $w_0\in W_0^{1,2}(\Omega,\mathbb{R}^k)$ and every $h_0\in L^2(\Omega,\mathbb{R}^k)$, there exists $\zeta(t)\in L^\infty(0,T_0;H)$ such that $\xi(t):=\widetilde{T}\cdot\big(\zeta(t)\big)\in W^{1,2}(0,T_0;X^*)$ and $\zeta(t)$ satisfies the equation

$$\begin{cases}\frac{d\xi}{dt}(t)+\Lambda\cdot\big(\zeta(t)\big)+\widetilde{T}\cdot F_t\big(\zeta(t)\big)=0 & \text{for a.e. } t\in(0,T_0),\\ \zeta(0)=\Big(w_0(x),\ -h_0(x)-\Upsilon\big(w_0(x),x,0\big)\Big),\end{cases} \tag{4.48}$$

where we assume that $\zeta(t)$ is H-weakly continuous on $[0,T_0]$, as stated in Lemma 2.2. We can rewrite (4.48) as follows. Let $\big(u(x,t),v(x,t)\big)=\zeta(t)$. Then, by (4.48), (4.37), (4.42), (4.45), (4.32) and [11, Lemma 2.2], we have

$$u(x,t)\in L^\infty\big(0,T_0;W_0^{1,2}(\Omega,\mathbb{R}^k)\big)\cap W^{1,2}\big(0,T_0;L^2(\Omega,\mathbb{R}^k)\big),$$
$$v(x,t)\in L^\infty\big(0,T_0;L^2(\Omega,\mathbb{R}^k)\big)\cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big),$$

$u(x,t)$ is $W_0^{1,2}(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $v(x,t)$ is $L^2(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $u(x,0)=w_0(x)$, $v(x,0)=-h_0(x)-\Upsilon\big(w_0(x),x,0\big)$ and in $\Omega\times(0,T_0)$ $\big(u(x,t),v(x,t)\big)$ solves

$$\begin{cases}\frac{du}{dt}(x,t)+v(x,t)+\Upsilon\big(u(x,t),x,t\big)=0\,,\\ \frac{dv}{dt}(x,t)+\Delta_x u(x,t)-\Theta\big(u(x,t),x,t\big)-div_x\,\Xi\big(u(x,t),x,t\big)=0\,.\end{cases} \tag{4.49}$$

In particular, $\frac{du}{dt}(x,t) \in L^\infty\big(0,T_0;L^2(\Omega,\mathbb{R}^k)\big) \cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big)$ and $\frac{du}{dt}(x,0) = h_0(x)$. Moreover, differentiating the equality

$$v(x,t) = -\frac{du}{dt}(x,t) - \Upsilon\big(u(x,t),x,t\big)$$

in the argument t and inserting the result into the second equation in (4.49), we finally deduce (4.27). □

4.5 Schrödinger type nonlinear systems.

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $T_0 > 0$. Furthermore, let $\Theta(a,b,x,t) : \mathbb{R}^k_a \times \mathbb{R}^k_b \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^k$ and $\Xi(a,b,x,t) : \mathbb{R}^k_a \times \mathbb{R}^k_b \times \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}^k$ be measurable functions. Moreover, assume that $\Theta(a,b,x,t)$ and $\Xi(a,b,x,t)$ are C^1 as a functions of the first two arguments a and b when (x,t) is fixed. Also assume that $\Theta(a,b,x,t)$, $\nabla_x\Theta(a,b,x,t)$, $\Xi(a,b,x,t)$ and $\nabla_x\Xi(a,b,x,t)$ are globally Lipschitz in the first two arguments a and b, and*

$$\Theta(0,0,x,t) \in L^2\big(\mathbb{R};W^{1,2}_0(\Omega,\mathbb{R}^k)\big) \text{ and } \Xi(0,0,x,t) \in L^2\big(\mathbb{R};W^{1,2}_0(\Omega,\mathbb{R}^k)\big).$$

Then, for each $w_0(x) \in W^{1,2}_0(\Omega,\mathbb{R}^k)$ and $h_0(x) \in W^{1,2}_0(\Omega,\mathbb{R}^k)$, there exists

$$u(x,t) \in L^\infty\big(0,T_0;W^{1,2}_0(\Omega,\mathbb{R}^k)\big) \cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big) \text{ and}$$
$$v(x,t) \in L^\infty\big(0,T_0;W^{1,2}_0(\Omega,\mathbb{R}^k)\big) \cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big)$$

such that $u(x,t)$ and $v(x,t)$ are $W^{1,2}_0(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $u(x,0) = w_0(x)$, $v(x,0) = h_0(x)$ and $\big(u(x,t),v(x,t)\big)$ is a solution of

$$\begin{cases} \frac{du}{dt}(x,t) - \Delta_x v(x,t) + \Theta\big(u(x,t),v(x,t),x,t\big) = 0 \text{ in } \Omega\times(0,T_0), \\ \frac{dv}{dt}(x,t) + \Delta_x u(x,t) + \Xi\big(u(x,t),v(x,t),x,t\big) = 0 \text{ in } \Omega\times(0,T_0). \end{cases} \tag{4.50}$$

Proof. Let $X_0 := \big\{\varphi \in W^{1,2}_0(\Omega,\mathbb{R}^k) \cap W^{3,2}_{loc}(\Omega,\mathbb{R}^k) : \Delta\varphi \in W^{1,2}_0(\Omega,\mathbb{R}^k)\big\}$, and endow X_0 with the norm

$$\|\varphi\|_{X_0} := \Big(\|\nabla\Delta\varphi\|^2_{L^2(\Omega,\mathbb{R}^{k\times N})} + \|\Delta\varphi\|^2_{L^2(\Omega,\mathbb{R}^k)} + \|\nabla\varphi\|^2_{L^2(\Omega,\mathbb{R}^{k\times N})} + \|\varphi\|^2_{L^2(\Omega,\mathbb{R}^k)}\Big)^{1/2} \tag{4.51}$$

for all $\varphi \in X_0 \subset W^{1,2}_0(\Omega,\mathbb{R}^k) \cap W^{3,2}_{loc}(\Omega,\mathbb{R}^k)$. So X_0 is a separable reflexive Banach space (in fact, a Hilbert space). Next let $H_0 := W^{1,2}_0(\Omega,\mathbb{R}^k)$ be endowed with the standard scalar product

$$\langle\phi_1,\phi_2\rangle_{H\times H} = \int_\Omega \big(\nabla\phi_1(x) : \nabla\phi_2(x) + \phi_1(x)\cdot\phi_2(x)\big)dx$$

(a Hilbert space) and $\mathcal{T}_0 \in \mathcal{L}(X_0; H_0)$ be the trivial embedding operator from $X_0 \subset W_0^{1,2}(\Omega, \mathbb{R}^k)$ into $H_0 = W_0^{1,2}(\Omega, \mathbb{R}^k)$. Then $\mathcal{T}_0$ is an injective inclusion with dense image. As before, in out notation, $\big\{W_0^{1,2}(\Omega, \mathbb{R}^k)\big\}^* = W_0^{1,2}(\Omega, \mathbb{R}^k)$.

