“GRAPH PAPER” TRACE CHARACTERIZATIONS OF FUNCTIONS OF FINITE ENERGY

By

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Abstract. We characterize functions of finite energy in the plane in terms of their traces on the lines that make up “graph paper” with squares of side length m^n for all n and certain $1/2$-order Sobolev norms on the graph paper lines. We also obtain analogous results for functions of finite energy on two classical fractals: the Sierpinski gasket and the Sierpinski carpet.

1 Introduction

Functions of finite energy play an important role in analysis and probability. On euclidean space or a domain in euclidean space, these are just the functions whose gradient in the distribution sense belongs to L^2, with the energy given by

$$\int |\nabla F|^2\, dx. \tag{1.1}$$

As such, they make up a homogeneous Sobolev space that we denote here as H^1. The more usual inhomogeneous Sobolev space is smaller, requiring that $F \in L^2$ as well [11, 15]. There are many ways to generalize the notion of finite energy to other contexts, for example, as the functions in the domain of a Dirichlet form [6]. In this paper, we consider only functions of finite energy in regions in the plane and on two classical fractals: the Sierpinski gasket [10, 17] and the Sierpinski carpet [2, 3].

It is well known that functions of finite energy in the plane (or in higher dimensions) do not have to be continuous, so the value $F(x, y)$ at a point is not meaningful. Nevertheless, the trace on a line, say $TF(x) = F(x, 0)$, is well defined and belongs to a certain $1/2$-order homogeneous Sobolev space that we denote here by $H^{1/2}(\mathbb{R})$, defined by the finiteness of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|f(x) - f(y)|^2}{|x - y|^2}dxdy, \tag{1.2}$$

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with a corresponding norm estimate. Of course, it is the norm estimate that is important, since it implies the existence of the trace by routine arguments. The result is sharp, meaning that there is an extension operator from $H^{1/2}(\mathbb{R})$ to $H^1(\mathbb{R}^2)$.

There are, in fact, two rather natural $1/2$-order Sobolev spaces on $\mathbb{R}$. The other one, which we denote by $\tilde{H}^{1/2}(\mathbb{R})$ is larger, and only requires the finiteness of an integral like (1.2) in which integration is restricted to the region $|x-y| \le 1$. We show that the trace of a function F of finite energy in the strip $\{(x, y) : 0 < y < 1\}$ only belongs to $\tilde{H}(\mathbb{R})$. In particular, this implies that there does not exist a Sobolev extension theorem from H^1 of the strip to $H^1(\mathbb{R}^2)$, even though such a result for inhomogeneous Sobolev spaces is well known and essentially trivial.

The trace of a function of finite energy on a single line does not, of course, determine the function. What about the trace of an infinite collection of lines that together form a dense subset of the plane? A simple example is the set of lines of "graph paper," where we take the graph paper squares to have side length m^n, where m is an integer ($m \ge 2$) and n varies over $\mathbb{Z}$, so the graph papers GP_{m^n} are nested. The main results of this paper are first a trace theorem that characterizes the traces of $H^1(\mathbb{R}^2)$ functions on GP_{m^n} in terms of a Sobolev space $H^{1/2}(\mathsf{GP}_{m^n})$ with a given norm, and then the characterization of $H^1(\mathbb{R}^2)$ in terms of a uniform bound on the norms of the traces on GP_{m^n} as $n \to -\infty$.

The trace theorem is discussed in Section 3 in the context of Sobolev spaces $H^{1/2}$ on **metric graphs** (graphs whose edges have specified length, [4]), as discussed in Section 2. Because the functions in these spaces need not be continuous, the key issue is to understand a kind of "gluing" condition at the vertices of the graph. It turns out that this condition was given in [16]. For the convenience of the reader, we give all the proofs in Section 2, although many of the results are already known, because they are usually treated in the context of inhomogeneous Sobolev spaces. In Section 4, we discuss the trace characterizations of $H^1(\mathbb{R}^2)$. In Section 5, we discuss the analogous results on the two fractals. It turns out that the trace theorems are already known [7, 8, 9], and the Sobolev spaces are H^β for values satisfying $1/2 < \beta < 1$. The spaces of functions of finite energy on these fractals consist of continuous functions, as do the trace spaces; so there is no difficulty defining the traces, and the "gluing" condition at vertices is simply continuity. Thus the fractal analog of the trace characterization is perhaps simpler than the theorem in the plane. We also characterize the traces on Julia sets of functions of finite energy in the unbounded component of the complement of the Julia set. We believe strongly that there is a great benefit to thinking about problems in both the smooth and the fractal contexts, and to looking for interactions in the ideas that emerge. We hope this paper gives some support to this point of view.

2 Metric graphs

Definition 2.1. A **metric graph** $G = (V, E, L_e)$ consists of a graph (V, E) with vertices V and edges E and a function that assigns a length L_e in $(0, \infty]$ to each edge $e \in E$.

For a metric graph $G = (V, E, L_e)$, define the homogeneous Sobolev norm

$$\begin{aligned} \|f\|^2_{H^{1/2}(G)} = &\sum_{e\in E} \int_0^{L_e} \int_0^{L_e} \frac{|f(e(x)) - f(e(y))|^2}{|x-y|^2} dxdy \\ &+ \sum_{e\sim e'} \int_0^{L} \frac{\left|f(e(x)) - f(e'(x))\right|^2}{x} dx, \end{aligned} \tag{2.1}$$

where in the second sum $L = \min(L_e, L_{e'})$ and the parameterizations of e and e' are chosen so that $e(0)$ and $e'(0)$ correspond to the intersection point. We define the Sobolev space $H^{1/2}(G)$ to be the equivalence classes (modulo constants) of locally L^2 functions for which the norm is finite. It is easy to see that $H^{1/2}(G)$ is a Hilbert space.

Example 1. Let $G = \mathbb{R}$, so G has no vertices and a single edge of infinite length. We need to modify (2.1) in this case to read

$$\|f\|^2_{H^{1/2}(\mathbb{R})} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|f(x) - f(y)|^2}{|x-y|^2} dxdy. \tag{2.2}$$

For this example, we also want to consider the smaller norm

$$\|f\|^2_{\tilde{H}^{1/2}(\mathbb{R})} = \iint_{|x-y|\le 1} \frac{|f(x) - f(y)|^2}{|x-y|^2} dxdy \tag{2.3}$$

and the corresponding larger Sobolev space $\tilde{H}^{1/2}(\mathbb{R})$.

We note that the space $H^{1/2}(\mathbb{R})$ is **Möbius invariant**, which means that $f \in H^{1/2}(\mathbb{R})$ if and only if $f \circ M \in H^{1/2}(\mathbb{R})$ with equal norms, for $M(x) = (ax+b)/(cx+d)$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$. Indeed, it suffices to verify this for translations $M(x) = x + b$, dilations $M(x) = ax$, and the inversion $M(x) = 1/x$, where it follows by a simple change of variable in the integral defining the norm. We note that the same statement is false for $\tilde{H}^{1/2}(\mathbb{R})$.

We may easily characterize these norms and spaces in terms of the Fourier transform $\hat{f}$. The finiteness of the norm easily implies that f is a tempered distribution, so $\hat{f}$ is well-defined as a tempered distribution; and the equivalence of the functions that differ by a constant means $\hat{f}$ is only defined up to the addition of an arbitrary multiple of the delta function. Note that there is no "canonical" choice of f and $\hat{f}$ within each equivalence class.

Theorem 2.2. a) *$f \in H^{1/2}(\mathbb{R})$ if and only if $\hat{f}$ may be identified with a function that is locally in L^2 in the complement of the origin with*

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 |\xi| d\xi < \infty; \tag{2.4}$$

in fact, (2.4) *is a constant multiple of* (2.2).

b) *$f \in \tilde{H}^{1/2}(\mathbb{R})$ if and only if $\hat{f}$ may be identified with a function that is locally in L^2 in the complement of the origin, with*

$$\int_{|\xi|\geq 1} |\hat{f}(\xi)|^2 |\xi|\, d\xi + \int_{|\xi|\leq 1} |\hat{f}(\xi)|^2 |\xi|^2 d\xi < \infty; \tag{2.5}$$

and (2.5) *is bounded above and below by a multiple of* (2.3).

