HARMONIC ANALYSIS ASSOCIATED WITH A DISCRETE LAPLACIAN*

By

Óscar Ciaurri, T. Alastair Gillespie, Luz Roncal, José L. Torrea, and Juan Luis Varona

Abstract. It is well known that the fundamental solution of

$$u_t(n,t) = u(n+1,t) - 2u(n,t) + u(n-1,t), \quad n \in \mathbb{Z},$$

with $u(n,0) = \delta_{nm}$ for every fixed $m \in \mathbb{Z}$ is given by $u(n,t) = e^{-2t}I_{n-m}(2t)$, where $I_k(t)$ is the Bessel function of imaginary argument. In other words, the heat semigroup of the discrete Laplacian is described by the formal series $W_t f(n) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m)$. This formula allows us to analyze some operators associated with the discrete Laplacian using semigroup theory. In particular, we obtain the maximum principle for the discrete fractional Laplacian, weighted $\ell^p(\mathbb{Z})$-boundedness of conjugate harmonic functions, Riesz transforms and square functions of Littlewood-Paley. We also show that the Riesz transforms essentially coincide with the so-called discrete Hilbert transform defined by D. Hilbert at the beginning of the twentieth century. We also see that these Riesz transforms are limits of the conjugate harmonic functions. The results rely on a careful use of several properties of Bessel functions.

1 Introduction

The purpose of this paper is to analyze several operators associated with the discrete Laplacian

$$\Delta_{\mathrm{d}} f(n) = f(n+1) - 2f(n) + f(n-1), \quad n \in \mathbb{Z},$$

in a way similar to the analysis of classical operators associated with the euclidean Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2}.$$

Among the operators that we study are the (discrete) fractional Laplacian, maximal heat and Poisson semigroups, the square function operator, Riesz transforms, and the conjugate harmonic function operator.

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We use the heat semigroup $W_t = e^{t\Delta_d}$ as a fundamental tool. This is not difficult because the fundamental solution of

$$u_t(n,t) = u(n+1,t) - 2u(n,t) + u(n-1,t), \quad n \in \mathbb{Z},$$

with $u(n,0) = \delta_{nm}$ for every fixed $m \in \mathbb{Z}$ is given by $u(n,t) = e^{-2t} I_{n-m}(2t)$ (see [6] and [7]), where $I_k(t)$ is the Bessel function of imaginary argument; for these functions, see [11, Chapter 5]. Consequently, the heat semigroup is given by the formal series

$$W_t f(n) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m). \tag{1}$$

Then, the function $u(n,t) = W_t f(n)$ defined in (1) is the solution of the *discrete* heat equation

$$\begin{cases} \frac{\partial}{\partial t} u(n,t) = u(n+1,t) - 2u(n,t) + u(n-1,t), \\ u(n,0) = f(n), \end{cases} \tag{2}$$

where u is the unknown function and the sequence $f = \{f(n)\}_{n\in\mathbb{Z}}$ is the initial datum at time $t = 0$. Other second order differential operators and associated discrete heat kernels, namely, the Toda lattice (see [7, 8, 10]), arise when dealing with equations connected with physics.

We first show that the heat semigroup is a positive, Markovian, diffusion semigroup; see Section 2. Then we prove a maximum principle for the fractional Laplacian; see Theorem 1 below. We can apply the general theory developed by E. M. Stein [18] to obtain the boundedness on $\ell^p := \ell^p(\mathbb{Z})$, $1 < p < \infty$, of the maximal heat and Poisson operators and the square function. However this general theory does not cover the boundedness on the space ℓ^1. Analyzing the kernel of these operators, we obtain results in $\ell^p(w) := \ell^p(\mathbb{Z}, w)$, $1 \le p < \infty$, where w is an appropriate weight; see Theorem 2.

The "first" order difference operators

$$Df(n) = f(n+1) - f(n) \quad \text{and} \quad \widetilde{D} f(n) = f(n) - f(n-1) \tag{3}$$

allow factorization of the discrete Laplacian as $\Delta_d = \widetilde{D}D$. Given a factorization $L = \widetilde{X}X$ for a general positive Laplacian L, the "Riesz transforms" are usually defined as $X(L)^{-1/2}$ and $\widetilde{X}(L)^{-1/2}$; see [22, 23]. However, in our case, the operator $(-\Delta_d)^{-1/2}$ is not well-defined. We overcome this difficulty by defining the "Riesz transforms"

$$\mathcal{R} = \lim_{\alpha\to(1/2)^-} D(-\Delta_d)^{-\alpha} \quad \text{and} \quad \widetilde{\mathcal{R}} = \lim_{\alpha\to(1/2)^-} \widetilde{D}(-\Delta_d)^{-\alpha}. \tag{4}$$

It turns out that these are convolution operators with the kernels $\{1/\pi(n+1/2)\}_{n\in\mathbb{Z}}$ and $\{1/\pi(n-1/2)\}_{n\in\mathbb{Z}}$. A close analysis of the semigroup and its kernel allows us to define the harmonic conjugate functions and to see that some Cauchy–Riemann type equations are satisfied; see Theorem 3 below. Moreover, we show that the Riesz transforms are the limits of the conjugate harmonic functions. We provide definitions of these operators and, using the Fourier transform and some abstract results on discrete distributions, prove that they are effectively bounded on ℓ^2. We refer the reader to [4] and [21] for details.

The operators that we consider in the paper are

(i) the fractional Laplacians $(-\Delta_{\mathrm{d}})^\sigma f$, $0<\sigma<1$;

(ii) the maximal heat semigroup $W^*f=\sup_{t\geq 0}|W_tf|$ with $W_t=e^{t\Delta_{\mathrm{d}}}f$, and the maximal Poisson semigroup $P^*f=\sup_{t\geq 0}|P_tf|$ with $P_t=e^{-t\sqrt{-\Delta_{\mathrm{d}}}}f$;

(iii) the square function $g(f)=(\int_0^\infty|t\partial_te^{t\Delta_{\mathrm{d}}}f|^2dt/t)^{1/2}$;

(iv) the Riesz transforms, as defined in (4);

(v) the conjugate harmonic function $Q_tf=\mathcal{R}P_tf$ and $\widetilde{Q}_tf=\widetilde{\mathcal{R}}P_tf$, $t\geq 0$.

The main results of this paper are Theorems 1, 2 and 3 below. The first one contains maximum and comparison principles for the fractional Laplacian.

Theorem 1. *Let* $0<\sigma<1$.

(i) Let $f\in\ell^2$ *be such that* $f\geq 0$ *and* $f(n_0)=0$ *for some* n_0. *Then*

$$(-\Delta_{\mathrm{d}})^\sigma f(n_0)\leq 0.$$

Moreover, $(-\Delta_{\mathrm{d}})^\sigma f(n_0)=0$ *only if* $f(n)=0$ *for all* $n\in\mathbb{Z}$.

(ii) Let $f,g\in\ell^2$ *be such that* $f\geq g$ *and* $f(n_0)=g(n_0)$ *for some* $n_0\in\mathbb{Z}$. *Then*

$$(-\Delta_{\mathrm{d}})^\sigma f(n_0)\leq(-\Delta_{\mathrm{d}})^\sigma g(n_0).$$

Moreover, $(-\Delta_{\mathrm{d}})^\sigma f(n_0)=(-\Delta_{\mathrm{d}})^\sigma g(n_0)$ *only if* $f(n)=g(n)$ *for all* $n\in\mathbb{Z}$.

In order to get mapping properties in ℓ^1 or $\ell^p(w)$ spaces, we use the vector-valued theory of Calderón–Zygmund operators in spaces of homogeneous type $(\mathbb{Z},\mu,|\cdot|)$. Here, $\mu(A)$ is the counting measure for a set $A\subseteq\mathbb{Z}$, and $|\cdot|$ is the absolute value function on $\mathbb{Z}$. A weight on $\mathbb{Z}$ is a sequence $w=\{w(n)\}_{n\in\mathbb{Z}}$ of nonnegative numbers. We recall that w is a discrete Muckenhoupt weight for ℓ^p and write $w\in A_p$ if there exists a constant $C<\infty$ such that for every pair of integers (M,N) with $M\leq N$,

$$\Big(\sum_{k=M}^N w(k)\Big)\Big(\sum_{k=M}^N w(k)^{-1/(p-1)}\Big)^{1/(p-1)}\leq C(N-M+1)^p,\quad 1<p<\infty,$$

and

$$\Big(\sum_{k=M}^{N} w(k)\Big) \sup_{k\in[M,N]} w(k)^{-1} \le C(N-M+1), \quad p=1;$$

see [9, Section 8].

