# IMPROVED STRICHARTZ ESTIMATES FOR A CLASS OF DISPERSIVE EQUATIONS IN THE RADIAL CASE AND THEIR APPLICATIONS TO NONLINEAR SCHRÖDINGER AND WAVE EQUATIONS

*By*

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**Abstract.** We prove some new Strichartz estimates for a class of dispersive equations with radial initial data. In particular, we obtain the full radial Strichartz estimates up to some endpoints for the Schrödinger equation. Using these estimates, we obtain some new results related to nonlinear problems, including small data scattering and large data LWP for the nonlinear Schrödinger and wave equations with radial critical initial data and the well-posedness theory for the fractional order Schrödinger equation in the radial case.

## 1 Introduction

In this paper, we study the Cauchy problem for a class of dispersive equations of the type

$$i\partial_t u = -\phi(\sqrt{-\Delta})u + f, \quad u(0,x) = u_0(x), \tag{1.1}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}$ is smooth away from origin, $u(t,x) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$, $n \geq 2$, is the unknown function, $f(t,x)$ is a given function (e.g., $f = |u|^p u$ in the nonlinear setting), and $\phi(\sqrt{-\Delta})u = \mathscr{F}^{-1}\phi(|\xi|)\mathscr{F}u$. Here, $\mathscr{F}$ denotes the spatial Fourier transform, and $\phi(|\xi|)$ is usually referred to as the **dispersion relation** of equation (1.1). Many dispersive equations, for instance, the Schrödinger equation ($\phi(r) = r^2$), the wave equation ($\phi(r) = r$), the Klein-Gordon equation ($\phi(r) = \sqrt{1+r^2}$), the beam equation ($\phi(r) = \sqrt{1+r^4}$), and the fourth-order Schrödinger equation ($\phi(r) = r^2 + r^4$) reduce to equations of type (1.1).

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In the pioneering work [30], Strichartz derived a priori estimates in space-time norm $L_t^q L_x^r$ of solutions of (1.1) by proving a Fourier restriction inequality. Later, his results were improved via a dispersive estimate and duality argument (cf. [16] and references therein). The dispersive estimate

$$\|e^{it\phi(\sqrt{-\Delta})}u_0\|_X \lesssim |t|^{-\theta}\|u_0\|_{X'}, \tag{1.2}$$

where $X'$ is the dual space of $X$, plays a crucial role. Applying (1.2) together with a standard argument (cf. [16]), we can immediately obtain the Strichartz estimates. For instance, we see from the explicit formula of the free Schrödinger solution that

$$\|e^{it\Delta}u_0\|_{L_x^\infty} \lesssim |t|^{-n/2}\|u_0\|_{L_x^1}.$$

In [10], the authors systematically studied the dispersive estimates for (1.1) by imposing certain asymptotic conditions on $\phi$.

As explained in [16], the full range of the non-retarded Strichartz estimates for the Schrödinger equation are known completely while that of the retarded estimates remain open. However, it is well known that if the initial data $u_0$ is radial, generalized Strichartz estimates exist; see, e.g., [25]. Shao [23] showed that the frequency localized non-retarded Strichartz estimates for the Schrödinger equation give a wider range of estimates. For example,

$$\|e^{it\Delta}P_k u_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \le C 2^{(\frac{n}{2}-\frac{n+2}{q})k}\|u_0\|_2 \tag{1.3}$$

if $q > (4n+2)/(2n-1)$ and $u_0$ is radial. The proof relies deeply on the radial assumption, which eliminates the bad-type evolution in the non-radial case (e.g., the Knapp counter-example). Similar estimates hold for the wave equation; see [22].

It is easy to see that equation (1.1) is rotationally-invariant. Thus it is natural to ask whether better Strichartz estimates are obtainable for radial initial data than those derived from the dispersive estimates. Moreover, for applications, it is important to have the mixed space-time estimate

$$\|e^{it\phi(\sqrt{-\Delta})}P_k u_0\|_{L_t^q L_x^r(\mathbb{R}^{n+1})} \le C_k\|u_0\|_2, \tag{1.4}$$

where $u_0$ is assumed to be radial.

The first purpose of this paper is to obtain the sharp range of (1.3) for equation (1.1) in the case $q = r$. Indeed, we simplify some proofs and overcome the difficulty caused by the lack of scaling invariance by adapting some ideas in [10]. Moreover, by dealing carefully with logarithmic divergence, we prove that (1.3) actually holds for $q = (4n+2)/(2n-1)$. Our second purpose is to apply the improved Strichartz estimates to the nonlinear equations including the nonlinear Schrödinger equation, nonlinear wave equation, and nonlinear fractional-order Schrödinger equation.

Using the Christ-Kiselev lemma (Lemma 3.3 below), we derive the retarded estimates from the non-retarded estimates in order to apply our results to nonlinear problems. For example, consider the nonlinear Schrödinger equation

$$iu_t + \Delta u = \mu |u|^p u, \quad u(0, x) = u_0(x),$$

the well-posedness theory of which was studied deeply during the past decades. We remark that even in the case that $L^2$ is subcritical in the sense of scaling, the threshold of regularity in $\dot{H}^s$ for strong well-posedness is $s \geq \max(0, s_c)$, where $s_c$ is the scaling critical regularity. This can be seen from the Galilean invariance (see [1, 3])

$$u(t, x) \to e^{-i|y|^2 t + iy\cdot x} u(t, x - 2ty), \quad y \in \mathbb{R}^d.$$

However, it is also easy to see that the radial assumption breaks down the Galilean invariance. Thus it is natural to expect to be able to go below $L^2$ in the radial case. This is indeed the case, as discussed in detail in Section 4.

In order to study the non-homogeneous case (e.g. Klein-Gordon equation), we treat the high frequency and the low frequency in different scales. As in [10], we assume that $\phi : \mathbb{R}^+ \to \mathbb{R}$ is smooth and satisfies some of the following conditions.

(H1) There exists $m_1 > 0$ such that

$$|\phi'(r)| \sim r^{m_1 - 1} \text{ and } |\phi^{(\alpha)}(r)| \lesssim r^{m_1 - \alpha}, \quad r \geq 1,$$

for all integers $\alpha \geq 2$.

(H2) There exists $m_2 > 0$ such that

$$|\phi'(r)| \sim r^{m_2 - 1} \textit{ and } |\phi^{(\alpha)}(r)| \lesssim r^{m_2 - \alpha}, \quad 0 < r < 1,$$

for all integers $\alpha \geq 2$.

(H3) There exists $\alpha_1$ such that

$$|\phi''(r)| \sim r^{\alpha_1 - 2}, \quad r \geq 1.$$

(H4) There exists $\alpha_2$, such that

$$|\phi''(r)| \sim r^{\alpha_2 - 2}, \quad 0 < r < 1.$$

**Remark 1.1.** Heuristically, (H1) and (H3) reflect the dispersive effect in high frequency. If $\phi$ satisfies (H1) and (H3), then $\alpha_1 \leq m_1$. Similarly, the dispersive effect in low frequency is described by (H2) and (H4). If $\phi$ satisfies (H2) and (H4), then $\alpha_2 \geq m_2$. The special case $\alpha_2 = m_2$ occurs in most instances.

For convenience, given $m_1, m_2, \alpha_1, \alpha_2 \in \mathbb{R}$ as in (H1)-(H4), we define

$$m(k) = \begin{cases} m_1, & \text{for } k \geq 0, \\ m_2, & \text{for } k < 0; \end{cases} \quad \text{and} \quad \alpha(k) = \begin{cases} \alpha_1 & \text{for } k \geq 0, \\ \alpha_2 & \text{for } k < 0. \end{cases} \tag{1.5}$$

Now we are ready to state our first result.

**Theorem 1.2.** *Suppose $n \geq 2$, $k \in \mathbb{Z}$, $\phi : \mathbb{R}^+ \to \mathbb{R}$ is smooth away from origin, and $u_0$ is spherically symmetric. If $\phi$ satisfies* (H1) *and* (H2)*, then for $2n/(n-1) < q \leq \infty$,*

$$\|S_\phi(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \lesssim 2^{(\frac{n}{2}-\frac{n+m(k)}{q})k}\|u_0\|_2, \tag{1.6}$$

*Furthermore, if $\phi$ also satisfies* (H3) *and* (H4)*, then for $(4n+2)/(2n-1) \leq q \leq 6$,*

$$\|S_\phi(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \lesssim 2^{(\frac{n}{2}-\frac{n+m(k)}{q})k+(\frac{1}{4}-\frac{1}{2q})(m(k)-\alpha(k))k}\|u_0\|_2, \tag{1.7}$$

*where $m(k), \alpha(k)$ are given by* (1.5)*, $P_k$ is the Littlewood-Paley projector, and $S_\phi(t) = e^{it\phi(\sqrt{-\Delta})}$ is the dispersive group, (defined later). The range of $q$ is optimal in the sense that* (1.6) *fails to hold if $q \leq 2n/(n-1)$ and* (1.7) *fails to hold if $q < (4n+2)/(2n-1)$.*

For the Schrödinger equation, $\phi(r) = r^2$ and $\phi$ satisfies (H1)-(H4) with $m(k) = \alpha(k) = 2$. The following corollary is then an immediate consequence of Theorem 1.2.

**Corollary 1.3.** *Assume $n \geq 2$, $k \in \mathbb{Z}$, $(4n+2)/(2n-1) \leq q \leq \infty$. Then there exists $C > 0$ such that*

$$\|e^{it\Delta}P_k u_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \leq C2^{(\frac{n}{2}-\frac{n+2}{q})k}\|u_0\|_2 \tag{1.8}$$

*for spherically symmetric $u_0 \in L^2(\mathbb{R}^n)$. Moreover, the range of $q$ is optimal in the sense that* (1.8) *fails to hold if $q < (4n+2)/(2n-1)$.*

**Remark 1.4.** Shao [23] proved (1.8) for $q > (4n+2)/(2n-1)$. For the wave equation, $\phi(r) = r$ and $\phi$ satisfies (H1)-(H2) with $m(k) \equiv 1$. Equation (1.6) then reduces to that given in [22]. Of interest is the fact that the range $q > 2n/(n-1)$ is optimal for the wave equation. It is also worth noting that if $q > 2n/(n-1)$, (1.6) gives a better bound than (1.7) since $k[m(k) - \alpha(k)] \geq 0$ in view of Remark 1.1.

We apply Theorem 1.2 to some concrete equations. Then, using the Christ-Kiselev lemma, we get the retarded Strichartz estimates. In view of the classical

Strichartz estimates, it is natural to want to know the sharp range of the mixed Strichartz estimates

$$\|S_\phi(t)P_k u_0\|_{L_t^q L_x^r(\mathbb{R}^{n+1})} \lesssim C(k)\|u_0\|_2.$$

For the purpose of finding the sharp range, we restrict ourselves to the simple case $\phi(r) = r^a$, $a > 0$; namely, we consider the estimate

$$\|e^{itD^a}P_k f\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \leq C2^{k(\frac{n}{2}-\frac{a}{q}-\frac{n}{r})}\|f\|_{L_x^2(\mathbb{R}^n)}, \tag{1.9}$$

where $D = \sqrt{-\Delta}$, $a > 0$. In this case, we have scaling invariance, and thus the proof is less complicated. Nevertheless, the proof can still be adapted to the general case.

**Theorem 1.5.** (a) *Assume $a = 1$ and $n \geq 3$. Then* (1.9) *holds for all radial functions $f \in L^2(\mathbb{R}^n)$ if and only if*

$$(q, r) = (\infty, 2) \quad or \quad 2 \leq q \leq \infty, \quad \frac{1}{q} + \frac{n-1}{r} < \frac{n-1}{2}.$$

(b) *Assume $0 < a \neq 1$ and $n \geq 2$. Then* (1.9) *holds for all radial functions $f \in L^2(\mathbb{R}^n)$ if*

$$\frac{4n+2}{2n-1} \leq q \leq \infty, \quad \frac{2}{q} + \frac{2n-1}{r} \leq n - \frac{1}{2} \quad or$$
$$2 \leq q < \frac{4n+2}{2n-1}, \quad \frac{2}{q} + \frac{2n-1}{r} < n - \frac{1}{2}.$$

*On the other hand,* (1.9) *fails to hold if*

$$q > 2 \quad or \quad \frac{2}{q} + \frac{2n-1}{r} > n - \frac{1}{2}.$$

**Remark 1.6.** The range of $(q, r)$ is indicated in Figure 1, where

$$B = \left(\frac{n-3}{2n-2}, \frac{1}{2}\right),\ C = \left(\frac{n-2}{2n-2}, \frac{1}{2}\right),\ D = \left(\frac{n-1}{2n}, \frac{n-1}{2n}\right),$$
$$B' = \left(\frac{n-2}{2n}, \frac{1}{2}\right),\ C' = \left(\frac{2n-3}{4n-2}, \frac{1}{2}\right),\ D' = \left(\frac{2n-1}{4n+2}, \frac{2n-1}{4n+2}\right).$$

The results for the wave equation ($a = 1$) are not new. The positive results appear in [20, 25, 29, 7]. A counter-example was given in [13].

On the other hand, for the Schrödinger equation, the results seem to be new. We see that the picture is almost complete, except that the segment $C'D'$ is unknown[1]. In view of the positive results on the non-radial endpoint in [16], we conjecture the following radial endpoint estimates.

[1] After this paper appeared on arXiv, there was further improvement showing that $C'D'$ is also allowed except the point $C'$; see [6, 15].

