ONE-DIMENSIONAL WAVE EQUATIONS DEFINED BY FRACTAL LAPLACIANS

By

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Abstract. We study one-dimensional wave equations defined by a class of fractal Laplacians. These Laplacians are defined by fractal measures generated by iterated function systems with overlaps, such as the well-known infinite Bernoulli convolution associated with the golden ratio and the three-fold convolution of the Cantor measure. The iterated function systems defining these measures do not satisfy the post-critically finite condition or the open set condition. Using second-order self-similar identities introduced by Strichartz et al., we discretize the equations and use the finite element and central difference methods to obtain numerical approximations of the weak solutions. We prove that the numerical solutions converge to the weak solution and obtain estimates for the rate of convergence.

1 Introduction

In this paper, we study approximations of the solution of the wave equation defined by a one-dimensional fractal measure. Such fractals have recently attracted considerable attention because of their relation to classical analysis and their numerous unusual properties. In such situations, classical approximation methods have to be modified to produce accurate results; see [3, 28, 39, 27] and the references therein. In this paper, we investigate the solution of the wave equation theoretically, and also provide numerical examples.

Our long term goal is to combine ideas of Strichartz, including the celebrated Strichartz estimates, with such recent results as [18] and [36], in a comprehensive study of wave equations on fractals and fractafolds. However, currently there are few mathematical tools available for studying wave equations on fractals, despite the fact that the existence of large gaps in the spectrum on many fractals, together

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with heat kernel estimates, implies that Fourier series on these fractals can have better convergence than in the classical case (and, as was noted by Strichartz in [38], "... is the first kind of example which improves on the corresponding results in smooth analysis"). Among the most recent results, the infinite wave propagation speed was recently proved on some *post-critically finite (p.c.f.)* (see [22]) fractals in the preprint [26] by Yin-Tat Lee. This question was open, even in the standard case of the Sierpiński gasket, since 1999; see [4, 37]. The proof in [26] relies partially on Kigami's p.c.f. assumptions (see [22] and references therein) and, more substantially, on certain heat kernel estimates. In general, heat kernel estimates on fractals have been a difficult and extensively studied, with the most relevant recent results and references contained in [23, 24, 17, 16]. It is not clear at present if the heat kernel estimates assumed in [26] can be verified for the fractal measures that we consider, but some preliminary results can be found in [42]. The most intuitive idea, essentially due to Strichartz, is that there is no reason why the wave propagation speed should be finite on fractals, because of the difference in time and Laplacian scalings. In our paper, we do not discuss the wave propagation speed directly, but rather develop approximating tools that may help in this study.

Let μ be a continuous positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) \subseteq [a, b]$. Let $L^2_\mu[a, b] := \{f : [a, b] \to \mathbb{R} : \int |f|^2 \, d\mu < \infty\}$; if μ is Lebesgue measure, we simply denote the space by $L^2[a, b]$. It is well known (see, e.g., [1, 19]) that μ defines a Dirichlet Laplacian Δ_μ on $L^2_\mu[a, b]$, described as follows. Let $H^1(a, b)$ be the Sobolev space of all functions in $L^2[a, b]$ whose weak derivatives belong to $L^2[a, b]$, with the inner product

$$(u, v)_{H^1(a,b)} := \int_a^b uv dx + \int_a^b u'v' dx.$$

Let $H^1_0(a, b)$ be the completion of $C^\infty_c(a, b)$ in $H^1(a, b)$; $H^1_0(a, b)$ and $H^1(a, b)$ are dense subspaces of $L^2_\mu[a, b]$. Define a quadratic form on $L^2_\mu[a, b]$,

$$\mathcal{E}(u, v) = \int_a^b u'v' dx, \tag{1.1}$$

with domain $\operatorname{Dom} \mathcal{E}$ equal to some dense subspace of $H^1_0(a, b)$; see Section 2. Since the embedding $\operatorname{Dom} \mathcal{E} \hookrightarrow L^2_\mu[a, b]$ is compact, $\mathcal{E}$ is closed and is, in fact, a Dirichlet form on $L^2_\mu[a, b]$. Thus there exists a nonnegative self-adjoint operator T on $L^2_\mu[a, b]$ such that $\operatorname{Dom} \mathcal{E} = \operatorname{Dom}(T^{1/2})$ and

$$\mathcal{E}(u, v) = (T^{1/2}u, T^{1/2}v)_\mu \quad \text{for all } u, v \in \operatorname{Dom} \mathcal{E},$$

where $(u, v)_\mu := \int_a^b uv d\mu$ denotes the inner product on $L^2_\mu[a, b]$; we also denote the corresponding norm by $\| \cdot \|_\mu$. We define $\Delta_\mu := -T$ and call it the **Dirichlet Laplacian with respect to μ.**

Let $u \in \operatorname{Dom}\mathcal{E}$ and $f \in L^2_\mu[a,b]$. It is known that $u \in \operatorname{Dom}(\Delta_\mu)$ and $\Delta_\mu u = f$ if and only if $\Delta u = f d\mu$ in the sense of distributions, i.e.,

$$\int_a^b u'v'\,dx = \int_a^b (-\Delta_\mu u)v\,d\mu \quad \text{for all } v \in C_c^\infty(a,b).$$

It is also known (see, e.g., [1, 19]) that there exists an orthonormal basis of eigenfunctions of Δ_μ, and the eigenvalues $\{\lambda_n\}$ are discrete and satisfy $0 \le \lambda_1 < \lambda_2 < \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$.

The operators Δ_μ and their generalizations have been studied in connection with spectral functions of the string and diffusion processes; see [8, 9, 21]. More recently, they have been studied in connection with fractal measures; see [10, 11, 12, 13, 14, 1, 19, 29, 30, 33, 31].

Our study of the operator Δ_μ is mainly motivated by the effort to extend the current theory of analysis on fractals to include iterated function systems (IFSs) with overlaps. Such IFSs do not satisfy the well-known post-critically finite condition or the open set condition. Nevertheless, under the assumption that μ satisfies a family of second-order self-similar identities, some results concerning Δ_μ can be obtained. In [2], the finite element method is used to compute numerical approximations of the eigenvalues and eigenfunctions; and in [31], formulas defining the spectral dimension of Δ_μ have been obtained for a class of measures that include the infinite Bernoulli convolution associated with the golden ratio and the three-fold convolution of the Cantor measure.

The main purpose of this paper is to study one-dimensional wave equations defined by a class of fractal Laplacians, subject to the Dirichlet boundary condition. More precisely, we study the following non-homogeneous *hyperbolic initial/boundary value problem (IBVP)*:

$$\begin{cases} u_{tt} - \Delta_\mu u = f & \text{on } [a,b]\times[0,T], \\ u = 0 & \text{on } \{a,b\}\times[0,T], \\ u = g,\ u_t = h & \text{on } [a,b]\times\{t=0\}. \end{cases} \tag{1.2}$$

The following existence and uniqueness result (see Definition 2.5 for the definition of a weak solution) follows easily from the general theory for wave equations in Hilbert spaces; see Section 2.

Theorem 1.1. *Assume $g \in \operatorname{Dom}\mathcal{E}$, $h \in L^2_\mu[a,b]$ and $f \in L^2(0,T;\operatorname{Dom}\mathcal{E})$. Then equation (1.2) has a unique weak solution.*

We are mainly interested in fractal measures μ. Let D be a non-empty compact subset of $\mathbb{R}^d$. A function $S: D \to D$ is called a **contraction** on D if there exists

$0 < c < 1$ such that

$$|S(x) - S(y)| \le c|x - y| \quad \text{for all } x, y \in D. \tag{1.3}$$

An **iterated function system (IFS)** on D is a finite collection of contractions on D. Each IFS $\{S_i\}_{i=1}^q$ defines a unique compact subset $F \subseteq D$, called the **invariant set** or **attractor**, which satisfies $F = \bigcup_{i=1}^q S_i(F)$. Also, to each set of probability weights $\{p_i\}_{i=1}^q$, where $p_i > 0$ and $\sum_{i=1}^q p_i = 1$, there exists a unique probability measure, called the **invariant measure**, satisfying the identity

$$\mu = \sum_{i=1}^q p_i \mu \circ S_i^{-1}; \tag{1.4}$$

see [20, 7]. We call S a contractive **similitude** if equality holds in (1.3). IFSs studied in this paper consist of contractive similitudes; they are of the form

$$S_i(x) = \rho_i R_i x + b_i, \quad i = 1, \dots, q; \tag{1.5}$$

here $0 < \rho_i < 1$, R_i is an orthogonal transformation, and $b_i \in \mathbb{R}^d$. For such an IFS, we call the corresponding invariant set F the **self-similar set** and the invariant measure μ the **self-similar measure**.

An IFS $\{S_i\}_{i=1}^q$ is said to satisfy the **open set condition (OSC)** if there exists a non-empty bounded open set U such that $\bigcup_i S_i(U) \subseteq U$ and $S_i(U) \cap S_j(U) = \varnothing$ for all $i \ne j$. An IFS which does not satisfy the OSC is said to have **overlaps**. For an IFS of contractive similitudes, it is known that if the linear parts of the IFS maps are commensurable, then the p.c.f. condition implies the OSC; see [5].