Our second result concerns mapping $\ell^p(w)$ properties for the maximal heat and Poisson semigroups and the square function operator.

Theorem 2. *Let $w \in A_p$, $1 \le p < \infty$. Then the operators W^*, P^*, and g are norms of vector-valued Calderón–Zygmund operators in the sense of the space of homogeneous type $(\mathbb{Z}, \mu, |\cdot|)$. Therefore, each of these operators is bounded from $\ell^p(w)$ into itself for $1 < p < \infty$ and also from $\ell^1(w)$ into weak-$\ell^1(w)$.*

Finally, Riesz transforms can be seen as limits of conjugate harmonic functions; the latter are bounded on $\ell^p(w)$ and satisfy some Cauchy–Riemann equations.

Theorem 3. *Let $f \in \ell^p(w)$, $1 \le p < \infty$, with $w \in A_p$.*

(i) *The operators $Q^*f = \sup_{t\ge 0} Q_t f$ and $\widetilde{Q}^*f = \sup_{t\ge 0} \widetilde{Q}_t f$ are bounded from $\ell^p(w)$, $1 < p < \infty$, into itself and from $\ell^1(w)$ into weak-$\ell^1(w)$.*

(ii) *the operators Q_t, $\widetilde{Q}_t$ and P_t satisfy the Cauchy–Riemann type equations*

$$\begin{cases} \partial_t(Q_t f) = -D(P_t f), \\ \widetilde{D}(Q_t f) = \partial_t(P_t f); \end{cases} \qquad \begin{cases} \partial_t(\widetilde{Q}_t f) = -\widetilde{D}(P_t f), \\ D(\widetilde{Q}_t f) = \partial_t(P_t f). \end{cases}$$

Moreover, $\partial^2_{tt} Q_t f(n) + \Delta_{\mathrm{d}} Q_t f(n) = 0$ and $\partial^2_{tt} \widetilde{Q}_t f(n) + \Delta_{\mathrm{d}} \widetilde{Q}_t f(n) = 0$.

(iii) *For $n \in \mathbb{Z}$, $\lim_{t\to 0} Q_t f(n) = \mathcal{R} f(n)$ and $\lim_{t\to 0} \widetilde{Q}_t f(n) = \widetilde{\mathcal{R}} f(n)$; the limits also holds in the $\ell^p(w)$ sense for $1 < p < \infty$.*

The study of discrete operators of harmonic analysis on ℓ^p was initiated by M. Riesz [15]. In addition to the $L^p(\mathbb{R})$ boundedness of the Hilbert transform, he showed the ℓ^p boundedness of its discrete analogue. Later, A. P. Calderón and A. Zygmund [3] noticed that $L^p(\mathbb{R}^d)$ boundedness of singular integrals implies the $\ell^p(\mathbb{Z}^d)$ boundedness of their discrete analogues. R. Hunt, B. Muckenhoupt, and R. Wheeden [9] proved weighted inequalities for the discrete Hilbert and discrete maximal operator in the one-dimensional case. In the last few years, several works have been developed for nonconvolution discrete analogues of continuous operators; some contributions were made by I. Arkhipov and K. I. Oskolkov [1], J. Bourgain [2], and E. M. Stein and S. Wainger [19, 20]; see also [13], where a brief history and a nice exposition of recent progress on discrete analogues can be found.

In this work, we show one-dimensional results on some operators of harmonic analysis. Results concerning multidimensional discrete fractional integrals and multidimensional discrete Riesz transforms will appear elsewhere.

The paper is organized as follows. In Section 2, we develop the theory of semigroups for the operator (1). In Section 3, we prove Theorem 1. Theorem 2 is proved in Sections 4 and 5. Sections 6 and 7 contain the definition on ℓ^2 of Riesz transforms and conjugate harmonic functions and the proof of Theorem 3.

Modified Bessel functions I_k are used often throughout the paper. They always involve rather sophisticated and technical computations, So, to make the paper more readable, we collect all the properties that we need concerning these functions in Appendix A.

2 The discrete heat and Poisson semigroups

Let

$$G(k,t) = e^{-2t} I_k(2t), \quad k \in \mathbb{Z}. \tag{5}$$

Observe that by (A.2), $G(-k,t) = G(k,t)$. Consider the operator

$$W_t f(n) = \sum_{m\in\mathbb{Z}} G(n-m,t) f(m), \quad t > 0. \tag{6}$$

We prove in Proposition 1 that $\{W_t\}_{t\geq 0}$ is a positive Markovian diffusion semigroup; see [18, Chapter 3, p. 65].

Some definitions are needed for our analysis. Let $\mathbb{T} \equiv \mathbb{R}/(2\pi\mathbb{Z})$ be the one-dimensional torus. We identify $\mathbb{T}$ with the interval $(-\pi, \pi]$ and identify functions on $\mathbb{T}$ with 2π-periodic functions on $\mathbb{R}$. Integration over $\mathbb{T}$ can be described in terms of Lebesgue integration over $(-\pi, \pi]$. We can define the Fourier transform $\mathcal{F}_{\mathbb{Z}}(f)(\theta) = \sum_n f(n)e^{in\theta}$, $\theta \in \mathbb{T}$, of a sequence $f \in \ell^1$. It is well known that the operator $f \mapsto \mathcal{F}_{\mathbb{Z}}(f)$ can be extended as an isometry from ℓ^2 into $L^2(\mathbb{T})$, and the inverse operator $\mathcal{F}_{\mathbb{Z}}^{-1}$ is given by

$$\mathcal{F}_{\mathbb{Z}}^{-1}(\varphi)(n) = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(\theta) e^{-in\theta}\, d\theta. \tag{7}$$

Proposition 1. *Let $f \in \ell^\infty$. The family $\{W_t\}_{t\geq 0}$ satisfies*

(i) $W_0 f = f$;
(ii) $W_{t_1} W_{t_2} f = W_{t_1+t_2} f$;
(iii) *if $f \in \ell^2$, then $W_t f \in \ell^2$ and $\lim_{t\to 0} W_t f = f$ in ℓ^2;*
(iv) *(contraction property) $\|W_t f\|_{\ell^p} \leq \|f\|_{\ell^p}$ for $1 \leq p \leq +\infty$;*
(v) *(positivity preserving) $W_t f \geq 0$ if $f \geq 0$, $f \in \ell^2$;*

(vi) *(Markovian property)* $W_t 1 = 1$.

Proof. Using the identity (A.6), we have

$$\left|\sum_{m\in\mathbb{Z}} e^{-2t} I_m(2t) f(n-m)\right| \leq \|f\|_{\ell^\infty} \sum_{m\in\mathbb{Z}} e^{-2t} I_m(2t) = \|f\|_{\ell^\infty}.$$

Therefore, W_t is well-defined in ℓ^∞. Now (A.3) gives (i), since

$$W_0 f(n) = \sum_{m\in\mathbb{Z}} G(n-m,0) f(m) = \sum_{m\in\mathbb{Z}} e^0 I_{n-m}(0) f(m) = f(n).$$

Concerning (ii), we use (A.4), giving

$$\begin{aligned}\sum_{k\in\mathbb{Z}} G(n-k,t_1) G(k-m,t_2) &= \sum_{k\in\mathbb{Z}} e^{-2t_1} I_{n-k}(2t_1) e^{-2t_2} I_{k-m}(2t_2) \\ &= e^{-2(t_1+t_2)} I_{n-m}(2(t_1+t_2)) = G(n-m,t_1+t_2).\end{aligned}$$

Then

$$\begin{aligned} W_{t_1} W_{t_2} f(n) &= W_{t_1}\left(\sum_{m\in\mathbb{Z}} G(\cdot - m, t_2) f(m)\right)(n) \\ &= \sum_{k\in\mathbb{Z}} G(n-k,t_1)\left(\sum_{m\in\mathbb{Z}} G(k-m,t_2) f(m)\right) \\ &= \sum_{m\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}} G(n-k,t_1) G(k-m,t_2)\right) f(m) \\ &= \sum_{m\in\mathbb{Z}} G(n-m,t_1+t_2) f(m) = W_{t_1+t_2} f(n), \end{aligned}$$

which proves (ii).