Proof. a) is, of course, well known and follows from the formal computation

$$\begin{aligned} \|f\|^2_{H^{1/2}(\mathbb{R})} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x+t) - f(x)|^2 dx \frac{dt}{t^2} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 |e^{2\pi i \xi t} - 1|^2 d\xi \frac{dt}{t^2} \\ &= c\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 |\xi|\, d\xi \end{aligned}$$

for $c = \int_{-\infty}^{\infty} |e^{2\pi i t} - 1|^2 (dt/t^2)$.

To prove b), we similarly compute

$$\begin{aligned} \|f\|^2_{\tilde{H}^{1/2}(\mathbb{R})} &= \int_{-1}^{1}\int_{-\infty}^{\infty} |f(x+t) - f(x)|^2\, dx \frac{dt}{t^2} \\ &= \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \left(\int_{-1}^{1} |e^{2\pi i \xi t} - 1|^2 \frac{dt}{t^2} \right) d\xi. \end{aligned}$$

Now

$$\int_{-1}^{1} |e^{2\pi i \xi t} - 1|^2 \frac{dt}{t^2} = |\xi| \int_{-|\xi|}^{|\xi|} |e^{2\pi i t} - 1|^2 \frac{dt}{t^2};$$

and for $|\xi| \geq 1$, the last integral is bounded above and below by a constant. On the other hand, for $|\xi| \leq 1$, the integrand is bounded above and below by a constant, so the integral is bounded above and below by the length of the interval. This shows the equivalence of (2.3) and (2.5).

The formal computation easily implies that any $f \in \tilde{H}^{1/2}(\mathbb{R})$ has a Fourier transform satisfying (2.5). To complete the proof, we need to show that any locally L^2 function $g(\xi)$ with

$$\int_{|\xi|\geq 1} |g(\xi)|^2 |\xi|\, d\xi + \int_{|\xi|\leq 1} |g(\xi)|^2 |\xi|^2 d\xi < \infty \tag{2.6}$$

is, in fact, the Fourier transform of a function in $\tilde{H}^{1/2}(\mathbb{R})$. Since the only problem is near the origin, we may assume that g is supported in $[-1, 1]$. Let $h(\xi) = \xi g(\xi)$. Note that $h \in L^2$ by (2.6). We define a distribution $\tilde{g}$ associated to g as follows. Note that $\langle \tilde{g}, \varphi \rangle = \int h(\xi)\varphi(\xi)(d\xi/\xi)$ is well-defined for any $\varphi \in \mathcal{S}$ with $\varphi(0) = 0$. Choose $\psi \in \mathcal{S}$ with $\psi(0) = 1$. Then $\varphi(\xi) = (\varphi(\xi) - \varphi(0)\psi(\xi)) + \varphi(0)\psi(\xi)$, with the first summand vanishing at the origin. We choose to have $\langle \tilde{g}, \psi \rangle = 0$, so our definition of $\tilde{g}$ is

$$\langle \tilde{g}, \varphi \rangle = \int h(\xi)\,(\varphi(\xi) - \varphi(0)\psi(\xi))\,\frac{d\xi}{\xi}. \tag{2.7}$$

It follows that $h(\xi) = \xi\tilde{g}(\xi)$ in the distribution sense. The inverse Fourier transform of $\tilde{g}$ is the function f. Note that f has a derivative in L^2, so it is continuous; and the formal computation shows that $f \in \tilde{H}^{1/2}(\mathbb{R})$. □

A trivial consequence of the theorem is that the space $\tilde{H}^{1/2}(\mathbb{R})$ is strictly larger than $H^{1/2}(\mathbb{R})$. On the other hand, $L^2(\mathbb{R}) \cap \tilde{H}^{1/2}(\mathbb{R}) = L^2(\mathbb{R}) \cap H^{1/2}(\mathbb{R})$.

Example 2. Let G be the graph with one vertex and two edges of infinite length meeting at the vertex. We may realize G as the real line with edges $(-\infty, 0]$ and $[0, \infty)$, and we write it as $\mathbb{R}^- \cup \mathbb{R}^+$. Then (2.1) can be written explicitly as

$$\begin{aligned} \|f\|^2_{H^{1/2}(\mathbb{R}^-\cup\mathbb{R}^+)} &= \int_{-\infty}^0 \int_{-\infty}^0 \frac{|f(x) - f(y)|^2}{|x-y|^2} dxdy \\ &\quad + \int_0^\infty \int_0^\infty \frac{|f(x) - f(y)|^2}{|x-y|^2} dxdy \\ &\quad + \int_0^\infty \frac{|f(x) - f(-x)|^2}{x} dx. \end{aligned} \tag{2.8}$$

Theorem 2.3. *The spaces $H^{1/2}(\mathbb{R}^- \cup \mathbb{R}^+)$ and $H^{1/2}(\mathbb{R})$ are identical with equivalent norms.*

Proof. This result is essentially contained in [16, Section III.3]. Let x, y stand for variables that are always positive. Since $\int_0^\infty dy/(x+y)^2 = 1/x$, we have

$$\int_0^\infty \frac{|f(x) - f(-x)|^2}{x}\, dx = \int_0^\infty \int_0^\infty \frac{|f(x) - f(-x)|^2}{(x+y)^2} dydx.$$

Writing $f(x) - f(-x) = (f(x) - f(y)) + (f(y) - f(-x))$, we have by the triangle inequality

$$\left(\int_0^\infty \frac{|f(x)-f(-x)|^2}{x}dx\right)^{1/2} \leq \left(\int_0^\infty\int_0^\infty \frac{|f(x)-f(y)|^2}{(x+y)^2}\,dydx\right)^{1/2} + \left(\int_0^\infty\int_0^\infty \frac{|f(y)-f(-x)|^2}{|x+y|^2}dydx\right)^{1/2} \leq 2\|f\|_{H^{1/2}(\mathbb{R})},$$

since $1/(x+y)^2 \leq 1/(x-y)^2$. This yields the bound of (2.8) by a multiple of $\|f\|_{H^{1/2}(\mathbb{R})}$. A similar argument gives

$$\left(\int_0^\infty\int_0^\infty \frac{|f(y)-f(-x)|^2}{|x+y|^2}dydx\right)^{1/2} \leq \left(\int_0^\infty\int_0^\infty \frac{|f(x)-f(y)|^2}{|x+y|^2}dydx\right)^{1/2} + \left(\int_0^\infty \frac{|f(x)-f(-x)|^2}{x}dx\right)^{1/2}$$

for the bound in the other direction. □

Example 3. Let G be the graph $\mathbb{Z}$; in other words, the vertices are the integers and the edges are $[k, k+1]$ for $k \in \mathbb{Z}$ of length 1. Then (2.1) is explicitly

$$\|f\|^2_{H^{1/2}(\mathbb{Z})} = \sum_{k\in\mathbb{Z}}\int_k^{k+1}\int_k^{k+1} \frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy + \sum_{k\in\mathbb{Z}}\int_0^1 \frac{|f(k+t)-f(k-t)|^2}{t}dt. \tag{2.9}$$

Theorem 2.4. *The spaces $H^{1/2}(\mathbb{Z})$ and $\tilde{H}^{1/2}(\mathbb{R})$ are identical with equivalent norms.*

Proof. The first term on the right side of (2.9) is clearly bounded by $\|f\|^2_{\tilde{H}^{1/2}(\mathbb{R})}$.