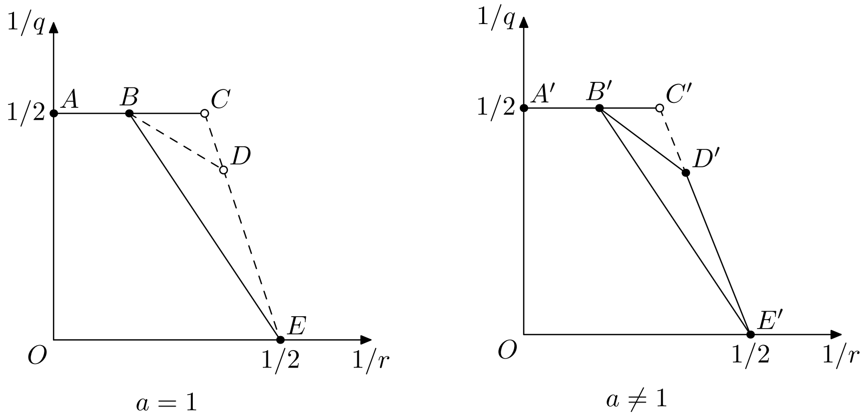


Figure 1. Range of $(q, r)$ for (1.9).

**Conjecture 1.7.** *Assume $n \geq 2$ and $0 < a \neq 1$. Then there exists a constant $C$ such that*

$$\|e^{itD^a} P_0 f\|_{L_t^2 L_x^{\frac{4n-2}{2n-3}}(\mathbb{R}\times\mathbb{R}^n)} \leq C\|f\|_{L_x^2(\mathbb{R}^n)}, \tag{1.10}$$

*for all radial functions $f \in L_x^2(\mathbb{R}^n)$.*

This is very similar to the endpoint Strichartz estimates in the non-radial case, which was studied in [16]. As expected, (1.10) is just "logarithmically far" from being proved. Indeed, for every $j \in \mathbb{N}$,

$$\|e^{itD^a} P_0 f\|_{L_t^2 L_x^{\frac{4n-2}{2n-3}}(\mathbb{R}\times\{|x|\sim 2^j\})} \leq C\|f\|_{L_x^2(\mathbb{R}^n)}.$$

However, we cannot adapt the method on $D'E'$ to overcome this logarithmical divergence; see Remark 2.14 below for further discussion of (1.10).

Using these Strichartz estimates, we study nonlinear problems and prove some new results, one of which is the following for the nonlinear Schrödinger equation.

**Theorem 1.8.** *Assume*

$$n \geq 2, \quad 0 < p < \frac{4}{n}, \quad s_{sch} = \frac{n}{2} - \frac{2}{p}, \quad \frac{1-n}{2n+1} \leq s_{sch} < 0,$$

*and $u_0$ is radial. If $\|u_0\|_{\dot{H}^{s_{sch}}} \leq \delta$ for some $\delta \ll 1$, there exists a unique global solution $u$ of*

$$iu_t + \Delta u = \mu|u|^p u, \quad u(0, x) = u_0(x)$$

*($\mu = \pm 1$) such that $u \in C(\mathbb{R} : \dot{H}^{s_{sch}}) \cap L_{t,x}^{\frac{p(n+2)}{2}}(\mathbb{R}\times\mathbb{R}^n)$. Moreover, there exist $u_\pm \in \dot{H}^{s_{sch}}$ such that $\|u - S(t)u_\pm\|_{\dot{H}^{s_{sch}}} \to 0$, as $t \to \pm\infty$.*

Our methods show that the index $(1-n)/(2n-1)$ is sharp for the critical global well-posedness (GWP). For further results in this direction, see Theorem 4.1 below. For the nonlinear wave equation, we prove the following result.

**Theorem 1.9.** *Assume*

$$n \geq 2, \quad 0 < p < \frac{4}{n-1}, \quad s_w = \frac{n}{2} - \frac{2}{p}, \quad \frac{1}{2n} < s_w < 1/2,$$

*and $u_0$ is radial. If $\|u_0\|_{\dot{H}^{s_w}} + \|u_1\|_{\dot{H}^{s_w-1}} \leq \delta$ for some $\delta \ll 1$, there exists a unique global solution $u$ of*

$$\partial_{tt}u - \Delta u = \mu |u|^p u, \quad (t,x) \in \mathbb{R}^{n+1}, \quad u(0) = u_0(x),\ u_t(0) = u_1(x)$$

*($\mu = \pm 1$) such that $u \in C(\mathbb{R} : \dot{H}^{s_w}) \cap C^1(\mathbb{R} : \dot{H}^{s_w-1}) \cap L_{t,x}^{\frac{2n+2}{n-2s_w}}(\mathbb{R} \times \mathbb{R}^n)$, and there exists $(u_\pm, v_\pm) \in \dot{H}^{s_w} \times \dot{H}^{s_w-1}$ such that*

$$\|u - W'(t)u_\pm\|_{\dot{H}^{s_w}} + \|u_t - W(t)v_\pm\|_{\dot{H}^{s_w-1}} \to 0 \quad \textit{as } t \to \pm\infty.$$

Our results also hold for more general nonlinearity, e.g., $F(u)$ with $F$ satisfying conditions such as (4.3). In [21], Lindblad and Sogge studied the semi-linear wave equation with the same nonlinearity but with general non-radial initial data. For example, for the nonlinearity $|u|^p$, they proved small data scattering in $\dot{H}^s \times \dot{H}^{s-1}$ with $s = \frac{n}{2} - \frac{2}{p-1}$ if $p \geq \frac{n+3}{n-1}$ and local well-posedness if $s \geq s(p,n)$ for some $s(p,n)$. Thus their results cover the case $s_w \geq 1/2$ in Theorem 1.9, which is the main reason why we restrict ourselves to the case $s_w < 1/2$. In the same paper [21], the authors actually showed that their results are sharp by constructing some counter-examples. However, the counter-examples for $s_w < 1/2$ don't work for the radial case. Our Theorem 1.9 improves their results in the radial case. Actually, we find a critical regularity in the radial case $s_0(n) < 1/2n$, which we discuss in detail in Theorem 4.4. In Section 4, we also study the nonlinear fractional order Schrödinger equation and establish the well-posedness theory in the radial case; cf. Theorem 4.7 below.

The fact that better well-posedness results hold in the radial case has been observed before; see [25, 8], [11, 12]. Our results generalize these results. In the non-radial case, with additional angular regularity, one can also go below $L^2$; see [8, 14] and the references therein. Actually, the results in [8] for the Schrödinger equation are more general than ours but with different resolution space. Our results for local well-posedness hold without change for the inhomogeneous data $u_0 \in H^s$; see Remark 4.3. It is then natural to ask whether (1.8) and (1.9) hold for non-radial functions with certain angular regularity.

Throughout this paper, $C > 1$ and $c < 1$ denote universal positive constants, which might be different at different places. We write $A \lesssim B$ to mean that there exists $C$ such that $A \leq CB$, and $A \sim B$ to mean that $A \lesssim B$ and $B \lesssim A$. We use $\hat{f}(\xi)$ and $\mathscr{F}(f)$ to denote the spatial Fourier transform of $f$ on $\mathbb{R}^n$ defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx.$$

We denote by $p'$ the **Hölder dual** of $p \in [1, \infty]$, i.e., $1/p + 1/p' = 1$. Unless stated otherwise, $\Phi : \mathbb{R} \to [0, 1]$ is a non-negative, smooth, even function such that $\operatorname{supp}\Phi \subseteq \{x : |x| \leq 2\}$. and $\Phi(x) = 1$ for $|x| \leq 1$, We set $\psi(x) = \Phi(x) - \Phi(2x)$ and let $P_k$ be the **Littlewood-Paley projector** for $k \in \mathbb{Z}$, namely,

$$P_k f = \mathscr{F}^{-1}\psi(2^{-k}|\xi|)\mathscr{F}f, \quad P_{\leq 0}f = \mathscr{F}^{-1}\Phi(|\xi|)\mathscr{F}f.$$

We denote by $S_\phi(t)$ the **evolution group** related to (1.1), defined as

$$S_\phi(t)u_0(x) = e^{it\phi(\sqrt{-\Delta})}u_0(x) = c_n \int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{it\phi(|\xi|)}\hat{u}_0(\xi)\,d\xi.$$

We use the Lebesgue spaces $L^p := L^p(\mathbb{R}^n)$, $\|\cdot\|_p := \|\cdot\|_{L^p}$ and the space-time norm $L^q_t L^r_x$ of $f$ on $\mathbb{R} \times \Omega$ defined by

$$\|f(t, x)\|_{L^q_t L^r_x(\mathbb{R}\times\Omega)} = \Big\| \|f(t, x)\|_{L^r_x(\Omega)} \Big\|_{L^q_t(\mathbb{R})},$$

where $\Omega \subset \mathbb{R}^n$. When $q = r$, we abbreviate $L^q_t L^r_x(\mathbb{R} \times \Omega)$ by $L^q_{t,x}(\mathbb{R} \times \Omega)$.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.2. In Section 3, we apply Theorem 1.2 to some concrete equations. In Section 4, we apply the improved Strichartz estimates to nonlinear problems.

## 2 Proofs of Theorem 1.2 and Theorem 1.5

First we prove Theorem 1.2 by adapting some ideas in [10] and [23]. However, there is a new difficulty for the endpoint case $q = (4n + 2)/(2n - 1)$ in (1.7) due to a logarithmic divergence. Fortunately, this logarithmic divergence can be overcome by using the double weight Hardy-Littlewood-Sobolev inequality. On the other hand, the logarithmic divergence for the endpoint $q = 2n/(n-1)$ in (1.6) is essential. We present the proof in the following three steps.

**Step 1**. Non-endpoint: $q > 2n/(n-1)$ in (1.6), $q > (4n+2)/(2n-1)$ in (1.7).

For $j \in \mathbb{Z}$, let $A_j := \{x \in \mathbb{R}^n : 2^{j-1} \leq |x| < 2^j\}$, $I_j = [2^{j-1}, 2^j)$. Fixing $k \in \mathbb{Z}$, we decompose $\|S_\phi(t)\Delta_k u_0(x)\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^n)}$ and get

$$\begin{aligned}\|S_\phi(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} &\le \sum_{j\in\mathbb{Z}} \|S_\phi(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}\times A_j)} \\ &= \sum_{j+k\le 1} \|S_\phi(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}\times A_j)} + \sum_{j+k\ge 2} \|S_\phi(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}\times A_j)}. \end{aligned} \tag{2.1}$$

The main task reduces to estimating $\|S_\phi(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}\times A_j)}$. It is easy to see that $S_\phi(t)P_k u_0$ is spherically symmetric in space if $u_0$ is radial. Thus we can rewrite it in an integral form related to the Bessel function. The two parts $j+k\le 1$ and $j+k\ge 2$ exploit different properties of Bessel functions. We give the estimates of the two parts in the following two propositions.

**Proposition 2.1.** *Assume $u_0\in L^2(\mathbb{R}^n)$, $u_0$ is radial, and $\phi$ satisfies* (H1) *and* (H2). *If $k, j\in\mathbb{Z}$ with $j+k\le 1$ and $2\le q\le\infty$, then*

$$\|S_\phi(t)P_k u_0(x)\|_{L^q_{t,x}(\mathbb{R}\times A_j)} \lesssim 2^{\frac{nj}{q}} 2^{(\frac{n}{2}-\frac{m(k)}{q})k} \|P_k u_0\|_{L^2}, \tag{2.2}$$

*where $m(k)$ is given by* (1.5).

**Proposition 2.2.** *Assume $u_0\in L^2(\mathbb{R}^n)$, $u_0$ is radial, and $\phi$ satisfies* (H1) *and* (H2). *If $k, j\in\mathbb{Z}$ with $j+k\ge 2$ and $2\le q\le\infty$, then*

$$\|S_\phi(t)P_k u_0(x)\|_{L^q_{t,x}(\mathbb{R}\times A_j)} \lesssim 2^{(\frac{n}{q}-\frac{n-1}{2})j} 2^{(\frac{1}{2}-\frac{m(k)}{q})k} \|P_k u_0\|_{L^2}. \tag{2.3}$$

*Furthermore, if $\phi$ also satisfies* (H3) *and* (H4), *then for $2\le q\le 6$,*

$$\|S_\phi(t)P_k u_0(x)\|_{L^q_{t,x}(\mathbb{R}\times A_j)} \lesssim 2^{(\frac{2n+1}{2q}-\frac{2n-1}{4})j} 2^{(\frac{-3m(k)+\alpha(k)+1}{2q}+\frac{m(k)-\alpha(k)+1}{4})k} \|P_k u_0\|_{L^2}, \tag{2.4}$$

*where $m(k)$, $\alpha(k)$ is given by* (1.5).

Postponing momentarily the proofs of Propositions 2.1 and 2.2, we show how these results can be used to complete the proof of Theorem 1.2 in the non-endpoint case.

**Proof of Theorem 1.2 (non-endpoint).** We may assume $q<\infty$. Assume first that $\phi$ satisfies (H1) and (H2). From (2.1), Proposition 2.1, and Proposition 2.2, we get

$$\begin{aligned}\|S_\phi(t)P_k u_0(x)\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \lesssim& \sum_{j+k\le 1} 2^{\frac{nj}{q}} 2^{(\frac{n}{2}-\frac{m(k)}{q})k} \|P_k u_0\|_{L^2} \\ &+ \sum_{j+k\ge 2} 2^{(\frac{n}{q}-\frac{n-1}{2})j} 2^{(\frac{1}{2}-\frac{m(k)}{q})k} \|P_k u_0\|_{L^2} \\ \lesssim& 2^{(\frac{n}{2}-\frac{m(k)}{q}-\frac{n}{q})k} \|P_k u_0\|_{L^2},\end{aligned}$$

since $q > 2n/(n-1)$, and thus $\frac{n}{q} - \frac{n-1}{2} < 0$. Thus (1.6) is proved.