For (iv), Minkowski's integral inequality yields

$$\begin{aligned} \|W_t f\|_{\ell^p} &= \left(\sum_{n\in\mathbb{Z}} |W_t f(n)|^p\right)^{1/p} = \left(\sum_{n\in\mathbb{Z}}\left|\sum_{m\in\mathbb{Z}} e^{-2t} I_m(2t) f(n-m)\right|^p\right)^{1/p} \\ &\leq \sum_{m\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}} |e^{-2t} I_m(2t) f(n-m)|^p\right)^{1/p} \\ &= \sum_{m\in\mathbb{Z}} e^{-2t} I_m(2t)\left(\sum_{n\in\mathbb{Z}} |f(n-m)|^p\right)^{1/p} \\ &= \sum_{m\in\mathbb{Z}} e^{-2t} I_m(2t) \|f\|_{\ell^p} = \|f\|_{\ell^p}, \end{aligned}$$

where we have used both (A.5) and (A.6).

Part (v) follows from (A.5).

Part (vi) is obtained using (A.6):

$$W_t 1(n) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{m-n}(2t)\cdot 1 = \sum_{m\in\mathbb{Z}} e^{-2t} I_m(2t) = 1$$

for every $n \in \mathbb{Z}$.

Finally, we prove (iii). We have already proved the boundedness in ℓ^2, and we need only care about the limit. Observe that we can write $W_t f(n) = (G(\cdot,t) * f)(n)$, where the convolution is performed on $\mathbb{Z}$ (i.e., $g * f(n) = \sum_m g(n-m)f(m)$). Moreover,

$$\mathcal{F}_{\mathbb{Z}}(G(\cdot,t) * f)(\theta) = \mathcal{F}_{\mathbb{Z}}(G(\cdot,t))(\theta)\mathcal{F}_{\mathbb{Z}}(f)(\theta).$$

We compute $\mathcal{F}_{\mathbb{Z}}(G(\cdot,t))(\theta)$. By (5) and the formula (see [14, p. 456])

$$\int_0^\pi e^{z\cos\theta}\cos m\theta\, d\theta = \pi I_m(z), \quad |\arg z| < \pi,$$

we have

$$\begin{aligned} G(m,t) = e^{-2t} I_m(2t) &= \frac{e^{-2t}}{\pi}\int_0^\pi e^{2t\cos\theta}\cos m\theta\, d\theta \\ &= \frac{e^{-2t}}{2\pi}\int_{-\pi}^\pi e^{2t\cos\theta}(\cos m\theta - i\sin m\theta)\, d\theta \\ &= \frac{e^{-2t}}{2\pi}\int_{-\pi}^\pi e^{2t\cos\theta} e^{-im\theta}\, d\theta. \end{aligned}$$

In view of the inversion formula (7), we conclude that

$$\mathcal{F}_{\mathbb{Z}}(G(\cdot,t))(\theta) = e^{-2t(1-\cos\theta)} = e^{-4t\sin^2\theta/2}.$$

Therefore,

$$\lim_{t\to 0}\|W_t f - f\|_{\ell^2} = \lim_{t\to 0}\|(e^{-4t\sin^2(\theta/2)} - 1)\mathcal{F}_{\mathbb{Z}}(f)(\theta)\|_{L^2(\mathbb{T})} = 0. \qquad \square$$

Remark 1. Observe that

$$\|W_t f - f\|_{\ell^2} = \|(e^{-4t\sin^2(\theta/2)} - 1)\mathcal{F}_{\mathbb{Z}}(f)(\theta)\|_{L^2(\mathbb{T})} \le Ct\|f\|_{\ell^2}, \quad 0 < t < 1/8.$$

In particular, for $f \in \ell^2$,

$$|W_t f(n) - f(n)| \le Ct\|f\|_{\ell^2} \tag{8}$$

for every $n \in \mathbb{Z}$.

Proposition 2. *Let $f \in \ell^\infty$. Then*

$$u(n,t) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m), \quad t > 0,\ n \in \mathbb{Z},$$

is a solution of equation (2).

Proof. We just use (A.9) and (A.3). □

Remark 2. It can be checked that the Poisson operator defined via the subordination formula

$$e^{-\beta t} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{t^2\beta^2}{4u}}\, du = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/(4v)}}{\sqrt{v}} e^{-v\beta^2}\, \frac{dv}{v}, \tag{9}$$

by

$$P_t f(n) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} W_{t^2/(4u)} f(n)\, du = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/(4v)}}{\sqrt{v}} W_v f(n)\, \frac{dv}{v} \tag{10}$$

satisfies the "Laplace" equation

$$\partial_{tt}^2 P_t f(n) + \Delta_{\mathrm{d}} P_t f(n) = 0.$$

3 Maximum principle for fractional powers of the discrete Laplacian

In this section, we prove Theorem 1. Given $0 < \sigma < 1$, define

$$(-\Delta_{\mathrm{d}})^\sigma f(n) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \big(e^{t\Delta_{\mathrm{d}}} f(n) - f(n)\big) \frac{dt}{t^{1+\sigma}}, \quad f \in \ell^2, \tag{11}$$

where Γ is the gamma function. Notice that, by (6), (5), and (A.6), the integrand of this operator can be written as

$$e^{t\Delta_{\mathrm{d}}} f(n) - f(n) = \sum_{m\in\mathbb{Z}} G(n-m,t)\big(f(m) - f(n)\big). \tag{12}$$

Proof of Theorem 1. Observe that $(-\Delta_{\mathrm{d}})^\sigma f$ is well-defined by formula (11); indeed, by (8) and (12), both integrals on the right-hand side of the decomposition

$$\int_0^\infty |e^{t\Delta_{\mathrm{d}}} f(n) - f(n)| \frac{dt}{t^{1+\sigma}} = \int_0^{1/8} |e^{t\Delta_{\mathrm{d}}} f(n) - f(n)| \frac{dt}{t^{1+\sigma}} + \int_{1/8}^\infty |e^{t\Delta_{\mathrm{d}}} f(n) - f(n)| \frac{dt}{t^{1+\sigma}},$$

are finite. Then

$$(-\Delta_{\mathrm{d}})^{\sigma} f(n_0) = \frac{1}{\Gamma(-\sigma)} \int_0^{\infty} \sum_{m\in\mathbb{Z}} G(n_0 - m, t)\big(f(m) - f(n_0)\big)\frac{dt}{t^{1+\sigma}}$$
$$= \frac{1}{\Gamma(-\sigma)} \int_0^{\infty} \sum_{m\in\mathbb{Z}} G(n_0 - m, t) f(m)\frac{dt}{t^{1+\sigma}}.$$

The positivity of $G(n, t)$ gives the maximum principle stated in (i).

The comparison principle for $(-\Delta)^{\sigma}$ in (ii) is an immediate consequence of the maximum principle.

4 Heat and Poisson semigroups as ℓ^{∞} norms of Calderón–Zygmund operators

In this section, we prove Theorem 2 for the operators W^* and P^*. The key point is obtaining estimates for the kernels, which are contained in Proposition 3 below. In the proof of that proposition, apart from facts concerning modified Bessel functions (see Appendix A), we frequently use the well-known fact that

$$\frac{\Gamma(z+r)}{\Gamma(z+t)} \sim z^{r-t}, \quad z > 0, \ r, t \in \mathbb{R}. \tag{13}$$

Also note that for $\eta > 0$ and $\gamma \geq 0$,

$$(1-r)^{\eta} r^{\gamma} \leq \left(\frac{\gamma}{\gamma+\eta}\right)^{\gamma}, \quad 0 < r < 1. \tag{14}$$

Proposition 3. *Let $T(m, t)$ be either the discrete heat kernel $G(m, t)$ or the Poisson kernel. Then*

$$\sup_{t\geq 0} |T(m, t)| \leq \frac{C_1}{|m| + 1} \qquad \text{and} \qquad \sup_{t\geq 0} |T(m+1, t) - T(m, t)| \leq \frac{C_2}{m^2 + 1},$$

where C_1 and C_2 are constants independent of $m \in \mathbb{Z}$.