Next, clearly, for every $h \in W^{-1,2}(\Omega, \mathbb{R}^k)$, there exists unique $H_h \in W_0^{1,2}(\Omega, \mathbb{R}^k)$ such that $\Delta H_h - H_h = h$. Then define $S_0 \in \mathcal{L}\big(W^{-1,2}(\Omega, \mathbb{R}^k), X_0^*\big)$ by

$$\langle \delta, S_0 \cdot h \langle_{X_0 \times X_0^*} = \int_\Omega \Big\{ \Big((\nabla \Delta)\delta(x) - \nabla \delta(x) \Big) : \nabla H_h(x) + \Big((\Delta \delta(x) - \delta(x) \Big) \cdot H_h(x) \Big\} dx \tag{4.52}$$

for all $\delta \in X_0$ and all $h \in W^{-1,2}(\Omega, \mathbb{R}^k)$. Then, since for every $\phi \in W_0^{1,2}(\Omega, \mathbb{R}^k)$ there exists unique $\delta_\phi \in X_0$ such that $\Delta \delta_\phi - \delta_\phi = \phi$, we deduce that S_0 is injective inclusion (i.e., $\ker S_0 = 0$). As before, $\{X_0, H_0, X_0^*\}$ is an evolution triple with the corresponding inclusion operators $\mathcal{T}_0 \in \mathcal{L}(X_0; H_0)$ and $\widetilde{\mathcal{T}}_0 \in \mathcal{L}(H_0; X_0^*)$, as defined in Definition 2.6, by

$$\langle \delta, \widetilde{\mathcal{T}}_0 \cdot \varphi \rangle_{X_0 \times X_0^*} := \langle \mathcal{T}_0 \cdot \delta, \varphi \rangle_{H_0 \times H_0} \tag{4.53}$$

for all $\varphi \in H_0$ and $\delta \in X_0$. However,

$$\begin{aligned}
\langle \mathcal{T}_0 \cdot \delta, \varphi \rangle_{H_0 \times H_0} &= \int_\Omega \Big(\nabla \delta(x) : \nabla \varphi(x) + \delta(x) \cdot \varphi(x) \Big) dx \\
&= \int_\Omega \Big(\delta(x) - \Delta \delta(x) \Big) \cdot \varphi(x)\, dx \\
&= \int_\Omega \Big(\delta(x) - \Delta \delta(x) \Big) \cdot \Big(\Delta H_{L \cdot \varphi}(x) - H_{L \cdot \varphi}(x) \Big) dx \\
&= \int_\Omega \Big\{ \Big((\nabla \Delta)\delta(x) - \nabla \delta(x) \Big) : \nabla H_{L \cdot \varphi}(x) + \Big((\Delta \delta(x) - \delta(x) \Big) \cdot H_{L \cdot \varphi}(x) \Big\} dx \\
&= \langle \delta, (S_0 \circ L) \cdot \varphi \rangle_{X_0 \times X_0^*}
\end{aligned} \tag{4.54}$$

for every $\varphi \in H_0$ and $\delta \in X_0$, where $L \in \mathcal{L}\big(W_0^{1,2}(\Omega, \mathbb{R}^k), W^{-1,2}(\Omega, \mathbb{R}^k)\big)$ is the trivial inclusion of $W_0^{1,2}(\Omega, \mathbb{R}^k)$ in $W^{-1,2}(\Omega, \mathbb{R}^k)$. Thus, plugging (4.59) into (4.53), we obtain

$$\widetilde{\mathcal{T}}_0 \cdot \varphi = S_0 \cdot (L \cdot \varphi) \tag{4.55}$$

for every $\varphi \in H_0$. Next set

$$X := \Big\{ \big(u(x), v(x)\big) : \ u(x) : \Omega \to \mathbb{R}^k, \ v(x) : \Omega \to \mathbb{R}^k, \ u(x) \in X_0, \ v(x) \in X_0 \Big\}; \tag{4.56}$$

and on X, consider the norm

$$\|z\|_X := \big(\|u\|_{X_0}^2 + \|v\|_{X_0}^2 \big)^{1/2} \tag{4.57}$$

for all $z = (u, v) \in X$. Then X is a separable reflexive Banach space. Next set

(4.58) $$H := \Big\{ \big(u(x), v(x)\big) : \ u(x) : \Omega \to \mathbb{R}^k, \ v(x) : \Omega \to \mathbb{R}^k, \ u(x) \in H_0, \ v(x) \in H_0 \Big\};$$

and on H, consider the scalar product

(4.59) $$\langle z_1, z_2 \rangle_{H \times H} := \langle u_1, u_2 \rangle_{H_0 \times H_0} + \langle v_1, v_2 \rangle_{H_0 \times H_0}$$
$$= \int_\Omega \Big\{ \nabla u_1(x) : \nabla u_2(x) + u_1(x) \cdot u_2(x) + \nabla v_1(x) : \nabla v_2(x) + v_1(x) \cdot v_2(x) \Big\} dx$$

for all $z_1 = (u_1, v_1)$, $z_2 = (u_2, v_2) \in H$. Then H is a Hilbert space. Furthermore, consider the operator $T \in \mathcal{L}(X, H)$ defined by

(4.60) $$T \cdot z = \big(\mathcal{T}_0 \cdot u, \mathcal{T}_0 \cdot v \big)$$

for all $z = (u, v) \in X$. Then T is an injective inclusion with dense image. Furthermore,

(4.61) $$X^* := \Big\{ \big(u, v\big) : u \in X_0^*, \ v \in X_0^* \Big\},$$

where

(4.62) $$\langle \delta, h \rangle_{X \times X^*} = \langle \delta_0, h_0 \rangle_{X_0 \times X_0^*} + \langle \delta_1, h_1 \rangle_{X_0 \times X_0^*}$$

for all $\delta = (\delta_0, \delta_1) \in X$ and all $h = (h_0, h_1) \in X^*$, and

(4.63) $$\|z\|_{X^*} := \big(\|u\|^2_{X_0^*} + \|v\|^2_{X_0^*} \big)^{1/2}$$

for all $z = (u, v) \in X^*$. Moreover, the corresponding operator $\widetilde{T} \in \mathcal{L}(H; X^*)$, defined as in (2.2), is defined by

(4.64) $$\widetilde{T} \cdot z = \big(\widetilde{\mathcal{T}_0} \cdot u, \widetilde{\mathcal{T}_0} \cdot v \big) = \big(S_0 \cdot (L \cdot u), S_0 \cdot (L \cdot v) \big)$$

for all $z = (u, v) \in H$. Thus $\{X, H, X^*\}$ is an evolution triple with the corresponding inclusion operators $T \in \mathcal{L}(X; H)$ and $\widetilde{T} \in \mathcal{L}(H; X^*)$ as defined in Definition 2.6.