For the second term, we note that an argument as in the proof of Theorem 2.3 gives the estimate

$$\int_0^1 \frac{|f(k+t)-f(k-t)|^2}{t}dt \leq c\int_{k-1}^{k+1}\int_{k-1}^{k+1} \frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy;$$

and summing over $k \in \mathbb{Z}$, we obtain

$$\sum_{k\in\mathbb{Z}}\int_0^1 \frac{|f(k+t)-f(k-t)|^2}{t}dt \leq c\iint_{|x-y|\leq 2} \frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy.$$

A straightforward estimate controls the integral over $1 \leq |x-y| \leq 2$ by a multiple of the integral over $|x-y| \leq 1$, so we have

$$\|f\|^2_{H^{1/2}(\mathbb{Z})} \leq c\|f\|^2_{\tilde{H}^{1/2}(\mathbb{R})}.$$

For the reverse estimate, we use an argument in the proof of Theorem 2.3 to obtain

$$\begin{aligned}\int_{k-1}^{k}\int_{k}^{k+1}\frac{|f(x)-f(y)|^2}{|x-y|^2}\,dxdy \le \int_{k-1}^{k}\int_{k-1}^{k}\frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy\\ +\int_{k}^{k+1}\int_{k}^{k+1}\frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy\\ +\int_{0}^{t}\frac{|f(k+t)-f(k-t)|^2}{t}dt\end{aligned}$$

and then sum over $k \in \mathbb{Z}$. □

Example 4. Let G be the square graph SQ_δ with side length δ. So SQ_δ has 4 vertices that we identify with the points $(0,0)$, $(\delta,0)$, (δ,δ), $(0,\delta)$ in the plane and 4 edges of length δ. Then

$$\begin{aligned}\|f\|^2_{H^{1/2}(\mathsf{SQ}_\delta)} &= \int_0^\delta\int_0^\delta\frac{|f(x,0)-f(y,0)|^2}{|x-y|^2}\,dxdy+\int_0^\delta\int_0^\delta\frac{|f(\delta,x)-f(\delta,y)|^2}{|x-y|^2}dxdy\\ &+\int_0^\delta\int_0^\delta\frac{|f(x,\delta)-f(y,\delta)|^2}{|x-y|^2}\,dxdy+\int_0^\delta\int_0^\delta\frac{|f(0,x)-f(0,y)|^2}{|x-y|^2}dxdy\\ &+\int_0^\delta|f(x,0)-f(0,x)|^2\,\frac{dx}{x}+\int_0^\delta|f(x,0)-f(\delta,\delta-x)|^2\frac{dx}{x}\\ &+\int_0^\delta|f(x,\delta)-f(\delta,x)|^2\,\frac{dx}{x}+\int_0^\delta|f(0,x)-f(\delta-x,\delta)|^2\frac{dx}{x}.\end{aligned} \tag{2.10}$$

Although the $H^{1/2}(\mathsf{SQ}_\delta)$ norm does not involve comparisons between values on opposite edges, it is not difficult to show bounds

$$\begin{aligned}\int_0^1|f(x,0)-f(x,\delta)|^2\,dx &\le c\delta\|f\|^2_{H^{1/2}(\mathsf{SQ}_\delta)}\\ \int_0^1|f(0,y)-f(\delta,y)|^2\,dy &\le c\delta\|f\|^2_{H^{1/2}(\mathsf{SQ}_\delta)}.\end{aligned} \tag{2.11}$$

Example 5. Let G be the graph paper graph GP_δ with vertices at $\{(j\delta,k\delta)\}$, $j,k \in \mathbb{Z}$ and horizontal and vertical edges of length δ joining $(j\delta,k\delta)$ with

$((j+1)\delta, k\delta)$ and $(j\delta, k\delta)$ with $(j\delta, (k+1)\delta)$. The norm is given by

$$\begin{aligned} \|f\|^2_{H^{1/2}(\mathsf{GP}_\delta)} &= \sum_j \sum_k \int_0^\delta \int_0^\delta \frac{|f(j\delta+x,k\delta)-f(j\delta+y,k\delta)|^2}{|x-y|^2} dxdy \\ &+ \sum_j \sum_k \int_0^\delta \int_0^\delta \frac{|f(j\delta,k\delta+x)-f(j\delta,k\delta+y)|^2}{|x-y|^2} dxdy \\ &+ \sum_j \sum_k \int_{-\delta}^\delta |f(j\delta+x,k\delta)-f(j\delta,k\delta+x)|^2 \frac{dx}{|x|} \\ &+ \sum_j \sum_k \int_0^\delta |f(j\delta+x,k\delta)-f(j\delta-x,k\delta)|^2 \frac{dx}{x} \\ &+ \sum_j \sum_k \int_0^\delta |f(j\delta,k\delta+x)-f(j\delta,k\delta-x)|^2 \frac{dx}{x}. \end{aligned} \tag{2.12}$$

Of course, we could get an equivalent norm by deleting the last two sums in (2.12), as they are controlled by the third sum. We may regard GP_δ as a countable union of square graphs SQ_δ; indeed, it is easily seen that $f \in H^{1/2}(\mathsf{GP}_\delta)$ if and only if the restriction of f to each of the square graphs is in $H^{1/2}(\mathsf{SQ}_\delta)$ with the sum of the squares of the norms $\|f\|^2_{H^{1/2}(\mathsf{SQ}_\delta)}$ finite, and this gives an equivalent norm.

3 Traces of functions of finite energy

Consider the homogeneous Sobolev space $H^1(\mathbb{R}^2)$ of functions with finite energy

$$\|F\|^2_{H^1(\mathbb{R})} = \int_{\mathbb{R}^2} |\nabla F(x,y)|^2 dxdy. \tag{3.1}$$

These form a Hilbert space modulo constants. Functions of finite energy do not have to be continuous, as the example $F(x,y) = \log|\log(x^2+y^2)|$ (multiplied by an appropriate cutoff function) shows. However, it is well known that these functions have well-defined traces on straight lines that are in $H^{1/2}(\mathbb{R})$, and $H^{1/2}(\mathbb{R})$ is the exact space of traces. Since the usual treatment of traces involves inhomogeneous Sobolev spaces, we give the proof for the convenience of the reader. We omit the routine step of actually defining the traces and just prove the norm estimates.

Theorem 3.1. *The trace map $T : H^1(\mathbb{R}^2) \to H^{1/2}(\mathbb{R})$ given formally by $TF(x) = F(x,0)$ is continuous; i.e.,*

$$\|TF\|_{H^{1/2}(\mathbb{R})} \le c\|F\|_{H^1(\mathbb{R}^2)}. \tag{3.2}$$

Moreover, there exists a continuous extension map $E : H^{1/2}(\mathbb{R}) \to H^1(\mathbb{R}^2)$ with $TEf = f$ and

$$\|Ef\|_{H^1(\mathbb{R}^2)} \le c\|f\|_{H^{1/2}(\mathbb{R})}. \tag{3.3}$$

Proof. We work on the Fourier transform side, where

$$\|F\|^2_{H^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (\xi^2 + \eta^2)|\hat{F}(\xi,\eta)|^2 d\xi d\eta \quad \text{and} \tag{3.4}$$

$$(Tf)^\wedge(\xi) = \int_{-\infty}^{\infty} \hat{F}(\xi,\eta) d\eta. \tag{3.5}$$

By Theorem 2.2, we have

$$\|Tf\|^2_{H^{1/2}(\mathbb{R})} = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \hat{F}(\xi,\eta)\, d\eta \right|^2 |\xi| d\xi.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \hat{F}(\xi,\eta)\, d\eta \right|^2 &\le \left(\int_{-\infty}^{\infty} (\xi^2+\eta^2)|\hat{F}(\xi,\eta)|^2\, d\eta \right) \left(\int_{-\infty}^{\infty} \frac{1}{\xi^2+\eta^2} d\eta \right) \\ &= \frac{\pi}{|\xi|} \left(\int_{-\infty}^{\infty} (\xi^2+\eta^2)|\hat{F}(\xi,\eta)|^2 d\eta \right), \end{aligned}$$

so

$$\begin{aligned} \|Tf\|^2_{H^{1/2}(\mathbb{R})} &\le \pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\xi^2+\eta^2)|\hat{F}(\xi,\eta)|^2 d\eta \\ &= \pi \|F\|^2_{H^1(\mathbb{R}^2)}, \end{aligned}$$

which proves(3.2).