Proof. We start with the heat kernel. Observe that by (A.1), (A.7) and (A.8)

$$\sup_{t\geq 0} G(0, t) = \sup_{t\geq 0} e^{-2t} I_0(2t) \leq C.$$

Moreover, by (A.2), we can assume that $m > 0$. By (5) and (A.10), we can write

$$\begin{aligned}
G(m,t) &= e^{-2t} I_m(2t) = e^{-2t} \frac{(2t)^m}{\sqrt{\pi}\, 2^m \Gamma(m+1/2)} \int_{-1}^{1} e^{-2ts}(1-s^2)^{m-1/2}\, ds \\
&= \frac{t^m}{\sqrt{\pi}\,\Gamma(m+1/2)} \int_{-1}^{1} e^{-2t(1+s)}(1-s^2)^{m-1/2}\, ds \\
&= \frac{2t^m}{\sqrt{\pi}\,\Gamma(m+1/2)} \int_0^t e^{-4w}\big(\tfrac{2w}{t}\big)^{m-1/2}\big(2\big(1-\tfrac{w}{t}\big)\big)^{m-1/2}\, \tfrac{dw}{t} \\
&= \frac{t^{-1/2} 4^m}{\sqrt{\pi}\,\Gamma(m+1/2)} \int_0^t e^{-4w} w^{m-1} w^{1/2}\big(1-\tfrac{w}{t}\big)^{m-1/2}\, dw \\
&\le \frac{4^m}{\sqrt{\pi}\,\Gamma(m+1/2) m^{1/2}} \int_0^t e^{-4w} w^{m-1}\, dw \le C \frac{\Gamma(m)}{\Gamma(m+1/2) m^{1/2}} \sim \frac{1}{m},
\end{aligned}$$

where we have used (14) with $r = w/t$, $\eta = m - 1/2$, and $\gamma = 1/2$, and (13).

Concerning smoothness estimates, using (A.14) and proceeding as in the growth estimate, we arrive at

$$\begin{aligned}
|DG(m,t)| &= \frac{2t^{-3/2} 4^m}{\sqrt{\pi}\,\Gamma(m+1/2)} \int_0^t e^{-4w} w^{m-1} w^{3/2}\big(1-\tfrac{w}{t}\big)^{m-1/2}\, dw \\
&\le 2 \frac{4^m}{\sqrt{\pi}\,\Gamma(m+1/2)(m+1)^{3/2}} \int_0^t e^{-4w} w^{m-1}\, dw \\
&\le C \frac{\Gamma(m)}{\Gamma(m+1/2)(m+1)^{3/2}} \sim \frac{1}{m^2},
\end{aligned}$$

where we have used (14) with $r = w/t$, $\eta = m - 1/2$, and $\gamma = 3/2$, and (13).

Using the subordination formula (10), we get that the Poisson kernel satisfies the desired estimates. □

Proof of Theorem 2 for W^* and P^*. Let $\sup_{t\ge 0}|T_t|$ be either W^* (with $T_t = W_t$) or P^* (with $T_t = P_t$). From Proposition 1, Stein's Maximal Theorem of diffusion semigroups (see [18, Chapter III, Section 2]) establishes that $\sup_{t\ge 0} T_t$ is bounded from ℓ^p, $1 < p < \infty$, into itself. Then the vector-valued operator $f \mapsto \widetilde{T} f = \{T_t f\}_t$ is bounded from ℓ^p into $\ell^p_{L^\infty}$ for $1 < p < \infty$ where, for a Banach space $\mathbb{B}$, $\ell^p_{\mathbb{B}}$ is the space of functions such that $\big(\sum_n \|f(n)\|^p_{\mathbb{B}}\big)^{1/p}$.

On the other hand, by Proposition 3, the kernel of the operator $\widetilde{T}$ satisfies the so-called Calderón–Zygmund estimates. Therefore, by the general theory of vector-valued Calderón–Zygmund operators in spaces of homogeneous type (see [16, 17]), the operator $\widetilde{T}$ is bounded from $\ell^p(w)$ into $\ell^p_{L^\infty}(w)$, for $1 < p < \infty$ and $w \in A_p$, and from $\ell^1(w)$ into weak-$\ell^1_{L^\infty}(w)$, for $w \in A_1$. □

As a standard corollary of Theorem 2 we have the following result.

Corollary 1. *Let* $\sup_{t\ge 0}|T_t|$ *be either* W^* *(with* $T_t = W_t$*) or* P^* *(with* $T_t = P_t$*). Then* $\lim_{t\to 0} T_t f(n) = f(n)$ *for every* n *and every function* $f \in \ell^p(w)$*, for* $w \in A_p$*,* $1 \le p < \infty$*.*

5 The discrete g-function operator

In this section, we prove Theorem 2 for the discrete g-function

$$gf(n) = \Big(\int_0^\infty |t\partial_t W_t f(n)|^2\,dt/t\Big)^{1/2}.$$

We first prove the following appropriate vector-valued kernel estimates.

Proposition 4. *Let* $\mathbb{B} = L^2\big((0,\infty), dt/t\big)$*. Let* $G(m,t)$ *be the discrete heat kernel. Then*

$$\|t\,\partial_t G(m,t)\|_{\mathbb{B}} \le \frac{C_1}{|m|+1} \quad \textit{and} \quad \|D(t\,\partial_t G(m,t))\|_{\mathbb{B}} \le \frac{C_2}{m^2+1},$$

where C_1 *and* C_2 *are constants independent of* $m \in \mathbb{Z}$*.*

Proof. By reasoning analogous to that in Proposition 3, we can assume that $m > 0$. We begin with the growth estimate. Observe that by (A.9) and (A.15), one has

$$\|t\partial_t G(m,t)\|_{\mathbb{B}} = \|te^{-2t}(I_{m+1}(2t) - 2I_m(2t) + I_{m-1}(2t))\|_{\mathbb{B}} \le C\frac{S_1^{1/2}+S_2^{1/2}}{\Gamma(m-1/2)},$$

where

$$S_1 := \int_0^\infty t\Big(t^{m-2}e^{-2t}\int_{-1}^1 e^{-2ts}s(1-s^2)^{m-3/2}\,ds\Big)^2\,dt$$

and

$$S_2 := \int_0^\infty t\Big(t^{m-1}e^{-2t}\int_{-1}^1 e^{-2ts}(1+s)^2(1-s^2)^{m-3/2}\,ds\Big)^2\,dt.$$

Concerning S_1, we have

$$
\begin{aligned}
S_1 &\le \int_0^\infty t^{2m-3}e^{-4t}\int_{-1}^1 e^{-2ts}(1-s^2)^{m-3/2}\,ds\int_{-1}^1 e^{-2tu}(1-u^2)^{m-3/2}\,du\,dt\\
&= \int_{-1}^1\int_{-1}^1 (1-s^2)^{m-3/2}(1-u^2)^{m-3/2}\int_0^\infty t^{2m-3}e^{-2t(2+s+u)}\,dt\,ds\,du\\
&= \frac{\Gamma(2m-2)}{2^{2m-2}}\int_{-1}^1\int_{-1}^1 \frac{(1-s^2)^{m-3/2}(1-u^2)^{m-3/2}}{(2+s+u)^{2m-2}}\,du\,ds\\
&= \Gamma(2m-2)\int_0^1\int_0^1 \frac{x^{m-3/2}(1-x)^{m-3/2}y^{m-3/2}(1-y)^{m-3/2}}{(x+y)^{2m-2}}\,dx\,dy\\
&\le \Gamma(2m-2)\int_0^1 y^{m-3/2}(1-y)^{m-3/2}\int_0^1 \frac{x^{m-3/2}}{(x+y)^{2m-2}}\,dx\,dy\\
&= \Gamma(2m-2)\int_0^1 (1-y)^{m-3/2}\int_0^{1/y}\frac{w^{m-3/2}}{(1+w)^{2m-2}}\,dw\,dy\\
&\le \frac{\Gamma(2m-2)\Gamma(m-1/2)\Gamma(m-3/2)}{\Gamma(2m-2)}\int_0^1 (1-y)^{m-3/2}\,dy\int_0^\infty \frac{w^{m-3/2}}{(1+w)^{2m-2}}\,dw\\
&= \frac{\Gamma(m-1/2)^2}{(m-1/2)^2},
\end{aligned}
$$

so that

$$\frac{S_1^{1/2}}{\Gamma(m-1/2)} \le \frac{C}{(m-1/2)}.$$

For S_2, we proceed analogously, and get

$$
\begin{aligned}
S_2 &\le \Gamma(2m)\frac{\Gamma(m+3/2)\Gamma(m-3/2)}{\Gamma(2m)}\int_0^1 y^{m+1/2}(1-y)^{m-3/2}\,dy\\
&= \frac{\Gamma(m+3/2)\Gamma(m-3/2)\Gamma(3)\Gamma(m-1/2)}{\Gamma(m+5/2)};
\end{aligned}
$$

hence

$$\frac{S_2^{1/2}}{\Gamma(m-1/2)} \le \frac{C}{(m-1/2)}.$$

We obtain the bound for $\|t\partial_t G(m,t)\|_{\mathbb{B}}$ by combining the estimates for S_1 and S_2.