Next define $\Lambda \in \mathcal{L}(H, X^*)$ by

(4.65) $$\Lambda \cdot z := \Big(- S_0 \cdot (\Delta v - v), S_0 \cdot (\Delta u - u) \Big)$$

for all $z = (u, v) \in H$ (i.e., $(\Delta u - u) \in W^{-1,2}(\Omega, \mathbb{R}^k)$, $(\Delta v - v) \in W^{-1,2}(\Omega, \mathbb{R}^k)$), where S_0 is defined in (4.52). Then, using (4.62), we deduce

(4.66) $$\langle h, \Lambda \cdot (T \cdot h) \rangle_{X \times X^*} = -\langle u, S_0 \cdot (\Delta v - v) \rangle_{X_0 \times X_0^*} + \langle v, S_0 \cdot (\Delta u - u) \rangle_{X_0 \times X_0^*}$$
$$= - \int_\Omega \Big\{ \Big((\nabla \Delta u)(x) - \nabla u(x) \Big) : \nabla v(x) + \Big(\Delta u(x) - u(x) \Big) \cdot v(x) \Big\} dx$$
$$+ \int_\Omega \Big\{ \Big((\nabla \Delta v)(x) - \nabla v(x) \Big) : \nabla u(x) + \Big(\Delta v(x) - v(x) \Big) \cdot u(x) \Big\} dx = 0$$

for all $h = (u, v) \in X$. Furthermore, for each $t \in [0, T_0]$, define the function $F_t(z) : H \to H$ by

$$F_t(z) := \Big(\Theta\big(u(x,t), v(x,t), x, t\big) - v(x),\ \Xi\big(u(x,t), v(x,t), x, t\big) + u(x)\Big) \tag{4.67}$$

for all $z = (u, v) \in H$

we have $\Theta(u(x,t), v(x,t), x, t)$, $\Xi(u(x,t), v(x,t), x, t) \in W_0^{1,2}(\Omega, \mathbb{R}^k)$ for a.e. t),

i.e.,

$$\begin{aligned}\langle F_t(z), z_0\rangle_{H\times H} = \int_\Omega \Big\{ &\Big(\nabla_x\Big\{\Theta\big(u(x,t), v(x,t), x, t\big)\Big\} - \nabla v(x)\Big) : \nabla u_0(x) \\ &+ \Big(\Theta\big(u(x,t), v(x,t), x, t\big) - v(x)\Big)\cdot u_0(x) \\ &+ \Big(\nabla_x\Big\{\Xi\big(u(x,t), v(x,t), x, t\big)\Big\} + \nabla u(x)\Big) : \nabla v_0(x) \\ &+ \Big(\Xi\big(u(x,t), v(x,t), x, t\big) + u(x)\Big)\cdot v_0(x)\Big\} dx\end{aligned} \tag{4.68}$$

for all $z = (u, v) \in H$ for and all $z_0 = (u_0, v_0) \in H$. Then

$$\|F_t(z)\|_H \le C\|z\|_H + f(t) \tag{4.69}$$

for all $z \in H$ and all $t \in [0, T_0]$, for some constant $C > 0$ and some $f(t) \in L^2(0, T_0; \mathbb{R})$. Furthermore, F_t satisfies the Lipschitz condition

$$\big\|\widetilde{T} \circ DF_t(z)\big\|_{\mathcal{L}(H;X^*)} \le C \tag{4.70}$$

for all $z \in H$ and all $t \in [0, T_0]$. Moreover, since the embedding of $H = W_0^{1,2}(\Omega, \mathbb{R}^k)$ in $L^2_{loc}(\Omega, \mathbb{R}^k)$ is compact, we obtain that if $z_n \rightharpoonup z_0$ weakly in H, then $z_n \to z_0$ strongly in $L^2_{loc}(\Omega, \mathbb{R}^k)$. Thus, by (4.69), we obtain $F_t(z_n) \rightharpoonup F_t(z_0)$ weakly in H. So F_t is weak to weak continuous in H. Then all the conditions of Corollary 3.2 satisfied; and by that corollary, for every $w_0 \in W_0^{1,2}(\Omega, \mathbb{R}^k)$ and every $h_0 \in W_0^{1,2}(\Omega, \mathbb{R}^k)$, there exists $\zeta(t) = \big(u(x,t), v(x,t)\big) \in L^\infty(0, T_0; H)$ such that $\xi(t) := \widetilde{T}\cdot\big(\zeta(t)\big) \in W^{1,2}(0, T_0; X^*)$ and $\zeta(t)$ satisfy the equation

$$\begin{cases} \frac{d\xi}{dt}(t) + \Lambda\cdot\zeta(t) + \widetilde{T}\cdot F_t\big(\zeta(t)\big) = 0 \quad \text{for a.e. } t \in (0, T_0), \\ \zeta(0) = \big(w_0(x), h_0(x)\big), \end{cases} \tag{4.71}$$

where we assume that $\zeta(t)$ is H-weakly continuous on $[0, T_0]$, as stated in Lemma 2.2. We can rewrite (4.71) as follows. Let $\big(u(x,t), v(x,t)\big) = \zeta(t)$. Then $u(x,t) \in L^\infty\big(0, T_0; W_0^{1,2}(\Omega, \mathbb{R}^k)\big)$, $v(x,t) \in L^\infty\big(0, T_0; W_0^{1,2}(\Omega, \mathbb{R}^k)\big)$, $u(x,t)$ and

$v(x,t)$ are $W_0^{1,2}(\Omega,\mathbb{R}^k)$-weakly continuous on $[0,T_0]$, $u(x,0) = w_0(x)$, $v(x,0) = h_0(x)$; and by (4.55) and the definitions of Λ and F_t, we obtain

$$
\begin{aligned}
(4.72)\quad & -\left\langle \frac{\partial\delta}{\partial t}(x,t), S_0\cdot u(x,t)\right\rangle_{X_0\times X_0^*} \\
&+\left\langle \delta(x,t), S_0\cdot\Big(-\Delta_x v(x,t)+\Theta\big(u(x,t),v(x,t),x,t\big)\Big)\right\rangle_{X_0\times X_0^*} = 0
\end{aligned}
$$

for all $\delta(x,t)\in C_c^1\big((0,T_0;X_0\big)$, and

$$
\begin{aligned}
(4.73)\quad & -\left\langle \frac{\partial\delta}{\partial t}(x,t), S_0\cdot v(x,t)\right\rangle_{X_0\times X_0^*} \\
&+\left\langle \delta(x,t), S_0\cdot\Big(\Delta_x u(x,t)+\Xi\big(u(x,t),v(x,t),x,t\big)\Big)\right\rangle_{X_0\times X_0^*} = 0
\end{aligned}
$$

for all $\delta(x,t)\in C_c^1\big((0,T_0;X_0\big)$. Then, by [11, Lemma 2.2], we obtain

$$
\frac{du}{dt}(x,t)\in L^2\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big) \text{ and } \frac{dv}{dt}(x,t)\in L^2\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big),
$$

and thus

$$
\begin{aligned}
&u(x,t)\in L^\infty\big(0,T_0;W_0^{1,2}(\Omega,\mathbb{R}^k)\big)\cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big) \text{ and}\\
&v(x,t)\in L^\infty\big(0,T_0;W_0^{1,2}(\Omega,\mathbb{R}^k)\big)\cap W^{1,2}\big(0,T_0;W^{-1,2}(\Omega,\mathbb{R}^k)\big).
\end{aligned}
$$

Moreover, $\big(u(x,t),v(x,t)\big)$ solves (4.50). □

4.6 Incompressible Navier-Stokes equations and magneto-hydrodynamics. Let $\Omega\subset\mathbb{R}^N$ be a domain. The initial-boundary value problem for the incompressible Navier-Stokes equations is as follows:

$$
(4.74)\quad
\begin{cases}
(i)\ \ \frac{\partial v}{\partial t}+\operatorname{div}_x(v\otimes v)+\nabla_x p = \nu_h\Delta_x v+f \text{ for all } (x,t)\in\Omega\times(0,T_0),\\
(ii)\ \ \operatorname{div}_x v = 0 \text{ for all } (x,t)\in\Omega\times(0,T_0),\\
(iii)\ \ v(x,t)=\gamma(x,t) \text{ for all } (x,t)\in\partial\Omega\times(0,T_0),\\
(iv)\ \ v(x,0)=v_0(x) \text{ for all } x\in\Omega.
\end{cases}
$$

Here, $v = v(x,t) : \Omega \times (0,T_0) \to \mathbb{R}^N$ is an unknown velocity, $p = p(x,t) : \Omega \times (0,T_0) \to \mathbb{R}$ is an unknown pressure associated with v, $\nu_h > 0$ is a given constant hydrodynamical viscosity, $f : \Omega \times (0,T_0) \to \mathbb{R}^N$ is a given force field, $\gamma = \gamma(x,t)$ is a given velocity on the boundary (which can be nontrivial for fluid driven by its boundary) and $v_0 : \Omega \to \mathbb{R}^N$ is a given initial velocity.