Conversely, given $f \in H^{1/2}(\mathbb{R})$, define $Ef = F$ by the Poisson integral

$$F(x,y) = \frac{|y|}{\pi} \int \frac{f(x-t)}{t^2+y^2} dt, \tag{3.6}$$

so that $TF = f$. Then

$$\hat{F}(\xi,\eta) = \frac{1}{\pi} \frac{\hat{f}(\xi)|\xi|}{\eta^2+|\xi|^2}. \tag{3.7}$$

By (3.4), we have

$$\begin{aligned} \|F\|^2_{H^1(\mathbb{R}^2)} &= \frac{1}{\pi^2} \int_{\mathbb{R}^2} \frac{|\hat{f}(\xi)|^2|\xi|^2}{\eta^2+\xi^2} d\xi d\eta \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 |\xi| d\xi, \end{aligned}$$

so we obtain (3.3) by Theorem 2.2. □

Note that we define the extension Ef to be harmonic in each half-plane $y > 0$ and $y < 0$. Since harmonic functions minimize energy, our extension achieves the minimum $H^1(\mathbb{R}^2)$ norm.

There is a virtually identical trace theorem for functions of finite energy in the half-plane, say $y > 0$, denoted $\mathbb{R}^2_+$. To see this, we have only to observe that an even reflection

$$RF(x, y) = F(x, -y) \quad \text{for } y < 0 \tag{3.8}$$

maps $H^1(\mathbb{R}^2_+)$ continuously to $H^1(\mathbb{R}^2)$.

Theorem 3.2. *The trace map $T : H^1(\mathbb{R}^2_+) \to H^{1/2}(\mathbb{R})$ given formally by $TF(x) = F(x, 0)$ is well-defined and bounded, and there exists a bounded extension map $E : H^{1/2}(\mathbb{R}) \to H^1(\mathbb{R}^2_+)$ with $TEf = f$. Moreover, the analogues of* (3.2) *and* (3.3) *hold.*

If we combine this with the well-known observation that energy is conformally invariant in the plane (not true in other dimensions, however), we obtain a powerful tool for obtaining trace theorems for other domains: find a conformal map between the domain and the half-space $\mathbb{R}^2_+$, and transfer the $H^{1/2}(\mathbb{R})$ norm from the boundary of $\mathbb{R}^2_+$ to the boundary of the domain, assuming the conformal map extends continuously to the boundary.

A simple example is the strip $S = \{(x, y) : 0 < y < \pi\}$. In complex variables notation, $\varphi(z) = \log z$ is the conformal map from $\mathbb{R}^2_+$ to S, with $\psi(z) = e^z$ its inverse. So $F \in H^1(S)$ if and only if $F \circ \varphi \in H^1(\mathbb{R}^2_+)$ with equal norms. Then $f(t) = F(\varphi(t)) \in H^{1/2}(\mathbb{R})$. In light of Theorem 2.2, this means

$$\begin{aligned}\int_0^\infty \int_0^\infty \frac{|F(\log t) - F(\log s)|^2}{|t-s|^2} dtds + \int_0^\infty \int_0^\infty \frac{|F(\log t + i\pi) - F(\log s + i\pi)|^2}{|t-s|^2} dtds \\ + \int_0^\infty |F(\log t) - F(\log t + i\pi)|^2 \frac{dt}{t} \\ \leq c\|F\|^2_{H^1(S)}.\end{aligned} \tag{3.9}$$

The change of variable $x = \log t$, $y = \log s$ transforms the left hand side of (3.9) into

$$\begin{aligned}&\int_{-\infty}^\infty \int_{-\infty}^\infty |F(x) - F(y)|^2 \frac{e^x e^y}{|e^x - e^y|^2} dxdy \\ &+ \int_{-\infty}^\infty \int_{-\infty}^\infty |F(x+y) - F(x+i\pi)|^2 \frac{e^x e^y}{|e^x - e^y|} dxdy \\ &+ \int_{-\infty}^\infty |F(x) - F(x+i\pi)|^2 dx.\end{aligned} \tag{3.10}$$

To simplify notation, we split the trace of F on the boundary of S into two pieces $T_0F(x) = F(x)$ and $T_1F(x) = F(x + i\pi)$, so that T_0F and T_1F are functions on $\mathbb{R}$.

Theorem 3.3. *If* $F \in H^1(S)$ *then* T_0F *and* T_1F *are in* $\tilde{H}^{1/2}(\mathbb{R})$ *and* $T_0F - T_1F \in L^2(\mathbb{R})$*, with*

$$\|T_0F\|^2_{\tilde{H}^{1/2}(\mathbb{R})} + \|T_1F\|^2_{\tilde{H}^{1/2}(\mathbb{R})} + \|T_0F - T_1F\|_2^2 \leq c\|F\|^2_{H^1(S)}. \tag{3.11}$$

Conversely, given f_0 *and* f_1 *in* $\tilde{H}^{1/2}(\mathbb{R})$ *with* $f_0 - f_1 \in L^2(\mathbb{R})$*, there exists* $F = E(f_0, f_1)$ *with* $T_0F = f_0$*,* $T_1F = f_1$*,* $F \in H^1(S)$ *with the reverse estimate of* (3.11) *holding.*

Proof. In view of (3.10), it suffices to show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(x) - F(y)|^2 \frac{e^x e^y}{|e^x e^y|}|e^x - e^y|^2 dxdy \tag{3.12}$$

is bounded above and below by a constant multiple of

$$\iint_{|x-y|\leq 1} \frac{|F(x) - F(y)|^2}{|x-y|^2} dxdy = \|F\|^2_{\tilde{H}^{1/2}(\mathbb{R})}. \tag{3.13}$$

Note that we may rewrite (3.12) as

$$\frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|F(x) - F(y)|^2}{|\sinh((x-y)/2)|^2} dxdy. \tag{3.14}$$

It is clear that (3.14) is bounded below by a multiple of (3.13). For the upper bound, we need $\iint_{|x-y|\geq 1} |f(x) - f(y)|^2 / |\sinh((x-y)/2)|^2\, dx\, dy$ to be bounded above by a multiple of (3.13). However, proving this is a routine exercise because of the exponential decay of $|\sinh((x-y)/2)|^{-2}$. □

It might seem perplexing that the trace space on each of the lines is larger than $H^{1/2}(\mathbb{R})$, since in particular this implies that there are functions in $H^1(S)$ that do not extend to $H^1(\mathbb{R}^2)$. However, it is easy to give an example of such a function: just take $F(x, y) = g(x)$, where $g(0) = 0$ for $x \leq 0$, $g(x) = 1$ for $x \geq 1$, and g is smooth in $[0, 1]$. Then ∇F has compact support in S; so $F \in H^{1/2}(S)$, but $g \notin H^{1/2}(\mathbb{R})$.