We are interested in one-dimensional self-similar measures defined by IFSs with overlaps. Such IFSs are not p.c.f. and are thus not covered by Kigami's theory. In order to discretize a wave equation and obtain numerical approximations of the weak solution, we assume that the corresponding self-similar measure satisfies a family of second-order self-similar identities, an idea introduced by Strichartz et al. [35]. Let $\{S_i\}_{i=1}^q$ be an IFS of contractive similitudes on $\mathbb{R}$, and let μ be the corresponding self-similar measure. Assume, in addition, that $\operatorname{supp}(\mu) = [a, b]$. Define an auxiliary IFS

$$T_j(x) = \rho^{n_j} x + d_j, \quad j = 1, 2, \dots, N, \tag{1.6}$$

where $n_j \in \mathbb{N}$ and $d_j \in \mathbb{R}$, and let

$$\rho := \max\{\rho^{n_j} : 1 \le j \le N\}. \tag{1.7}$$

The measure μ is said to satisfy a family of **second-order self-similar identities** (or simply **second-order identities**) with respect to $\{T_j\}_{j=1}^N$ (see [25]) if

(1) $\mathrm{supp}(\mu) \subseteq \bigcup_{j=1}^{N} T_j(\mathrm{supp}(\mu))$; and

(2) for each Borel subset $A \subseteq \mathrm{supp}(\mu)$ and $0 \leq i, j \leq N$, $\mu(T_iT_jA)$ can be expressed as a linear combination of $\{\mu(T_kA) : k = 1, \dots, N\}$, i.e.,

$$\mu(T_iT_jA) = \sum_{k=0}^{N} d_k \mu(T_kA),$$

where $d_k = d_k(i, j)$ are independent of A. In matrix form,

$$\text{(1.8a)} \qquad \begin{bmatrix} \mu(T_1T_jA) \\ \vdots \\ \mu(T_NT_jA) \end{bmatrix} = M_j \begin{bmatrix} \mu(T_1A) \\ \vdots \\ \mu(T_NA) \end{bmatrix}, \quad j = 1, \dots, N;$$

or, equivalently,

$$\text{(1.8b)} \qquad \mu(T_iT_jA) = \boldsymbol{e}_i M_j \begin{bmatrix} \mu(T_1A) \\ \vdots \\ \mu(T_NA) \end{bmatrix}, \quad i, j = 1, \dots, N,$$

where $\boldsymbol{e}_i$ is the ith row of the $N \times N$ identity matrix and M_j is some $N \times N$ matrix independent of A.

We assume that $\{T_j\}_{j=0}^{N}$ satisfies the OSC. The m-th level iteration of the auxiliary IFS $\{T_j\}_{j=0}^{N}$ induces a partition V_m of $\mathrm{supp}(\mu) = [a, b]$. Moreover, the μ measure of each subinterval in the partition can be computed in terms of a matrix product. This provides us with a good way of discretizing the wave equation.

Letting $f(x,t) = 0$, multiplying the first equation in (1.2) by $v \in \mathrm{Dom}\,\mathcal{E}$, integrating both sides with respect to $d\mu$, and then integrating by parts, we obtain

$$\text{(1.9)} \qquad -\int_a^b u_x(x,t)\, v'(x) dx = \int_a^b u_{tt}(x,t)\, v(x) d\mu,$$

where $u_x(x,t)$ is the weak partial derivative of u with respect to x and u_{tt} is the weak second partial derivative with respect to t.

Theorem 1.2. *Let μ be a self-similar measure defined by a one-dimensional IFS of contractive similitudes on $\mathbb{R}$ as in (1.4) and (1.5). Assume that* $\mathrm{supp}(\mu) = [a, b]$ *and that μ satisfies a family of second-order self-similar identities. Then the finite element method for the equation (1.9) discretizes it to a system of second-order ordinary differential equations (3.9), which has a unique solution (and can be solved numerically).*

Based on this result, we solve the homogeneous IBVP (1.2) numerically for three different measures, namely, the weighted Bernoulli-type measure, the infinite Bernoulli convolution associated with the golden ratio, and the three-fold convolution of the Cantor measure. We show that the approximate solutions converge to the actual weak solution and obtain a rate of convergence.

Theorem 1.3. *Assume the hypotheses of Theorem 1.2, let $f = 0$ in equation (1.2), and fix $t \in [0, T]$. Then the approximate solutions u^m obtained by the finite element method converge in $L^2_\mu[a, b]$ to the actual weak solution u. Moreover, there exists a constant $C > 0$ such that for all $m \geq 1$,*

$$\left\|u^m - u\right\|_\mu \leq \left(C\sqrt{T}\, \|u_{tt}\|_{2,\mathrm{Dom}\,\mathcal{E}} + 2\, \|u\|_{\mathrm{Dom}\,\mathcal{E}}\right)\rho^{m/2}.$$

This paper is organized as follows. We summarize some basic classical results, definitions, and notation in Section 2. In Section 3, we use the finite element and central difference methods to obtain numerical approximations to the corresponding homogeneous IBVP (1.2), proving Theorem 1.2. In Section 4, we apply our numerical methods to the above-mentioned measures, and illustrate some numerical results. In Section 5, we prove the convergence of the approximation scheme and obtain Theorem 1.3 as a corollary.

2 Preliminaries

In this section, we summarize notation, definitions, and preliminary results used throughout the rest of the paper. We denote the topological dual of a Banach space X by X'. For $v \in X'$ and $u \in X$ we denote by $\langle v, u\rangle = \langle v, u\rangle_{X',X} := v(u)$ the dual pairing of X' and X.

A function $s : [0, T] \to X$ is called **simple** if it has the form

$$s(t) = \sum_{m=1}^{N} \chi_{E_m}(t) u_m \quad \text{for } t \in [0, T], \tag{2.1}$$

where each E_m is a Lebesgue measurable subset of $[0, T]$, $u_m \in X$ for $m = 1, \ldots, N$, and χ_{E_m} is the characteristic function on E_m. A function $u : [0, T] \to X$ is **strongly measurable** if there exist simple functions $s_n : [0, T] \to X$ such that

$$s_n(t) \to u(t) \quad \text{as } n \to \infty \text{ for Lebesgue a.e. } t \in [0, T].$$

A function $u : [0, T] \to X$ is **weakly measurable** if for each $v \in X'$, the mapping $t \mapsto \langle v, u(t)\rangle$ is Lebesgue measurable.

A function $u : [0, T] \to X$ is **almost separably valued** if there exists a subset $E \subseteq [0, T]$ with zero Lebesgue measure such that the set $\{u(t) : t \in [0, T]\backslash E\}$

is separable. By a theorem of Pettis [32], a function $u : [0, T] \to X$ is strongly measurable if and only if it is weakly measurable and almost separably valued. Since every subset of a separable Banach space X is separable, the two concepts of measurability coincide, and we can use the term **measurable** without ambiguity.

Definition 2.1. Let X be a separable Banach space with norm $\|\cdot\|_X$. Define $L^p(0, T; X)$ to be the space of all measurable functions $u : [0, T] \to X$ satisfying

(a) $\|u\|_{L^p(0,T;X)} := \big(\int_0^T \|u(t)\|_X^p \, dt\big)^{1/p} < \infty$, if $1 \le p < \infty$, and

(b) $\|u\|_{L^\infty(0,T;X)} := \operatorname{ess\,sup}_{0 \le t \le T} \|u(t)\|_X < \infty$, if $p = \infty$.

When the interval $[0, T]$ is understood, we abbreviate these norms as $\|u\|_{p,X}$ and $\|u\|_{\infty,X}$.

Remark 2.1. For each $1 \le p \le \infty$, $L^p(0, T; X)$ is a Banach space; moreover, $L^{p_2}(0, T; X) \subseteq L^{p_1}(0, T; X)$ if $0 \le p_1 \le p_2 \le \infty$. If $(X, (\cdot, \cdot)_X)$ is a separable Hilbert space, then $L^2(0, T; X)$ is a Hilbert space with the inner product

$$(u, v)_{L^2(0,T;X)} := \int_0^T \big(u(t), v(t)\big)_X dt.$$

Definition 2.2. Let X be a Banach space and $u \in L^1(0, T; X)$. We say that $v \in L^1(0, T; X)$ is the **weak derivative** of u, written $u_t = v$, if

$$\int_0^T \phi_t(t)u(t)\, dt = -\int_0^T \phi(t)u_t(t)dt$$

for all scalar test functions $\phi \in C_c^\infty(0, T)$.

Definition 2.3. Let X be a Banach space and X' its dual. We say a sequence $\{u_m\}_{m=1}^\infty \subseteq X$ **converges weakly to** $u \in X$, written $u_m \rightharpoonup u$, if $\langle v, u_m\rangle \to \langle v, u\rangle$ for each bounded linear functional $v \in X'$.

For the more general definition of derivatives of distributions with values in a Hilbert space, we refer the reader to [41, Section 25].

The notion of a Gelfand triple [15], defined below, plays an important role in our investigation of the wave equation.

Definition 2.4. Let V and H be separable Hilbert spaces with the continuous injective dense embedding $\iota : V \hookrightarrow H$. Identifying H with its dual H', we obtain the continuous and dense embedding $V \hookrightarrow H \cong H' \hookrightarrow V'$. Assume, in addition, that the dual pairing between V and V' is compatible with the inner product on H in the sense that $\langle v, u\rangle_{V',V} = (v, u)_H$ for all $u \in V \subset H$ and $v \in H \cong H' \subset V'$. The triple (V, H, V') is called a **Gelfand triple** (The pair (H, V) is also called a **rigged Hilbert space**.)

Since V is itself a Hilbert space, it is isomorphic with its dual V'. However, in general, this isomorphism is not the same as the composition $\iota^*\iota : V \subset H = H' \hookrightarrow V'$, where ι^* is the adjoint of ι.