Now assume $\phi$ also satisfies (H3) and (H4). Then

$$\|S_\phi(t)P_k u_0(x)\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \lesssim \sum_{j+k\le 1} 2^{\frac{nj}{q}} 2^{(\frac{n}{2}-\frac{m(k)}{q})k}\|P_k u_0\|_{L^2} + \sum_{j+k\ge 2} 2^{(\frac{2n+1}{2q}-\frac{2n-1}{4})j} 2^{(\frac{-3m(k)+\alpha(k)+1}{2q}-\frac{m(k)-\alpha(k)+1}{4})k}\|P_k u_0\|_{L^2}.$$

Note that if $q > (4n+2)/(2n-1)$, then $\frac{2n+1}{2q} - \frac{2n-1}{4} < 0$. Thus we can sum over $j$ and bound the quantity above by

$$C\left[2^{(\frac{n}{2}-\frac{n+m(k)}{q})k+(\frac{1}{4}-\frac{1}{2q})[m(k)-\alpha(k)]k} + 2^{(\frac{n}{2}-\frac{n+m(k)}{q}))k}\right]\|P_k u_0\|_{L^2},$$

which is sufficient for (1.7) since $(\frac{1}{4}-\frac{1}{2q})[m(k)-\alpha(k)]k \ge 0$ in view of Remark 1.1. □

It remains to prove Propositions 2.1 and 2.2. The proofs rely heavily on the radial properties, in particular, the Fourier-Bessel function

$$J_m(r) = \frac{(r/2)^m}{\Gamma(m+1/2)\pi^{1/2}}\int_{-1}^{1} e^{irt}(1-t^2)^{m-1/2}dt, \qquad m > -1/2.$$

We first recall two properties of $J_m(r)$.

**Lemma 2.3** (Properties of Bessel functions). *For $0 < r < \infty$ and $m > -1/2$,*
(i) $J_m(r) \le Cr^m$,
(ii) $J_m(r) \le Cr^{-1/2}$.

For the proof of Lemma 2.3, we refer the reader to [27, p. 338].

It is well known that if $f(x) = g(|x|)$ is radial, the Fourier transform of $f$ is also radial (cf. [26]) and

$$\hat{f}(\xi) = 2\pi\int_0^\infty g(s)s^{n-1}(s|\xi|)^{-1/(n-2)}J_{(n-2)/2}(s|\xi|)ds. \tag{2.5}$$

Thus, if $\widehat{u_0}(\xi) = h(|\xi|)$ is radial, $S_\phi(t)P_k u_0 = F(t,|x|)$ and

$$F(t,|x|) = 2\pi\int_0^\infty e^{it\phi(s)}\psi_k(s)h(s)s^{n-1}(s|x|)^{-1/(n-2)}J_{(n-2)/2}(s|x|)ds, \tag{2.6}$$

where $\psi_k(x) = \psi(x/2^k)$. Proving Propositions 2.1 and 2.2 reduces to a one-dimensional problem involving Bessel functions. We use the following local smoothing effect type estimate.

**Lemma 2.4.** *Suppose $k \in \mathbb{Z}$, $\varphi \in L^2(\mathbb{R})$, and $\phi$ satisfies* (H1) *and* (H2). *Then for $2 \le q \le \infty$,*

$$\left\| \int_{\mathbb{R}} \psi_k(s)\varphi(s)e^{-it\phi(s)}ds \right\|_{L_t^q} \lesssim 2^{(\frac{1}{2}-\frac{m(k)}{q})k} \|\psi_k\varphi\|_{L^2},$$

*where $m(k)$ is defined in* (1.5).

**Proof.** It is easy to see that $\phi$ is invertible in the support of $\psi_k$. We denote the inverse of $\phi$ by $\phi^{-1}$. By the change of variable $a = \phi(s)$, we obtain

$$\left\| \int_{\mathbb{R}} \psi_k(s)\varphi(s)e^{-it\phi(s)}\, ds \right\|_{L_t^q} = \left\| \int_{\mathbb{R}} \psi_k(\phi^{-1}(a))e^{-ita} \frac{\varphi(\phi^{-1}(a))}{|\phi'(\phi^{-1}(a))|} da \right\|_{L_t^q}.$$

Then, from the Hausdorff-Young inequality and change of variable $s = \phi(a)$, we see that this is bounded by

$$C \left\| \psi_k(\phi^{-1}(a)) \frac{\varphi(\phi^{-1}(a))}{|\phi'(\phi^{-1}(a))|} \right\|_{L_a^{q'}} = C \left\| \psi_k(s) \frac{\varphi(s)}{|\phi'(s)|^{\frac{1}{q}}} \right\|_{L_s^{q'}},$$

From the condition, we have $\phi'(s) \sim 2^{(m(k)-1)k}$ in the support of $\psi_k$. By Hölder's inequality, we can then bound this by

$$C2^{\frac{-m(k)+1}{q}k} 2^{(\frac{1}{q'}-\frac{1}{2})k} \|\psi_k(s)\varphi(s)\|_{L_s^2} = C2^{(\frac{1}{2}-\frac{m(k)}{q})k} \|\psi_k\varphi\|_{L^2}.$$

□

**Lemma 2.5** (Strichartz estimate)**.** *Suppose $\varphi \in L^2(\mathbb{R})$ and $\phi$ satisfies either* (H3) *or* (H4). *Then, for $k \in \mathbb{Z}$,*

$$\left\| \int_{\mathbb{R}} \psi_k(s)\varphi(s)e^{irs-it\phi(s)}\, ds \right\|_{L_t^6 L_r^6} \lesssim 2^{(\frac{1}{3}-\frac{\alpha(k)}{6})k} \|\psi_k\varphi\|_{L^2},$$

*where $\alpha(k)$ is defined in* (1.5).

**Proof.** Since $\phi$ satisfies (H3) and (H4), [10, Theorem 1] gives the decay estimate

$$\left\| \int_{\mathbb{R}} \psi_k(s)\varphi(s)e^{irs-it\phi(s)}ds \right\|_{L_r^\infty} \lesssim |t|^{-\frac{1}{2}} 2^{(1-\frac{\alpha(k)}{2})} \|\mathscr{F}^{-1}[\psi_k\varphi]\|_{L^1}.$$

The result now follows immediately from [10, Proposition 1]; also see [16]. □

**Proof of Proposition 2.1.** It follows from (2.6), Lemma 2.3(i), and Lemma 2.4 that

$$\begin{aligned}
\|S_\phi(t)P_k u_0(x)\|_{L_{t,x}^q(\mathbb{R}\times A_j)} &\lesssim \|F(t,r) r^{\frac{n-1}{q}}\|_{L_t^q L_{I_j}^q} \\
&\lesssim 2^{(\frac{1}{2}-\frac{m(k)}{q})k} \|\psi_k(s)h(s)s^{n-1} r^{\frac{n-1}{q}}\|_{L_{r\in I_j}^q L_s^2} \\
&\lesssim 2^{\frac{nj}{q}} 2^{(\frac{n}{2}-\frac{m(k)}{q})k} \|\psi_k(s)h(s)s^{\frac{n-1}{2}}\|_{L_s^2},
\end{aligned}$$

completing the proof of Proposition 2.1, since $\|\psi_k(s)h(s)s^{\frac{n-1}{2}}\|_{L^2_s} = \|P_k u_0\|_{L^2}$. □

It remains to prove Proposition 2.2. We use the decay properties of the Bessel function at $\infty$, more precisely,

$$J_{\frac{n-2}{2}}(r) = \frac{e^{i(r-(n-1)\pi/4)} + e^{-i(r-(n-1)\pi/4)}}{2r^{1/2}} + d_n r^{\frac{n-2}{2}} e^{-ir} E_+(r) - e_n r^{\frac{n-2}{2}} e^{ir} E_-(r), \tag{2.7}$$

where $E_\pm(r) \lesssim r^{-(n+1)/2}$ if $r \geq 1$ and $d_n, e_n$ are constants; see [27]. We insert (2.7) into (2.6) and then divide $F(t,|x|)$ into two parts: a main term and an error term, namely,

$$F(t,|x|) = M(t,|x|) + E(t,|x|), \tag{2.8}$$

where

$$\begin{aligned} M(t,r) &= c_n r^{-(n-1)/2} \int_{\mathbb{R}} \psi_k(s)h(s)s^{(n-1)/2} e^{i(rs-t\phi(s))} ds \\ &\quad + \bar{c}_n r^{-(n-1)/2} \int_{\mathbb{R}} \psi_k(s)h(s)s^{\frac{n-1}{2}} e^{-i(rs+t\phi(s))} ds, \\ E(t,r) &= c_1 \int_{\mathbb{R}} \psi_k(s)h(s)s^{n-1} e^{-it\phi(s)-irs} E_+(rs) ds \\ &\quad - c_2 \int_{\mathbb{R}} \psi_k(s)h(s)s^{n-1} e^{-it\phi(s)+irs} E_-(rs) ds. \end{aligned}$$

First we estimate the error term $E(t,|x|)$.

**Lemma 2.6.** *Assume $\phi$ satisfies* (H1) *and* (H2). *If $j+k \geq 2$ and $2 \leq q \leq \infty$, then*

$$\|E(t,|x|)\|_{L^q_{t,x}(\mathbb{R}\times A_j)} \lesssim 2^{(-\frac{n+1}{2}+\frac{n}{q})j} 2^{-(\frac{1}{2}+\frac{m(k)}{q})k} \|P_k u_0\|_{L^2}. \tag{2.9}$$

**Proof.** As in the proof of Proposition 2.1, we have

$$\begin{aligned} \|E(t,|x|)\|_{L^q_{t,x}(\mathbb{R}\times A_j)} &\lesssim \|E(t,r) r^{\frac{n-1}{q}}\|_{L^q_t L^q_{I_j}} \\ &\lesssim 2^{(\frac{1}{2}-\frac{m(k)}{q})k} \|\psi_k(s)F(s)s^{n-1} r^{\frac{n-1}{q}} E_\pm(rs)\|_{L^q_{r\in I_j} L^2_s} \\ &\lesssim 2^{-(\frac{1}{2}+\frac{m(k)}{q})k} 2^{j(\frac{n}{q}-\frac{n+1}{2})} \|\psi_k(s)F(s)s^{\frac{n-1}{2}}\|_{L^2_s}, \end{aligned}$$

where we have used the fact that $|E_\pm(r)| \lesssim r^{-(n+1)/2}$. □

Next we estimate the main term $M(t,|x|)$.

**Lemma 2.7.** (a) *Assume $\phi$ satisfies* (H1) *and* (H2). *If $j+k \geq 2$, then*

$$\|M(t,|x|)\|_{L^2_{t,x}(\mathbb{R}\times A_j)} \lesssim 2^{j/2} 2^{\frac{1-m(k)}{2}k} \|P_k u_0\|_{L^2}, \tag{2.10}$$

$$\|M(t,|x|)\|_{L^\infty_{t,x}(\mathbb{R}\times A_j)} \lesssim 2^{-j(n-1)/2} 2^{k/2} \|P_k u_0\|_{L^2}. \tag{2.11}$$

(b) *Assume $\phi$ satisfies* (H3) *and* (H4). *If $j+k\geq 2$, then*

$$\|M(t,|x|)\|_{L^6_{t,x}(\mathbb{R}\times A_j)} \lesssim 2^{-\frac{n-1}{3}j}2^{(\frac{1}{3}-\frac{\alpha(k)}{6})k}\|P_k u_0\|_{L^2}. \tag{2.12}$$

**Proof.** From symmetry, it suffices to estimate the first term in $M(t,|x|)$. From Lemma 2.4 with $q=2$, we obtain

$$\begin{aligned}\|M(t,|x|)\|_{L^2_{t,x}(\mathbb{R}\times A_j)} &\lesssim \|M(t,r)r^{\frac{n-1}{2}}\|_{L^2_t L^2_{I_j}}\\ &\lesssim \left\|\int_{\mathbb{R}} \psi_k(s)h(s)s^{\frac{n-1}{2}}e^{i(rs-t\phi(s))}ds\right\|_{L^2_{I_j}L^2_t}\\ &\lesssim 2^{j/2}2^{-\frac{m(k)-1}{2}k}\|\psi_k(s)h(s)s^{\frac{n-1}{2}}\|_{L^2_s},\end{aligned}$$

which gives the first inequality, as desired. Similarly,

$$\begin{aligned}\|M(t,|x|)\|_{L^\infty_{t,x}(\mathbb{R}\times A_j)} &\lesssim \|M(t,r)\|_{L^\infty_t L^\infty_{I_j}}\\ &\lesssim 2^{-j(n-1)/2}\left\|\int_{\mathbb{R}} \psi_k(s)h(s)s^{\frac{n-1}{2}}e^{i(rs-t\phi(s))}ds\right\|_{L^\infty_{I_j}L^\infty_t}\\ &\lesssim 2^{-j(n-1)/2}2^{k/2}\|\psi_k(s)h(s)s^{\frac{n-1}{2}}\|_{L^2_s},\end{aligned}$$

To prove (b), observe that Lemma 2.5 gives

$$\begin{aligned}\|M(t,|x|)\|_{L^6_{t,x}(\mathbb{R}\times A_j)} &\lesssim \|M(t,r)r^{\frac{n-1}{6}}\|_{L^6_t L^6_{I_j}}\\ &\lesssim 2^{-(n-1)j/3}\left\|\int_{\mathbb{R}} \psi_k(s)h(s)s^{\frac{n-1}{2}}e^{i(rs-t\phi(s))}ds\right\|_{L^6_t L^6_{I_j}}\\ &\lesssim 2^{-(n-1)j/3}2^{(\frac{1}{3}-\frac{\alpha(k)}{6})k}\|\psi_k(s)h(s)s^{\frac{n-1}{2}}\|_{L^2_s}.\end{aligned}$$

□

Now we are ready to prove Proposition 2.2.