Another simple example is the first quadrant $Q = \{(x, y) : x > 0 \text{ and } y > 0\}$. Then $\varphi(z) = \sqrt{z}$ is the conformal map of $\mathbb{R}^2_+$ to Q, with inverse $\psi(z) = z^2$. Again it is convenient to split the trace into two parts mapping to functions on $\mathbb{R}_+$, namely $T_0F(x) = F(x, 0)$ and $T_1F(x) = F(0, x)$. Since $F \in H^1(Q)$ if and only if $F \circ \varphi \in H^1(\mathbb{R}^2_+)$, again by Theorem 2.2, we have the expression

$$\int_0^{\infty}\int_0^{\infty} \frac{|T_0F(\sqrt{t}) - T_0F(\sqrt{s})|^2}{|t-s|^2} dsdt + \int_0^{\infty}\int_0^{\infty} \frac{|T_1F(\sqrt{t}) - T_1F(\sqrt{s})|^2}{|t-s|^2} dsdt$$
$$+ \int_0^{\infty} |T_0F(\sqrt{t}) - T_1F(\sqrt{t})|^2 \frac{dt}{t} \tag{3.15}$$

for the trace norm. With the substitutions $t = x^2$, $s = y^2$, this becomes

$$
\begin{aligned}
&4\int_0^\infty \int_0^\infty \frac{|T_0F(x) - T_0F(y)|^2}{|x-y|^2} \frac{xy}{|x+y|^2} dxdy \\
&\qquad + 4\int_0^\infty \int_0^\infty \frac{|T_1F(x) - T_1F(y)|^2}{|x-y|^2} \frac{xy}{|x+y|^2} dxdy \\
&\qquad + 2\int_0^\infty |T_1F(x) - T_1F(x)|^2 \frac{dx}{x}.
\end{aligned} \tag{3.16}
$$

It is easy to see that if $f_0, f_1 \in H^{1/2}(\mathbb{R}_+)$ and

$$
\int_0^\infty |f_0(x) - f_1(x)|^2 \frac{dx}{x} < \infty, \tag{3.17}
$$

then there exists $F \in H^1(Q)$ with $T_0F = f_0$ and $T_1F = f_1$, because $xy/|x+y|^2$ is bounded. In other words, the function

$$
f(x) = \begin{cases} f_0(x) & \text{if } x > 0, \\ f_1(x) & \text{if } x < 0 \end{cases}
$$

is in $H^{1/2}(\mathbb{R})$, and $\|F\|_{H^1(Q)} \le c\|f\|_{H^{1/2}(\mathbb{R})}$. It is possible to show the converse statement as well, but this involves some technicalities, since $xy/|x+y|^2$ is not bounded below. It is easier to observe that $F \in H^1(Q)$ may be extended by even reflection across the axes to a function in $H^1(\mathbb{R}^2)$. Thus the even reflections of T_0F and T_1F must be in $H^{1/2}(\mathbb{R}^2)$, so T_0F and T_1F must be in $H^{1/2}(\mathbb{R}_+)$; and we already have (3.17) for $f_0 = T_0F$, $f_1 = T_1F$.

A direct proof of (3.17) is possible, but involves technicalities.

Another simple example is the unit disk D, with the conformal mapping $\varphi(z) = (1-z)/(1+z)$ of $\mathbb{R}^2_+$ to D. The trace space of $H^1(D)$ is $H^{1/2}(C)$ for C the unit circle with norm

$$
\|f\|^2_{H^{1/2}(C)} = \int_0^{2\pi} \int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\theta'})|^2}{4\left|\sin \frac{1}{2}(\theta - \theta')\right|^2} d\theta d\theta'. \tag{3.18}
$$

Of course, $2\left|\sin \frac{1}{2}(\theta - \theta')\right|$ is exactly the chordal distance $|e^{i\theta} - e^{i\theta'}|$. It is interesting to observe that exactly the same trace space arises from the exterior of the circle $\{|z| > 1\}$, as $z \mapsto 1/\overline{z}$ is an anticonformal map of D to this exterior domain that agrees with $\varphi(z)$ on the circle. Similarly, for a circle C_r of radius r, the analog of (3.18) is

$$
\|f\|^2_{H^{1/2}(C_r)} = \int_0^{2\pi} \int_0^{2\pi} \frac{|f(re^{i\theta}) - f(re^{i\theta'})|^2}{4\left|r \sin \frac{1}{2}(\theta - \theta')\right|^2} rd\theta rd\theta'. \tag{3.19}
$$

Of course, it is not necessary to use a conformal map φ. A Lipschitz map or even a quasiconformal map changes the H^1 norm by a bounded amount. So for the interior SQ_δ° of the square SQ_δ, the trace space of $H^1(\mathsf{SQ}_\delta^\circ)$ is $H^{1/2}(\mathsf{SQ}_\delta)$ with norm given by (2.10), since one can "square the circle" with a Lipschitz map.

Next we consider traces on infinite collections of lines. First consider the horizontal line collection $\mathsf{HLC} = \{(x, n\pi) : x \in \mathbb{R}, n \in \mathbb{Z}\}$. For a function F in $H^1(\mathbb{R}^2)$, define the traces $T_nF(x) = F(x, \pi n)$.

Theorem 3.4. *A set of functions $\{f_n\}$ on $\mathbb{R}$ consists of the traces $f_n = T_nF$ of a function $F \in H^1(\mathbb{R}^2)$ if and only if $f_n \in \tilde{H}^{1/2}(\mathbb{R})$ and $f_n - f_{n+1} \in L^2(\mathbb{R})$ with*

$$\sum_n \|f_n\|^2_{\tilde{H}^{1/2}(\mathbb{R})} + \sum_n \|f_n - f_{n+1}\|^2_{L^2(\mathbb{R})} < \infty \tag{3.20}$$

and the corresponding norm equivalence holds.

Proof. Basically, we just have to apply Theorem 3.3 to each of the strips $\{n\pi < y < (n+1)\pi\}$ and sum (3.11) over all the strips. To do this, we simply observe that a function belongs to $H^1(\mathbb{R}^2)$ if and only if its restriction to each strip is in H^1 of that strip, the traces agree on neighboring strips, and the sum of the energies is finite. $\square$

There is something a bit unsettling about this result. We know that $f_n = T_nF$ actually belongs to the smaller space $H^{1/2}(\mathbb{R})$ for $F \in H^1(\mathbb{R}^2)$, yet this space plays no role in the characterization (3.20). An indirect consequence of the theorem is that if $\{f_n\}$ is a family of functions satisfying (3.20), then each f_n is indeed in $H^{1/2}(\mathbb{R})$. It should be possible to prove this directly, but again this seems rather technical. Note that we only get a uniform bound for $\|f_n\|^2_{H^{1/2}(\mathbb{R})}$. The following example shows that we cannot do too much better than this (most likely $\|f_n\|^2_{H^{1/2}(\mathbb{R})} = o(1)$).

Consider the function $F(x, y) = (1+x^2+y^2)^{-\alpha}$ for $\alpha > 0$. A direct computation shows that $|\nabla F(x, y)| \le 2\alpha(1+x^2+x^y)^{-\alpha-1/2}$, so $F \in H^1(\mathbb{R}^2)$. Now

$$T_nF(x) = (1+\pi^2n^2+x^2)^{-\alpha} = (1+\pi^2n^2)^{-\alpha} g\left(\frac{x}{\sqrt{1+\pi^2n^2}}\right)$$

for $g(x) = (1+x^2)^{-\alpha}$. It is easy to see that $g \in H^{1/2}(\mathbb{R})$; so by dilation invariance of the $H^{1/2}(\mathbb{R})$ norm, we see that $\|T_nF\|^2_{H^{1/2}(\mathbb{R})} = c(1+\pi^2n^2)^{-2\alpha}$. Thus $\sum \|T_nF\|^2_{H^{1/2}(\mathbb{R})} = \infty$ for $\alpha \le 1/4$.

Next we consider the trace on the graph paper graph GP_δ.

Theorem 3.5. *The trace space of $H^1(\mathbb{R}^2)$ on GP_δ is exactly $H^{1/2}(\mathsf{GP}_\delta)$ with norm given by* (2.12).

Proof. We simply use the trace theorem of H^1 on each δ-square that makes up GP_δ and add. $\square$

In place of square graph paper, we could consider the **triangular graph paper** TGP_δ consisting of the tiling of the plane by equilateral triangles of side length δ. The analog of Theorem 3.5 holds with essentially the same proof.