Throughout the rest of this section, μ denotes a finite positive Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) \subseteq [a, b]$ and $\mu(a, b) > 0$, where $-\infty < a < b < \infty$. It is known (see, e.g., [19, 29]) that there exists a constant $C > 0$ such that

$$\int_a^b |u|^2 \, d\mu \le C \int_a^b |\nabla u|^2 dx \quad \text{for all } u \in C_c^\infty(a, b). \tag{2.2}$$

This condition implies that each equivalence class $u \in H_0^1(a, b)$ contains a unique (in the $L^2_\mu[a, b]$ sense) member $\bar{u}$ that belongs to $L^2_\mu[a, b]$ and satisfies both of the following conditions:

(1) there exists a sequence $\{u_n\}$ in $C_c^\infty(a, b)$ such that $u_n \to \bar{u}$ in $H_0^1(a, b)$ and $u_n \to \bar{u}$ in $L^2_\mu[a, b]$;

(2) $\bar{u}$ satisfies the inequality in (2.2).

We call $\bar{u}$ the $L^2_\mu[a, b]$**-representative** of u. Under condition (2.2), the mapping $\iota : H_0^1(a, b) \to L^2_\mu[a, b]$ defined by $\iota(u) = \bar{u}$. is a bounded linear operator but not necessarily injective, because it is possible for a non-zero function $u \in H_0^1(a, b)$ to have an $L^2_\mu[a, b]$-representative that has zero $L^2_\mu[a, b]$-norm. To deal with this situation, we consider the subspace $\mathcal{N} := \{u \in H_0^1(a, b) : \|\iota(u)\|_\mu = 0\}$. The continuity of ι implies that $\mathcal{N}$ is a closed subspace of $H_0^1(a, b)$. Let $\mathcal{N}^\perp$ be its orthogonal complement in $H_0^1(a, b)$. Clearly, $\iota : \mathcal{N}^\perp \to L^2_\mu[a, b]$ is injective, and we can identify $\mathcal{N}^\perp$ and $\iota(\mathcal{N}^\perp)$. The subspace $\mathcal{N}^\perp$ is dense in $L^2_\mu[a, b]$; see [19]. Throughout this paper, we let $\operatorname{Dom} \mathcal{E} := \mathcal{N}^\perp$ and $\|\cdot\|_{\operatorname{Dom} \mathcal{E}} = \|\cdot\|_{H_0^1(a,b)}$. In general, the equivalence classes represented by $u \in H_0^1(a, b)$ and $\bar{u} \in L^2_\mu[a, b]$ are different. Corollary 2.1 below says that the continuous representative of u lies in the intersection of these two equivalence classes. We frequently identify $\bar{u}$ and u without mention.

Proposition 2.2. *Let $u \in H_0^1(a, b)$ and $\{\phi_n\} \subset C_c^\infty(a, b)$ be such that $\phi_n \to u$ in $H_0^1(a, b)$. Then there exists a subsequence $\{\phi_{n_k}\}$ such that $\phi_{n_k} \to u_c$ everywhere in $[a, b]$, where u_c is the continuous representative of the equivalence class of u in $H_0^1(a, b)$.*

Proof. Let $\{\phi_{n_k}\}$ be a subsequence converging pointwise Lebesgue a.e. to u_c on (a, b). Let $x \in (a, b)$ and $\epsilon > 0$ be arbitrary. Since ϕ_n is convergent, there exists $C > 0$ such that

$$\|\phi_n\|_{\operatorname{Dom} \mathcal{E}} \le C \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

Also, by the continuity of u_c, there exists $0 < \delta_\epsilon < (\epsilon/(3C))^2$ such that

$$|u_c(x) - u_c(y)| < \frac{\epsilon}{3} \tag{2.4}$$

for all $y \in [a, b]$, with $|y - x| < \delta_\epsilon$. Hence,

$$\left|\phi_{n_k}(x) - u_c(x)\right| \le \left|\phi_{n_k}(x) - \phi_{n_k}(y)\right| + \left|\phi_{n_k}(y) - u_c(y)\right| + |u_c(y) - u_c(x)|\,. \tag{2.5}$$

The first term can be estimated by using (2.3) as follows:

$$\begin{aligned}\left|\phi_{n_k}(x) - \phi_{n_k}(y)\right| &= \left|\int_y^x \phi'_{n_k}(s)\,ds\right| \le \Big(\int_y^x \left|\phi'_{n_k}(s)\right|^2 ds\Big)^{1/2} |x-y|^{1/2}\\ &\le \left\|\phi_{n_k}\right\|_{\mathrm{Dom}\,\mathcal{E}} |x-y|^{1/2} \le C\frac{\epsilon}{3C} \le \frac{\epsilon}{3}.\end{aligned} \tag{2.6}$$

Substituting (2.4) and (2.6) into (2.5), we get

$$\left|\phi_{n_k}(x) - u_c(x)\right| \le \frac{\epsilon}{3} + \left|\phi_{n_k}(y) - u_c(y)\right| + \frac{\epsilon}{3}$$

for all $y \in (x - \delta_\epsilon, x + \delta_\epsilon)$.

Finally, let $y \in (a, b)$ satisfy $\lim_{k\to\infty} \phi_{n_k}(y) = u_c(y)$. Then, for sufficiently large k, we have $\left|\phi_{n_k}(y) - u_c(y)\right| < \epsilon/3$, and hence $\left|\phi_{n_k}(x) - u_c(x)\right| < \epsilon$. Thus, $\lim_{k\to\infty} \phi_{n_k}(x) = u_c(x)$ for all $x \in [a, b]$. □

Corollary 2.1. *Let $u \in H_0^1(a, b)$, and let $\bar{u}$ be its unique $L^2_\mu[a, b]$ representative. Then u_c lies in the equivalence class of $\bar{u}$ in $L^2_\mu[a, b]$.*

Corollary 2.2. *If* $\mathrm{supp}(\mu) = [a, b]$, *then $\iota : H_0^1(a, b) \to L^2_\mu[a, b]$ is injective. Consequently,* $\mathrm{Dom}\,\mathcal{E} = H_0^1(a, b)$.

Proof. Let $u \in H_0^1(a, b)$ be such that $\iota(u) = 0$. Then $\bar{u} = u_c = 0$ in $L^2_\mu[a, b]$. Since $\mathrm{supp}(u) = [a, b]$, we have $u_c \equiv 0$ on $[a, b]$. Thus, $u = 0$ Lebesgue-a.e. on $[a, b]$. □

We denote both the classical and weak derivatives of a function $\varphi : (a, b) \to \mathbb{R}$ by φ'. For $u \in L^2(0, T; X)$, where X is $H_0^1(a, b)$, or $L^2_\mu[a, b]$, etc., we denote by $u_x(x, t)$ (or ∇u) the classical or weak derivative of u with respect to x at $t \in [0, T]$.

The spaces $\mathrm{Dom}\,\mathcal{E}$, $L^2_\mu[a, b]$, $(\mathrm{Dom}\,\mathcal{E})'$ form a Gelfand triple:

$$\mathrm{Dom}\,\mathcal{E} \hookrightarrow L^2_\mu[a, b] \cong (L^2_\mu[a, b])' \hookrightarrow (\mathrm{Dom}\,\mathcal{E})',$$

where we identify $L^2_\mu[a, b]$ with $(L^2_\mu[a, b])'$. The embedding $L^2_\mu[a, b] \hookrightarrow (\mathrm{Dom}\,\mathcal{E})'$ is given by

$$w \in L^2_\mu[a, b] \mapsto (w, \cdot)_\mu \in (L^2_\mu[a, b])' \subset (\mathrm{Dom}\,\mathcal{E})'.$$

Definition 2.5 (see, e.g., [6, 41])**.** Let $g \in \operatorname{Dom}\mathcal{E}$, $h \in L^2_\mu[a,b]$, and $f \in L^2(0,T;\operatorname{Dom}\mathcal{E})$. A function $u \in L^2(0,T;\operatorname{Dom}\mathcal{E})$, with $u_t \in L^2(0,T;L^2_\mu[a,b])$ and $u_{tt} \in L^2(0,T;(\operatorname{Dom}\mathcal{E})')$ is a **weak solution** of IBVP (1.2) if

(i) $\langle u_{tt}, v\rangle + \mathcal{E}(u,v) = (f,v)_\mu$ for each $v \in \operatorname{Dom}\mathcal{E}$ and Lebesgue a.e. $t \in [0,T]$;

(ii) $u(,0) = g$ and $u_t(,0) = h$.

Here, $\langle\cdot,\cdot\rangle$ denotes the pairing between $(\operatorname{Dom}\mathcal{E})'$ and $\operatorname{Dom}\mathcal{E}$.

Remark 2.3. (a). In (i) above, if $u_{tt} \in \operatorname{Dom}\mathcal{E}$ or $u_{tt} \in L^2_\mu[a,b]$, then $\langle u_{tt}, v\rangle = (u_{tt}, v)_\mu$, as in the definition of Gelfand triple.

(b). Given the Gelfand triple $\operatorname{Dom}\mathcal{E} \hookrightarrow L^2_\mu[a,b] \hookrightarrow (\operatorname{Dom}\mathcal{E})'$, for $u \in L^2(0,T;\operatorname{Dom}\mathcal{E})$ we have $u \in L^2(0,T;L^2_\mu[a,b])$ and $u \in L^2(0,T;(\operatorname{Dom}\mathcal{E})')$, and thus $u \in L^1(0,T;L^2_\mu[a,b])$ and $u \in L^1(0,T;(\operatorname{Dom}\mathcal{E})')$. Therefore, it makes sense to require the weak derivatives of u to satisfy $u_t \in L^1(0,T;L^2_\mu[a,b])$, $u_{tt} \in L^1(0,T;(\operatorname{Dom}\mathcal{E})')$, $u_t \in L^2(0,T;L^2_\mu[a,b])$, and $u_{tt} \in L^2(0,T;(\operatorname{Dom}\mathcal{E})')$.