The initial-boundary value problem for the incompressible magneto-hydrodynamics is as follows:

$$
(4.75)\quad \begin{cases}
(i)\ \frac{\partial v}{\partial t} + \operatorname{div}_x(v\otimes v) - \operatorname{div}_x(b\otimes b) + \nabla_x p = \nu_h \Delta_x v + f \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } (x,t)\in\Omega\times(0,T_0), \\
(ii)\ \frac{\partial b}{\partial t} + \operatorname{div}_x(b\otimes v) - \operatorname{div}_x(v\otimes b) = \nu_m \Delta_x b \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } (x,t)\in\Omega\times(0,T_0), \\
(iii)\ \operatorname{div}_x v = 0 \text{ for all } (x,t)\in\Omega\times(0,T_0), \\
(iv)\ \operatorname{div}_x b = 0 \text{ for all } (x,t)\in\Omega\times(0,T_0), \\
(v)\ v(x,t) = 0 \text{ for all } (x,t)\in\partial\Omega\times(0,T_0), \\
(vi)\ b\cdot\boldsymbol{n} = 0 \text{ for all } (x,t)\in\partial\Omega\times(0,T_0), \\
(vii)\ \sum_{j=1}^N \left(\frac{\partial b_i}{\partial x_j} - \frac{\partial b_j}{\partial x_i}\right)\boldsymbol{n}_j = 0 \text{ for all } (x,t)\in\partial\Omega\times(0,T_0) \\
\qquad\qquad\qquad\qquad\qquad \text{for all } i = 1,2,\dots N, \\
(viii)\ v(x,0) = v_0(x) \text{ for all } x\in\Omega, \\
(ix)\ b(x,0) = b_0(x) \text{ for all } x\in\Omega.
\end{cases}
$$

Here, $v = v(x,t) : \Omega \times (0,T_0) \to \mathbb{R}^N$ is an unknown velocity, $b = b(x,t) : \Omega \times (0,T_0) \to \mathbb{R}^N$ is an unknown magnetic field, $p = p(x,t) : \Omega \times (0,T_0) \to \mathbb{R}$ is an unknown total pressure (hydrodynamical+magnetic), $\nu_h > 0$ and $\nu_m > 0$ are given constant hydrodynamical and magnetic viscosities, $f : \Omega \times (0,T_0) \to \mathbb{R}^N$ is a given force field, $v_0 : \Omega \to \mathbb{R}^N$ is a given initial velocity, $b_0 : \Omega \to \mathbb{R}^N$ is a given initial magnetic field and $\boldsymbol{n}$ is a normal to $\partial\Omega$.

Next, for constant $\lambda \in \{0, 1\}$, consider the system

$$
(4.76)\quad \begin{cases}
\frac{\partial v}{\partial t} + \operatorname{div}_x(v \otimes v) - \lambda \operatorname{div}_x(b \otimes b) + \nabla_x p = \nu_h \Delta_x v + f \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } (x, t) \in \Omega \times (0, T_0), \\
\frac{\partial b}{\partial t} + \lambda \operatorname{div}_x(b \otimes v) - \lambda \operatorname{div}_x(v \otimes b) = \nu_m \Delta_x b \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } (x, t) \in \Omega \times (0, T_0), \\
\operatorname{div}_x v = 0 \text{ for all } (x, t) \in \Omega \times (0, T_0), \\
\operatorname{div}_x b = 0 \text{ for all } (x, t) \in \Omega \times (0, T_0), \\
v(x, t) = \gamma(x, t) \text{ for all } (x, t) \in \partial\Omega \times (0, T_0), \\
b \cdot \boldsymbol{n} = 0 \text{ for all } (x, t) \in \partial\Omega \times (0, T_0), \\
\sum_{j=1}^N \left(\frac{\partial b_i}{\partial x_j} - \frac{\partial b_j}{\partial x_i}\right)\boldsymbol{n}_j = (\lambda/\nu_m)\big(\gamma \cdot \boldsymbol{n}\big)b \text{ for all } (x, t) \in \partial\Omega \times (0, T_0) \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } i = 1, 2, \dots N, \\
v(x, 0) = v_0(x) \text{ for all } x \in \Omega, \\
b(x, 0) = b_0(x) \text{ for all } x \in \Omega.
\end{cases}
$$

For $\lambda = 1$ and $\gamma \equiv 0$, this system coincides with (4.75). On the other hand, if (v, b, p) is a solution of (4.76) with $\lambda = 0$, then (v, p) is a solution of (4.74).

If there exists a sufficiently regular function $r = r(x, t) : \Omega \times (0, T_0) \to \mathbb{R}^N$ such that $r(x, t) = \gamma(x, t)\ \forall (x, t) \in \partial\Omega \times (0, T_0)$ and $\operatorname{div}_x r \equiv 0$, then choose one and define the new unknown function $u(x, t) := v(x, t) - r(x, t)$ and its initial value $u_0(x) := v_0(x) - r(x, 0)$. Then we can rewrite (4.76) in the terms of (u, b, p) as

$$
(4.77)\quad \begin{cases}
\frac{\partial u}{\partial t} + \operatorname{div}_x \big(u \otimes u + r \otimes u + u \otimes r - \lambda b \otimes b\big) + \nabla_x p = \nu_h \Delta_x u + \hat{f} \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } (x, t) \in \Omega \times (0, T_0), \\
\frac{\partial b}{\partial t} + \lambda \operatorname{div}_x(b \otimes u - u \otimes b + b \otimes r - r \otimes b) = \nu_m \Delta_x b \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } (x, t) \in \Omega \times (0, T_0), \\
\operatorname{div}_x u = 0 \text{ for all } (x, t) \in \Omega \times (0, T_0), \\
\operatorname{div}_x b = 0 \text{ for all } (x, t) \in \Omega \times (0, T_0), \\
u = 0 \text{ for all } (x, t) \in \partial\Omega \times (0, T_0), \\
b \cdot \boldsymbol{n} = 0 \text{ for all } (x, t) \in \partial\Omega \times (0, T_0), \\
\sum_{j=1}^N \left(\frac{\partial b_i}{\partial x_j} - \frac{\partial b_j}{\partial x_i}\right)\boldsymbol{n}_j = (\lambda/\nu_m)\big(r \cdot \boldsymbol{n}\big)b \text{ for all } (x, t) \in \partial\Omega \times (0, T_0) \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } i = 1, 2, \dots N, \\
u(x, 0) = u_0(x) \text{ for all } x \in \Omega, \\
b(x, 0) = b_0(x) \text{ for all } x \in \Omega,
\end{cases}
$$

where $\hat{f} := f + \Delta_x r - \partial_t r - div_x(r \otimes r)$. We prove the existence of a solution of the system (4.77) for $\lambda = 0$ and $\lambda = 1$.

We need some preliminaries.