We now pass to the smoothness estimates. By (5) and (A.16), we have

$$
\begin{aligned}
\|D(t\partial_t G(m,t))\|_{\mathbb{B}} &= \|te^{-2t}(I_{m+2}(2t)-3I_{m+1}(2t)+3I_m(2t)-I_{m-1}(2t))\|_{\mathbb{B}}\\
&\le C\,\frac{T_1^{1/2}+T_2^{1/2}+T_3^{1/2}}{\Gamma(m-1/2)},
\end{aligned}
$$

where

$$\begin{aligned} T_1 &= \int_0^\infty t\Big(t^{m-3}e^{-2t}\int_{-1}^1 e^{-2ts}s(1-s^2)^{m-3/2}\,ds\Big)^2\,dt,\\ T_2 &= \int_0^\infty t\Big(t^{m-2}e^{-2t}\int_{-1}^1 e^{-2ts}s(1+s)(1-s^2)^{m-3/2}\,ds\Big)^2\,dt, \end{aligned}$$

and

$$T_3 = \int_0^\infty t\Big(t^{m-1}e^{-2t}\int_{-1}^1 e^{-2ts}(1+s)^3(1-s^2)^{m-3/2}\,ds\Big)^2\,dt.$$

To treat each term, we follow the same procedure as in the growth estimates. We get

$$\begin{aligned} T_1 &\le \Gamma(2m-4)\frac{\Gamma(m-1/2)\Gamma(m-7/2)}{\Gamma(2m-4)}\int_0^1 y^2(1-y)^{m-3/2}\,dy\\ &= \frac{\Gamma(m-7/2)\Gamma(3)\Gamma(m-1/2)^2}{\Gamma(m+5/2)};\\ T_2 &\le \Gamma(2m-2)\frac{\Gamma(m+1/2)\Gamma(m-5/2)}{\Gamma(2m-2)}\int_0^1 y^2(1-y)^{m-3/2}\,dy\\ &= \frac{\Gamma(m+1/2)\Gamma(m-5/2)\Gamma(3)\Gamma(m-1/2)}{\Gamma(m+5/2)}; \end{aligned}$$

and

$$\begin{aligned} T_3 &\le \Gamma(2m)\frac{\Gamma(m+5/2)\Gamma(m-5/2)}{\Gamma(2m)}\int_0^1 y^4(1-y)^{m-3/2}\,dy\\ &= \frac{\Gamma(m+5/2)\Gamma(m-5/2)}{\Gamma(m-1/2)^2}\,\frac{\Gamma(5)\Gamma(m-1/2)}{\Gamma(m+9/2)}. \end{aligned}$$

The desired bound follows. □

Proof of Theorem 2 for g**.** Since W_t is a diffusion semigroup, g is bounded from ℓ^2 into itself; see [18, p. 74]. In order to extend this result to weighted inequalities and to the range $1 \le p < \infty$, we observe that g can be viewed as a vector-valued operator taking values in the Banach space $\mathbb{B} = L^2((0,\infty), dt/t)$, so that $gf = \|t\partial_t W_t f\|_{\mathbb{B}}$. Indeed, $gf(n) = \|S_t f(n)\|_{\mathbb{B}}$, where

$$S_t f(n) = \sum_{m\in\mathbb{Z}} t\frac{\partial}{\partial t}G(n-m,t)f(n).$$

Now, by Proposition 4 and the general theory of vector-valued Calderón–Zygmund operators, Theorem 2 holds for the square function g. □

6 Riesz transforms: ℓ^2 definition and mapping properties in $\ell^p(w)$ spaces

In this section, we define and study the discrete Riesz transforms and see that they coincide essentially with the discrete Hilbert transform.

A first attempt to define the discrete Riesz transform $\mathcal{R}$ could be by means of the heat semigroup, as was done with the fractional Laplacian and the square function operator. In this way, we introduce the discrete fractional integral $(-\Delta_d)^{-\alpha}$ for $0 < \alpha < 1/2$. We use the formula

$$(-\Delta_{\mathrm{d}})^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{t\Delta_{\mathrm{d}}} t^\alpha \,\frac{dt}{t}. \tag{15}$$

However, it follows from (5) and asymptotics (A.8) for $t \to \infty$ that the integral in (15) is not absolutely convergent for $\alpha = 1/2$. Thus, although formally we can write $\mathcal{R} = D(-\Delta_d)^{-1/2}$, we cannot define the Riesz transform in ℓ^2 using (15) with $\alpha = 1/2$. Instead, in order to define properly these operators in ℓ^2, we need to develop an abstract theory of discrete distributions and Fourier transform (see [4, 21] for details). Recall the definition of the Fourier transform $\mathcal{F}_{\mathbb{Z}}$ given in Section 2. We write $\mathcal{F}_{\mathbb{T}}(\varphi)(n) := \mathcal{F}_{\mathbb{Z}}^{-1}(\varphi)(n)$. Analogously, we can keep the notation $\mathcal{F}_{\mathbb{Z}}(f)(\theta) := \mathcal{F}_{\mathbb{T}}^{-1}(f)(\theta)$. Denote by $\mathcal{S}(\mathbb{Z})$ the class of sequences $\{c_n\}_{n\in\mathbb{Z}}$ such that for each $k \in \mathbb{N}$ there exists C_k with $\sup_{n\in\mathbb{Z}} |n|^k |c_n| < C_k$. The Fourier transform produces an isomorphism between the space $\mathcal{S}(\mathbb{Z})$ with the canonical family of seminorms and the space of $C^\infty(\mathbb{T})$ functions (each function in $C^\infty(\mathbb{T})$ is associated with the sequence of its Fourier coefficients). This isomorphism allows us to define the Fourier transform on the spaces of distributions $\big(\mathcal{S}(\mathbb{Z})\big)'$ and $\big(C^\infty(\mathbb{T})\big)'$ as follows:

$$\langle \mathcal{F}_{\mathbb{Z}}(\Lambda), \varphi\rangle_{\mathbb{T}} = \langle \Lambda, \mathcal{F}_{\mathbb{T}}(\varphi)\rangle_{\mathbb{Z}}, \quad \Lambda \in \big(\mathcal{S}(\mathbb{Z})\big)' \text{ and } \varphi \in C^\infty(\mathbb{T}), \tag{16}$$

and

$$\langle \mathcal{F}_{\mathbb{T}}(\Phi), f\rangle_{\mathbb{Z}} = \langle \Phi, \mathcal{F}_{\mathbb{Z}}(f)\rangle_{\mathbb{T}}, \quad \Phi \in \big(C^\infty(\mathbb{T})\big)' \text{ and } f \in \mathcal{S}(\mathbb{Z}); \tag{17}$$

see [4, Chapter 12].

We also need to define some different actions on sets of distributions. The convolution of a distribution $\Lambda \in \big(\mathcal{S}(\mathbb{Z})\big)'$ with a function $f \in \mathcal{S}(\mathbb{Z})$ is given by

$$\langle \Lambda * f, g\rangle_{\mathbb{Z}} = \langle \Lambda, \tilde{f} * g\rangle_{\mathbb{Z}}, \quad f, g \in \mathcal{S}(\mathbb{Z}) \text{ and } \tilde{f}(n) = f(-n). \tag{18}$$

The multiplication of a distribution $\mathcal{U} \in \big(C^\infty(\mathbb{T})\big)'$ by a function $\psi \in C^\infty(\mathbb{T})$ is given by

$$\langle \mathcal{U}\psi, \varphi\rangle_{\mathbb{T}} = \langle \mathcal{U}, \psi\varphi\rangle_{\mathbb{T}}, \quad \psi, \varphi \in C^\infty(\mathbb{T}). \tag{19}$$

Observe that by (16), (18), (19) and some standard properties of Fourier transforms, we get

$$\begin{aligned}\langle \mathcal{F}_{\mathbb{Z}}(\Lambda * f), \varphi\rangle_{\mathbb{T}} &= \langle \Lambda * f, \mathcal{F}_{\mathbb{T}}(\varphi)\rangle_{\mathbb{Z}} = \langle \Lambda, \tilde{f} * \mathcal{F}_{\mathbb{T}}(\varphi)\rangle_{\mathbb{Z}} \\ &= \langle \Lambda, \mathcal{F}_{\mathbb{T}}[\mathcal{F}_{\mathbb{T}}^{-1}(\tilde{f})\varphi]\rangle_{\mathbb{Z}} = \langle \mathcal{F}_{\mathbb{Z}}(\Lambda), \mathcal{F}_{\mathbb{T}}^{-1}(\tilde{f})\varphi\rangle_{\mathbb{T}} \\ &= \langle \mathcal{F}_{\mathbb{Z}}(\Lambda)\mathcal{F}_{\mathbb{T}}^{-1}(\tilde{f}), \varphi\rangle_{\mathbb{T}} = \Big\langle \mathcal{F}_{\mathbb{Z}}(\Lambda)\mathcal{F}_{\mathbb{Z}}(f), \varphi\rangle_{\mathbb{T}}.\end{aligned}$$