Let V, H be Hilbert spaces, V separable. Assume that the embedding $V \hookrightarrow H$ is continuous, injective, and dense, so that $V \hookrightarrow H \hookrightarrow V'$ form a Gelfand triple; see [41, Section 17]. Let $0 < T < \infty$, and assume that for $t \in [0,T]$, $a(t,\varphi,\psi)$ is a continuous sesquilinear form, i.e.,

$$|a(t,\varphi,\psi)| \le c\|\varphi\|_V\|\psi\|_V, \quad \text{for all } \varphi,\psi \in V, \tag{2.7}$$

where $c > 0$ is a constant independent of t. Then there exists a representation operator $L(t) : V \to V'$, such that for each t, $L(t)$ is linear and continuous, with $a(t;\varphi,\psi) = (L(t)\varphi,\psi)_H$.

Assume that for all $\varphi,\psi \in V$ the function $t \mapsto a(t;\varphi,\psi)$ is continuously differentiable for $t \in [0,T]$, i.e.,

$$a(t;\varphi,\psi) \in C^1[0,T], \quad \text{for all } \varphi,\psi \in V, \tag{2.8}$$

where

$$\left|\frac{d}{dt}a(t;\varphi,\psi)\right| \le c\|\varphi\|_V\|\psi\|_V, \quad \text{for all } t \in [0,T],$$

where c is independent of t.

Assume further that $a(t;\varphi,\psi)$ is **antisymmetric**, i.e.,

$$a(t;\varphi,\psi) = \overline{a(t;\varphi,\psi)}, \quad \text{for all } \varphi,\psi \in V. \tag{2.9}$$

Finally, assume V**-coercion**, i.e., there exist constants $\alpha,\beta > 0$ such that

$$a(t;\varphi,\varphi) + \beta\|\varphi\|_H^2 \ge \alpha\|\varphi\|_V^2, \quad \text{for all } t \in [0,T] \text{ and } \varphi \in V. \tag{2.10}$$

The proof of Theorems 2.3 and 2.4 stated below can be found in [41, Sections 29–30].

Theorem 2.3. *Let V, H be Hilbert spaces, where V is separable. Assume that the embedding $V \hookrightarrow H$ is injective, continuous, and dense, so that $V \hookrightarrow H \hookrightarrow V'$ form a Gelfand triple. Assume conditions (2.7)–(2.10) above hold. Then, for each $f \in L^2(0,T;H)$, $0 < T < \infty$, and initial conditions $u_0 \in V$, $u_1 \in H$, there exists a unique function $u(t) \in L^2(0,T;V)$ with $du/dt \in L^2(0,T;H)$ such that*

$$\frac{d^2u}{dt^2} + L(t)u = f \quad \text{for } t \in [0,T], \quad u(0) = u_0, \quad \frac{du(0)}{dt} = u_1, \tag{2.11}$$

in the sense that

$$\left(\frac{d^2u}{dt^2}, \varphi\right)_H + (L(t)u, \varphi)_H = (f, \varphi)_H, \quad \text{for all } \varphi \in V.$$

Definition 2.6. Let V be a Hilbert space. For each integer $k \geq 0$, define the Sobolev space

$$W_2^k(0,T;V) := \left\{u : (0,T) \to V \text{ measurable} : \frac{d^n u}{dt^n} \in L^2(0,T;V) \text{ for } 0 \leq n \leq k\right\},$$

where the differentiation is in the distributional sense. Equip $W_2^k(0,T;V)$ with the norm

$$\|u\|_k^2 := \sum_{n=0}^{k} \int_0^T \left\|\frac{d^n u}{dt^n}\right\|_V^2 dt.$$

As shown in the following theorem, the smoothness of the solution of equation (2.11) increases with that of f.

Theorem 2.4. *Assume the hypotheses of Theorem 2.3 and that $a(\varphi, \psi)$ and L are independent of t. Consider the hyperbolic equation*

$$\frac{d^2u}{dt^2} + Lu = f \quad \text{for } t \in (0,T), \tag{2.12}$$

with the initial conditions

$$u(0) = u_0, \quad \frac{du(0)}{dt} = u_1. \tag{2.13}$$

Assume, in addition, that $f \in W_2^k(0,T;H)$, $k \geq 1$, and

$$u_0, u_1, f''(0), \dots, f^{(k-3)}(0) \in V \quad \text{and} \quad f^{(k-2)}(0) - Lf^{(k-3)}(0) \in H.$$

Then the solution u of (2.12) and (2.13) satisfies

$$u \in W_2^{k-1}(0,T;V), \quad \frac{d^k u(t)}{dt^k} \in L^2(0,T;H), \quad \frac{d^{k+1}u(t)}{dt^{k+1}} \in L^2(0,T;V').$$

Proof of Theorem 1.1. In order to apply Theorem 2.3, we let $V = \mathrm{Dom}\,\mathcal{E}$, $H = L^2_\mu[a,b]$, and set $a(t;u,v) = \mathcal{E}(u,v)$, which is independent of t. Then, for all $u, v \in \mathrm{Dom}\,\mathcal{E}$,

$$\begin{aligned}|a(t;u,v)| &= \Big|\int u'v'dx\Big| \le \Big(\int |u'|^2 dx\Big)^{1/2}\Big(\int |v'|^2 dx\Big)^{1/2} \\ &= \|u\|_{\mathrm{Dom}(\mathcal{E})}\|v\|_{\mathrm{Dom}(\mathcal{E})},\end{aligned}$$

so that condition (2.7) holds. Also, $\mathcal{E}$ is bilinear. Thus, there exists a representation operator $L : \mathrm{Dom}\,\mathcal{E} \to (\mathrm{Dom}\,\mathcal{E})'$ such that $\mathcal{E}(u,v) = (Lu,v)_{L^2_\mu[a,b]}$ which satisfies $L = -\Delta_\mu$ on $\mathrm{Dom}\,(-\Delta_\mu)$.

Conditions (2.8) and (2.9) clearly hold, since $t \mapsto a(t;u,v) = \mathcal{E}(u,v)$ is constant in time and real-valued,.

Finally, for all $t \in [0,T]$ and $u \in V$,

$$a(t;u,u) + \|u\|^2_{L^2_\mu[a,b]} \ge \mathcal{E}(u,u) = \|u\|^2_{\mathrm{Dom}\,\mathcal{E}};$$

and thus $\mathrm{Dom}\,\mathcal{E}$-coercion (condition (2.10)) holds with $\alpha = \beta = 1$. Theorem 1.1 now follows from Theorem 2.3. □

As a consequence of Theorem 2.4, we have the following regularity result for solutions of homogeneous wave equations in our setting.

Theorem 2.5. *Assume the hypotheses of Theorem 1.1 and, in addition, that $h \in \mathrm{Dom}\,\mathcal{E}$ and $f = 0$. Then the solution of the homogeneous equation (1.2) satisfies*

$$\begin{gathered}u \in W_2^{k-1}(0,T;\mathrm{Dom}\,\mathcal{E}), \qquad \frac{d^k u}{dt^k} \in L^2(0,T;L^2_\mu[a,b]), \\ \frac{d^{k+1}u}{dt^{k+1}} \in L^2(0,T;(\mathrm{Dom}\,\mathcal{E})'), \quad k \ge 1.\end{gathered}$$

3 The finite element method

In this section, we set $f = 0$ in equation (1.2) and use the finite element method to solve the homogeneous IBVP. We consider only self-similar measures μ (see (1.4)) defined by an IFS $\{S_i\}_{i=1}^q$ of contractive similitudes of the form

$$S_i(x) = \rho x + b_i, \quad i = 1, \dots, q.$$

In addition, we assume that μ satisfies a family of second-order self-similar identities with respect to an auxiliary IFS $\{T_j\}_{j=1}^N$ of the form (1.6). Assume also that $\{T_j\}_{j=1}^N$ satisfies the OSC.

For each multi-index $J = (j_1, \dots, j_m) \in \{1, \dots, N\}^m$, we denote by $T_J[a, b]$ the interval $[x_{i-1}, x_i]$, where the index i is obtained directly from J (see [2]) as

$$i = i(J) := (j_1 - 1)N^{m-1} + (j_2 - 1)N^{m-2} + \cdots + (j_m - 1)N^0 + 1.$$

For example, if $J = (1, \dots, 1)$, then $i(J) = 1$; and if $J = (N, \dots, N)$, then $i(J) = N^m$. We call $T_J[a, b]$ a **level-m subinterval**. It follows that

$$T_{J_i}[a, b] := T_J[a, b] = [x_{i-1}, x_i] \tag{3.1a}$$

and

$$T_{J_i}(x) := T_J(x) = (x_i - x_{i-1})\frac{x - a}{b - a} + x_{i-1}. \tag{3.1b}$$

We apply the finite element method to approximate the weak solution $u(x, t)$ satisfying (1.9) by

$$u^m(x, t) = \sum_{j=0}^{N^m} \beta_j(t)\phi_j(x) \tag{3.2}$$

where, for $j = 0, 1, \dots, N^m$, $\beta_j(t) = \beta_{m,j}(t)$ are functions to be determined and $\phi_j(x) := \phi_{m,j}(x)$ are the standard piecewise linear **finite element basis functions** (also called **tent functions**) defined by

(3.3)
$$\phi_j(x) := \begin{cases} (x - x_{j-1})/(x_j - x_{j-1}) & \text{if} \quad x_{j-1} \le x \le x_j, \; j = 1, \dots, N^m, \\ (x - x_{j+1})/(x_j - x_{j+1}) & \text{if} \quad x_j \le x \le x_{j+1}, \; j = 0, \dots, N^m - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We require $u^m(x, t)$ to satisfy the integral form of the homogeneous wave equation

$$\int_a^b u_{tt}^m(x, t)\phi_i(x)\, d\mu = -\int_a^b \nabla u^m(x, t)\phi_i'(x)dx, \quad \text{for } i = 0, 1, \dots, N^m, \tag{3.4}$$

where $u_{tt}^m := (u^m)_{tt}$.