Definition 4.1. Let $\Omega \subset \mathbb{R}^N$ be an open set.

- We denote by $\mathcal{V}_N = \mathcal{V}_N(\Omega)$ the space $\{\varphi \in C_c^\infty(\Omega, \mathbb{R}^N) : div\, \varphi = 0\}$ and by $L_N = L_N(\Omega)$ the closure of $\mathcal{V}_N$ in $L^2(\Omega, \mathbb{R}^N)$. We endow L_N with the scalar product $\langle \varphi_1, \varphi_2 \rangle_{B_N} := \int_\Omega \varphi_1 \cdot \varphi_2\, dx$ and the norm $\|\varphi\| := \big(\int_\Omega |\varphi|^2 dx\big)^{1/2}$.
- We denote by $V_N = V_N(\Omega)$ the closure of $\mathcal{V}_N$ in $W_0^{1,2}(\Omega, \mathbb{R}^N)$ and endow V_N with the scalar product $\langle \varphi_1, \varphi_2 \rangle_{V_N} := \int_\Omega \big(\nabla \varphi_1 : \nabla \varphi_2 + \varphi_1 \cdot \varphi_2\big) dx$ and the norm $\|\varphi\| := \big(\int_\Omega |\nabla \varphi|^2 dx + \int_\Omega |\varphi|^2 dx\big)^{1/2}$.
- We let

$$C_c^\infty(\overline{\Omega}, \mathbb{R}^N) := \big\{\varphi : \Omega \to \mathbb{R}^N : \text{ there exists } \bar{\varphi} \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N) \text{ such that } \bar{\varphi}(x) = \varphi(x) \text{ for all } x \in \Omega\big\}.$$

Furthermore, given $\varphi \in \mathcal{D}'(\Omega, \mathbb{R}^N)$, let

$$rot_x \varphi := \left\{\frac{\partial \varphi_i}{\partial x_j} - \frac{\partial \varphi_j}{\partial x_i}\right\}_{1 \le i, j \le N} = \big(\nabla_x f\big) - \big(\nabla_x f\big)^T \in \mathcal{D}'\big(\Omega, \mathbb{R}^{N \times N}\big), \tag{4.78}$$

define the linear space

$$B'_N = B'_N(\Omega) := \Big\{\varphi \in L_N : \ rot_x \varphi \in L^2\big(\Omega, \mathbb{R}^{N \times N}\big)\Big\}, \tag{4.79}$$

and endow B'_N with the scalar product

$$\langle \varphi_1, \varphi_2 \rangle_{B'_N} := \int_\Omega \big(\varphi_1 \cdot \varphi_2 + (1/2) rot_x \varphi_1 \cdot rot_x \varphi_2\big)\, dx$$

and the corresponding norm $\|\varphi\|_{B'_N} := \big(\langle \varphi, \varphi \rangle_{B'_N}\big)^{1/2}$. Then B'_N is a Hilbert space. Moreover, clearly B'_N is continuously embedded in $W^{1,2}_{loc}(\Omega, \mathbb{R}^N) \cap L_N$. We also denote by $B_N = B_N(\Omega)$ the closure of $B'_N(\Omega) \cap C_c^\infty(\overline{\Omega}, \mathbb{R}^N)$ in $B'_N(\Omega)$ and endow B_N with the norm of $B'_N(\Omega)$. (Clearly, B_N is a subset of B'_N, and if the boundary of the domain Ω is sufficiently regular, then B_N and B'_N coincide.)

Proposition 4.5. *For each* $r \in L^2\big(0, T_0; W^{1,2}(\Omega, \mathbb{R}^N)\big) \cap L^\infty$, $f \in L^2\big(0, T_0; L^2(\Omega, \mathbb{R}^N)\big)$, $g \in L^2\big(0, T_0; L^2(\Omega, \mathbb{R}^{N \times N})\big)$, $\nu_h > 0$, $\nu_m > 0$, $\lambda \in \{0, 1\}$, $v_0(\cdot) \in L_N$ *and* $b_0(\cdot) \in L_N$, *there exist* $u(x,t) \in L^2(0, T_0; V_N) \cap L^\infty(0, T_0; L_N)$ *and* $b(x,t) \in L^2(0, T_0; B_N) \cap L^\infty(0, T_0; L_N)$ *such that* $u(\cdot, t)$ *and* $b(\cdot, t)$ *are* L_N*-weakly continuous in* t *on* $[0, T_0]$, $u(x, 0) = v_0(x)$, $b(x, 0) = b_0(x)$, *and* $u(x,t)$ *and* $b(x,t)$ *satisfy*

$$\int_0^{T_0}\int_\Omega \Big\{ \Big(u(x,t)\otimes u(x,t) + r(x,t)\otimes u(x,t) + u(x,t)\otimes r(x,t) - \lambda b(x,t)\otimes b(x,t) + g(x,t)\Big) : \nabla_x \psi(x,t) - f(x,t)\cdot\psi(x,t) + u(x,t)\cdot\partial_t\psi(x,t)\Big\}\,dxdt \tag{4.80}$$
$$= \int_0^{T_0}\int_\Omega \nu_h \nabla_x u(x,t) : \nabla_x\psi(x,t)\,dxdt - \int_\Omega \upsilon_0(x)\cdot\psi(x,0)\,dx$$

for every $\psi(x,t)\in C_c^1\big(\Omega\times[0,T_0),\mathbb{R}^N\big)\cap C^1\big([0,T_0];V_N\big)$ *and*

$$\int_0^{T_0}\int_\Omega \Big\{ \lambda\Big(b(x,t)\otimes u(x,t) - u(x,t)\otimes b(x,t) + b(x,t)\otimes r(x,t) - r(x,t)\otimes b(x,t)\Big) : \nabla_x\phi(x,t) + b(x,t)\cdot\partial_t\phi(x,t)\Big\}\,dxdt \tag{4.81}$$
$$= \int_0^{T_0}\int_\Omega \frac{\nu_m}{2} rot_x b(x,t) : rot_x\phi(x,t)\,dxdt - \int_\Omega b_0(x)\cdot\phi(x,0)\,dx,$$

for every $\phi(x,t)\in C_c^1\big(\mathbb{R}^N\times[0,T_0),\mathbb{R}^N\big)\cap C^1\big([0,T_0];B_N\big)$*; i.e.,*

$$\begin{cases} \frac{\partial u}{\partial t} + \mathrm{div}_x\big(u\otimes u + r\otimes u + u\otimes r - \lambda b\otimes b\big) + \nabla_x p \\ \qquad\qquad = \nu_h\Delta_x u - f - \mathrm{div}_x\, g \ \ \textit{for all } (x,t)\in\Omega\times(0,T_0), \\ \frac{\partial b}{\partial t} + \lambda\,\mathrm{div}_x(b\otimes u - u\otimes b + b\otimes r - r\otimes b) = \nu_m\Delta_x b \\ \qquad\qquad\qquad \textit{for all } (x,t)\in\Omega\times(0,T_0), \\ \mathrm{div}_x\, u = 0 \ \textit{for all } (x,t)\in\Omega\times(0,T_0), \\ \mathrm{div}_x\, b = 0 \ \textit{for all } (x,t)\in\Omega\times(0,T_0), \\ u = 0 \ \textit{for all } (x,t)\in\partial\Omega\times(0,T_0), \\ b\cdot\boldsymbol{n} = 0 \ \textit{for all } (x,t)\in\partial\Omega\times(0,T_0), \\ rot_x b\cdot\boldsymbol{n} = (\lambda/\nu_m)\big(r\cdot\boldsymbol{n}\big)b \ \textit{for all } (x,t)\in\partial\Omega\times(0,T_0), \\ u(x,0) = u_0(x) \ \textit{for all } x\in\Omega, \\ b(x,0) = b_0(x) \ \textit{for all } x\in\Omega. \end{cases} \tag{4.82}$$