**Proof of Proposition 2.2.** Since $\phi$ satisfies (H1) and (H2), interpolating (2.10) and (2.11), we get

$$\|M(t,|x|)\|_{L^q_{t,x}(\mathbb{R}\times A_j)} \lesssim 2^{(\frac{n}{q}-\frac{n-1}{2})j}2^{(\frac{1}{2}-\frac{m(k)}{q})k}\|P_k u_0\|_{L^2} \tag{2.13}$$

for $2\leq q\leq\infty$. From Lemma 2.6, we then obtain

$$\begin{aligned}\|S_\phi(t)P_k u_0(x)\|_{L^q_{t,x}(\mathbb{R}\times A_j)} &\lesssim \|E(t,|x|)\|_{L^q_{t,x}(\mathbb{R}\times A_j)} + \|M(t,|x|)\|_{L^q_{t,x}(\mathbb{R}\times A_j)}\\ &\lesssim 2^{(\frac{n}{q}-\frac{n-1}{2})j}2^{(\frac{1}{2}-\frac{m(k)}{q})k}\|P_k u_0\|_{L^2}\end{aligned}$$

for $2\leq q\leq\infty$. Moreover, interpolating (2.10) and (2.12) yields

$$\|M(t,|x|)\|_{L^q_{t,x}(\mathbb{R}\times A_j)} \lesssim 2^{(\frac{2n+1}{2q}-\frac{2n-1}{4})j}2^{(\frac{-3m(k)+\alpha(k)+1}{2q}+\frac{m(k)-\alpha(k)+1}{4})k}\|P_k u_0\|_{L^2} \tag{2.14}$$

for $2 \le q \le 6$ if $\phi$ also satisfies (H3) and (H4). Thus, in view of Lemma 2.6 and (2.14), the left-hand side of (2.4) is bounded by

$$\begin{aligned}\|S_\phi(t)P_k u_0(x)\|_{L^q_{t,x}(\mathbb{R}\times A_j)} &\lesssim \|E(t,|x|)\|_{L^q_{t,x}(\mathbb{R}\times A_j)} + \|M(t,|x|)\|_{L^q_{t,x}(\mathbb{R}\times A_j)}\\ &\lesssim (C_1(k,j)+C_2(k,j))\|P_k u_0\|_{L^2},\end{aligned}$$

where

$$\begin{aligned}C_1(k,j) &= 2^{(-\frac{n+1}{2}+\frac{n}{q})j}2^{-(\frac{1}{2}+\frac{m(k)}{q})k},\\ C_2(k,j) &= 2^{(\frac{2n+1}{2q}-\frac{2n-1}{4})j}2^{(\frac{-3m(k)+\alpha(k)+1}{2q}+\frac{m(k)-\alpha(k)+1}{4})k}.\end{aligned}$$

It remains to prove $C_1(k,j) \le C_2(k,j)$. A simple calculation yields

$$\begin{aligned}\frac{C_2(k,j)}{C_1(k,j)} &= 2^{(\frac{2n+1}{2q}-\frac{2n-1}{4}+\frac{n+1}{2}-\frac{n}{q})j+(\frac{-3m(k)+\alpha(k)+1}{2q}+\frac{m(k)-\alpha(k)+1}{4}+\frac{1}{2}+\frac{m(k)}{q})k}\\ &= 2^{(j+k)(\frac{1}{2q}+\frac{3}{4})+(\frac{1}{4}-\frac{1}{2q})(m(k)-\alpha(k))k}.\end{aligned}$$

It is easy to see that

$$(j+k)\Big(\frac{1}{2q}+\frac{3}{4}\Big)+\Big(\frac{1}{4}-\frac{1}{2q}\Big)(m(k)-\alpha(k))k \ge 1,$$

since $j+k \ge 2$ and $(m(k)-\alpha(k))k \ge 0$ in view of Remark 1.1. □

**Step 2.** Endpoint: $q = (4n+2)/(2n-1)$ in (1.7).

From Step 1, we see that in this case the sum over $j \ge 2-k$ does not converge. To overcome this, we do not decompose for large $j$. The main tools are the van der Corput Lemma [27] and the double weight Hardy-Littlewood-Sobolev inequalities [28].

**Lemma 2.8** (van der Corput [27])**.** *Assume $\psi \in C_0^\infty(\mathbb{R})$ and $P \in C^2(\mathbb{R})$ is a real-valued function satisfying $|P''(\xi)| \ge \lambda$ in the support of $\psi$. Then*

$$\Big|\int e^{iP(\xi)}\psi(\xi)d\xi\Big| \le C\lambda^{-1/2}(\|\psi\|_\infty + \|\psi'\|_1).$$

**Lemma 2.9** (double weight Hardy-Littlewood-Sobolev inequalities [28])**.** *If $1 < r, s < \infty$, $1/r+1/s \ge 1$, $0 < \lambda < d$, $\alpha+\beta \ge 0$ and*

$$1-\frac{1}{r}-\frac{\lambda}{d} < \frac{\alpha}{d} < 1-\frac{1}{r},\quad \frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{d} = 2,$$

*then*

$$\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{f(x)g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dxdy\Big| \le C_{\alpha,\beta,s,\lambda,d}\|f\|_r\|g\|_s.$$

Now we proceed to prove (1.7) for $q = (4n+2)/(2n-1)$. Obviously,

$$\begin{aligned}\|S_\phi(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} &\le \sum_{j\le 1-k} \|S_\phi(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}\times A_j)} + \|S_\phi(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}\times\{|x|\ge 2^{1-k}\})}\\ &:= I + II.\end{aligned}$$

From Step 1, we see that the term $I$ is bounded, as desired. It remains to bound the term $II$. Using (2.25), we get

$$\begin{aligned}II &\le \|M(t,|x|)\|_{L^q_{t,x}(\mathbb{R}\times\{|x|\ge 2^{1-k}\})} + \|E(t,|x|)\|_{L^q_{t,x}(\mathbb{R}\times\{|x|\ge 2^{1-k}\})}\\ &:= II_1 + II_2.\end{aligned}$$

From Step 1, we see that the term $II_2$ is bounded as desired. Thus, it remains to bound the term $II_1$. From symmetry, it suffices to prove

$$\begin{aligned}&\left\| 1_{[2^{1-k},\infty)}(r) r^{(\frac{1}{q}-\frac{1}{2})(n-1)} \int_{\mathbb{R}} \psi_k(s) h(s) s^{\frac{n-1}{2}} e^{i(rs-t\phi(s))} ds \right\|_{L^q_{t,r}}\\ &\qquad\qquad \lesssim 2^{(\frac{n}{2}-\frac{n+m(k)}{q})k+(\frac{1}{4}-\frac{1}{2q})(m(k)-\alpha(k))k} \|h(s)s^{\frac{n-1}{2}}\|_2,\end{aligned}$$

which follows from the estimate

$$\left\| |r|^{(\frac{1}{q}-\frac{1}{2})(n-1)} \int_{\mathbb{R}} \psi_0(s) h(s) e^{i(rs-t2^{-km(k)}\phi(2^k s))} ds \right\|_{L^q_{t,r}} \lesssim 2^{(\frac{1}{4}-\frac{1}{2q})(m(k)-\alpha(k))k} \|h\|_2. \tag{2.15}$$

It remains to prove (2.15). Since $\psi_0(s)$ is supported in $\{s \sim 1\}$, from (H1)-(H4), we see that $\phi_k(s) = 2^{-km(k)}\phi(2^k s)$ has an inverse, which we denote by $\eta_k = \phi_k^{-1} : range(\phi_k) \to \{s \sim 1\}$. Moreover,

$$|\eta_k'| \sim 1, \quad |\eta_k''| \sim 2^{k(\alpha(k)-m(k))}. \tag{2.16}$$

By a change of variable $s = \eta_k(\mu)$, we conclude that (2.15) is equivalent to

$$\left\| |r|^{(\frac{1}{q}-\frac{1}{2})(n-1)} \int_{\mathbb{R}} \psi_0(\eta_k(\mu)) h(\mu) e^{i(r\eta_k(\mu)-t\mu)} d\mu \right\|_{L^q_{t,r}} \lesssim 2^{(\frac{1}{4}-\frac{1}{2q})(m(k)-\alpha(k))k} \|h\|_2 \tag{2.17}$$

For $f \in L^2(\mathbb{R})$, define the operator

$$Tf(x,t) = |x|^{(\frac{1}{q}-\frac{1}{2})(n-1)} \int_{\mathbb{R}} \psi_0(\eta_k(\mu)) f(\mu) e^{i(x\eta_k(\mu)-t\mu)} d\mu.$$

It suffices to prove $\|T\|_{L^2\to L^q_{t,x}} \lesssim 2^{(\frac{1}{4}-\frac{1}{2q})(m(k)-\alpha(k))k}$. By duality, we have

$$T^*g(\mu) = \psi_0(\eta_k(\mu)) \int_{\mathbb{R}\times\mathbb{R}} e^{-i(x\eta_k(\mu)-t\mu)} |x|^{(\frac{1}{q}-\frac{1}{2})(n-1)} g(x,t) dx dt.$$

By the $TT^*$ arguments, it suffices to prove

$$\|TT^*g\|_{L^q} \lesssim 2^{(\frac{1}{2}-\frac{1}{q})(m(k)-\alpha(k))k}\|g\|_{L^{q'}}.$$

By definition,

$$\begin{aligned} TT^*g(x,t) &= |x|^{(\frac{1}{q}-\frac{1}{2})(n-1)}\int \psi_0^2(\eta_k(\mu))e^{-i(y\eta_k(\mu)-\tau\mu)}|y|^{(\frac{1}{q}-\frac{1}{2})(n-1)} \\ &\qquad \times g(y,\tau)e^{i(x\eta_k(\mu)-t\mu)}d\mu dy d\tau \\ &= |x|^{(\frac{1}{q}-\frac{1}{2})(n-1)}\int_{\mathbb{R}^2} K(x-y,t-\tau)|y|^{(\frac{1}{q}-\frac{1}{2})(n-1)}g(y,\tau)dyd\tau, \end{aligned}$$

where

$$K(x-y,t-\tau) = \int \psi_0^2(\eta_k(\mu))e^{i[(x-y)\eta_k(\mu)-(t-\tau)\mu]}d\mu.$$

Using Plancherel's equality, we get

$$\left\|\int K(x-y,t-\tau)g(y,\tau)d\tau\right\|_{L_t^2} \lesssim \|g(y,\cdot)\|_{L^2}.$$

On the other hand, it follows from the van der Corput Lemma and (2.16) that

$$|K(x-y,t-\tau)| \lesssim 2^{\frac{k(m(k)-\alpha(k))}{2}}|x-y|^{-1/2}.$$

Thus, by interpolation, we have

$$\left\|\int K(x-y,t-\tau)g(y,\tau)d\tau\right\|_{L_t^q} \lesssim 2^{k(m(k)-\alpha(k))(\frac{1}{2}-\frac{1}{q})}|x-y|^{-(\frac{1}{2}-\frac{1}{q})}\|g(u,\cdot)\|_{L^{q'}}.$$

Minkowski's inequality then gives

$$\begin{aligned} \|TT^*g\|_{L^q_{x,t}} &\lesssim 2^{k(m(k)-\alpha(k))(\frac{1}{2}-\frac{1}{q})} \\ &\qquad \left\| |x|^{(\frac{1}{q}-\frac{1}{2})(n-1)}\int |y|^{(\frac{1}{q}-\frac{1}{2})(n-1)}\|g(y,\cdot)\|_{L^{q'}}|x-y|^{-(\frac{1}{2}-\frac{1}{q})}dy\right\|_{L_x^q}. \end{aligned}$$

To complete the proof, it suffices to show that

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{g(y)f(x)}{|x|^{(\frac{1}{2}-\frac{1}{q})(n-1)}|y|^{(\frac{1}{2}-\frac{1}{q})(n-1)}|x-y|^{(\frac{1}{2}-\frac{1}{q})}}dxdy\right| \lesssim \|g\|_{L^{q'}}\|f\|_{L^{q'}}. \tag{2.18}$$

However, this follows immediately from Lemma 2.9; indeed, it is easy to verify the hypotheses of that lemma with $q = (4n+2)/(2n-1)$, $\alpha = \beta = (\frac{1}{2}-\frac{1}{q})(n-1)$, $\lambda = \frac{1}{2}-\frac{1}{q}$, $r = s = q'$, $d = 1$.

**Step 3.** Sharpness.

It remains to prove that the range of $q$ is optimal. We prove that

$$\|e^{it\sqrt{-\Delta}}P_0u_0\|_{L^q_{t,x}}\lesssim\|u_0\|_2$$

fails to hold if $q \le 2n/(n-1)$ and

$$\|e^{it\Delta}P_0u_0\|_{L^q_{t,x}}\lesssim\|u_0\|_2$$

fails to hold if $q < (4n+2)/(2n-1)$. As for the first claim, we see from the proof in Step 1, that it suffices to disprove the following statement: for $q = 2n/(n-1)$,

$$\left\|r^{\frac{n-1}{q}}r^{-\frac{n-1}{2}}\int_{\mathbb{R}}\psi_0(s)h(s)\cos(rs-(n-1)\pi/4)e^{its}ds\right\|_{L^q_{t,r\ge2}}\lesssim\|h\|_2. \tag{2.19}$$

To this end, take $h(s) = 1_{[0,10]}(s)$. From the fact that for $r \gg 1$

$$\left\|\int_{\mathbb{R}}\psi_0(s)\cos(rs-\frac{(n-1)\pi}{4})e^{its}ds\right\|_{L^q_{|t-r|\le1}}\gtrsim\|c\hat{\psi}_0(t+r)+\bar{c}\hat{\psi}_0(t-r)\|_{L^q_{|t-r|\le1}}\gtrsim1,$$

we obtain

$$\left\|r^{\frac{n-1}{q}}r^{-\frac{n-1}{2}}\int_{\mathbb{R}}\psi_0(s)h(s)\cos(rs-(n-1)\pi/4)e^{its}ds\right\|_{L^q_{t,r\ge2}}=\infty.$$

Thus (2.19) fails to hold if $q = 2n/(n-1)$.