As $u^m(a, t) = u^m(b, t) = 0$, we have $\beta_0(t) = \beta_{N^m}(t) = 0$. Using this and substituting (3.2) into (3.4) gives

$$\sum_{j=1}^{N^m-1} \beta_j''(t) \int_a^b \phi_i(x)\phi_j(x)\, d\mu = -\sum_{j=1}^{N^m-1} \beta_j(t) \int_a^b \phi_i'(x)\phi_j'(x)dx, \tag{3.5}$$

for $1 \le i \le N^m - 1$. We define the **mass matrix** $\mathbf{M} = (M_{ij})$ and **stiffness matrix** $\mathbf{K} = (K_{ij})$, respectively, by

$$M_{ij} = \int_a^b \phi_i(x)\phi_j(x)d\mu, \quad 1 \le i, j \le N^m - 1, \tag{3.6a}$$

and

$$K_{ij} = -\int_a^b \phi_i'(x)\phi_j'(x)dx, \quad 1 \le i, j \le N^m - 1. \tag{3.6b}$$

Both **M** and **K** are tridiagonal and of order $(N^m - 1) \times (N^m - 1)$. Let

$$\mathbf{w}(t) := \begin{bmatrix} w_1(t) \\ \vdots \\ w_{N^m-1}(t) \end{bmatrix} = \begin{bmatrix} \beta_1(t) \\ \vdots \\ \beta_{N^m-1}(t) \end{bmatrix}.$$

Then (3.5) can be put into matrix form as

$$\mathbf{M}\mathbf{w}'' = -\mathbf{K}\mathbf{w}. \tag{3.7}$$

This gives us a system of second-order linear ODEs with constant coefficients. We need two initial conditions. The initial condition $u(x, 0) = g(x)$ for $a \le x \le b$ can be approximated by its linear interpolant $\tilde{g}(x) = \sum_{i=1}^{N^m-1} g(x_i)\phi_i(x)$. Therefore, we set $w_i(0) = g(x_i)$ and $w_i'(0) = h(x_i)$. This leads to the initial conditions

$$\mathbf{w}(0) = \mathbf{w}_0 = \begin{bmatrix} g(x_1) \\ \vdots \\ g(x_{N^m-1}) \end{bmatrix}, \qquad \mathbf{w}'(0) = \mathbf{w}_0' = \begin{bmatrix} h(x_1) \\ \vdots \\ h(x_{N^m-1}) \end{bmatrix}. \tag{3.8}$$

Consequently, we get the linear system

$$\begin{cases} \mathbf{M}\dfrac{d^2\mathbf{w}}{dt^2} = -\mathbf{K}\mathbf{w}, \quad t > 0 \\ \mathbf{w}(0) = \mathbf{w}_0, \quad \mathbf{w}'(0) = \mathbf{w}_0'. \end{cases} \tag{3.9}$$

We describe how to compute **M**; the matrix **K** can be computed directly. From the definition of the ϕ_i's and (3.1), we have

$$M_{i,i} = \frac{1}{(b-a)^2}\left(\int_a^b (x-a)^2 d\mu \circ T_{J_i} + \int_a^b (b-x)^2 d\mu \circ T_{J_{i+1}}\right), \quad 1 \le i \le N^m - 1, \tag{3.10a}$$

$$M_{i,i-1} = \frac{1}{(b-a)^2}\int_a^b (x-a)(b-x)d\mu \circ T_{J_i}, \quad 2 \le i \le N^m - 1, \tag{3.10b}$$

$$M_{i,i+1} = \frac{1}{(b-a)^2}\int_a^b (x-a)(b-x)d\mu \circ T_{J_{i+1}}, \quad 1 \le i \le N^m - 2. \tag{3.10c}$$

Define

$$\mathfrak{I}_{k,j} := \int_a^b x^k \, d\mu \circ T_j, \quad k = 0, 1, 2, \; j = 1, \dots, N \tag{3.11a}$$

$$\mathfrak{J}_{k,j} := \int_{T_j[a,b]} x^k \, d\mu, \quad k = 0, 1, 2, \; j = 1, \dots, N. \tag{3.11b}$$

We regard the $\mathfrak{I}_{k,j}$ and $\mathfrak{J}_{k,j}$ as known constants. In fact, for all examples we study, they can be computed exactly; see Section 4. A sufficient condition for computing them explicitly is given in [2].

Lemma 3.1. *The matrix* $\mathbf{M}$ *is completely determined by the integrals* $\mathfrak{I}_{k,j}$ *or, equivalently,* $\mathfrak{J}_{k,j}$*, where* $k = 0, 1, 2$ *and* $j = 1, \dots, N$.

Proof. For $J = (j_1, \dots, j_m) \in \{1, \dots, N\}^m$, iterating (1.8) shows that for every Borel subset $A \subseteq \operatorname{supp}(\mu)$,

$$\mu(T_J A) = c_J \begin{bmatrix} \mu(T_1 A) \\ \vdots \\ \mu(T_N A) \end{bmatrix}, \tag{3.12}$$

where $c_J := [c_J^1, \dots, c_J^N] := \boldsymbol{e}_{j_1} M_{j_2} \cdots M_{j_m}$, i.e.,

$$\mu(T_J A) = \sum_{j=1}^{N} c_J^i \mu(T_j A). \tag{3.13}$$

In view of the fact that $\mathbf{M}$ is tridiagonal and the expressions for $M_{i,i}$, $M_{i,i-1}$, and $M_{i,i+1}$, the entries of $\mathbf{M}$ are completely determined by the integrals

$$\int_a^b x^k \, d\mu \circ T_J, \quad k = 0, 1, 2, \quad J \in \{1, \dots, N\}^m,$$

which, by virtue of (3.13), can be written as $\sum_{j=1}^N c_J^i \int_a^b x^k \, d\mu \circ T_j$. This proves that $\mathbf{M}$ is determined by the $\mathfrak{I}_{k,j}$.

Finally, since

$$\int_a^b x^k \, d\mu \circ T_j = \int_{T_j[a,b]} (T_j^{-1} x)^k \, d\mu \quad \text{and} \quad \int_{T_j[a,b]} x^k \, d\mu = \int_a^b (T_j x)^k \, d\mu \circ T_j,$$

$\mathbf{M}$ is also determined by the $\mathfrak{J}_{k,j}$. □

The system in (3.9) has a unique solution if $\mathbf{M}$ is invertible.

Proposition 3.1. *Assume that* $\operatorname{supp}(\mu) = [a, b]$. *Then the mass matrix* $\mathbf{M}$ *is invertible. Consequently, (3.9) has a unique solution* $\mathbf{w}(t)$. *Moreover,* $\beta_j(t) \in C^2(0, T)$ *for* $j = 1, \dots, N^m - 1$.

Proof. If the mass matrix **M** is not invertible, there exists a nonzero piecewise linear function with zero L^2_μ norm, which implies that the measure μ does not have a full support. □

Proof of Theorem 1.2. This follows by combining the derivations above, Lemma 3.1, and Proposition 3.1. □

We now give another sufficient condition for the matrix **M** to be invertible. Define

$$(3.14)\qquad \begin{aligned} p_1(x) &:= \frac{(x-a)^2}{(b-a)^2}, & p_2(x) &:= \frac{(x-a)(2x-a-b)}{(b-a)^2},\\ p_3(x) &:= \frac{(b-x)(a+b-2x)}{(b-a)^2}, & p_4(x) &:= \frac{(b-x)^2}{(b-a)^2}, \end{aligned}$$

then

$$(3.15a)\qquad M_{1,1} - M_{1,2} = \int_a^b p_1\, d\mu \circ T_{J_1} + \int_a^b p_2\, d\mu \circ T_{J_2},$$

$$(3.15b)\qquad M_{i,i} - M_{i,i-1} - M_{i,i+1} = \int_a^b p_2\, d\mu \circ T_{J_i} + \int_a^b p_3\, d\mu \circ T_{J_{i+1}}, \quad 2 \le i \le N^m - 2,$$

$$(3.15c)\qquad M_{N^m-1,N^m-1} - M_{N^m-1,N^m-2} = \int_a^b p_3\, d\mu \circ T_{J_{N^m-1}} + \int_a^b p_4\, d\mu \circ T_{J_{N^m}}.$$

Recall that an $n \times n$ complex matrix $A = (a_{ij})$ is **strictly diagonally dominant** if

$$(3.16)\qquad |a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \quad \text{for } 1 \le i \le n.$$

It is well known that every $n \times n$ strictly diagonally dominant complex matrix is invertible; see, e.g., [40].

Proposition 3.2. *Let* **M** *be the mass matrix defined in (3.6) and* $p_i, i = 1, \ldots, 4$*, be defined as in (3.14). Assume that*

$$\int_a^b p_1 d\mu \circ T_{J_1} + \int_a^b p_2 d\mu \circ T_{J_2} > 0,$$

$$\int_a^b p_3 d\mu \circ T_{J_{N^m-1}} + \int_a^b p_4 d\mu \circ T_{J_{N^m}} > 0,$$

$$\int_a^b p_2 d\mu \circ T_{J_i} + \int_a^b p_3 d\mu \circ T_{J_{i+1}} > 0, \quad \textit{for all } 2 \le i \le N^m - 2.$$

Then **M** *is strictly diagonally dominant, and thus invertible. Hence the conclusions of Proposition 3.1 hold.*

For the infinite Bernoulli convolution associated with the golden ratio, as well as the three-fold convolution of the Cantor measure (see Section 4), it can be shown that $\mathbf{M}$ is strictly diagonally dominant. We omit the details.