Moreover, if either $\lambda = 0$ *and* Ω *is bounded or* $r(x,t)\equiv 0$*, then* $u(x,t)$ *and* $b(x,t)$ *satisfy the energy inequality*

$$\begin{aligned} &\frac12\int_\Omega \big|u(x,\tau)\big|^2 dx + \frac12\int_\Omega \big|b(x,\tau)\big|^2 dx + \int_0^\tau\int_\Omega \nu_h\big|\nabla_x u(x,t)\big|^2\,dxdt \\ &\quad + \int_0^\tau\int_\Omega \frac{\nu_m}{2}\big|rot_x b(x,t)\big|^2\,dxdt \le \frac12\int_\Omega\big|\upsilon_0(x)\big|^2 dx + \frac12\int_\Omega \big|b_0(x)\big|^2 dx \\ &\quad + \int_0^\tau\int_\Omega \Big(\Big\{g(x,t) + r(x,t)\otimes u(x,t) + u(x,t)\otimes r(x,t)\Big\} : \nabla_x u(x,t) \\ &\quad + \lambda\big\{b(x,t)\otimes r(x,t)\big\} : rot_x b(x,t) - f(x,t)\cdot u(x,t)\Big)\,dxdt \end{aligned} \tag{4.83}$$

for all $\tau \in [0, T_0]$.

Proof. Fix $\nu_h > 0$, $\nu_m > 0$, $\lambda \in \{0,1\}$, $f \in L^2\big(0, T_0; L^2(\Omega, \mathbb{R}^N)\big)$ $g \in L^2\big(0, T_0; L^2(\Omega, \mathbb{R}^{N\times N})\big)$, $r \in L^2\big(0, T_0; W^{1,2}(\Omega, \mathbb{R}^N)\big) \cap L^\infty$, $v_0(\cdot) \in L_N$ and $b_0(\cdot) \in L_N$. Next define the space U'_N as a closure of $\mathcal{V}_N$ with respect to the norm

$$\|\varphi\|_{U'_N} := \|\varphi\|_{V_N} + \sup_{x\in\Omega} |\varphi(x)| + \sup_{x\in\Omega} |\nabla\varphi(x)| \tag{4.84}$$

and the space D'_N as a closure of $B_N \cap C_c^\infty(\overline{\Omega}, \mathbb{R}^N)$ with respect to the norm

$$\|\varphi\|_{D'_N} := \|\varphi\|_{B_N} + \sup_{x\in\Omega} |\varphi(x)| + \sup_{x\in\Omega} |\nabla\varphi(x)|. \tag{4.85}$$

Then, clearly, U'_N and D'_N are separable Banach spaces, which, however, are not reflexive. On the other hand, by [11, Lemma A.2], there exist separable Hilbert spaces U_N and D_N and bounded linear inclusion operators $A_1 \in \mathcal{L}(U_N; U'_N)$ and $A_2 \in \mathcal{L}(D_N; D'_N)$ such that A_1 and A_2 are injective, the image of A_1 is dense in U'_N and the image of A_2 is dense in B'_N. On the other hand, clearly, U'_N is trivially embedded in V_N and the trivial embedding operator $I_1 \in \mathcal{L}(U'_N; V_N)$ is injective and has dense range in V_N. Similarly, D'_N is trivially embedded in B_N, and the trivial embedding operator $I_2 \in \mathcal{L}(D'_N; B_N)$ is injective and has dense range in B_N. Therefore,

$$Q_1 := I_1 \circ A_1 \in \mathcal{L}(U_N; V_N) \quad \text{and} \quad Q_2 := I_2 \circ A_2 \in \mathcal{L}(D_N; B_N)\,, \tag{4.86}$$

are injective and have dense ranges in V_N and B_N respectively. Next define $P_1 \in \mathcal{L}(V_N; L_N)$ as the trivial inclusion of V_N into L_N and $P_2 \in \mathcal{L}(B_N; L_N)$ as the trivial inclusion of B_N into L_N. Then, clearly, P_1 and P_2 are injective and have dense ranges in L_N. Finally, define

$$\mathcal{T}_1 := P_1 \circ Q_1 \in \mathcal{L}(U_N; L_N) \quad \text{and} \quad \mathcal{T}_2 := P_2 \circ Q_2 \in \mathcal{L}(D_N; L_N). \tag{4.87}$$

Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are injective and have dense ranges in L_N. Next set

$$X := \Big\{ \big(\psi, \varphi\big) :\ \psi \in U_N,\ \varphi \in D_N \Big\}, \tag{4.88}$$

and on X consider the norm

$$\|x\|_X := \big(\|\psi\|^2_{U_N} + \|\varphi\|^2_{D_N}\big)^{1/2} \tag{4.89}$$

for all $x = (\psi, \varphi) \in X$. Thus X is a separable reflexive Banach space. Similarly, set

$$Z := \Big\{ \big(\psi, \varphi\big) :\ \psi : \Omega \to \mathbb{R}^N,\ \varphi : \Omega \to \mathbb{R}^N,\ \psi \in V_N,\ \varphi \in B_N \Big\}, \tag{4.90}$$

and on Z consider the norm

$$\|z\|_Z := \left(\|\psi\|^2_{V_N} + \|\varphi\|^2_{B_N}\right)^{1/2} \tag{4.91}$$

for all $z = (\psi, \varphi) \in Z$. Thus Z is also a separable reflexive Banach space. Finally, set

$$H := \left\{\left(\psi, \varphi\right) : \ \psi : \Omega \to \mathbb{R}^N, \ \varphi : \Omega \to \mathbb{R}^N, \ \psi \in L_N, \ \varphi \in L_N\right\} \tag{4.92}$$

and on H, consider the scalar product

$$\begin{aligned} \langle h_1, h_2\rangle_{H\times H} &:= \langle \psi_1, \psi_2\rangle_{L_N\times L_N} + \langle \varphi_1, \varphi_2\rangle_{L_N\times L_N} \\ &= \int_\Omega \left\{\psi_1(x)\cdot\psi_2(x) + \varphi_1(x)\cdot\varphi_2(x)\right\}dx \end{aligned} \tag{4.93}$$

for all $h_1 = (\psi_1, \varphi_1), h_2 = (\psi_2, \varphi_2) \in H$. Then H is a Hilbert space. Furthermore, define $Q \in \mathcal{L}(X, Z)$ by

$$Q\cdot h = \left(Q_1\cdot\psi, Q_2\cdot\varphi\right) \tag{4.94}$$

for all $h = (\psi, \varphi) \in X$. Similarly, define $P \in \mathcal{L}(Z, H)$ by

$$P\cdot z = \left(P_1\cdot\psi, P_2\cdot\varphi\right) \tag{4.95}$$

for all $z = (\psi, \varphi) \in Z$, and $T \in \mathcal{L}(X, H)$ by

$$T\cdot h = \left(\mathcal{T}_1\cdot\psi, \mathcal{T}_2\cdot\varphi\right) \tag{4.96}$$

for all $h = (\psi, \varphi) \in X$. Thus, clearly, $T = P\circ Q$, and T is an injective inclusion with dense image. Furthermore,

$$X^* := \left\{\left(\psi, \varphi\right) : \ \psi \in (U_N)^*, \ \varphi \in (D_N)^*\right\}, \tag{4.97}$$

where

$$\langle \delta, h\rangle_{X\times X^*} = \langle \delta_0, h_0\rangle_{U_N\times(U_N)^*} + \langle \delta_1, h_1\rangle_{D_N\times(D_N)^*} \tag{4.98}$$

for all $\delta = (\delta_0, \delta_1) \in X$ and all $h = (h_0, h_1) \in X^*$. Thus $\{X, H, X^*\}$ is an evolution triple with the corresponding inclusion operators $T \in \mathcal{L}(X; H)$ and $\widetilde{T} \in \mathcal{L}(H; X^*)$, as defined in Definition 2.6.