4 The graph paper trace characterization

In this section, we fix an integer $m \geq 2$ and consider the sequence of graph paper graphs GP_{m^n}, thought of as the unions of the edges, or equivalently, the countable union of horizontal and vertical lines in the plane with m^n separation. These are nested subsets of the plane, $\mathsf{GP}_{m^n} \subset \mathsf{GP}_{m^{n'}}$ if $n' < n$, and we are interested in the limit as $n \to -\infty$, so the graph paper gets increasingly finer.

We let T_n denote the trace map from functions defined on $\mathbb{R}^2$ to GP_{m^n}. By the nesting property, we may also consider T_n to be defined on functions on $\mathsf{GP}_{m^{n'}}$ with $n' < n$. Our goal is to characterize functions in $H^1(\mathbb{R}^2)$ by their traces T_nF.

Theorem 4.1. a) *Let $F \in H^1(\mathbb{R}^2)$. Then $T_nF \in H^{1/2}(\mathsf{GP}_{m^n})$ for all n with uniformly bounded norms, and*

$$\sup_{n\in\mathbb{Z}} \|T_nF\|^2_{H^{1/2}(\mathsf{GP}_{m^n})} \leq c\|F\|^2_{H^1(\mathbb{R}^2)}. \tag{4.1}$$

b) *Let $\{f_n\} \subset H^{1/2}(\mathsf{GP}_{m^n})$ be a sequence of functions with uniformly bounded norms satisfying the consistency condition $T_nf_{n'} = f_n$ for $n' < n$. Then there exists $F \in H^1(\mathbb{R}^2)$ such that $T_nF = f_n$ and*

$$\|F\|^2_{H^1(\mathbb{R}^2)} \leq c \sup_{n\in\mathbb{Z}} \|f_n\|^2_{H^{1/2}(\mathsf{GP}_{m^n})}. \tag{4.2}$$

Proof. Part a) is an immediate consequence of Theorem 3.5.

To prove b), we define F_n to be the harmonic extension of f_n into each of the graph paper spaces. Since these harmonic extensions minimize energy, we have $F_n \in H^1(\mathbb{R}^2)$ and

$$\|F_n\|_{H^1(\mathbb{R}^2)} \leq c\|f_n\|_{H^{1/2}(\mathsf{GP}_{m^n})},$$

again by Theorem 3.5. Thus there exists a subsequence $n_j \to -\infty$ such that F_{n_j} converges in the weak topology of $H^1(\mathbb{R}^2)$ to a function F satisfying (4.2). It remains to show that the weak convergence respects traces, so that $T_nF_{n_j} = f_n$ for all n_j implies $T_nF = f_n$.

But the equality of traces on GP_{m^n} is the same as equality of traces on each of the lines that make up GP_{m^n}; and since all lines are essentially equivalent, it suffices to show that $F_{n_j}(x, 0)$ converges weakly in $H^{1/2}(\mathbb{R})$ to $F(x, 0)$. This is most easily seen on the Fourier transform side, where both $H^1(\mathbb{R}^2)$ and $H^{1/2}(\mathbb{R})$ are just weighted L^2 spaces.

The weak convergence $F_{n_j} \to F$ in $H^1(\mathbb{R}^2)$ says

$$\iint \hat{F}_{n_j}(\xi, \eta)G(\xi, \eta)(\xi^2 + \eta^2)\, d\xi d\eta \to \iint \hat{F}(\xi, \eta)G(\xi, \eta)(\xi^2 + \eta^2)d\xi d\eta \tag{4.3}$$

for every $G \in L^2\left((\xi^2 + \eta^2)d\xi d\eta\right)$. The weak convergence $F_{n_j}(x, 0) \to F(x, 0)$ requires that we show

$$\int \left(\int \hat{F}_{n_j}(\xi, \eta)\, d\eta\right) H(\xi)|\xi|\, d\xi \to \int \left(\int \hat{F}(\xi, \eta)\, d\eta\right) H(\xi)|\xi| d\xi \tag{4.4}$$

for every $H \in L^2(|\xi|\, d\xi)$. So given H, choose

$$G(\xi, \eta) = \frac{|\xi| H(\xi)}{\xi^2 + \eta^2}. \tag{4.5}$$

Since

$$\begin{aligned}\iint |G(\xi, \eta)|^2(\xi^2 + \eta^2)\, d\xi d\eta &= \int \left(\int \frac{|\xi|^2}{\xi^2 + \eta^2} d\eta\right) |H(\xi)|^2 d\xi \\ &= \pi \int |H(\xi)|^2 |\xi| d\xi,\end{aligned}$$

we may use the choice of G in (4.2). But then (4.3) and (4.4) are identical. □

This result localizes in several ways. For example, if $F \in H^1(\mathbb{R}^2)$ and we wish to estimate the amount of energy that is contained in an open set Ω, i.e.,

$$\int_\Omega |\nabla F|^2 dxdy, \tag{4.6}$$

we just have to take the sum of the terms in (2.12) that correspond to edges contained in Ω. Denote this sum by $\|T_n F\|^2_{H^{1/2}(\Omega \cap \mathsf{GP}_{m^n})}$. Then (4.6) is bounded above and below by a constant times

$$\sup_{n \in \mathbb{Z}} \|T_n F\|^2_{H^{1/2}(\Omega \cap \mathsf{GP}_{m^n})}. \tag{4.7}$$

We obtain the same norm equivalence if we only assume $F \in H^1(\Omega)$, meaning (4.6) is finite. (Note that this does not say anything about the trace of F on the boundary of Ω.) Also, we may start by assuming that $F \in H^1_{\text{loc}}(\mathbb{R}^2)$, meaning that (4.6) is finite whenever Ω is bounded, and obtain the norm equivalence of (4.6) and (4.7).

The same result holds with GP_{m^n} replaced with TGP_{m^n}.

It is clear that we may replace the sup in (4.2) and (4.7) by the lim sup as $n \to -\infty$. It is not clear that a limit has to exist however, since we only have estimates above and below, rather than identity, for our norms.

We can also characterize functions of finite energy by their traces on pencils of parallel lines of equal separation, i.e., the horizontal lines in GP_{m^n}. Denote this by PP_{m^n}. We use Theorem 3.4, but the norms defined by (3.20) are not dilation invariant. That means we want to define $\tilde{H}^{1/2}(\mathsf{PP}_{m^n})$ by the finiteness of

$$
\begin{aligned}
(4.8) \qquad & \sum_{k\in\mathbb{Z}} \iint_{|x-y|\le m^n} \frac{|f(x, km^n) - f(y, km^n)|^2}{|x-y|^2} dxdy \\
& \qquad + \sum_{k\in\mathbb{Z}} m^{-n} \int_{-\infty}^{\infty} |f(x, (k+1)m^n) - f(x, km^n)|^2 dx,
\end{aligned}
$$

and we define this to be $\|f\|^2_{\tilde{H}^{1/2}(\mathsf{PP}_{m^n})}$. Then the analog of Theorem 4.1 holds with T_nF equal to the trace on PP_{m^n} and $H^{1/2}(\mathsf{GP}_{m^n})$ replaced by $\tilde{H}^{1/2}(\mathsf{PP}_{m^n})$. The proof is essentially the same, using the scaled version of Theorem 3.4 with (4.8) in place of (3.20).