To prove the second claim, note that similarly, it suffices to disprove the following statement; for $q < (4n+2)/(2n-1)$,

$$\left\|r^{\frac{n-1}{q}}r^{-\frac{n-1}{2}}\int_{\mathbb{R}}\psi_0(s)h(s)\cos(rs-(n-1)\pi/4)e^{its^2}ds\right\|_{L^q_{t,r\ge2}}\lesssim\|h\|_2. \tag{2.20}$$

To this end, fix $j$ sufficiently large and take $h(s) = 2^{j/2}1_{|s-1|\lesssim2^{-j}}$. Then $\|h\|_2 = 1$. For $t > 0$, the main contribution of $r^{\frac{n-1}{q}}r^{-\frac{n-1}{2}}\int_{\mathbb{R}}h(s)\cos(rs-(n-1)\pi/4)e^{its^2}ds$ is

$$c_nr^{\frac{n-1}{q}}r^{-\frac{n-1}{2}}\int_{\mathbb{R}}h(s)e^{-irs}e^{its^2}ds = c_n2^{j/2}r^{\frac{n-1}{q}}r^{-\frac{n-1}{2}}\int_{\mathbb{R}}1_{|s|\le2^{-j}}(s)e^{-irs}e^{its^2}e^{i2ts}ds.$$

Thus the left-hand side of (2.20) is larger than

$$\left\|2^{j/2}r^{\frac{n-1}{q}}r^{-\frac{n-1}{2}}\int_{\mathbb{R}}1_{|s|\le2^{-j}}(s)e^{-irs}e^{its^2}e^{i2ts}ds\right\|_{L^q_{r\sim2^{2j},|r-2t|\lesssim2^j}}\gtrsim2^{j(\frac{2n+1}{q}-\frac{2n-1}{2})},$$

which is unbounded if $q < (4n+2)/(2n-1)$. This completes the proof of Theorem 1.2.

Before proving Theorem 1.5, we give the following maximal function estimates, which generalize the results in [18] for $a \ge 2$ to $a > 0$.

**Lemma 2.10.** *Assume $a > 0$ and $k \geq 0$. Then*

$$\Big\|\int_{\mathbb{R}} e^{it|\xi|^a} e^{ix\xi}\eta_0(\xi/2^k)f(\xi)d\xi\Big\|_{L_x^2 L^\infty_{|t|\lesssim 1}} \lesssim B(a,k)\|f\|_2, \tag{2.21}$$

*where*

$$B(a,k) = \begin{cases} 2^{ak/4} & \text{for } a \neq 1, \\ 2^{k/2} & \text{for } a = 1. \end{cases}$$

*Moreover, these bounds are sharp.*

**Proof.** By the change of variables $\xi = 2^k\eta$, $x = 2^{-k}y$, we see that (2.21) is equivalent to

$$\Big\|\int_{\mathbb{R}} e^{it|\xi|^a} e^{ix\xi}\eta_0(\xi)f(\xi)d\xi\Big\|_{L_x^2 L^\infty_{|t|\lesssim 2^{ka}}} \lesssim B(a,k)\|f\|_2. \tag{2.22}$$

By $TT^*$ methods, (2.22) is equivalent to

$$\Big\|\int_{\mathbb{R}^2}\Big[\int_{\mathbb{R}} e^{i(t-t')|\xi|^a} e^{i(x-x')\xi}\eta_0(\xi)d\xi\Big] g(t',x')dt'dx'\Big\|_{L_x^2 L^\infty_{|t|\lesssim 2^{ka}}} \lesssim B(a,k)^2\|g\|_{L_x^2 L^1_{|t|\lesssim 2^{ka}}}. \tag{2.23}$$

Set $K_a(x-x',t-t') = \int_{\mathbb{R}} e^{i(t-t')|\xi|^a} e^{i(x-x')\xi}\eta_0(\xi)d\xi$. Since $|t-t'|\lesssim 2^{ka}$, by the method of stationary phase and the van der Corput Lemma, it is easy to see that for $a \neq 1$,

$$|K_a(x-x',t-t')|\lesssim (1+|x-x'|)^{-1/2}1_{|x-x'|\lesssim 2^{ka}} + |x-x'|^{-4}1_{|x-x'|\gg 2^{ka}}$$

and, for $a = 1$,

$$|K_1(x-x',t-t')|\lesssim 1\cdot 1_{|x-x'|\lesssim 2^k} + |x-x'|^{-4}1_{|x-x'|\gg 2^k}.$$

Using these bounds and Young's inequality, we get

$$\begin{aligned}\Big\|\int_{\mathbb{R}^2} K_a(x-x',t-t')g(t',x')dt'dx'\Big\|_{L_x^2 L^1_{|t|\lesssim 2^{ka}}} &\lesssim \|K_a\|_{L^1_x L^\infty_t}\|g\|_{L_x^2 L^1_{|t|\lesssim 2^{ka}}} \\ &\lesssim B(a,k)^2\|g\|_{L_x^2 L^1_{|t|\lesssim 2^{ka}}}.\end{aligned}$$

Thus we obtain the desired bounds.

It remains to show that these bounds are sharp. First we consider $a = 1$. For $f$ supported in $\{\xi > 0\}$, we have

$$L.H.S \text{ of } (2.22) \gtrsim \Big\|\int_{\mathbb{R}} e^{-ix\xi}e^{ix\xi}\eta_0(\xi)f(\xi)d\xi\Big\|_{L^2_{|x|\lesssim 2^k}} \gtrsim 2^{k/2},$$

which shows the sharpness of the bound $2^{k/2}$. Now consider $a \neq 1$. Take $f = \theta^{-1/2} 1_{|\xi-1|\lesssim\theta}$, $\theta = 2^{-ka/2}$. Then $\|f\|_2 \sim 1$, and

$$\begin{aligned} L.H.S \text{ of } (2.22) &\gtrsim \theta^{-1/2} \left\| \int_{|\xi|\lesssim\theta} e^{it(\xi+1)^a} e^{-it} e^{ix\xi} d\xi \right\|_{L^2_{|x|\lesssim\theta^{-2}} L^\infty_{|t|\lesssim 2^{ka}}} \\ &\gtrsim \theta^{-1/2} \left\| \int_{|\xi|\lesssim\theta} e^{-ix(\xi+1)^a/a} e^{ix/a} e^{ix\xi} d\xi \right\|_{L^2_{|x|\lesssim\theta^{-2}}} \\ &\gtrsim \theta^{-1/2} = 2^{ka/4}, \end{aligned}$$

where in the last inequality we have used the fact that $|(\xi+1)^a - 1 - a\xi| \lesssim \xi^2$. □

We now present the proof of Theorem 1.5 in the cases $a \neq 1$ and $a = 1$, $n \geq 3$.

**Case 1:** $a \neq 1$.

Since (1.9) is trivial if $(q, r) = (\infty, 2)$, by Bernstein's inequality, Riesz-Thorin interpolation, and the classical Strichartz estimates, it suffices to prove (1.9) for $(q, r) = (2, r)$, where $(4n-2)/(2n-3) < r < 2n/(n-2)$.

By the scaling transform $(t, x) \to (\lambda^a t, \lambda x)$, we may clearly assume $k = 0$. By the classical Strichartz estimates

$$\|e^{itD^a} P_0 f\|_{L^2_t L_x^{\frac{2n}{n-2}}} \leq C\|f\|_{L^2_x}$$

(see [16] for $n \geq 3$ and [31] for $n = 2$), we see from Hölder's inequality that it suffices to prove

$$\|e^{itD^a} P_0 f\|_{L^2_t L^r_{|x|\geq 10}} \leq C\|f\|_{L^2_x}. \tag{2.24}$$

As before, we divide $u_a(t, |x|) = e^{itD^a} P_0 f$ into a main term and an error term, viz.,

$$u_a(t, |x|) = M_a(t, |x|) + E_a(t, |x|), \tag{2.25}$$

where

$$\begin{aligned} M_a(t, r) &= c_n r^{-\frac{n-1}{2}} \int_{\mathbb{R}} \psi_0(s) g(s) s^{\frac{n-1}{2}} e^{i(rs - ts^a)} ds \\ &\quad + \bar{c}_n r^{-\frac{n-1}{2}} \int_{\mathbb{R}} \psi_0(s) g(s) s^{\frac{n-1}{2}} e^{-i(rs + ts^a)} ds, \\ E_a(t, r) &= c_1 \int_{\mathbb{R}} \psi_0(s) g(s) s^{n-1} e^{-its^a - irs} E_+(rs) ds \\ &\quad - c_2 \int_{\mathbb{R}} \psi_0(s) g(s) s^{n-1} e^{-its^a + irs} E_-(rs) ds. \end{aligned}$$

First we bound the main term $M_a$.

**Lemma 2.11.** *Assume $a \neq 1$, $a > 0$, $j \geq 2$, and $2 \leq r \leq \infty$. Then*

$$\|M_a(t,|x|)\|_{L_t^2L_x^r(\mathbb{R}\times A_j)} \lesssim 2^{j(\frac{2n-1}{2r}-\frac{2n-3}{4})}\|f\|_{L^2}. \tag{2.26}$$

**Proof.** This was proved for $r = 2$ in Lemma 2.7. By Riesz-Thorin interpolation, it suffices to prove for $r = \infty$. By the definition of $M_a$ and symmetry, it suffices to show

$$2^{-\frac{(n-1)j}{2}}\left\|\int_{\mathbb{R}}\eta(s)g(s)e^{i(rs^{1/a}-ts)}ds\right\|_{L_t^2L_r^\infty(\mathbb{R}\times I_j)} \lesssim 2^{-\frac{(2n-3)j}{4}}\|g\|_2, \tag{2.27}$$

where $\eta(s)$ is a bump function on $\{s \sim 1\}$. By the change of variables $\xi = s2^{aj}$, $t = 2^{aj}x$, we see that it suffices to prove

$$2^{-\frac{(n-1)j}{2}}\left\|\int_{\mathbb{R}}\eta(\xi/2^{aj})g(\xi)e^{i(t\xi^{1/a}-x\xi)}d\xi\right\|_{L_x^2L_{|t|\leq 2}^\infty} \lesssim 2^{-\frac{(2n-3)j}{4}}\|g\|_2, \tag{2.28}$$

which reduces to a maximal function estimate associated to the dispersion $\xi^{1/a}$. Since $a \neq 1$, (2.28) follows immediately from Lemma 2.10. □

Next we estimate the error term $E_a(t,|x|)$. Although this term certainly has better estimates than the main term, the following rough estimates suffice for our purpose.

**Lemma 2.12.** *Assume $a \neq 1$, $j \geq 2$, and $2 \leq r \leq 2n/(n-2)$. Then*

$$\|E_a(t,|x|)\|_{L_t^2L_x^r(\mathbb{R}\times A_j)} \lesssim 2^{-\frac{j}{2}(\frac{n}{r}-\frac{n-2}{2})}\|f\|_{L^2}. \tag{2.29}$$

**Proof.** Th was proved for $r = 2$ in Lemma 2.6. For $r = 2n/(n-2)$, we have

$$\|E_a(t,|x|)\|_{L_t^2L_x^{\frac{2n}{n-2}}(\mathbb{R}\times A_j)} \leq \|u_a(t,|x|)\|_{L_t^2L_x^{\frac{2n}{n-2}}} + \|M_a(t,|x|)\|_{L_t^2L_x^{\frac{2n}{n-2}}(\mathbb{R}\times A_j)} \lesssim \|f\|_2,$$

where we have used the classical endpoint Strichartz estimates and Lemma 2.7. □

We are ready to prove (2.24). Indeed, since $(4n-2)/(2n-3) < r < 2n/(n-2)$, by Lemmas 2.7 and 2.6, we can sum over $j \geq 1$ to obtain

$$\|e^{itD^a}P_0f\|_{L_t^2L_{|x|\geq 10}^r} \leq \sum_{j=1}^{\infty}\|M_a(t,|x|)\|_{L_t^2L_x^r(\mathbb{R}\times A_j)} + \sum_{j=1}^{\infty}\|E_a(t,|x|)\|_{L_t^2L_x^r(\mathbb{R}\times A_j)} \lesssim \|f\|_2.$$

**Case 2:** $a = 1$ and $n \geq 3$.

As in Case 1, it suffices to prove (1.9) for $(q,r) = (2,r)$, where

$$\frac{2n-2}{n-2} < r < \frac{2n-2}{n-3}.$$

Using the decomposition (2.25) and the following lemma, we immediately obtain (1.9).

**Lemma 2.13.** *Assume* $j \geq 2$ *and* $2 \leq r \leq \infty$, $2 \leq q \leq (2n-2)/(n-3)$. *Then*

$$\|M_1(t,|x|)\|_{L_t^2 L_x^r(\mathbb{R}\times A_j)} \lesssim 2^{j(\frac{n-1}{r}-\frac{n-2}{2})}\|f\|_{L^2},$$
$$\|E_1(t,|x|)\|_{L_t^2 L_x^q(\mathbb{R}\times A_j)} \lesssim 2^{-(\frac{n-1}{2q}-\frac{n-3}{4})}\|f\|_{L^2}.$$

The proof follows exactly as the proof of two lemmas above; thus we omit it.

Finally, we show sharpness. The condition $q \geq 2$ is necessary since (1.9) is time-translation invariant. The same counter-example used to show the sharpness of (2.20) shows that the condition

$$\frac{2}{q}+\frac{2n-1}{r} \leq n-\frac{1}{2}$$

is necessary.