Next, we discuss the solution of the linear system (3.7). Let $\mathbf{w}_n := \mathbf{w}(t_n)$, $n \geq -1$, and use the central difference method to solve the IVP (3.9). (The value of $\mathbf{w}_{-1}$ is defined below.)

We approximate the derivatives as

$$\frac{d^2\mathbf{w}(t_n)}{dt^2} \approx \frac{\mathbf{w}_{n+1} - 2\mathbf{w}_n + \mathbf{w}_{n-1}}{(\Delta t)^2} \quad \text{and} \quad \mathbf{w}'(t_n) \approx \frac{\mathbf{w}_{n+1} - \mathbf{w}_{n-1}}{2\Delta t}. \tag{3.17}$$

Substituting (3.17) into (3.7) yields

$$\frac{\mathbf{w}_{n+1} - 2\mathbf{w}_n + \mathbf{w}_{n-1}}{(\Delta t)^2} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{w}_n, \quad \text{i.e.,} \quad \mathbf{w}_{n+1} = (2\mathbf{I} - (\Delta t)^2\mathbf{M}^{-1}\mathbf{K})\mathbf{w}_n - \mathbf{w}_{n-1}.$$

Moreover, using

$$\mathbf{w}_1 = (2\mathbf{I} - (\Delta t)^2\mathbf{M}^{-1}\mathbf{K})\mathbf{w}_0 - \mathbf{w}_{-1} \quad \text{and} \quad \mathbf{w}_0' = \frac{\mathbf{w}_1 - \mathbf{w}_{-1}}{2\Delta t},$$

we get

$$\mathbf{w}_1 = \left(\mathbf{I} - \frac{(\Delta t)^2}{2}\mathbf{M}^{-1}\mathbf{K}\right)\mathbf{w}_0 + (\Delta t)\mathbf{w}_0'. \tag{3.18}$$

Therefore, equation (3.7) becomes

$$\begin{cases} \mathbf{w}_{n+1} = (2\mathbf{I} - (\Delta t)^2\mathbf{M}^{-1}\mathbf{K})\mathbf{w}_n - \mathbf{w}_{n-1}, & n = 1, 2, \ldots, \\ \mathbf{w}_0 = \mathbf{w}(t_0) = \mathbf{w}(0), \\ \mathbf{w}_1 = \mathbf{w}(t_1) = \left(\mathbf{I} - \frac{1}{2}(\Delta t)^2\mathbf{M}^{-1}\mathbf{K}\right)\mathbf{w}_0 + (\Delta t)\mathbf{w}_0', \\ t_n = n\Delta t. \end{cases} \tag{3.19}$$

To solve this system, fix Δt and substitute the initial conditions $\mathbf{w}_0$ and $\mathbf{w}_0'$ from (3.8) into (3.18) to get $\mathbf{w}_1$. Then substitute $\mathbf{w}_0$ and $\mathbf{w}_1$ into the first equation in (3.19) to find $\mathbf{w}_2$. Then $\mathbf{w}_{n+1}$ can be computed recursively.

4 Fractal measures defined by iterated function systems

In this section, we solve the homogeneous IBVP (1.2) numerically for three different measures: a weighted Bernoulli-type measure, the infinite Bernoulli convolution associated with the golden ratio, and the three-fold convolution of the Cantor measure. The first measure is defined by a p.c.f. IFS, whereas the second and third are defined by IFSs with overlaps.

We assume the hypotheses of Section 3. In order to solve (3.9) or (3.19), we need to compute the matrix $\mathbf{M}$ (the matrix $\mathbf{K}$ can be computed easily). In light of Lemma 3.1, it suffices to compute the integrals $\mathcal{I}_{k,j}$, $k = 0, 1, 2$, $j = 1, \dots, N$, as defined in (3.11). We find the exact values of these integrals for the measures in this section. The following integration formula is used repeatedly. For every continuous function φ on $\operatorname{supp}(\mu) = [a, b]$,

$$\int_a^b \varphi \, d\mu = \sum_{i=1}^{q} p_i \int_a^b \varphi \circ S_i d\mu. \tag{4.1}$$

Substituting the values of $\mathcal{I}_{k,j}$ into (3.10), we obtain the matrix $\mathbf{M}$. This allows us to solve equation (3.19).

4.1 Weighted Bernoulli-type measure.

A weighted Bernoulli-type measure μ is defined by the IFS

$$S_1(x) = \frac{1}{2}x, \qquad S_2(x) = \frac{1}{2}x + \frac{1}{2},$$

together with probability weights p, $1 - p$. Thus

$$\mu = p\mu \circ S_1^{-1} + (1 - p)\mu \circ S_2^{-1}.$$

For each Borel subset $A \subseteq [0, 1]$,

$$\begin{bmatrix} \mu(S_1 S_i A) \\ \mu(S_2 S_i A) \end{bmatrix} = M_i \begin{bmatrix} \mu(S_1 A) \\ \mu(S_2 A) \end{bmatrix}, \quad i = 1, 2,$$

where

$$M_1 = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1-p & 0 \\ 0 & 1-p \end{bmatrix}$$

Let $J = j_1 j_2 \cdots j_m$, $j_i = 1$ or 2. Then

$$\mu(S_J A) = c_J \begin{bmatrix} \mu(S_1 A) \\ \mu(S_2 A) \end{bmatrix}, \quad \text{where } c_J = \boldsymbol{e}_{j_1} M_{j_2} \cdots M_{j_m} = (c_J^1, c_J^2).$$

Since the IFS satisfies the open set condition, it is straightforward to evaluate the integrals $\mathcal{I}_{k,j}$; we omit the details. In view of [1], we choose the weight $p = 2 - \sqrt{3}$ in Figure 1.

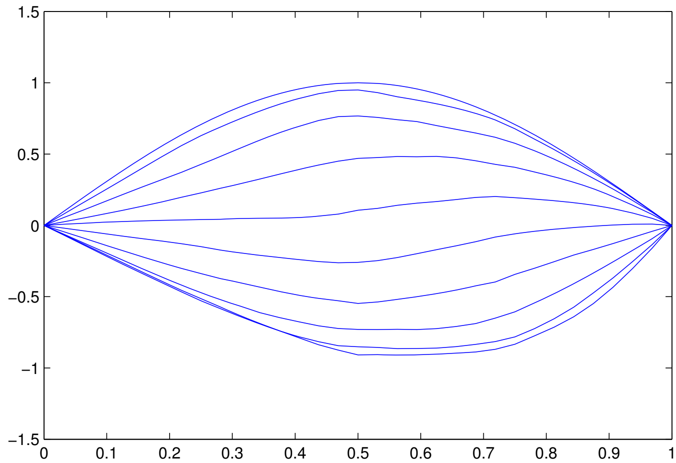
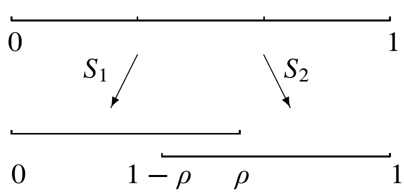


Figure 1. The weighted Bernoulli-type measure associated with the weights $p = 2-\sqrt{3}$ and $1-p = \sqrt{3}-1$. The initial data $g = \sin(\pi x)$ and $h = 0$ are used, and the time step Δt in equation (3.19) is taken to be 0.001. From top to bottom, the values of t are 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9. Animations for this and other graphs in the paper have been created and uploaded to the webpage `http://homepages.uconn.edu/fractals/wave/`.

4.2 Infinite Bernoulli convolution associated with the golden ratio. The infinite Bernoulli convolution associated with the golden ratio is defined by the IFS

$$S_1(x) = \rho x, \quad S_2(x) = \rho x + (1-\rho), \quad \rho = \frac{\sqrt{5}-1}{2}.$$



For each $0 < p < 1$, we call the corresponding self-similar measure

$$\mu = p\mu \circ S_1^{-1} + (1-p)\mu \circ S_2^{-1}$$

a **weighted infinite Bernoulli convolution associated with the golden ratio**. If $p = 1/2$, we get the classical one.