5 Fractals

The **Sierpinski gasket** (SG) is the self-similar fractal defined by the identity

$$
\mathsf{SG} = \bigcup_{i=0}^{2} \Phi_i(\mathsf{SG}), \tag{5.1}
$$

where Φ_i are the homothety maps of the plane $\Phi_i(x) = (x + q_i)/2$ and $\{q_0, q_1, q_2\}$ are the vertices of an equilateral triangle with side length 1. SG is the unique nonempty compact subset of the plane satisfying (5.1). The mappings $\{\Phi_i\}$ comprise what is called an **iterated function system**, and the iterates of the mappings are denoted $\Phi_w = \Phi_{w_1} \circ \cdots \circ \Phi_{w_m}$ where $w = (w_1, \ldots, w_m)$ is a word of length $|w| = m$ and $w_j \in \{0, 1, 2\}$ for each j. Then, iterating (5.1), we obtain

$$
\mathsf{SG} = \bigcup_{|w|=m} \Phi_w(\mathsf{SG}), \tag{5.2}
$$

which expresses SG as a union of 3^m miniature gaskets (called **m-cells**) that are similar to SG with similarity ratio 2^{-m}. Note that SG has the **post-critically finite (PCF)** property that distinct m-cells can intersect only at the vertices $\Phi_w q_i$. For this reason, we refer to $\{q_i\}$ as the **boundary** of SG, and $\{\Phi_w q_i\}$ as the **boundary** of the m-cell $\Phi_m(\mathsf{SG})$, although these are not boundaries in the topological sense.

We may approximate SG by the metric graphs $\mathsf{SG}_m = \mathsf{SG} \cap \mathsf{TG}_{2^{-m}}$. Here, the vertices are $\{\Phi_w q_i\}$, for $|w| = m$ and $i = 0, 1, 2$, the edges are $\{\Phi_w e_{ij}\}$ for $|w| = m$, where e_{ij} is the edge of the original triangle joining q_i and q_j, and $\Phi_w e_{ij}$ has length 2^{-m}. Denote by

$$E_m(f) = \sum_{i \neq j} \sum_{|w|=m} |f(\Phi_w q_i) - f(\Phi_w q_j)|^2 \tag{5.3}$$

the unrenormalized graph energy on SG_m. Kigami (see [10, 17]) defines an energy on SG by

$$\mathcal{E}(f) = \lim_{m \to \infty} \left(\frac{5}{3}\right)^m E_m(f). \tag{5.4}$$

The renormalization factor $(5/3)^m$ may be explained as follows. The sequence $\{(5/3)^m E_m(f)\}$ is always nondecreasing, and there exists a 3-dimensional space of harmonic functions for which it is constant. We can then define $\operatorname{dom}\mathcal{E}$, the space of functions of finite energy, as those functions for which (5.4) is finite. This is a space of continuous functions on SG that forms an infinite dimensional Hilbert space (after modding out by the constants) with norm $\mathcal{E}(f)^{1/2}$. This energy satisfies the self-similar identity

$$\mathcal{E}(f) = \sum_{i=0}^{2} \left(\frac{5}{3}\right) \mathcal{E}(f \circ \Phi_i) \tag{5.5}$$

as well as the axioms for a local regular Dirichlet form [6]. Up to a constant multiple, it is the only Dirichlet form with these properties. It is also symmetric with respect to the D_3 symmetry group of the triangle. This energy forms the basic building block for a whole theory of analysis on SG, including a theory of Laplacians. We do not use this wider theory here, but direct the curious reader to [10, 17] for details.

Since the functions in $\operatorname{dom}\mathcal{E}$ are continuous, there is no problem defining traces T_m on SG_m. The problem of characterizing the trace space $T_n(\mathsf{SG})$ on the boundary of the triangle has been solved by Jonsson [8, 9] (see [7] for a different proof) in terms of Sobolev spaces of order β, with $\beta = \frac{1}{2} + \frac{\log 5/3}{\log 4}$. Note that $1/2 < \beta < 1$. For any metric graph G, we define $H^\beta(G)$ (for any β in the above range) to be the space of continuous functions such that

$$\|F\|^2_{H^\beta(G)} = \sum_{e \in E} \int_0^{L_e} \int_0^{L_e} \frac{|F(e(x)) - F(e(y))|^2}{|x - y|^{1+2\beta}} dx dy < \infty. \tag{5.6}$$

Note that in contrast to (2.1), there is no term comparing values on intersecting edges, since the continuity condition takes care of the comparison (this idea is also used in [16]). We then have the following result, analogous to Theorem 3.1.

Proposition 5.1 ([7, 8, 9]). *The trace map T_0 is continuous from* $\operatorname{dom}\mathcal{E}$ *to* $H^\beta(\mathsf{SG}_0)$, $\beta = \frac{1}{2} + \frac{\log 5/3}{\log 4}$, *and*

$$\|T_0 F\|^2_{H^\beta(\mathsf{SG}_0} \le c\mathcal{E}(F). \tag{5.7}$$

Moreover, there exists a continuous linear extension map $E_0 : H^\beta(\mathsf{SG}_0) \to \operatorname{dom}\mathcal{E}$ such that $T_0E_0 f = f$ and

$$\mathcal{E}(E_0 f) \le c\|f\|^2_{H^\beta(\mathsf{SG}_0)}. \tag{5.8}$$

We note that [7, 8, 9] use a slightly different, but equivalent, norm for $H^\beta(\mathsf{SG}_0)$.

Next, we need to obtain the analogous statement for the trace map T_m to SG_m. We note that energy is additive for continuous functions, and in view of the self-similarity (5.5) iterated,

$$\mathcal{E}(F) = \sum_{|w|=m} \left(\frac{5}{3}\right)^m \mathcal{E}(F \circ \Phi_w). \tag{5.9}$$

Applying (5.8) to $F \circ \Phi_w$, we have

$$\sum_{|w|=m} \left(\frac{5}{3}\right)^m \|T_0 F \circ \Phi_w\|^2_{H^\beta(\mathsf{SG}_0)} \le c \sum_{|w|=m} \left(\frac{5}{3}\right)^m \mathcal{E}(F \circ \Phi_w) = c\mathcal{E}(F), \tag{5.10}$$

by (5.9). Now we observe that $\mathsf{SG}_m = \bigcup_{|w|=m} \Phi_w(\mathsf{SG}_0)$, and this is a disjoint union of edges, since each edge is just a side of a triangle $\Phi_w(\mathsf{SG}_0)$ for some w with $|w| = m$.

Consider one of these edges, $\Phi_w(e_{ij})$. It is parameterized by x in the interval $[0, 2^{-m}]$, and the contribution (5.6) is

$$\begin{aligned} &\int_0^{2^{-m}} \int_0^{2^{-m}} \frac{|F(e(x)) - F(e(y))|^2}{|x-y|^{1+2\beta}} dxdy \\ &\qquad = \frac{4^m}{2^{1+2\beta}} \int_0^1 \int_0^1 \frac{|F(\Phi_w(e_{ij}(x))) - F(\Phi_w(e_{ij}(y)))|^2}{|x-y|^{1+2\beta}} dxdy \end{aligned} \tag{5.11}$$

after a change of variables. Summing all the contributions over all the edges in SG_m yields

$$\|T_m F\|^2_{H^\beta(\mathsf{SG}_m)} = \sum_{|w|=m} \frac{4^m}{2^{(1+2\beta)m}} \|T_0 F \circ \Phi_w\|^2_{H^\beta(\mathsf{SG}_0)}, \tag{5.12}$$

by (5.11). But the choice of β makes $4/2^{1+2\beta} = 5/3$, so (5.12) combined with (5.10) yields

$$\|T_m F\|^2_{H^\beta(\mathsf{SG}_m)} \le c\mathcal{E}(F). \tag{5.13}$$

This is the exact analog of (5.7).

Theorem 5.2. *The trace map T_m is continuous from* $\operatorname{dom}\mathcal{E}$ *to* $H^\beta(\mathsf{SG}_m)$ *for β as in Proposition 5.1, and the estimate* (5.13) *holds. Moreover, there exists a continuous linear extension map $E_m : H^\beta(\mathsf{SG}_m) \to$* $\operatorname{dom}\mathcal{E}$ *with $T_m E_m f = f$ and*

$$\mathcal{E}(E_m f) \le c\|f\|^2_{H^\beta(\mathsf{SG}_m)}. \tag{5.14}$$

Proof. We have already established (5.13). To define the extension map E_m, we set

$$E_m(f) = \Phi_w^{-1} E_0(f \circ \Phi_w) \quad \text{on } \Phi_w(\mathsf{SG}). \tag{5.15}$$

Note that $E_m(f)$ is continuous, because at the boundary points of the m-cells that make up SG_m we have $E_m(f) = f$. The same reasoning that yields (5.13) from (5.7) also leads from (5.8) to (5.14). □

Next we have the analog of Theorem 4.1.