**Remark 2.14.** From the proof of Theorem 1.5, we see that to prove Conjecture 1.7 for $a=2$, it suffices to prove

$$\left\| r^{-1/(2n-1)} \int_{\mathbb{R}} \psi_0(s)g(s)e^{i(rs-ts^2)}ds \right\|_{L_t^2 L_{\{r\geq 1\}}^{\frac{4n-2}{2n-3}}} \lesssim \|g\|_2.$$

However, we have not been able to prove this last inequality.

## 3 Strichartz estimates in the radial case

In this section, we apply Theorem 1.2 to some dispersive equations. Since we do not have decay estimates, we use the Christ-Kiselev Lemma to derive retarded linear estimates. First we prove a duality property for radial function.

**Lemma 3.1.** *Assume* $1 \leq p \leq \infty$, $1 = 1/p + 1/p'$, *and radial* $f \in L^p(\mathbb{R}^n)$. *Then*

$$\|f\|_{L^p(\mathbb{R}^n)} = \sup\left\{ \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| : \ g \in L^{p'}(\mathbb{R}^n),\ g \textit{ is radial and } \|g\|_{L^{p'}} \leq 1 \right\}. \tag{3.1}$$

**Proof.** Denote the right-hand side of (3.1) by $B$. Obviously, $B \leq \|f\|_{L^p(\mathbb{R}^n)}$; thus it suffices to show $\|f\|_{L^p(\mathbb{R}^n)} \leq B$. By duality, we have

$$\begin{aligned}
\|f\|_{L^p(\mathbb{R}^n)} &= \sup_{g\in L^{p'},\|g\|_{L^{p'}}=1} \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \\
&= \sup_{g\in L^{p'},\|g\|_{L^{p'}}=1} \left| \int_0^\infty \int_{\mathbb{S}^{n-1}} f(r)g(rx')r^{n-1}drd\sigma(x') \right| \\
&= \sup_{g\in L^{p'},\|g\|_{L^{p'}}=1} \left| \int_{\mathbb{R}^n} f(x)\tilde{g}(x)dx \right|,
\end{aligned}$$

where we have set $\tilde{g}(x) = |\mathbb{S}^{n-1}|^{-1}\int_{\mathbb{S}^{n-1}} g(|x|x')d\sigma(x')$. It is easy to see from Hölder's inequality that $\tilde{g}$ is radial and $\|\tilde{g}\|_{L^{p'}} \le 1$. Thus we get $\|f\|_{L^p(\mathbb{R}^n)} \le B$ as desired. □

Obviously, Lemma 3.1 holds similarly for functions $f(t,x)$ spherically symmetric in $x$, e.g., $f \in L_t^pL_x^q$. As a corollary, we can apply Lemma 3.1 to obtain dual version estimates of the linear estimates in the radial case.

**Lemma 3.2.** *Assume* $1 \le q, r \le \infty$, $1/q + 1/q' = 1/r + 1/r' = 1$, $k \in \mathbb{Z}$. *If for all radial* $u_0 \in L^2(\mathbb{R}^n)$

$$\|S_\phi(t)P_ku_0\|_{L_t^qL_x^r} \lesssim C(k)\|u_0\|_{L^2},$$

*then for all* $f \in L_t^{q'}L_x^{r'}$ *spherically symmetric in space,*

$$\left\|\int_{\mathbb{R}} S_\phi(-t)[P_kf(t,\cdot)](x)dt\right\|_{L^2(\mathbb{R}^n)} \lesssim C(k)\|f\|_{L_t^{q'}L_x^{r'}}.$$

The following lemma is very useful for deriving retarded estimates from non-retarded estimates. The estimate we need is the following. For its proof, we refer the readers to [24].

**Lemma 3.3** (Christ-Kiselev [4]). *Assume* $1 \le p_1, q_1, p_2, q_2 \le \infty$ *with* $p_1 > p_2$. *If*

$$\left\|\int_{\mathbb{R}} S_\phi(t-s)(P_kf(s))(x)ds\right\|_{L_t^{p_1}L_x^{q_1}} \lesssim C(k)\|f\|_{L_t^{p_2}L_x^{q_2}}$$

*holds for all* $f \in L_t^{p_2}L_x^{q_2}$ *spherically symmetric in space, then*

$$\left\|\int_0^t S_\phi(t-s)(P_kf(s))(x)ds\right\|_{L_t^{p_1}L_x^{q_1}} \lesssim C(k)\|f\|_{L_t^{p_2}L_x^{q_2}}$$

*holds with the same bound* $C(k)$, *for all* $f \in L_t^{p_2}L_x^{q_2}$ *spherically symmetric in space.*

We are now ready to give some new Strichartz estimates for some concrete equations. First note that from Minkowski's inequality and the Littlewood-Paley square function theorem, we get

$$\|f\|_{L_t^qL_x^r} \lesssim \|\|P_kf\|_{L_t^qL_x^r}\|_{l_k^2}, \quad \|\|P_kf\|_{L_t^{q'}L_x^{r'}}\|_{l_k^2} \lesssim \|f\|_{L_t^{q'}L_x^{r'}} \tag{3.2}$$

for $2 \le q, r < \infty$. We apply (3.2) to obtain Strichartz estimates on the whole space.

**1: Schrödinger equation**

$$\begin{aligned} &i\partial_tu + \Delta u = F, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ &u(0) = u_0(x). \end{aligned} \tag{3.3}$$

By Duhamel's principle, $u = S(t)u_0 - i\int_0^t S(t-\tau)F(\tau)d\tau$, where $S(t) = e^{-it\Delta}$, which corresponds to $\phi(r) = r^2$. We see that $\phi$ satisfies (H1)-(H4) with $m_1 = m_2 = \alpha_1 = \alpha_2 = 2$. Thus, by Theorem 1.2,

$$\|S(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \lesssim 2^{(\frac{n}{2}-\frac{n+2}{q})k}\|u_0\|_2 \tag{3.4}$$

for $q \geq (4n-2)/(2n-1)$ and radial $u_0$.

**Definition 3.4.** Suppose $n \geq 2$. The exponent pair $(q, r)$ is said to be **n-D radial Schrödinger-admissible** if $q, r \geq 2$ and

$$\frac{4n+2}{2n-1} \leq q \leq \infty \quad \text{and} \quad \frac{2}{q} + \frac{2n-1}{r} \leq n - \frac{1}{2} \tag{3.5}$$
$$\text{or}$$
$$2 \leq q < \frac{4n+2}{2n-1} \quad \text{and} \quad \frac{2}{q} + \frac{2n-1}{r} < n - \frac{1}{2}.$$

For $n \geq 3$, the n-D radial Schrödinger-admissible pairs are described in Figure 1 ($a \neq 1$).

**Proposition 3.5** (Schrödinger Strichartz estimate)**.** *Suppose $n \geq 2$ and $u$, $u_0$, and $F$ are spherically symmetric and satisfy equation* (3.3). *Then*

$$\|u\|_{L^q_t L^r_x} + \|u\|_{C(\mathbb{R}:\dot{H}^\gamma)} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}, \tag{3.6}$$

*if $\gamma \in \mathbb{R}$, $(q, r)$ and $(\tilde{q}, \tilde{r})$ are both n-D radial Schrödinger-admissible, either $(\tilde{q}, \tilde{r}, n) \neq (2, \infty, 2)$ or $(q, r, n) \neq (2, \infty, 2)$, and $(q, r, n)$ and $(\tilde{q}, \tilde{r}, n)$ satisfy the "gap" condition*

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - \gamma, \quad \frac{2}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2} + \gamma.$$

**Proof.** The case $F = 0$ follows from Theorem 1.5[2]. Now assume $F \neq 0$, $(q, r)$ and $(\tilde{q}, \tilde{r})$ are both n-D radial Schrödinger admissible, $(\tilde{q}, \tilde{r}, n) \neq (2, \infty, 2)$, and $(q, r, n)$ and $(\tilde{q}, \tilde{r}, n)$ satisfy the "gap" condition. If $\gamma = 0$, the result follows from known estimates [16]. If $\gamma \neq 0$, by scaling, it suffices to prove

$$\left\|\int_0^t S(t-s)P_0F(s)ds\right\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}. \tag{3.7}$$

In view of the Christ-Kiselev Lemma, since either $q, r > 2$ or $\tilde{q}, \tilde{r} > 2$, it suffices to prove

$$\left\|\int_{\mathbb{R}} S(t-s)P_0F(s)ds\right\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}, \tag{3.8}$$

which follows immediately from non-retarded linear estimates and Lemma 3.2. □

[2]For $(q, r, n) = (2, \infty, 2)$, it was proved similarly for the wave equation in [31].

**Remark 3.6.** We can take $\gamma < 0$, which means there is a smoothing effect in the non-retarded Strichartz estimates. This only holds in the radial case. There are also smoothing effects in some retarded estimates; but we only derive estimates which do not have a smoothing effect.

**2. Wave equation**

$$\begin{aligned} &\partial_{tt}u - \Delta u = F, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ &u(0) = u_0(x),\ u_t(0) = u_1(x). \end{aligned} \tag{3.9}$$

By Duhamel's principle, $u = W'(t)u_0 + W(t)u_1 - \int_0^t W(t-\tau)F(\tau)d\tau$, where

$$W(t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}, \quad W'(t) = \cos(t\sqrt{-\Delta}).$$

This reduces to $W_\pm(t) := e^{\pm it(-\Delta)^{1/2}}$, which corresponds to $\phi(r) = r$. Then we see that $\phi$ satisfies (H1) and (H2) with $m_1 = m_2 = 1$. Thus by Theorem 1.2,

$$\|W_\pm(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \lesssim 2^{(\frac{n}{2}-\frac{n+1}{q})k}\|u_0\|_2. \tag{3.10}$$

for $q > 2n/(n-1)$ and radial $u_0$

**Definition 3.7.** Suppose $n \geq 2$. The exponent pair $(q,r)$ is said to be **n-D radial wave-admissible** if $q, r \geq 2$, and one of the following holds:

(1) $n = 2$, $(q,r) \in A_2 = \{(q,r) : \frac{1}{q} + \frac{1}{r} < \frac{1}{2}, q > 4\} \cup \{(4,\infty), (\infty,2)\}$;
(2) $n \geq 3$, $(q,r) \in A_{\geq 3} = \{(q,r) : q \geq 2, \frac{1}{q} + \frac{n-1}{r} < \frac{n-1}{2}\} \cup \{(\infty,2)\}$.

For $n \geq 4$, the n-D radial wave-admissible pairs are described in the Figure 1 ($a = 1$).

**Proposition 3.8** (Wave Strichartz estimate)**.** *Suppose $n \geq 2$ and $u, u_0, u_1, F$ are spherically symmetric and satisfy equation* (3.9). *Then*

$$\|u\|_{L^q_t L^r_x} + \|u\|_{C([0,T]:\dot{H}^\gamma)} + \|\partial_t u\|_{C([0,T]:\dot{H}^{\gamma-1})} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|u_1\|_{\dot{H}^{\gamma-1}} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \tag{3.11}$$

*if $\gamma \in \mathbb{R}$, $(q,r)$ and $(\tilde{q},\tilde{r})$ are both n-D radial wave-admissible, $(\tilde{q},\tilde{r},n) \neq (2,\infty,3)$, and $(q,r,n)$ and $(\tilde{q},\tilde{r},n)$ satisfy the "gap" condition*

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma, \quad \frac{1}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2} - 1 + \gamma.$$

The proof is similar to that of Proposition 3.5; hence we omit it.

**3. Klein-Gordon equation**

$$\begin{aligned} &\partial_{tt}u - \Delta u + u = F, \\ &u(0) = u_0(x),\ u_t(0) = u_1(x). \end{aligned} \tag{3.12}$$

By Duhamel's principle, $u = K'(t)u_0 + K(t)u_1 - \int_0^t K(t-\tau)F(\tau)d\tau$, where

$$K(t) = \omega^{-1}\sin(t\omega), \quad K'(t) = \cos(t\omega), \quad \omega = \sqrt{I-\Delta}.$$

This reduces to the semi-group $K_{\pm}(t) := e^{\pm it(I-\Delta)^{1/2}}$, which corresponds to $\phi(r) = (1+r^2)^{1/2}$. A simple calculation yields

$$\phi'(r) = \frac{r}{(1+r^2)^{\frac{1}{2}}}, \quad \phi''(r) = \frac{1}{(1+r^2)^{\frac{3}{2}}},$$

which shows that $\phi$ satisfies (H1)-(H4) with $m_1 = 1$, $\alpha_1 = -1$, $m_2 = \alpha_2 = 2$. Thus, by Theorem 1.2, for $q \geq (4n+2)/(2n-1)$ and radial $u_0$,

$$\|K_{\pm}(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \lesssim C(q,k)\|u_0\|_2, \tag{3.13}$$

where

$$C(q,k) = \begin{cases} 2^{(\frac{n}{2}-\frac{n+2}{q})k} & \text{for } k \leq 0; \\ 2^{(\frac{n}{2}-\frac{n+1}{q})k} & \text{for } k \geq 0,\ 2n/(n-1) < q \leq \infty; \\ 2^{(\frac{n}{2}-\frac{n+1}{q})k+(\frac{1}{2}-\frac{1}{q})k} & \text{for } k \geq 0,\ (4n+2)/(2n-1) \leq q \leq 2n/(n-1). \end{cases}$$

**4. Beam equation**

$$\begin{aligned} &\partial_{tt}u + \Delta^2 u + u = F, \\ &u(0) = u_0(x),\ u_t(0) = u_1(x). \end{aligned} \tag{3.14}$$

By Duhamel's principle, $u = B'(t)u_0 + B(t)u_1 - \int_0^t B(t-\tau)F(\tau)d\tau$, where

$$B(t) = \omega^{-1}\sin(t\omega), \quad B'(t) = \cos(t\omega), \quad \omega = \sqrt{I+\Delta^2}.$$

This reduces to the semi-group $B_{\pm}(t) := e^{\pm it(I+\Delta^2)^{1/2}}$, which corresponds to $\phi(r) = (1+r^4)^{1/2}$. A simple calculation yields

$$\phi'(r) = \frac{2r^3}{(1+r^4)^{1/2}}, \quad \phi''(r) = \frac{6r^2+2r^6}{(1+r^4)^{3/2}},$$

which shows that $\phi$ satisfies (H1) and (H2) with $m_1 = \alpha_1 = 2$, $m_2 = \alpha_2 = 4$. Thus, by Theorem 1.2, for $q \geq (4n+2)/(2n-1)$ and radial $u_0$,

$$\|B_{\pm}(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \lesssim B(q,k)\|u_0\|_2, \tag{3.15}$$

where

$$B(q,k) = \begin{cases} 2^{(\frac{n}{2}-\frac{n+4}{q})k} & \text{for } k \leq 0; \\ 2^{(\frac{n}{2}-\frac{n+2}{q})k} & \text{for } k \geq 0. \end{cases}$$

**5. Fractional-order Schrödinger equation**

$$
\begin{aligned}
&i\partial_t u + (-\Delta)^{\frac{\sigma}{2}} u = F,\\
&u(0) = u_0(x),
\end{aligned}
\tag{3.16}
$$

where $1 < \sigma < 2$. By Duhamel's principle, $u = S_\sigma(t)u_0 + \int_0^t S_\sigma(t-\tau)F(\tau)d\tau$, where $S_\sigma(t) = e^{-it\phi(\sqrt{-\Delta})}$ with $\phi(r) = r^\sigma$. By simple calculation, we see that $\phi$ satisfies (H1)-(H4) with $m_1 = \alpha_1 = m_2 = \alpha_2 = \sigma$. Thus, by Theorem 1.2,

$$
\|S_\sigma(t)P_k u_0\|_{L^q_{t,x}(\mathbb{R}^{n+1})} \lesssim 2^{(\frac{n}{2}-\frac{n+\sigma}{q})k}\|u_0\|_2 \tag{3.17}
$$

for $q \geq (4n+2)/(2n-1)$ and radial $u_0$.

