A NOTE ON THE SCHRÖDINGER MAXIMAL FUNCTION

By

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Abstract. It is shown that control of the Schrödinger maximal function $\sup_{0<t<1} |e^{it\Delta} f|$ for $f \in H^s(\mathbb{R}^n)$ requires $s \geq n/2(n+1)$.

1 Introduction

Recall that the solution of the linear Schrödinger equation

$$\begin{cases} iu_t - \Delta u = 0 \\ u(x,0) = f(x) \end{cases} \tag{1.1}$$

with $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ is given by

$$e^{it\Delta} f(x) = (2\pi)^{-n/2} \int e^{i(x.\xi + t|\xi|^2)} \hat{f}(\xi) d\xi. \tag{1.2}$$

Assuming f belongs to the space $H^s(\mathbb{R}^n)$ for suitable s, when does the almost convergence property

$$\lim_{t\to 0} e^{it\Delta} f = f \ \text{ a.e.} \tag{1.3}$$

hold? This problem was brought up in Carleson's paper [C], where convergence was proved for $s \geq 1/4$ when $n = 1$. Dahlberg and Kenig [D-K] showed that this result is sharp. In higher dimension, the question of identifying the optimal exponent s has been studied by several authors, and our state of knowledge may be summarized as follows. For $n = 2$, the strongest result to date appears in [L] and asserts (1.3) for $f \in H^s(\mathbb{R}^2)$, $s > 3/8$. More generally, for $n \geq 2$, (1.3) was shown to hold for $f \in H^s(\mathbb{R}^n)$, $s > (2n-1)/4n$; see [B].

In the opposite direction, for $n \geq 2$, the condition $s \geq n/2(n+2)$ was proven to be necessary; see [L-R] and also [D-G] for a different approach based on pseudo-conformal transformation. Here we show the following stronger statement.

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Proposition 1. *Let $n \geq 2$ and $s < n/2(n+1)$. Then there exist sequences $R_k \to \infty$ and $f_k \in L^2(\mathbb{R}^n)$ with $\hat{f}_k$ supported in the annulus $|\xi| \sim R_k$, such that $\|f_k\|_2 = 1$ and*

$$\lim_{k\to\infty} R_k^{-s}\Big\| \sup_{0<t<1} |e^{it\Delta} f_k(x)|\Big\|_{L^1(B(0,1))} = \infty. \tag{1.4}$$

There is some evidence that the exponent $n/2(n+1)$ could be the optimal one, though limited to multi-linear considerations appearing in [B]. Of course, the $n = 1$ case coincides with the [D-K] result, while for $n = 2$, the above proposition leaves a gap between $1/3$ and $3/8$. It may be also worth pointing out that for $n = 2$, in some sense, our example fits a scenario where the arguments from [B] require the $s > 3/8$ condition.

2 Proof of Proposition 1

Let $x = (x_1, \dots, x_n) = (x_1, x') \in B(0,1) \subset \mathbb{R}^n$, and let $\varphi : \mathbb{R} \to \mathbb{R}_+$, and $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}_+$ satisfy supp $\hat{\varphi} \subset [-1,1]$, supp $\hat{\Phi} \subset B(0,1)$, $\hat{\varphi}, \hat{\Phi}$ smooth, and $\varphi(0) = \Phi(0) = 1$. Set $D = R^{(n+2)/2(n+1)}$, and define

$$f(x) = e(Rx_1)\varphi(R^{1/2}x_1)\Phi(x')\prod_{j=2}^{n}\Big(\sum_{\frac{R}{2D}<\ell_j<\frac{R}{D}} e^{iD\ell_j x_j}\Big), \tag{2.1}$$

where $\ell = (\ell_2, \dots, \ell_n) \in \mathbb{Z}^{n-1}$. Hence

$$\|f\|_2 \sim R^{-1/4}\Big(\frac{R}{D}\Big)^{(n-1)/2} \quad \text{and} \quad \text{supp } \hat{f} \subset [|\xi| \sim R]. \tag{2.2}$$