Next, define $\Phi(h) : Z \to [0, +\infty)$ by

$$\Phi(h) := \frac{1}{2}\int_\Omega \left(\nu_h\left|\nabla_x\psi(x)\right|^2 + \frac{\nu_m}{2}\left|rot_x\varphi(x)\right|^2 + \left|\psi(x)\right|^2 + \left|\varphi(x)\right|^2\right)dx$$

for all $h = (\psi, \varphi) \in Z = \big(V_N, B_N\big)$. So the mapping $D\Phi(h) : Z \to Z^*$ is linear and monotone. Furthermore, for each $t \in [0, T_0]$, define $\Theta_t(\sigma) : H \to (U_N)^*$ by

$$\begin{aligned}(4.99)\quad &\Big\langle \delta, \Theta_t(\sigma)\Big\rangle_{U_N\times(U_N)^*}\\ &:= -\int_\Omega \bigg\{\Big(w(x)\otimes w(x)+r(x,t)\otimes w(x,t)+w(x,t)\otimes r(x,t)-\lambda b(x)\otimes b(x)\Big)+g(x,t)\bigg\}\\ &\qquad : \nabla\{A_1\cdot\delta\}(x)\,dx\\ &\qquad\qquad + \int_\Omega \Big(f(x,t)-w(x)\Big)\cdot\{A_1\cdot\delta\}(x)\,dx\end{aligned}$$

for all $\sigma = (w, b) \in L_N \oplus L_N \equiv H$ and all $\delta \in U_N$. Next, for each $t \in [0, T_0]$, define $\Xi_t(\sigma) : H \to (D_N)^*$ by

$$\begin{aligned}(4.100)\quad &\Big\langle \delta, \Xi_t(\sigma)\Big\rangle_{D_N\times(D_N)^*}\\ &:= -\int_\Omega \lambda\Big(b(x)\otimes w(x) - w(x)\otimes b(x) + b(x)\otimes r(x,t) - r(x,t)\otimes b(x)\Big)\\ &\qquad : \nabla\{A_2\cdot\delta\}(x)\,dx - \int_\Omega b(x)\cdot\{A_2\cdot\delta\}(x)\,dx\end{aligned}$$

for all $\sigma = (w, b) \in L_N \oplus L_N \equiv H$ and all $\delta \in D_N$. Finally, for each $t \in [0, T_0]$, define $F_t(\sigma) : H \to X^*$ by

$$(4.101)\qquad F_t(\sigma) := \big(\Theta_t(\sigma), \Xi_t(\sigma)\big)$$

for all $\sigma \in H$. Then $F_t(\sigma)$ is Gâteaux differentiable at every $\sigma \in H$, and the derivative DF_t of $F_t(\sigma)$ satisfies the condition

$$(4.102)\qquad \|DF_t(\sigma)\|_{\mathcal{L}(H;X^*)} \le C\big(\|\sigma\|_H + 1\big)$$

for all $\sigma \in H$ and all $t \in [0, T_0]$, for some constant $C > 0$. Moreover,

$$\begin{aligned}(4.103)\quad &\Big\langle \delta, F_t(T\cdot\delta)\Big\rangle_{X\times X^*} = \Big\langle \psi, \Theta_t(T\cdot\delta)\Big\rangle_{U_N\times(U_N)^*} + \Big\langle \varphi, \Xi_t(T\cdot\delta)\Big\rangle_{D_N\times(D_N)^*}\\ &= -\int_\Omega \bigg\{\Big(w(x)\otimes w(x)+r(x,t)\otimes w(x,t)+w(x,t)\otimes r(x,t)-\lambda b(x)\otimes b(x)\Big)+g(x,t)\bigg\}\\ &\qquad : \nabla w(x)\,dx\\ &+\int_\Omega \Big(f(x,t)-w(x)\Big)\cdot w(x)\,dx - \int_\Omega \lambda\Big(b(x)\otimes w(x)-w(x)\otimes b(x)+b(x)\otimes r(x,t)-r(x,t)\otimes b(x)\Big)\\ &\qquad : \nabla b(x)\,dx - \int_\Omega b(x)\cdot b(x)\,dx\end{aligned}$$

where $w = A_1 \cdot \psi$, $b = A_2 \cdot \varphi$ for all $\delta = (\psi, \varphi) \in U_N \oplus D_N = X$ and all $t \in [0, T_0]$. Thus, since $w = A_1 \cdot \psi \in U_N'$ and $b = A_2 \cdot \varphi \in D_N'$, we can rewrite (4.103) as

$$\begin{aligned}(4.104)\quad \Big\langle \delta, F_t(T \cdot \delta)\Big\rangle_{X \times X^*} &= \int_\Omega \Big(f(x,t) \cdot w(x) - g(x,t) : \nabla w(x)\Big)\,dx \\ &\quad - \int_\Omega \Big(\big|w(x)\big|^2 + \big|b(x)\big|^2\Big)\,dx \\ - \int_\Omega \bigg(\Big\{r(x,t) \otimes w(x) + w(x) \otimes r(x,t)\Big\} &: \nabla w(x) + \lambda \Big\{b(x) \otimes r(x,t)\Big\} : rot_x b(x)\bigg)dx \\ - \int_\Omega \frac{1}{2}\bigg\{w(x) \cdot \nabla_x \big|w(x)\big|^2 + \lambda w(x) \cdot \nabla_x \big|b(x)\big|^2 &- 2\lambda b(x) \cdot \nabla_x \Big(w(x) \cdot b(x)\Big)\bigg\}\,dx,\end{aligned}$$

where $w = A_1 \cdot \psi$, $b = A_2 \cdot \varphi$ for all $\delta = (\psi, \varphi) \in U_N \oplus D_N = X$ and all $t \in [0, T_0]$. On the other hand ,$w(x), b(x) \in L_N$, and thus $\mathrm{div}_x\{\chi_\Omega w\} = \mathrm{div}_x\{\chi_\Omega b\}$ in the sense of distributions (here, χ_Ω is characteristic function of the set Ω). Thus the last integral in (4.104) vanishes; and therefore, since $r(x,t) \in L^\infty$, we obtain

$$\begin{aligned}(4.105)\quad \Big\langle \delta, F_t(T \cdot \delta)\Big\rangle_{X \times X^*} &= \int_\Omega \Big(f(x,t) \cdot w(x) - g(x,t) : \nabla w(x)\Big)\,dx \\ &\quad - \int_\Omega \Big(\big|w(x)\big|^2 + \big|b(x)\big|^2\Big)\,dx \\ - \int_\Omega \bigg(\Big\{r(x,t) \otimes w(x) + w(x) \otimes r(x,t)\Big\} &: \nabla w(x) + \lambda \Big\{b(x) \otimes r(x,t)\Big\} : rot_x b(x)\bigg)dx \\ &\geq -C\Big(\big\|Q \cdot \delta\big\|_Z + 1\Big)\Big(\big\|T \cdot \delta\big\|_H + 1\Big) - \mu(t),\end{aligned}$$

where $w = A_1 \cdot \psi$, $b = A_2 \cdot \varphi$ for all $\delta = (\psi, \varphi) \in X$ and all $t \in [0, T_0]$. Here, $\mu(t) \in L^1(0, T_0; \mathbb{R})$ is some non-negative function.