**Proposition 3.9.** *Suppose $n \geq 2$ and $u, u_0, F$ are spherically symmetric in space and satisfy equation* (3.16). *Then*

$$
\|u\|_{L^q_t L^r_x} + \|u\|_{C(\mathbb{R}:\dot{H}^\gamma)} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}, \tag{3.18}
$$

*if $\gamma \in \mathbb{R}$, $(q, r)$ and $(\tilde{q}, \tilde{r})$ are both n-D radial Schrödinger-admissible, $(\tilde{q}, \tilde{r}, n) \neq (2, \infty, 2)$, and $(q, r, n)$ and $(\tilde{q}, \tilde{r}, n)$ satisfy the "gap" condition*

$$
\frac{\sigma}{q} + \frac{n}{r} = \frac{n}{2} - \gamma, \quad \frac{\sigma}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2} + \gamma.
$$

The proof is similar to that of Proposition 3.5 except for the case$(q, r, n) = (2, \infty, 2)$, which needs to be handled separately. This particular case follows similarly as for the Schrödinger equation in [31]. We omit the details.

In particular, taking $\gamma = 0$, we get a family of Strichartz estimates without loss of regularity.

**Corollary 3.10.** *Suppose $n \geq 2$, $2n/(2n-1) < \sigma \leq 2$, and $u, u_0, F$ are spherically symmetric in space and satisfy equation* (3.16). *Then*

$$
\|u\|_{L^q_t L^r_x} + \|u\|_{C(\mathbb{R}:L^2)} \lesssim \|u_0\|_{L^2} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \tag{3.19}
$$

*if $(q, r)$ and $(\tilde{q}, \tilde{r}) \in \{(q, r) : q, r \geq 2, \frac{\sigma}{q} + \frac{n}{r} = \frac{n}{2}\}$ and $(\tilde{q}, \tilde{r}, n) \neq (2, \infty, 2)$.*

These estimates without loss of derivative hold only in the radial case. We present the Knapp-counterexample to show that the general non-radial Strichartz estimates have loss of derivative for $1 < \sigma < 2$.

Assume that for a general non-radial function $f$,

$$
\left\| \int_{\mathbb{R}^d} e^{it|\xi|^\sigma} e^{ix\xi} \eta_0(\xi)\hat{f}(\xi)d\xi \right\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2}. \tag{3.20}
$$

Take $D = \{\xi = (\xi_1, \xi') \in \mathbb{R}^d : |\xi_1 - 1| \lesssim \delta, |\xi'| \le \delta\}$ and let $\hat{f} = 1_D(\xi)$. Then $\|f\|_2 \sim \delta^{d/2}$, and

$$\int_{\mathbb{R}^d} e^{it|\xi|^\sigma} e^{ix\xi} \eta_0(\xi)\hat{f}(\xi)d\xi = e^{i(t+x_1)} \int_D e^{it(|\xi|^\sigma - \xi_1^\sigma)} e^{it(\xi_1^\sigma - 1 - \sigma(\xi_1 - 1))} e^{i(t\sigma + x_1)(\xi_1 - 1)} e^{ix'\xi'} d\xi.$$

Since in $D$ $||\xi|^\sigma - \xi_1^\sigma| \lesssim |\xi'|^2 \lesssim \delta^2$ and $|\xi_1^\sigma - 1 - \sigma(\xi_1 - 1)| \lesssim |\xi_1 - 1|^2 \lesssim \delta^2$, it follows that

$$\left| \int_{\mathbb{R}^d} e^{it|\xi|^\sigma} e^{ix\xi} \eta_0(\xi)\hat{f}(\xi)d\xi \right| \sim |D|$$

for $|t| \lesssim \delta^{-2}$, $|t\sigma + x_1| \lesssim \delta^{-1}$, and $|x'| \lesssim \delta^{-1}$. Therefore, (3.20) implies $\delta^{-\frac{2}{q} - \frac{d}{r} + \frac{d}{2}} \lesssim 1$. Taking $\delta \ll 1$ then implies immediately that

$$\frac{2}{q} + \frac{d}{r} \le \frac{d}{2}.$$

## 4 Applications to nonlinear equations

In this section, we apply the improved Strichartz estimates to certain nonlinear equations, viz., the nonlinear Schrödinger equation and nonlinear wave equation. These equations have been studied extensively.

**4.1 Non-linear Schrödinger equations.** First we consider the semi-linear Schrödinger equation

(4.1) $$i\partial_t u + \Delta u = \mu |u|^p u, \quad u(0) = u_0(x),$$

where $u(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C},\ n \ge 2, u_0 \in \dot{H}^s, p > 0, \mu = \pm 1$. It is easy to see that equation (4.1) is invariant under the scaling transformation

$$u(t,x) \to \lambda^{2/p} u(\lambda^2 t, \lambda x),\ u_0(x) \to \lambda^{2/p} u_0(\lambda x), \quad \lambda > 0.$$

Then the space $\dot{H}^{s_{sch}}$ ($s_{sch} = \frac{n}{2} - \frac{2}{p}$) is the critical space for (4.1) in the sense of scaling, i.e., $\|\lambda^{2/p} u_0(\lambda \cdot)\|_{\dot{H}^{s_{sch}}} = \|u_0\|_{\dot{H}^{s_{sch}}}$. In particular, if $p < 4/n$, then $s_{sch} < 0$, which is our main concern.

The well-posedness and scattering for the nonlinear Schrödinger equation (4.1) have been studied extensively. We refer the reader to [19] for a survey. It is well known that the threshold of $\dot{H}^s$-well-posedness for (4.1) is $s \ge \max(0, s_{sch})$. Our next result concerns the well-posedness and scattering in the radial case.

**Theorem 4.1.** *Assume $n \geq 2$, $0 < p < 4/n$, $s_{sch} = \frac{n}{2} - \frac{2}{p}$, $\frac{1-n}{2n+1} \leq s_{sch} < 0$, and $u_0$ is radial. Then we have the following.*

(1) *(small data scattering): If $\|u_0\|_{\dot H^{s_{sch}}} \leq \delta$ for some $\delta \ll 1$, there exists a unique global solution $u$ of* (4.1) *such that*

$$u \in C(\mathbb{R} : \dot H^{s_{sch}}) \cap L_{t,x}^{\frac{p(n+2)}{2}}(\mathbb{R} \times \mathbb{R}^n),$$

*and $u_\pm \in \dot H^{s_{sch}}$ such that $\|u - S(t)u_\pm\|_{\dot H^{s_{sch}}} \to 0$ as $t \to \pm\infty$.*

(2) *(large data local well-posedness): If $u_0 \in \dot H^s$ for some $s_{sch} \leq s < 0$, there exist $T > 0$ and a unique solution $u \in C((-T,T) : \dot H^s) \cap L_{t,x}^{\frac{2(n+2)}{n-2s}}((-T,T) \times \mathbb{R}^n)$ of* (4.1).

**Proof.** The proof is quite standard. The main point is to choose the resolution space. By Duhamel's principle,

$$u = \Phi_{u_0}(u) = S(t)u_0 + \mu \int_0^t S(t-s)(|u|^{\frac{4}{n-2s_{sch}}} u)(s)ds.$$

First, we show part (1). Take[3]

$$q = r = \frac{2(n+2)}{n-2s_{sch}}, \quad \tilde q = \tilde r = \frac{2(n+2)}{n+2s_{sch}}.$$

It is easy to verify that $(q,r)$, $(\tilde q, \tilde r)$ satisfy the hypotheses of Proposition 3.5 with $\gamma = s_{sch}$. Thus, applying Proposition 3.5, we obtain

$$\begin{aligned}\|\Phi_{u_0}(u)\|_{L^q_{t,x}} + \|D^{s_{sch}}\Phi_{u_0}(u)\|_{L^\infty_t L^2_x} &\lesssim \|S(t)u_0\|_{L^q_t L^q_x} + \||u|^{\frac{4}{n-2s_{sch}}}u\|_{L^{\tilde q'}_{t,x}} \\ &\lesssim \|D^{s_{sch}}u_0\|_{L^2} + \|u\|^{1+\frac{4}{n-2s_{sch}}}_{L_{t,x}^{\frac{(n-2s_{sch}+4)\tilde q'}{n-2s_{sch}}}}.\end{aligned}$$

Note that $\tilde q' = 2(n+2)/(n-2s_{sch}+4)$; therefore, $(n-2s_{sch}+4)\tilde q'/(n-2s_{sch}) = q$. Thus part (1) follows from standard fixed point arguments; see[2].

Next, we show part (2). Local well-posedness for equation (4.1) in $\dot H^{s_{sch}}$ follows from the fact that for $q = 2(n+2)/(n-2s_{sch}) < \infty$,

$$\lim_{T\to 0} \|S(t)u_0\|_{L^q_{t\in[-T,T]}L^q_x} = 0.$$

Now we assume $s_{sch} < s < 0$. Take $q = r = 2(n+2)/(n-2s)$ and

$$\frac{1}{\tilde q} = \frac{n+2s}{2n+4} - \frac{2n(s-s_{sch})}{(n+2)(n-2s_{sch})}, \quad \frac{1}{\tilde r} = \frac{n+2s}{2n+4} + \frac{4s-4s_{sch}}{(n+2)(n-2s_{sch})}.$$

[3]The choice of index was determined by a collection of linear equation or inequalities. The choice is not unique, and we choose the simplest one here. We remark further on this for the wave equation.

It is easy to verify that $(q, r)$, $(\tilde{q}, \tilde{r})$ satisfy the hypotheses of Proposition 3.5 with $\gamma = s$, and $(p+1)\tilde{r}' = q$. Thus, applying Proposition 3.5, we obtain

$$\begin{aligned}\|\Phi_{u_0}(u)\|_{L^q_{t,x}} + \|D^s\Phi_{u_0}(u)\|_{L^\infty_t L^2_x} &\lesssim \|D^s u_0\|_{L^2} + \| |u|^{\frac{4}{n-2s_{sch}}} u\|_{L^{\tilde{q}'}_{t\in[-T,T]} L^{\tilde{r}'}_x} \\ &\lesssim \|D^s u_0\|_{L^2} + T^\theta \|u\|^{1+4/(n-2s_c)}_{L^{(n-2s_{sch}+4)\tilde{r}'/(n-2s_{sch})}_{t,x}}.\end{aligned}$$

for some $\theta > 0$. Thus part (2) also follows from standard fixed-point arguments.□

**Remark 4.2.** In part (2) of Theorem 4.1, the existence time $T$ depends only on $\|u_0\|_{\dot{H}^s}$ for $s > s_{sch}$, but on the profile of $u_0$ for $s = s_{sch}$.

Actually, we can say more than Theorem 4.1. Using a similar proof, we can obtain for $s_{sch} < (1-n)/(2n+1)$ (namely $0 < p < (8n+4)/(2n^2+3n-2)$) large data local well-posedness for (4.1) with $u_0 \in \dot{H}^s$ for $s > s_0$, where

$$s_0 = \begin{cases} \frac{1-n}{2n+1}, & \text{for } \frac{2}{n} \le p < \frac{8n+4}{2n^2+3n-2}; \\ \frac{np-n^2p}{2(-1+2n+np)}, & \text{for } p \le \frac{2}{n}. \end{cases} \tag{4.2}$$

Actually, $s_0$ is determined by the conditions

$$\begin{aligned} &2 \le q, r, \tilde{q}, \tilde{r} \le \infty, \\ &\frac{2}{q} + \frac{2n-1}{r} = n - \frac{1}{2}, \\ &\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - \gamma, \\ &\frac{2}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2} + \gamma, \\ &(p+1)\tilde{r}' = r, \tilde{q} = \infty. \end{aligned}$$

Then we can also obtain $(q, r)$, $(\tilde{q}, \tilde{r})$ for $s > s_0$, which can be used to prove local well-posedness as in the proof of Theorem 4.1.