Another fact that we need is that a linear operator $L : \big(\mathcal{S}(\mathbb{Z})\big)' \to \big(\mathcal{S}(\mathbb{Z})\big)'$ is bounded if and only if the operator $\mathcal{F}_{\mathbb{Z}} \circ L \circ \mathcal{F}_{\mathbb{Z}}^{-1}$ is bounded from $\big(C^\infty(\mathbb{T})\big)'$ into $\big(C^\infty(\mathbb{T})\big)'$. This is illustrated by the commutative diagram

$$\begin{array}{ccc} \big(\mathcal{S}(\mathbb{Z})\big)' & \xrightarrow{\ L\ } & \big(\mathcal{S}(\mathbb{Z})\big)' \\ \Big\downarrow{\scriptstyle \mathcal{F}_{\mathbb{Z}}} & & \Big\downarrow{\scriptstyle \mathcal{F}_{\mathbb{Z}}} \\ \big(C^\infty(\mathbb{T})\big)' & \xrightarrow{\ L\ } & \big(C^\infty(\mathbb{T})\big)' \end{array} \tag{20}$$

Now we are in position to define properly the Riesz transforms.

Proposition 5. *The discrete Riesz transforms* $\mathcal{R}$ *and* $\widetilde{\mathcal{R}}$ *defined in* (4) *are operators acting on* $\mathcal{S}(\mathbb{Z})$ *by the convolution with the sequences* $\big\{\frac{1}{\pi(n+1/2)}\big\}_{n\in\mathbb{Z}}$ *and* $\big\{\frac{1}{\pi(n-1/2)}\big\}_{n\in\mathbb{Z}}$, *respectively.*

Proof. We check that the operators we need to define the Riesz transforms can be seen as operators acting on $C^\infty(\mathbb{T})$.

Recall from Section 2 that the heat semigroup W_t is given by convolution with the sequence $\{G(m,t)\}_{m\in\mathbb{Z}} = \{e^{-2t}I_m(2t)\}_{m\in\mathbb{Z}}$, for each t. We also showed that there exists a constant C such that $G(m,t) \le C/(|m|+1)$; see Proposition 3. Hence

$$\begin{aligned}\big|\langle W_t f, g\rangle_{\mathbb{Z}}\big| = \big|\langle G(\cdot,t), \tilde{f} * g\rangle_{\mathbb{Z}}\big| &= \Big|\sum_{n\in\mathbb{Z}} G(n,t)\tilde{f} * g(n)\Big| \\ &\le C\sum_{n\in\mathbb{Z}} \frac{1}{|n|+1}|\tilde{f} * g(n)|.\end{aligned}$$

This inequality guarantees the boundedness of W_t from $\mathcal{S}(\mathbb{Z})$ into $\big(\mathcal{S}(\mathbb{Z})\big)'$. As a consequence of the computations made in the proof of Proposition 1(iii), we can understand the heat semigroup as the operator of multiplication by $e^{-4t\sin^2(\cdot)/2}$, which is bounded from $C^\infty(\mathbb{T})$ into $\big(C^\infty(\mathbb{T})\big)'$. Also, since $e^{-4t\sin^2(\cdot)/2} \in C^\infty(\mathbb{T})$, it is bounded from $C^\infty(\mathbb{T})$ into itself.

Let us continue with the fractional integral $(-\Delta_{\mathrm{d}})^{-\alpha}$ defined in (15) for $0 < \alpha < 1/2$. In our case, the action of the operator $e^{t\Delta_{\mathrm{d}}}$ is defined by

$e^{t\Delta_{\mathrm{d}}}(\varphi)(\theta) = e^{-4t\sin^2\frac{\theta}{2}}\varphi(\theta)$. Hence

$$(-\Delta_{\mathrm{d}})^{-\alpha}\varphi(\theta) = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-4t\sin^2\frac{\theta}{2}}\varphi(\theta)t^\alpha\,\frac{dt}{t} = \Big(4\sin^2\frac{\theta}{2}\Big)^{-\alpha}\varphi(\theta).$$

Since the function $(4\sin^2\theta/2)^{-\alpha}$ is integrable (recall that $0<\alpha<1/2$), $(-\Delta_{\mathrm{d}})^{-\alpha}$ is bounded from $C^\infty(\mathbb{T})$ into $C^\infty(\mathbb{T})$.

The last operator we are interested in is the first difference operator D defined in (3), acting on $\mathcal{S}(\mathbb{Z})$. Observe that

$$\mathcal{F}_{\mathbb{Z}}(Df)(\theta) = \sum_{n\in\mathbb{Z}} f(n+1)e^{in\theta} - \sum_{n\in\mathbb{Z}} f(n)e^{in\theta} = (e^{-i\theta}-1)\mathcal{F}_{\mathbb{Z}}(f)(\theta);$$

i.e., the operator $\mathcal{F}_{\mathbb{Z}}\circ D\circ\mathcal{F}_{\mathbb{Z}}^{-1}$ is given by the multiplier $(e^{-i\theta}-1)$. As a consequence, the operator $\mathcal{R}^\alpha = \mathcal{F}_{\mathbb{Z}}\circ D(-\Delta_{\mathrm{d}})^{-\alpha}\circ\mathcal{F}_{\mathbb{Z}}^{-1}$, $0<\alpha<1/2$, (bounded from $C^\infty(\mathbb{T})$ into itself) is associated with the multiplier

$$\begin{aligned}(e^{-i\theta}-1)\Big(4\sin^2\frac{\theta}{2}\Big)^{-\alpha} &= (e^{-i\theta}-1)\Big(\Big|2\sin\frac{\theta}{2}\Big|^2\Big)^{-\alpha} = e^{-i\theta/2}\frac{(e^{-i\theta/2}-e^{i\theta/2})}{2^{2\alpha}|\sin\theta/2|^{2\alpha}}\\ &= e^{-i\theta/2}\frac{-2i\,\sin\theta/2}{2^{2\alpha}|\sin\theta/2|^{2\alpha}}.\end{aligned}$$

Hence, for $0\le\theta\le 2\pi$, we have $\lim_{\alpha\to(1/2)^-}(e^{-i\theta}-1)\Big(4\sin^2\frac{\theta}{2}\Big)^{-\alpha} = -ie^{-i\theta/2}$. Therefore, the operator $\mathcal{R} = \lim_{\alpha\to(1/2)^-}\mathcal{R}^\alpha$, defined as the multiplication by the function $-ie^{-i\theta/2}$, is an operator bounded from $C^\infty(\mathbb{T})$ into $C^\infty(\mathbb{T})$. On the other hand,

$$\frac{1}{2\pi}\int_0^{2\pi} -ie^{-i\theta/2}e^{-in\theta}\,d\theta = -\frac{1}{2\pi}\int_0^{2\pi} ie^{-i(n+\frac{1}{2})\theta}\,d\theta = \frac{1}{\pi(n+\frac{1}{2})}.$$

By the diagram (20) above, we conclude that $\mathcal{R}$ is an operator acting on $\mathcal{S}(\mathbb{Z})$ by convolution with the sequence $\big\{\frac{1}{\pi(n+1/2)}\big\}_{n\in\mathbb{Z}}$.

Using an analogous reasoning with $\widetilde{D}$, it can be checked that $\widetilde{\mathcal{R}}$, defined on $C^\infty(\mathbb{T})$ as the operator of multiplication by the function $-ie^{i\theta/2}$, corresponds to the operator of convolution with the sequence $\big\{\frac{1}{\pi(n-1/2)}\big\}_{n\in\mathbb{Z}}$ acting on $\mathcal{S}(\mathbb{Z})$. □

Observe that Proposition 5 implies that both the operators $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are bounded in ℓ^p, $1<p<\infty$, and they are well-defined in ℓ^1. Moreover, since the kernels $\big\{\frac{1}{\pi(n+1/2)}\big\}_{n\in\mathbb{Z}}$ and $\big\{\frac{1}{\pi(n-1/2)}\big\}_{n\in\mathbb{Z}}$ obviously satisfy the Calderón–Zygmund estimates, we have the following (well-known) result.