Clearly,

$$\begin{aligned} e^{it\Delta} f(x) = &\iint \hat{\varphi}(\lambda)\hat{\Phi}(\xi') \\ &\times \Big\{\sum_\ell e\big((R+\lambda R^{1/2})x_1 + (\xi' + D\ell).x' + (R+\lambda R^{1/2})^2 t + |\xi' + D\ell|^2 t\big)\Big\} d\lambda d\xi', \end{aligned}$$

where $e(z) = e^{iz}$. Taking $|t| < c/R$, $|x| < c$, for suitable constant $c > 0$, one gets

$$\begin{aligned} |e^{it\Delta} f(x)| &\sim \Big|\int \hat{\varphi}(\lambda)\Big\{\sum_\ell e(\lambda R^{1/2}x_1 + D\ell.x' + 2\lambda R^{3/2}t + D^2|\ell|^2 t)\Big\} d\lambda\Big| \\ &\sim \varphi\big(R^{1/2}(x_1 + 2Rt)\big)\Big|\sum_\ell e(D\ell.x' + D^2|\ell|^2 t)\Big|. \end{aligned} \tag{2.3}$$

Specify, further, $t = -(x_1/2R) + \tau$ with $|\tau| < R^{-3/2}/10$ in order to ensure that the first factor in (2.3) is ~ 1. For this choice of t, the second factor becomes

$$\Big|\sum_\ell e\Big(D\ell.x' - \frac{D^2}{2R}|\ell|^2 x_1 + D^2|\ell|^2\tau\Big)\Big| = \prod_{j=2}^{n}\Big|\sum_{\frac{R}{2D}<\ell_j<\frac{R}{D}} e\big(\ell_j y_j + \ell_j^2(y_1 + s)\big)\Big| \tag{2.4}$$

with

$$y' = Dx' (\mathrm{mod}\, 2\pi), \quad y_1 = -\frac{D^2}{2R} x_1 (\mathrm{mod}\, 2\pi), \tag{2.5}$$

and where $s = D^2\tau$ is subject to the condition

$$|s| \lesssim D^2 R^{-3/2} = R^{-(n-1)/2(n+1)}. \tag{2.6}$$

We view $y = (y_1, y')$ as a point in the n-torus $\mathbb{T}^n$. Next, define the following subset $\Omega \subset \mathbb{T}^n$:

$$\Omega = \bigcup_{q \sim R^{\frac{n-1}{2(n+1)}}, a} \left\{ (y_1, y') : \left| y_1 - 2\pi \frac{a_1}{q} \right| < cR^{-\frac{n-1}{2(n+1)}} \text{ and } \left| y' - 2\pi \frac{a'}{q} \right| < c\frac{D}{R} \right\} \tag{2.7}$$

with $a = (a_1, a')$ (mod q) and $(a_1, q) = 1$. Hence

$$|\Omega| \sim R^{\frac{n-1}{2(n+1)}} R^{n\frac{n-1}{2(n+1)}} R^{-\frac{n-1}{2(n+1)}} \left(\frac{D}{R}\right)^{n-1} \sim 1,$$

and we take $x \in B(0, 1)$ for which y given by (2.5) belongs to Ω. Clearly, this gives a set of measure at least $c_1 > 0$.

We evaluate (2.4) for $y \in \Omega$. Let $q \sim R^{(n-1)/2(n+1)}$ and (a_1, a') (mod q) satisfy the approximations stated in (2.7), and set $s = 2\pi a_1/q - y_1$ for which (2.6) holds. Clearly, for $j = 2, \ldots, n$, by the quadratic Gauss sum evaluation,

$$\begin{aligned}\Bigl|\sum_{\frac{R}{2D} < \ell_j < \frac{R}{D}} e\bigl(\ell_j y_j + \ell_j^2 (y_1 + s)\bigr)\Bigr| &\sim \Bigl|\sum_{\frac{R}{2D} < \ell_j < \frac{R}{D}} e\Bigl(2\pi \frac{a_j}{q} \ell_j + 2\pi \frac{a_1}{q} \ell_j^2\Bigr)\Bigr| \\ &\sim R^{\frac{1}{2(n+1)}} \Bigl|\sum_{\ell_j = 0}^{q-1} e\Bigl(2\pi \frac{a_j}{q} \ell_j + 2\pi \frac{a_1}{q} \ell_j^2\Bigr)\Bigr| \\ &\sim R^{\frac{1}{2(n+1)}} q^{\frac{1}{2}} \sim R^{\frac{1}{4}}\end{aligned}$$

and

$$(2.4) \sim R^{\frac{n-1}{4}}. \tag{2.8}$$

Recalling (2.2), we obtain for $x \in B(0, 1)$ in a set of measure $c_1 > 0$ that

$$\sup_{0<t<1} \frac{|e^{it\Delta} f(x)|}{\|f\|_2} \gtrsim R^{\frac{n-1}{4}} R^{\frac{1}{4}} \left(\frac{D}{R}\right)^{\frac{n-1}{2}} = R^{\frac{n}{2(n+1)}}. \tag{2.9}$$

The claim in the proposition follows.

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