Theorem 5.3. a) *Let $F \in$* $\operatorname{dom}\mathcal{E}$. *Then $T_m F \in H^\beta(\mathsf{SG}_m)$ for all m with uniformly bounded norms, and*

$$\sup_m \|T_m F\|^2_{H^\beta(\mathsf{SG}_m)} \le c\mathcal{E}(F). \tag{5.16}$$

b) *Let $\{f_m\} \subset H^\beta(\mathsf{SG}_m)$ be a sequence of functions with uniformly bounded norms satisfying the consistency condition $T_m f_{m'} = f_m$ if $m \le m'$. Then there exists $F \in$* $\operatorname{dom}\mathcal{E}$ *such that $T_m F = f_m$ and*

$$\mathcal{E}(F) \le c \sup_m \|f_m\|^2_{H^\beta(\mathsf{SG}_m)}. \tag{5.17}$$

Proof. a) is an immediate consequence of (5.13).

To prove b), construct a sequence of functions $\{F_m\}$ by taking the harmonic (energy minimizing) extension of f_m from SG_m to SG. Then, by (5.14), the sequence $\{F_m\}$ is uniformly bounded in $\operatorname{dom}\mathcal{E}$. A quantitative version of the continuity of functions in $\operatorname{dom}\mathcal{E}$ implies that the sequence $\{F_m\}$ is also uniformly equicontinuous. Thus, by passing to a subsequence twice, if necessary, we can find a subsequence $\{F_{m_j}\}$ that converges both weakly in the Hilbert space $\operatorname{dom}\mathcal{E}$ and uniformly to a function F in $\operatorname{dom}\mathcal{E}$ with the estimate (5.17) holding. Because the convergence is pointwise and the consistency condition holds, we have $T_{m_j}F = F_{m_j} = f_{m_j}$ on SG_{m_j}. Therefore, $T_m F = f_m$. □

The second example of a fractal we consider is the **Sierpinski carpet** (SC), again defined by the self-similar identity

$$\mathsf{SC} = \bigcup_{i=1}^{8} \Phi_i(\mathsf{SC}), \tag{5.18}$$

where now Φ_i are the homothety maps of the plane with contraction ratio $1/3$ mapping the unit square into 8 of the 9 subsquares of side length $1/3$ (all except the central subsquare). This self-similar fractal is not PCF, so the method of Kigami cannot be used to construct an energy. Nevertheless, two approaches due to Barlow and Bass and to Kusuoka and Zhou [2] were given in the late 1980's. Recently, it was shown [3] that, up to a constant multiple, there is a unique self-similar energy; so the two approaches yield the same energy. Once again, all functions in $\operatorname{dom}\mathcal{E}$ are continuous. The self-similar identity for the energy here is

$$\mathcal{E}(F) = \sum_{i=1}^{8} r\mathcal{E}(F \circ \Phi_i), \tag{5.19}$$

where r is a constant (slightly larger than 1.25), whose exact value has not yet been determined.

Again we may approximate SC by a sequence $\{\mathsf{SC}_m\}$ of metric graphs, with $\mathsf{SC}_m = \mathsf{SC} \cap \mathsf{GP}_{3^{-m}}$. Thus, the edges of SC_m have length 3^{-m} and are of the form $\Phi_w(e_i)$ with $|w| = m$, where e_1, e_2, e_3, e_4 are the boundary edges of the unit square. Again let T_m denote the trace map onto SC_m. The trace space for T_0 has been identified by Hino and Kumagai [7] as the Sobolev space $H^\beta(\mathsf{SC}_0)$ with $\beta = 1/2 + \log r/\log 9$. Note that again $1/2 < \beta < 1$.

Proposition 5.4 ([7]). *The trace map T_0 is continuous from* $\operatorname{dom}\mathcal{E}$ *to $H^\beta(\mathsf{SC}_0)$ for $\beta = 1/2 + \log r/\log 9$ with*

$$\|T_0 F\|^2_{H^\beta(\mathsf{SC}_0)} \leq c\mathcal{E}(F). \tag{5.20}$$

Moreover, there exists a continuous linear extension map $E_0 : H^\beta(\mathsf{SC}_0) \to$ $\operatorname{dom}\mathcal{E}$ *with $T_0 E_0 f = f$ and*

$$\mathcal{E}(E_0 f) \leq c\|f\|^2_{H^\beta(\mathsf{SC}_0)}. \tag{5.21}$$

We now claim that the analogs of Theorem 5.2 and 5.3 hold for SC in place of SG. The proof is essentially the same; the only detail that needs to be checked is the dilation argument. In this case, the contribution to (5.6) from the edge $e = F_w(e_1)$ is

$$\begin{aligned}
&\int_0^{3^{-m}}\int_0^{3^{-m}} \frac{|F(e_i(x)) - F(e_i(y))|^2}{|x-y|^{1+2\beta}}dxdy \\
&\qquad = \frac{9^m}{3^{1+2\beta}}\int_0^1\int_0^1 \frac{|F(\Phi_w(e_i(x))) - F(\Phi_w(e_i(y)))|^2}{|x-y|^{1+2\beta}}dxdy,
\end{aligned} \tag{5.22}$$

after a change of variable, analogous to (5.11). We note that $9/3^{1+2\beta} = r$ in this case, so summing (5.22) yields the analog of (5.13) as a consequence of (5.20). The rest of the arguments are the same.

For our final fractal example we consider the classical Julia sets of complex polynomials. Fix a polynomial $P(z)$ (of degree at least 2) and let $\mathcal{J}$ denote its Julia set. We assume $\mathcal{J}$ is connected. In many cases (see [12]), it is possible to parameterize $\mathcal{J}$ by the unit circle as follows. Let Ω denote the unbounded component of the complement of $\mathcal{J}$ in $\mathbb{C}$, so $\Omega \cup \{\infty\}$ is simply connected, and let φ be a conformal map from $\{z : |z| > 1\}$ to Ω. In many cases, φ extends continuously to the boundary circle, and this maps C onto $\mathcal{J}$ (usually not one-to-one). Although there is usually no useful formula for φ, in many cases it is possible to describe explicitly the points on C that are identified under φ. There have been a number of papers that utilize this parametrization to construct an energy on $\mathcal{J}$ [13, 1, 5, 14].

Here we deal with a different problem: how to characterize the traces on $\mathcal{J}$ of functions of finite energy on Ω. The answer is almost immediate from the methods of Section 3. We know that $F \in H^1(\Omega)$ if and only if $F \circ \varphi \in H^1(|z| > 1)$, and the space of traces of $F \circ \varphi$ on C is exactly $H^{1/2}(C)$. Thus the space of traces of F on $\mathcal{J}$, which we should denote $H^{1/2}(\mathcal{J})$, is characterized by the finiteness of

$$\|F\|^2_{H^{1/2}(\mathcal{J})} = \int_0^{2\pi} \int_0^{2\pi} \frac{|F(\varphi(e^{i\theta})) - F(\varphi(e^{i\theta'}))|^2}{4\sin^2 \frac{1}{2}(\theta - \theta')} d\theta d\theta'. \tag{5.23}$$

One could perhaps hope for a more direct characterization in terms of an integral involving $|F(z) - F(z')|^2$ as z and z' vary over $\mathcal{J}$. This would involve choosing a measure on $\mathcal{J}$ (there are more than one natural choices) and finding the appropriate denominator in terms of a distance from z to z' on $\mathcal{J}$. Good luck!

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