Corollary 2. *Let $w\in A_p$, $1\le p<\infty$. Then the operators $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are bounded from $\ell^p(w)$ into itself and from $\ell^1(w)$ into weak-$\ell^1(w)$.*

7 Riesz transforms as limits of "harmonic" functions

In this section, we prove Theorem 3. Recall the conjugate harmonic operators

$$Q_t f = \mathcal{R} P_t f \quad \text{and} \quad \widetilde{Q}_t f = \widetilde{\mathcal{R}} P_t f.$$

First, we show that these operators are well defined and satisfy several properties for good functions.

Proposition 6. *Let Q_t and $\widetilde{Q}_t$ be defined as above and f be a compactly supported function.*

(i) *The operators Q_t, $\widetilde{Q}_t$, and P_t satisfy the Cauchy–Riemann type equations*

$$\begin{cases} \partial_t(Q_t f) = -D(P_t f), \\ \widetilde{D}(Q_t f) = \partial_t(P_t f), \end{cases} \qquad \begin{cases} \partial_t(\widetilde{Q}_t f) = -\widetilde{D}(P_t f), \\ D(\widetilde{Q}_t f) = \partial_t(P_t f); \end{cases}$$

moreover, $\partial^2_{tt} Q_t f(n) + \Delta_{\mathrm{d}} Q_t f(n) = 0$ *and* $\partial^2_{tt} \widetilde{Q}_t f(n) + \Delta_{\mathrm{d}} \widetilde{Q}_t f(n) = 0$.

(ii) *For $n \in \mathbb{Z}$, $\lim_{t\to 0} Q_t f(n) = \mathcal{R} f(n)$ and $\lim_{t\to 0} \widetilde{Q}_t f(n) = \widetilde{\mathcal{R}} f(n)$.*

(iii) $\left|\mathcal{F}_{\mathbb{Z}}(Q_t f)(\theta)\right| + \left|\mathcal{F}_{\mathbb{Z}}(\widetilde{Q}_t f)(\theta)\right| \le C|\mathcal{F}_{\mathbb{Z}}(f)(\theta)|$.

Proof. It is clear from the results in Sections 2 and 6 that the operators Q_t and $\widetilde{Q}_t$ can be defined as operators of multiplication by $-ie^{-i\theta/2}e^{-2t|\sin\theta/2|}$ and $-ie^{i\theta/2}e^{-2t|\sin\theta/2|}$, respectively. The conclusions are then obvious. □

Remark 3. By conjugate harmonic functions we mean functions $Q_t f(n)$ and $\widetilde{Q}_t f(n)$ that are harmonic conjugate functions of the harmonic function $P_t f(n)$ in the variables (t, n); see Remark 2.

Proof of Theorem 3. We prove the theorem for Q_t only, since the proof for $\widetilde{Q}_t$ is similar. In Section 6, we saw that the "natural" Riesz transforms associated with the operator Δ_{d} are bounded on the spaces $\ell^p(w)$; see Corollary 2. This, together with Theorem 2, shows that the conjugate harmonic functions are well-defined for all functions in $\ell^p(w)$, $1 \le p < \infty$, and that $\lim_{t\to 0} Q_t f = \mathcal{R} f$ in the $\ell^p(w)$ sense and pointwise for $1 < p < \infty$. In order to get the appropriate results in $\ell^1(w)$, we need to work a bit harder. We prove that the operator $\sup_{t\ge 0} |Q_t f|$ can be viewed as the norm of vector-valued Calderón–Zygmund operators whose kernels satisfy standard estimates; (i) is be a direct consequence.

Taking the derivative with respect to β in the subordination formula (9), we have

$$\frac{1}{\beta} e^{-\beta t} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/(4v)}}{\sqrt{v}} e^{-v\beta^2}\, dv. \tag{21}$$

Then, for $\theta \in [0, 2\pi]$, we obtain

$$
\begin{aligned}
\mathcal{F}_{\mathbb{Z}}(\mathcal{R}P_t f)(\theta) &= -ie^{-i\theta/2} e^{-t|2\sin\theta/2|}\mathcal{F}_{\mathbb{Z}}(f)(\theta) \\
&= e^{-i\theta/2}\frac{2i\sin\theta/2}{2|\sin\theta/2|} e^{-t|2\sin\theta/2|}\mathcal{F}_{\mathbb{Z}}(f)(\theta) \\
&= (e^{-i\theta}-1)\frac{1}{(4\sin^2\theta/2)^{1/2}} e^{-t|2\sin\theta/2|}\mathcal{F}_{\mathbb{Z}}(f)(\theta) \\
&= \Big(\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-t^2/(4v)}(e^{-i\theta}-1)e^{-4v\sin^2(\theta/2)}\frac{dv}{v^{1/2}}\Big)\mathcal{F}_{\mathbb{Z}}(f)(\theta),
\end{aligned} \tag{22}
$$

where we have used (21) in the last identity. Formula (22) allows us to write, for functions $f \in S(\mathbb{Z})$,

$$
\{Q_t f(n)\}_{t\geq 0} = \{DL^{-1/2}P_t f(n)\}_{t\geq 0} = \Big\{\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-t^2/(4v)} DW_v f(n)\frac{dv}{v^{1/2}}\Big\}_{t\geq 0}. \tag{23}
$$

Now we show that the kernel associated with the operator (23) satisfies the Calderón–Zygmund estimates; we denote this kernel by $Q(m,t)$. We consider only $m > 0$, as in previous cases. Reproducing the arguments given in the proof of Proposition 3, we get

$$
\begin{aligned}
|Q(m,t)| &= \frac{4^m\,2}{\pi\Gamma(m+1/2)} \\
&\quad\times \int_0^\infty e^{-t^2/(4v)}\int_0^v e^{-4w}w^{m-1}\Big(\frac{w}{v}\Big)^{3/2}\Big(1-\frac{w}{v}\Big)^{m-1/2}dw\,\frac{dv}{v^{1/2}}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\int_0^\infty e^{-t^2/(4v)}\int_0^v e^{-4w}w^{m-1}\Big(\frac{w}{v}\Big)^{3/2}\Big(1-\frac{w}{v}\Big)^{m-1/2}dw\,\frac{dv}{v^{1/2}} \\
&\quad= \int_0^\infty e^{-t^2/(4v)}\int_0^1 e^{-4vs}s^{m-1}s^{3/2}(1-s)^{m-1/2}\,ds\,v^{m-1/2}\,dv \\
&\quad= \int_0^1 s^{m-1}s^{3/2}(1-s)^{m-1/2}\int_0^\infty e^{-t^2/(4v)}e^{-4vs}v^{m-1/2}\,dv\,ds \\
&\quad= \int_0^1 s^{m-1}s^{3/2}(1-s)^{m-1/2}\int_0^\infty e^{-t^2 s/r}e^{-r}\Big(\frac{r}{4s}\Big)^{m-1/2}\frac{dr}{4s}\,ds \\
&\quad\leq \int_0^1 \frac{s^{m-1}s^{3/2}(1-s)^{m-1/2}}{(4s)^{m-1/2}s}\int_0^\infty e^{-r}r^{m-1/2}\,dr\,ds \\
&\quad= C4^{-m}\int_0^1 (1-s)^{m-1/2}ds\,\Gamma(m+1/2) \\
&\quad= C4^{-m}\frac{\Gamma(1)\Gamma(m+1/2)}{\Gamma(m+3/2)}\Gamma(m+1/2) \sim C4^{-m}\frac{1}{m}\Gamma(m+1/2).
\end{aligned}
$$

Hence we have proved that $\sup_{t\geq 0}|Q(m,t)| \leq C/(|m|+1)$, where C is a constant independent of m.