Next consider a sequence of open sets $\{\Omega_j\}_{j=1}^\infty$ such that for every $j \in \mathbb{N}$, Ω_j is compactly embedded in Ω_{j+1} and $\bigcup_{j=1}^\infty \Omega_j = \Omega$. Then set $Z_j := L^2\big(\Omega_j, \mathbb{R}^N\big)$ and define $\overline{L}_j \in \mathcal{L}(L_N, Z_j)$ by

$$\overline{L}_j \cdot \big(h(x)\big) := h(x) \llcorner \Omega_j \in L^2\big(\Omega_j, \mathbb{R}^N\big) = Z_j$$

for all $h(x) \in L_N(\Omega)$. Thus, by the standard embedding theorems for Sobolev spaces, the operators $\overline{L}_j \circ P_1 \in \mathcal{L}(V_N, Z_j)$ and $\overline{L}_j \circ P_2 \in \mathcal{L}(B_N, Z_j)$ are compact for every j. Moreover, if $\{\sigma_n\}_{n=1}^\infty \subset H$ is a sequence such that $\sigma_n = (h_n, w_n) \rightharpoonup \sigma_0 = (h_0, w_0)$ weakly in H and $\overline{L}_j \cdot h_n \to \overline{L}_j \cdot h_0$ and $\overline{L}_j \cdot w_n \to \overline{L}_j \cdot w_0$ strongly in Z_j as $n \to +\infty$ for every j, then $h_n \to h_0$ and $w_n \to w_0$ strongly in $L^2_{loc}(\Omega, \mathbb{R}^N)$; and thus, by (4.101) and (4.102), $F_t(\sigma_n) \rightharpoonup F_t(\sigma_0)$ weakly in X^*.

Thus all the conditions of Theorem 3.4 are satisfied. Applying that theorem, we deduce that there exists a function $h(t) \in L^2\big(0, T_0; Z\big)$ such that $\sigma(t) := P \cdot h(t)$

belongs to $L^\infty(0,T_0;H)$, $\gamma(t) := \widetilde{T}\cdot\sigma(t)$ belongs to $W^{1,2}(0,T_0;X^*)$ and $h(t)$ is a solution of

$$\begin{cases} \frac{d\gamma}{dt}(t) + F_t\big(\sigma(t)\big) + Q^*\cdot D\Phi\big(h(t)\big) = 0 \quad \text{for a.e. } t\in(0,T_0),\\ \sigma(0) = \big(v_0(x), b_0(x)\big), \end{cases} \tag{4.106}$$

where we assume that $\sigma(t)$ is H-weakly continuous on $[0,T_0]$ and $Q^*\in\mathcal{L}(Z^*,X^*)$ is the adjoint to Q. Then, by the definitions of Φ and F_t, we have that $h(x,t) := \big(u(x,t), b(x,t)\big)$ satisfies that $u(x,t)\in L^2(0,T_0;V_N)\cap L^\infty(0,T_0;L_N)$ and $b(x,t)\in L^2(0,T_0;B_N)\cap L^\infty(0,T_0;L_N)$, $u(\cdot,t)$ and $b(\cdot,t)$ are L_N-weakly continuous in t on $[0,T_0]$, $u(x,0)=v_0(x)$, $b(x,0)=b_0(x)$ and $u(x,t)$ and $b(x,t)$ satisfy

$$\begin{aligned} \int_0^{T_0}\int_\Omega\bigg\{\Big(&u(x,t)\otimes u(x,t) + r(x,t)\otimes u(x,t) + u(x,t)\otimes r(x,t)\\ &- \lambda b(x,t)\otimes b(x,t) + g(x,t)\Big) : \nabla_x\big\{A_1\cdot\psi(t)\big\}(x)\\ &- f(x,t)\cdot\big\{A_1\cdot\psi(t)\big\}(x) + u(x,t)\cdot\big\{A_1\cdot\partial_t\psi(t)\big\}(x)\bigg\}\,dxdt\\ =& \int_0^{T_0}\int_\Omega \nu_h\nabla_x u(x,t) : \nabla_x\big\{A_1\cdot\psi(t)\big\}(x)\,dxdt\\ &- \int_\Omega v_0(x)\cdot\big\{A_1\cdot\psi(0)\big\}(x)\,dx, \end{aligned} \tag{4.107}$$

for every $\psi(t)\in C^1\big([0,T_0];U_N\big)$ such that $\psi(T_0)=0$ and

$$\begin{aligned} \int_0^{T_0}\int_\Omega\bigg\{\lambda\Big(&b(x,t)\otimes u(x,t) - u(x,t)\otimes b(x,t) + b(x,t)\otimes r(x,t)\\ &- r(x,t)\otimes b(x,t)\Big) : \nabla_x\big\{A_2\cdot\phi(t)\big\}(x) + b(x,t)\cdot\big\{A_2\cdot\partial_t\phi(t)\big\}(x)\bigg\}\,dxdt\\ =& \int_0^{T_0}\int_\Omega \frac{\nu_m}{2} rot_x b(x,t) : rot_x\big\{A_2\cdot\phi(t)\big\}(x)\,dxdt\\ &- \int_\Omega b_0(x)\cdot\big\{A_2\cdot\phi(0)\big\}(x)\,dx, \end{aligned} \tag{4.108}$$

for every $\phi(t)\in C^1\big([0,T_0];D_N\big)$ such that $\phi(T_0)=0$. Thus since the image of A_1 is dense in U'_N and the image of A_2 is dense in D'_N, we deduce that $u(x,t)$ and $b(x,t)$ are solutions of (4.80) and (4.81).

Next, by (4.105) and the definition of Φ, we have

$$(4.109)\quad \Big\langle \delta, Q^* \cdot D\Phi(Q\cdot\delta) + F_t(T\cdot\delta)\Big\rangle_{X\times X^*}$$
$$= \int_\Omega \left(\nu_h\big|\nabla_x w(x)\big|^2 + \frac{\nu_m}{2}\big|rot_x b(x)\big|^2 + \int_\Omega \Big(f(x,t)\cdot w(x) - g(x,t) : \nabla w(x)\Big)\, dx\right.$$
$$- \int_\Omega \left(\Big\{r(x,t)\otimes w(x) + w(x)\otimes r(x,t)\Big\} : \nabla w(x) + \lambda\Big\{b(x)\otimes r(x,t)\Big\} : rot_x b(x)\right) dx,$$

where $w = A_1\cdot\psi$, $b = A_2\cdot\varphi$ for all $\delta = (\psi,\varphi)\in X$ and all $t\in[0,T_0]$. However, if Ω is bounded, the embedding operator P_1 is compact. On the other hand, either $\lambda = 0$ and Ω is bounded, or $r(x,t)\equiv 0$. Thus, by (4.109) together with Theorem 3.4, we finally deduce (4.83). □

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