The measure μ_p satisfies a family of second-order self-similar identities. This was first pointed out by Strichartz et al. [35]. Define

$$T_1(x) = \rho^2 x, \qquad T_2(x) = \rho^3 x + \rho^2, \qquad T_3(x) = \rho^2 x + \rho.$$

Then μ satisfies the following second-order identities [25]. For each Borel subset $A \subseteq [0, 1]$,

$$\begin{bmatrix} \mu(T_1T_iA) \\ \mu(T_2T_iA) \\ \mu(T_3T_iA) \end{bmatrix} = M_i \begin{bmatrix} \mu(T_1A) \\ \mu(T_2A) \\ \mu(T_3A) \end{bmatrix} \qquad i = 1, 2, 3,$$

where M_1, M_2, M_3 are, respectively,

$$\begin{bmatrix} p^2 & 0 & 0 \\ (1-p)p^2 & (1-p)p & 0 \\ 0 & 1-p & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & p^2 & 0 \\ 0 & (1-p)p & 0 \\ 0 & (1-p)^2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & p & 0 \\ 0 & (1-p)p & (1-p)^2 p \\ 0 & 0 & (1-p)^2 \end{bmatrix}.$$

We can make use of this to compute the measure of suitable subintervals of [0, 1]. In fact, letting $J = j_1 \cdots j_m$, $j_i = 1, 2$ or 3, we see that for every Borel subset $A \subseteq [0, 1]$,

$$\mu(T_J A) = c_J \begin{bmatrix} \mu(T_1A) \\ \mu(T_2A) \\ \mu(T_3A) \end{bmatrix}, \quad \text{where} \quad c_J = \boldsymbol{e}_{j_1} M_{j_2} \cdots M_{j_m} = (c_J^1, c_J^2, c_J^3).$$

Moreover, by using (4.1), we can evaluate the integrals $\mathcal{I}_{k,j}$ in (3.11). For $p = 1/2$, the results are

(4.2)

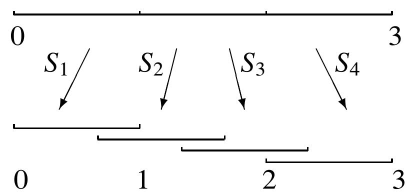
$$\begin{array}{lll} \displaystyle\int_0^1 d\mu \circ T_1 = \frac{1}{3} & \displaystyle\int_0^1 d\mu \circ T_2 = \frac{1}{3} & \displaystyle\int_0^1 d\mu \circ T_3 = \frac{1}{3} \\ \displaystyle\int_0^1 x\, d\mu \circ T_1 = \frac{1}{6(3\rho - 1)} & \displaystyle\int_0^1 x\, d\mu \circ T_2 = \frac{1}{6} & \displaystyle\int_0^1 x\, d\mu \circ T_3 = \frac{1}{6(3\rho^2 + 3)} \\ \displaystyle\int_0^1 x^2\, d\mu \circ T_1 = \frac{5\rho + 4}{6(\rho + 8)} & \displaystyle\int_0^1 x^2\, d\mu \circ T_2 = \frac{\rho + 5}{6(\rho + 8)} & \displaystyle\int_0^1 x^2\, d\mu \circ T_3 = \frac{2 - \rho}{6(\rho + 8)}. \end{array}$$

We can thus calculate the entries of the mass matrix **M** and solve the linear system (3.9). The result is shown in Figure 2.

4.3 Three-fold convolution of the Cantor measure. The three-fold convolution of the Cantor measure μ also satisfies a family of second-order identities. It is defined by the IFS

$$S_i(x) = \frac{1}{3}x + \frac{2}{3}(i - 1), \quad \text{for} \quad i = 1, 2, 3, 4,$$

which does not satisfy the OSC.



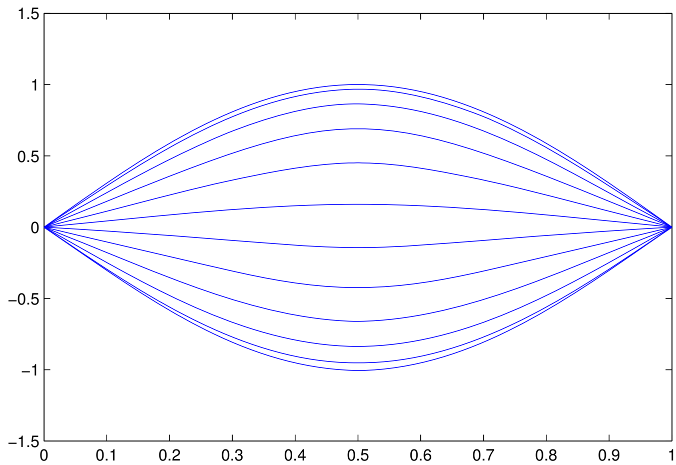


Figure 2. Infinite Bernoulli convolution associated with the golden ratio. The initial data $g = \sin(\pi x)$ and $h = 0$ are used. The time step Δt in equation (3.19) is taken to be 0.001. From top to bottom, the values of t are $0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1$.

The measure μ satisfies the self-similar identity

$$\mu = \frac{1}{8}\mu \circ S_1^{-1} + \frac{3}{8}\mu \circ S_2^{-1} + \frac{3}{8}\mu \circ S_3^{-1} + \frac{1}{8}\mu \circ S_4^{-1}.$$

Define

$$T_1(x) = \frac{1}{3}x, \quad T_2(x) = \frac{1}{3}x + 1, \quad T_3(x) = \frac{1}{3}x + 2.$$

Then μ satisfies the following second-order identities [25]. For every Borel subset $A \subseteq [0, 3]$,

$$\begin{bmatrix} \mu(T_{1j}A) \\ \mu(T_{2j}A) \\ \mu(T_{3j}A) \end{bmatrix} = M_j \begin{bmatrix} \mu(T_1A) \\ \mu(T_2A) \\ \mu(T_3A) \end{bmatrix}, \quad j = 1, 2, 3,$$

where the coefficient matrices M_j are given by

$$M_1 = \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad M_2 = \frac{1}{8}\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_3 = \frac{1}{8}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $J = j_1 \cdots j_m$, $j_i = 1, 2$ or 3. Then

$$\mu(T_J A) = c_J \begin{bmatrix} \mu(T_1 A) \\ \mu(T_2 A) \\ \mu(T_3 A) \end{bmatrix}, \quad \text{where} \quad c_J = \boldsymbol{e}_{j_1} M_{j_2} \cdots M_{j_m} = (c_J^1, c_J^2, c_J^3).$$

The integrals $\mathcal{I}_{k,j}$ in (3.11) are

(4.3)

$$\begin{aligned}
&\int_0^3 d\mu \circ T_1 = \frac{1}{5} && \int_0^3 d\mu \circ T_2 = \frac{3}{5} && \int_0^3 d\mu \circ T_3 = \frac{1}{5} \\
&\int_0^3 x\, d\mu \circ T_1 = \frac{27}{70} && \int_0^3 x\, d\mu \circ T_2 = \frac{9}{10} && \int_0^3 x\, d\mu \circ T_3 = \frac{3}{14} \\
&\int_0^3 x^2\, d\mu \circ T_1 = \frac{5517}{6440} && \int_0^3 x^2\, d\mu \circ T_2 = \frac{11943}{6440} && \int_0^3 x^2\, d\mu \circ T_3 = \frac{63}{184}.
\end{aligned}$$

Again, using these values, we can compute $\mathbf{M}$ and solve (3.9); see Figure 3.

5 Convergence of numerical approximations

In this section, we prove the convergence of the numerical approximations of the homogeneous IBVP (1.2). Some of our results are obtained by modifying similar ones in [34]; see also [2].

We assume the same setup as in Section 3 unless stated otherwise. Let V_m be the set of end-points of all the level-m subintervals, and arrange its elements so that $V_m = \{x_i : i = 0, 1, \ldots, N^m\}$ with $x_i < x_{i+1}$ for $i = 0, 1, \ldots, N^m - 1$, $x_0 = a$ and $x_{N^m} = b$. Let S^m be the space of continuous piecewise-linear functions on $[a, b]$ with nodes V_m, and let $S_D^m := \{u \in S^m : u(a) = u(b) = 0\}$ be the subspace of S^m consisting of functions satisfying the Dirichlet boundary condition. Then $\dim S^m = \#V_m = N^m + 1$ and $\dim S_D^m = \#V_m - 2 = N^m - 1$.

We choose the basis of S^m consisting of the tent functions $\{\phi_i\}_{i=0}^{N^m}$ defined in (3.3) and choose the basis $\{\phi_i\}_{i=1}^{N^m-1}$ for S_D^m.

Definition 5.1. Let V_m be defined as above and $\{\phi_i\}_{i=0}^{N^m}$ be defined as in (3.3). The linear map $\mathcal{P}_m : \operatorname{Dom} \mathcal{E} \to S_D^m$ defined by

$$\mathcal{P}_m v := \sum_{i=1}^{N^m-1} v(x_i)\phi_i(x), \quad v \in \operatorname{Dom} \mathcal{E},$$

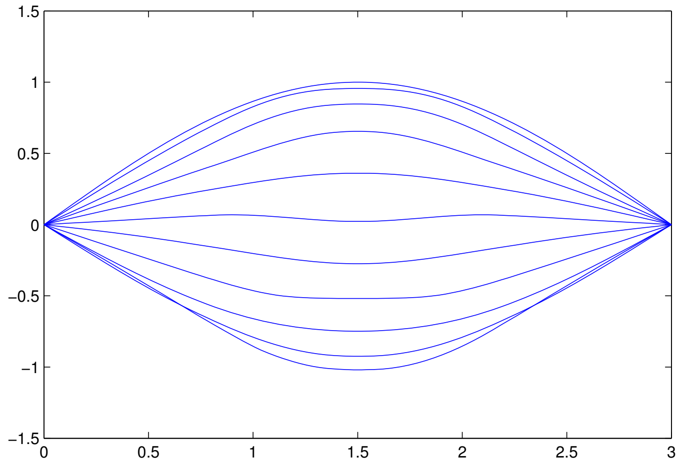


Figure 3. Three-fold convolution of the Cantor measure. The initial data $g = \sin(\pi x/3)$ and $h = 0$ are used, and $\Delta t = 0.001$. From top to bottom, the values of t are 0.0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0.

is called the **Rayleigh-Ritz projection** with respect to V_m.

$\mathcal{P}_m v$ is the piecewise linear interpolant of the values of v on V_m.