It remains to prove the smoothness of the kernel, i.e.,

$$\sup_{t\geq 0}|Q(m+1,t)-Q(m,t)| \leq \frac{C}{m^2+1},$$

where C is a constant independent of m. Using (A.15), we have

$$\begin{aligned}
|DQ(m,t)| &= \left|\int_0^\infty e^{-\frac{t^2}{4v}} e^{-2v}(I_{m+2}(2v)-2I_{m+1}(2v)+I_m(2v))\frac{dv}{v^{1/2}}\right| \\
&= \frac{1}{\sqrt{\pi}\,\Gamma(m+1/2)}\left|\int_0^\infty e^{-\frac{t^2}{4v}} v^m \left(\frac{1}{v}\int_{-1}^1 e^{-2v(1+s)} s(1-s^2)^{m-1/2}\,ds\right.\right. \\
&\quad \left.\left. + \int_{-1}^1 e^{-2v(1+s)}(1+s)^2(1-s^2)^{m-1/2}\,ds\right) dv\right| \\
&\leq \frac{4^m}{\sqrt{\pi}\,\Gamma(m+1/2)}\left(\int_0^1 u^{m-1/2}(1-u)^{m-1/2}\int_0^\infty e^{-\frac{t^2}{4v}} v^{m-3/2} e^{-4vu}\,dv\,du\right. \\
&\quad \left. + \int_0^1 u^{m+3/2}(1-u)^{m-1/2}\int_0^\infty e^{-\frac{t^2}{4v}} v^{m-1/2}e^{-4vu}\,dv\,du\right) =: I_1+I_2.
\end{aligned}$$

To estimate I_1 and I_2, we proceed as in the growth estimates, obtaining

$$I_1 \leq \frac{C}{\sqrt{\pi}\,\Gamma(m+1/2)}\frac{\Gamma(1)\Gamma(m+1/2)}{\Gamma(m+3/2)}\Gamma(m-1/2) \sim (m+1/2)^{-2}$$

and

$$I_2 \leq \frac{C}{\sqrt{\pi}\,\Gamma(m+1/2)}\frac{\Gamma(2)\Gamma(m+1/2)}{\Gamma(m+5/2)}\Gamma(m+1/2) \sim (m+1/2)^{-2}.$$

This completes the proof of (i).

To prove (ii) and (iii), observe that for $f \in \mathcal{S}(\mathbb{Z})$, $Q_t f$ can be defined alternatively as

$$Q_t f = \int_0^t DP_s f\,ds - \mathcal{R}f.$$

Applying Fourier transform, we see that this last definition is valid for every function in $\ell^p(w)$, $1 \leq p < \infty$, $w \in A_p$. Moreover, it can be checked that Q_t satisfies (ii) and (iii). □

A Appendix: Technical results on the modified Bessel functions of the first kind

Let I_k be the modified Bessel function of the first kind and order $k \in \mathbb{Z}$, defined as

$$\text{(A.1)} \qquad I_k(t) = i^{-k}J_k(it) = \sum_{m=0}^\infty \frac{1}{m!\,\Gamma(m+k+1)}\left(\frac{t}{2}\right)^{2m+k}.$$

Since k is an integer (and $1/\Gamma(n)$ is taken to equal zero if $n = 0, -1, -2, \ldots$), I_k is defined on the whole real line (even on the whole complex plane, on which it an entire function). We list several properties of I_k. Most of them can be found in [11, Chapter 5] and [12].

(i) For each $k \in \mathbb{Z}$,

$$I_{-k}(t) = I_k(t) \tag{A.2}$$

Also, from (A.1), it is clear that

$$I_0(0) = 1 \quad \text{and} \quad I_k(0) = 0 \quad \text{for} \quad k \neq 0. \tag{A.3}$$

(ii) The identity

$$I_r(t_1 + t_2) = \sum_{k\in\mathbb{Z}} I_k(t_1) I_{r-k}(t_2) \quad \text{for} \quad r \in \mathbb{Z}, \tag{A.4}$$

is called **Neumann's identity** (see [5, Chapter II, formula (7.10)]) and is an easy consequence of the generating function $e^{\frac{1}{2}t(u+u^{-1})} = \sum_{k\in\mathbb{Z}} u^k I_k(t)$, which sometimes serves as definition of I_k; see, e.g., [12, formula 10.35.1].

(iii) The modified Bessel function I_k satisfies

$$I_k(t) \geq 0 \tag{A.5}$$

for every $k \in \mathbb{Z}$ and $t \geq 0$, and

$$\sum_{k\in\mathbb{Z}} e^{-2t} I_k(2t) = 1. \tag{A.6}$$

Clearly, it follows from (A.1) that there exist constants $C, c > 0$, such that

$$ct^k \leq I_k(t) \leq Ct^k \quad \text{for } t \to 0^+. \tag{A.7}$$

Moreover, it is well known (see [11]) that

$$I_k(t) = Ce^t t^{-1/2} + R_k(t), \tag{A.8}$$

where $|R_k(t)| \leq C_k e^t t^{-3/2}$, for $t \to \infty$.

(iv) The modified Bessel function $I_k(t)$ satisfies

$$\frac{\partial}{\partial t} I_k(t) = \frac{1}{2}(I_{k+1}(t) + I_{k-1}(t));$$

and from this, it follows immediately that

$$\frac{\partial}{\partial t}(e^{-2t} I_k(2t)) = e^{-2t}(I_{k+1}(2t) - 2I_k(2t) + I_{k-1}(2t)). \tag{A.9}$$

(iv) The identity

$$I_\nu(z) = \frac{z^\nu}{\sqrt{\pi}\,2^\nu \Gamma(\nu+1/2)} \int_{-1}^{1} e^{-zs}(1-s^2)^{\nu-1/2}\,ds, \quad |\arg z| < \pi,\ \nu > -\frac{1}{2} \tag{A.10}$$

is valid for every real number $\nu > -1/2$. It is known as Schläfli's integral representation of Poisson type for modified Bessel functions; see [11, (5.10.22)]. Integrating the integral in (A.10) by parts once, twice and three times, we get, respectively,

$$I_\nu(z) = -\frac{z^{\nu-1}}{\sqrt{\pi}\,2^{\nu-1}\Gamma(\nu-1/2)} \int_{-1}^{1} e^{-zs} s(1-s^2)^{\nu-3/2}\,ds, \quad \nu > 1/2, \tag{A.11}$$

$$I_\nu(z) = \frac{z^{\nu-2}}{\sqrt{\pi}\,2^{\nu-2}\Gamma(\nu-3/2)} \int_{-1}^{1} e^{-zs} \frac{1+zs}{z} s(1-s^2)^{\nu-5/2}\,ds, \quad \nu > 3/2, \tag{A.12}$$

and

$$\begin{aligned} I_\nu(z) = &-\frac{z^{\nu-3}}{\sqrt{\pi}\,2^{\nu-3}\Gamma(\nu-5/2)} \\ &\times \int_{-1}^{1} e^{-zs}\, \frac{s(s^2z^2+3sz+3)}{z^2}(1-s^2)^{\nu-7/2}\,ds. \quad \nu > 5/2. \end{aligned} \tag{A.13}$$

Combining (A.10) and (A.11), we get, for $\nu > -1/2$,

$$I_{\nu+1}(z) - I_\nu(z) = -\frac{z^\nu}{\sqrt{\pi}\,2^\nu\Gamma(\nu+1/2)} \int_{-1}^{1} e^{-zs}(1+s)(1-s^2)^{\nu-1/2}\,ds. \tag{A.14}$$

Combining (A.10), (A.11) and (A.12), we obtain, for $\nu > -1/2$,

$$\begin{aligned} I_{\nu+2}(z) - 2I_{\nu+1}(z) + I_\nu(z) = &\frac{z^\nu}{\sqrt{\pi}\,2^\nu\Gamma(\nu+1/2)} \\ &\times \Big(\frac{2}{z}\int_{-1}^{1} e^{-zs}s(1-s^2)^{\nu-1/2}\,ds + \int_{-1}^{1} e^{-zs}(1+s)^2(1-s^2)^{\nu-1/2}\,ds\Big). \end{aligned} \tag{A.15}$$

Combining (A.10), (A.11), (A.12), and (A.13) we obtain, for $\nu > -1/2$,

$$\begin{aligned} I_{\nu+3}(z) - 3I_{\nu+2}(z) + 3I_{\nu+1}(z) - I_\nu(z) = &\frac{z^\nu}{\sqrt{\pi}2^\nu\Gamma(\nu+1/2)} \\ \times \Big(\frac{3}{z^2}\int_{-1}^{1} e^{-zs}s(1-s^2)^{\nu-1/2}\,ds &+ \frac{3}{z}\int_{-1}^{1} e^{-zs}s(1+s)(1-s^2)^{\nu-1/2}\,ds \\ + \int_{-1}^{1} e^{-zs}(1+s)^3(1-s^2)^{\nu-1/2}\,ds\Big). \end{aligned} \tag{A.16}$$

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Óscar Ciaurri, Luz Roncal, and Juan Luis Varona
Departamento de Matemáticas y Computación
Universidad de La Rioja
26004 Logroño, Spain
email: {oscar.ciaurri,luz.roncal,jvarona}@unirioja.es

T. Alastair Gillespie
School of Mathematics and Maxwell Institute for Mathematical Sciences
University of Edinburgh
Edinburgh EH9 3JZ, Scotland, UK
email: t.a.gillespie@ed.ac.uk

José L. Torrea
Departamento de Matemáticas, Facultad de Ciencias
Universidad Autónoma de Madrid
28049 Madrid, Spain
email: joseluis.torrea@uam.es

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