Lemma 5.1. *For $m \geq 1$ and $v \in \operatorname{Dom}\mathcal{E}$, $\mathcal{P}_m v$ is the component of v in the subspace S_D^m, $v - \mathcal{P}_m v$ vanishes on the boundary $\{a, b\}$, and $\mathcal{E}(v - \mathcal{P}_m v, w) = 0$ for all $w \in S_D^m$.*

Proof. See, e.g., [34]. □

Lemma 5.2. *Assume the hypotheses of Lemma 5.1. Then $v|_{V_m} = \mathcal{P}_m v|_{V_m}$ for all $v \in \operatorname{Dom}\mathcal{E}$,*

Proof. Similar to that of [2, Lemma 5.3]. □

Denote by $\|V_m\| := \max\{x_i - x_{i-1} : 1 \leq i \leq m\}$ the norm of a partition V_m.

Lemma 5.3. *Assume the hypotheses of Lemma 5.1, and let $v \in \operatorname{Dom}\mathcal{E}$. Then*

$$|\mathcal{P}_m v(x) - v(x)| \leq 2\,\|V_m\|^{1/2}\,\|v\|_{\operatorname{Dom}\mathcal{E}} \quad \text{for all } x \in [a, b].$$

In particular,

$$\|\mathcal{P}_m v - v\|_{\mu} \leq 2\|V_m\|^{1/2}\|v\|_{\operatorname{Dom}\mathcal{E}}.$$

Proof. First note that since v is absolutely continuous and belongs to $\operatorname{Dom}\mathcal{E}$,

$$(5.1)\quad |v(x)-v(y)| = \left|\int_y^x v'(s)\,ds\right| \le |x-y|^{1/2}\,\|v\|_{H_0^1(a,b)}, \quad \text{for all } x, y \in [a,b].$$

Now let $i \in \{1,\dots,N^m\}$ be such that $x \in [x_{i-1}, x_i]$. Then, by (5.1) and Lemma 5.2,

$$\begin{aligned} |\mathcal{P}_m v(x) - v(x)| &\le |\mathcal{P}_m v(x) - v(x_{i-1})| + |v(x_{i-1}) - v(x)| \\ &\le |v(x_i) - v(x_{i-1})| + |v(x_{i-1}) - v(x)| \\ &\le 2\|V_m\|^{1/2}\,\|v\|_{\operatorname{Dom}\mathcal{E}}. \end{aligned}$$

□

Throughout the rest of this section,

$$(5.2)\qquad g, h \in \operatorname{Dom}\mathcal{E} \quad \text{and} \quad f = 0,$$

and u is the solution of the corresponding homogeneous IBVP (1.2). According to Theorem 2.5,

$$(5.3)\qquad u \in W_2^k(0,T;\operatorname{Dom}\mathcal{E}) \quad \text{for all } k \ge 0.$$

In particular, $u_{tt} \in \operatorname{Dom}\mathcal{E}$ and

$$(5.4)\qquad (u_{tt}, v)_\mu + \mathcal{E}(u,v) = 0 \quad \text{for all } v \in \operatorname{Dom}\mathcal{E}.$$

As in Section 3, we let

$$u^m(x,t) = \sum_{i=1}^{N^m-1} \beta_i(t)\phi_i(x).$$

Finally, define

$$e(x,t) = e^m(x,t) := \mathcal{P}_m u(x,t) - u^m(x,t).$$

Lemma 5.4. *Let g, h, f, u, u^m, e be as above.*

(a) *u^m satisfies*

(i) $(u^m_{tt}, v^m)_\mu + \mathcal{E}(u^m, v^m) = 0$ *for all* $v^m \in S^m_D$,

(ii) $u^m(x,0) = \sum_{i=1}^{N^m-1} g(x_i)\phi_i(x)$ *and* $u^m_t(x,0) = \sum_{i=1}^{N^m-1} h(x_i)\phi_i(x)$;

(b)

$$(5.5)\qquad (e_{tt}, e_t)_\mu + \mathcal{E}(e, e_t) = (\mathcal{P}_m u_{tt} - u_{tt}, e_t)_\mu.$$

Proof. The proof of (a) follows from the derivations in Section 3; we omit the details.

As for (b), by definition and the fact that $u \in W_2^k(0,T;\operatorname{Dom}\mathcal{E})$ for $k \geq 0$, the functions e_t, e_{tt}, and $(\mathcal{P}_m u)_{tt} = \mathcal{P}_m u_{tt}$ all belong to $\mathcal{S}_D^m$.

Substituting e_t for v in (5.4) and for v^m in (a)(i), and then subtracting the resulting equations, we get $(u_{tt} - u_{tt}^m, e_t)_\mu + \mathcal{E}(u - u^m, e_t) = 0$. Equivalently,

$$(u_{tt} - \mathcal{P}_m u_{tt} + \mathcal{P}_m u_{tt} - u_{tt}^m, e_t)_\mu + \mathcal{E}(u - \mathcal{P}_m u + \mathcal{P}_m u - u^m, e_t) = 0,$$

which implies

$$(\mathcal{P}_m u_{tt} - u_{tt}^m, e_t)_\mu + \mathcal{E}(\mathcal{P}_m u - u^m, e_t) = (\mathcal{P}_m u_{tt} - u_{tt}, e_t)_\mu,$$

because $\mathcal{E}(u - \mathcal{P}_m u, e_t) = 0$ (Lemma 5.1). Identity (5.5) now follows from the definition of $e(t)$. □

Theorem 5.5. *Assume the hypotheses of Lemma 5.4, and let ρ be as in (1.7). Then there exists a constant $C > 0$ such that*

$$\left\|\mathcal{P}_m u - u^m\right\|_\mu \leq C\sqrt{T}\rho^{m/2}\, \|u_{tt}\|_{2,\operatorname{Dom}\mathcal{E}}\,.$$

Proof. Let $E(t) := \frac{1}{2}(e_t, e_t)_\mu + \frac{1}{2}\mathcal{E}(e,e) = \frac{1}{2}\|e_t\|_\mu^2 + \frac{1}{2}\|e\|_{\operatorname{Dom}\mathcal{E}}^2$. Then

$$\|e_t\|_\mu \leq \sqrt{2}\sqrt{E(t)}, \tag{5.6}$$

$$\|e\|_{\operatorname{Dom}\mathcal{E}} \leq \sqrt{2}\sqrt{E(t)}, \tag{5.7}$$

and

$$E(t) \leq \frac{1}{2}\big(\|e_t\|_\mu + \|e\|_{\operatorname{Dom}\mathcal{E}}\big)^2. \tag{5.8}$$

The left-hand side of (5.5) equals

$$\frac{1}{2}\big(\|e_t\|_\mu^2\big)_t + \frac{1}{2}\big(\|e\|_{\operatorname{Dom}\mathcal{E}}^2\big)_t = E_t(t). \tag{5.9}$$

For the right-hand side of (5.5), we apply the Cauchy-Schwarz inequality and (5.6) to get

$$E_t(t) = (\mathcal{P}_m u_{tt} - u_{tt}, e_t)_\mu \leq \|\mathcal{P}_m u_{tt} - u_{tt}\|_\mu\, \|e_t\|_\mu \leq \|\mathcal{P}_m u_{tt} - u_{tt}\|_\mu \sqrt{2}\sqrt{E(t)}. \tag{5.10}$$

Since $E(t) \geq 0$ with $E(0) = 0$, we can assume that $E(t) > 0$ on some interval $(\alpha, \beta) \subset [0,T]$ with $\alpha < \beta$ and $E(\alpha) = 0$. (Otherwise, by the continuity of $E(t)$, we have $E(s) = 0$ for all $s \in [0,T]$ and (5.11) below still holds.) It follows from (5.10) that

$$\frac{E_t(t)}{\sqrt{E(t)}} \leq \sqrt{2}\|\mathcal{P}_m u_{tt} - u_{tt}\|_\mu, \quad \alpha < s < \beta,$$

and thus

$$2\sqrt{E(s)} \le \sqrt{2}\int_{\alpha}^{\beta} \|\mathcal{P}_m u_{tt} - u_{tt}\|_{\mu}\, dt, \quad \alpha \le s \le \beta. \tag{5.11}$$

From (5.7) and (5.11), we have

$$\|e(s)\|_{\mathrm{Dom}\,\mathcal{E}} \le \sqrt{2}\sqrt{E(s)} \le \sqrt{2}\int_{\alpha}^{\beta} \|\mathcal{P}_m u_{tt} - u_{tt}\|_{\mu}\, dt \le \sqrt{2}\int_{0}^{T} \left\|\mathcal{P}_m u_{tt} - u_{tt}\right\|_{\mu} dt,$$

which holds for all $s \in [0, T]$. Thus, combining condition (2.2), Lemma 5.3, and the above estimations yields

$$\begin{aligned}\|e(s)\|_{\mu} \le& C\, \|e(s)\|_{\mathrm{Dom}\,\mathcal{E}} \le C\sqrt{T}\Big(\int_0^T \|\mathcal{P}_m u_{tt} - u_{tt}\|_{\mu}^2\, dt\Big)^{1/2} \\ \le& C\sqrt{T}\Big(\int_0^T \big(2\|V_m\|^{1/2}\, \|u_{tt}\|_{\mathrm{Dom}\,\mathcal{E}}\big)^2\, dt\Big)^{1/2} \quad \text{(Lemma 5.3)} \\ \le& 2C\sqrt{T}\rho^{m/2}\, \|u_{tt}\|_{2,\mathrm{Dom}\,\mathcal{E}}\,,\end{aligned}$$

which holds for all $s \in [0, T]$. This completes the proof. □

Proof of Theorem 1.3. For fixed $t \in [0, T]$,

$$\left\|u^m - u\right\|_{\mu} \le \left\|u^m - \mathcal{P}_m u\right\|_{\mu} + \|\mathcal{P}_m u - u\|_{\mu}\,.$$

Theorem 1.3 now follows by combining Lemma 5.3 and Theorem 5.5. □

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