The conclusions obtained above hold for general nonlinear terms $F(u)$, for example, if $F$ satisfies

$$\begin{aligned} |F(u)| &\lesssim |u|^{p+1}, \\ |u||F'(u)| &\sim |F(u)|. \end{aligned} \tag{4.3}$$

We describe the regularity $s$ for $\dot{H}^s$ local well-posedness and nonlinear increasing rate $p+1$ in Figure 3.

**Remark 4.3.** Part (2) in Theorem 4.1 also holds for data $u_0 \in H^s$. Indeed, we simply construct the resolution space

$$\|u\|_{Y_T} = \|P_{\le 0}u\|_{L^\infty_{[-T,T]}L^2} + \|P_{\ge 1}u\|_{L^q_{|t|\le T,x}}.$$

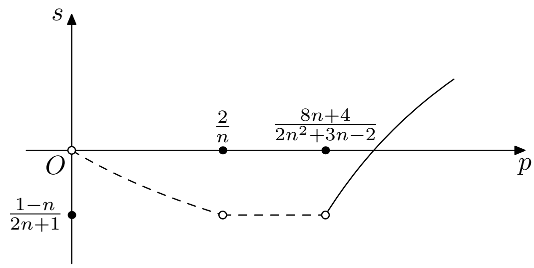


Figure 2. $\dot{H}^s$ well-posedness for NLS

**4.2 Nonlinear wave equations.** Next, we consider the semi-linear wave equations

$$(4.4)\qquad \begin{aligned} &\partial_{tt}u - \Delta u = \mu|u|^p u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n,\\ &u(0) = u_0(x),\ u_t(0) = u_1(x), \end{aligned}$$

where $u(t,x) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, $\mu = \pm 1$, $u_0 \in \dot{H}^s$, $u_1 \in \dot{H}^{s-1}$. It is easy to see that equation (4.4) is invariant under the scaling transformation

$$u(t,x) \to \lambda^{2/p}u(\lambda t, \lambda x),\ u_0(x) \to \lambda^{2/p}u_0(\lambda x), u_1(x) \to \lambda^{(2+p)/p}u_1(\lambda x), \quad \lambda > 0.$$

Then the space $\dot{H}^{s_w}\times\dot{H}^{s_w-1}$ ($s_w = \frac{n}{2} - \frac{2}{p}$) is the critical space for (4.4) in the sense of scaling, i.e., $\|\lambda^{2/p}u_0(\lambda\cdot)\|_{\dot{H}^{s_w}} = \|u_0\|_{\dot{H}^{s_w}}$.

The well-posedness and scattering for equation (4.4) have been studied deeply. We refer the reader to [9, 21, 25, 17] and the references therein. In this section, we study the well-posedness theory for (4.4) in $\dot{H}^s\times\dot{H}^{s-1}$ with radial initial data. As indicated in the Introduction, sharp results at the critical regularity were obtained in [21] for $s_w \geq 1/2$. Thus we restrict ourselves to the case $s_w < 1/2$ and find a threshold $s_0(n)$ for the critical GWP in the radial case

$$(4.5)\qquad s_0(n) = \begin{cases} \frac{5-\sqrt{17}}{4} & \text{for } n = 2,\\ \frac{12-\sqrt{129}}{6} & \text{for } n = 3,\\ \frac{n^2+3n-3-\sqrt{n^4+6n^3-n^2-14n+9}}{4n-4} & \text{for } n \geq 4. \end{cases}$$

It seems that this is the optimal regularity obtainable by our methods.

**Theorem 4.4.** *Assume that $n \geq 2$, $0 < p < 4/(n-1)$, $s_w = n/2 - 2/p$, $s_0(n) < s_w < 1/2$ with $s_0(n)$ given by (4.5), and $u_0$ is radial.*

(1) *If $\|u_0\|_{\dot{H}^{s_w}} + \|u_1\|_{\dot{H}^{s_w-1}} \leq \delta$ for some $\delta \ll 1$, there exist a unique global solution $u$ to (4.4) such that*

$$u \in C(\mathbb{R} : \dot{H}^{s_w}) \cap C^1(\mathbb{R} : \dot{H}^{s_w-1}) \cap L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n),$$

*where $(q,r)$ are given in the proof, and $(u_\pm, v_\pm) \in \dot H^{s_w} \times \dot H^{s_w-1}$ such that*

$$\|u - W'(t)u_\pm\|_{\dot H^{s_w}} + \|u_t - W(t)v_\pm\|_{\dot H^{s_w-1}} \to 0, \ \text{as } t \to \pm\infty.$$

(2) *If $u_0 \in \dot H^s$ for some $s_w \le s < 1/2$, there exist $T > 0$ and a unique solution $u$ to* (4.4) *defined on $(-T,T)$ such that*

$$u \in C((-T,T):\dot H^s) \cap C^1((-T,T):\dot H^{s-1}) \cap L_t^q L_x^r((-T,T)\times\mathbb{R}^n),$$

*where $(q,r)$ is the index given by part (1) for $s_w = s$.*

**Proof.** By Duhamel's principle,

$$u = \Phi_{u_0,u_1}(u) = W'(t)u_0 + W(t)u_1 + \mu\int_0^t W(t-s)(|u|^{\frac{4}{n-2s_w}}u)(s)ds.$$

First we show part (1) and explain how $s_0$ is obtained. The main issue is to choose the admissible pairs $(q,r)$, $(\tilde q, \tilde r)$ so that we can close the contraction argument[4]. By the choice of $(q,r)$ and $(\tilde q,\tilde r)$, we should have

$$\begin{aligned}\|\Phi_{u_0,u_1}(u)\|_{L_t^qL_x^r} &\lesssim \|W'(t)u_0\|_{L_t^qL_x^r} + \|W(t)u_1\|_{L_t^qL_x^r} + \||u|^{\frac{4}{n-2s_w}}u\|_{L_t^{\tilde q'}L_x^{\tilde r'}}\\ &\lesssim \|D^{s_w}u_0\|_{L^2} + \|D^{s_w-1}u_1\|_{L^2} + \|u\|_{L_t^qL_x^r}^{1+\frac{4}{n-2s_w}}.\end{aligned}$$

These inequalities hold if $(q,r)$, $(\tilde q,\tilde r)$ satisfy

$$\begin{aligned}&(q,r), (\tilde q,\tilde r) \text{ is n-D radial wave admissible},\\ &\frac1q + \frac nr = \frac n2 - s_w,\\ &\frac1{\tilde q} + \frac n{\tilde r} = \frac n2 - 1 + s_w,\\ &(p+1)\tilde r' = r, \ (p+1)\tilde q' = q.\end{aligned} \tag{4.6}$$

Given a solution to (4.6), part (1) follows from standard arguments. Therefore, it remains to find a solution of (4.6). We give explicit solutions in each of the possible cases.

Case 1: $1/2n < s_w \le 1/2$.

$$(q,r) = \left(\frac{2n+2}{n-2s_w}, \frac{2n+2}{n-2s_w}\right), \quad (\tilde q,\tilde r) = \left(\frac{2n+2}{n+2s_w-2}, \frac{2n+2}{n+2s_w-2}\right).$$

Case 2: $s_0 < s_w \le (2n)^{-1}$.

[4]The ideas for the Schrödinger equations are the same. However, the choice of the index for the wave equations is more complicated.

Case 2a: $n = 2$.

$$(q, r) = \left(\frac{3 - s_w}{(1 - s_w)^2}, \frac{3 - s_w}{1 - s_w}\right), \quad (\tilde{q}, \tilde{r}) = \left(\frac{1}{s_w}, \infty\right).$$

Case 2b: $n = 3$. There exists $0 < \theta \ll 1$ such that

$$\left(\frac{1}{q}, \frac{1}{r}\right) = \left(2s_w - 3\theta, \frac{1}{2} - s_w + \theta\right), \quad (\tilde{q}, \tilde{r}) = \left(\frac{q}{q - p - 1}, \frac{r}{r - p - 1}\right).$$

Case 2c: $n \geq 4$.

$$(q, r) = \left(\frac{2n + 8 - 4s_w}{n - 2s_w}, \frac{2n^2 + 8n - 4ns_w}{n^2 + 3n - 4ns_w + 4s_w^2 - 6s_w}\right), \ (\tilde{q}, \tilde{r}) = \left(2, \frac{2n}{n + 2s_w - 3}\right).$$

Part (1) is proved.

Next we show part (2). Local well-posedness in $\dot{H}^{s_w}$ follows from the fact that for the choice of $(q, r)$ in the proof of part (1),

$$\lim_{T \to 0} \|W'(t)u_0\|_{L^q_{t \in [-T,T]} L^r_x} + \|W(t)u_1\|_{L^q_{t \in [-T,T]} L^r_x} = 0.$$

Now we assume $s_w < s < 1/2$. The proof is very similar to that for the Schrödinger equations. We take $(q, r)$ corresponding to $s$ in part (1) and then take $(\tilde{q}, \tilde{r})$ to complete the argument. We omit the details. □

**Remark 4.5.** As with the Schrödinger equation, if $s_w \leq s_0(n)$, i.e., $p \leq 4/(n - 2s_0(n))$, we are unable to prove well-posedness in $\dot{H}^s \times \dot{H}^{s-1}$ down to $s = s_w$. However, we can improve the well-posedness results in [21]. We mention only the case $n \geq 4$. If $3/n < p \leq 4/(n - 2s_0(n))$, then large data local well-posedness holds in $\dot{H}^s \times \dot{H}^{s-1}$ for $s > s_2$ with

$$s_2 = \frac{np - 3}{2np + 2n - 2}.$$

Indeed, take $\tilde{q} = 2$, $\tilde{r} = 2n/(n - 3 + 2s)$, and $(q, r)$ such that

$$\frac{1}{q} = \frac{n}{2} - \frac{n}{r} - s, \quad \frac{1}{r} = \frac{1}{p + 1} - \frac{1}{(p + 1)\tilde{r}}.$$

With this choice, we can prove local well-posedness using arguments similar to those in the proof of Theorem 4.4.

The same results hold for general nonlinear terms $F(u)$ satisfying (4.3). We describe the regularity $s$ for $\dot{H}^s \times \dot{H}^{s-1}$ local well-posedness and nonlinear increasing rate $p + 1$ for (4.4) in Figure 3.

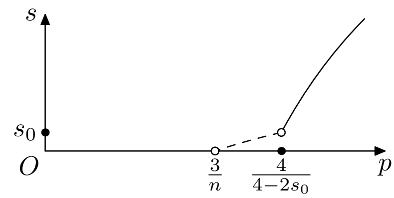


Figure 3. $\dot{H}^s \times \dot{H}^{s-1}$ well-posedenss for NLW.

**4.3 Nonlinear fractional-order Schrödinger equation.** In this section, we apply the improved Strichartz estimates to the nonlinear fractional-order Schrödinger equation

$$i\partial_t u + (\sqrt{-\Delta})^\sigma u = \mu |u|^p u, \quad u(0) = u_0(x), \tag{4.7}$$

where $u(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$, $n \geq 2$, $1 < \sigma < 2$, $\mu = \pm 1$, $u_0 \in \dot{H}^s$. To the best of our knowledge, there are few results concerning the well-posedness for (4.7). The main reason is that the usual Strichartz estimates derived by the decay estimates have a loss of derivative except the trivial one $L_t^\infty L_x^2$. Thus one may need to use other methods, for example, local smoothing effect methods, and the $X^{s,b}$ space. These methods should certainly provide some results, at least when $p$ is an even integer.

However, in the radial case, we obtain additional Strichartz estimates for (4.7), some of which do not have a loss in derivative. Our idea is to exploit these estimates. The equation (4.7) has the following two symmetries.

(i) *Scaling invariance*: for all $\lambda > 0$, (4.7) is invariant under the transformation

$$u(t,x) \to \lambda^{\sigma/p} u(\lambda^\sigma t, \lambda x), \ u_0(x) \to \lambda^{\sigma/p} u_0(\lambda x).$$

(ii) *Conservation laws*: if $u$ is a smooth solution of (4.7), then

$$\frac{d}{dt}\int_{\mathbb{R}^n} |u|^2 dx = 0 \quad \text{(conservation of mass)},$$

$$\frac{d}{dt}\int_{\mathbb{R}^n} \left||\nabla|^{\sigma/2} u\right|^2 - \frac{\mu}{p+2}|u|^{p+2} dx = 0 \quad \text{(conservation of energy)}.$$

Then we see the space $\dot{H}^{s_c}$, where $s_c = \frac{n}{2} - \frac{\sigma}{p}$, is critical in the sense of scaling; $\mu = -1$ is the defocusing case, while $\mu = 1$ corresponds to the focusing case. We require the following lemma.

**Lemma 4.6** (Fractional chain rule [5]). *Suppose $G \in C^1(\mathbb{C})$, $s \in (0,1]$, and $1 < p, p_1, p_2 < \infty$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then*

$$\||\nabla|^s G(u)\|_p \lesssim \|G'(u)\|_{p_1} \||\nabla|^s u\|_{p_2}.$$

In view of the conservation laws, we only consider the nonlinear terms between mass-critical and energy-critical, namely, $2\sigma/n \le p \le 2\sigma/(n-\sigma)$. First we consider the critical $\dot{H}^s$ well-posedness theory of (4.7). For simplicity of notation, we set $S_\sigma(t) = e^{it(\sqrt{-\Delta})^\sigma}$.

**Theorem 4.7.** *Assume $n \ge 2$, $2n/(2n-1) \le \sigma < 2$, $p \ge \frac{2\sigma}{n}$ $s_c = \frac{n}{2} - \frac{\sigma}{p}$, and $u_0 \in H^{s_c}$ is radial. Then the IVP* (4.7) *has the following properties.*

(1) *Small data scattering: if $\|u_0\|_{\dot{H}^{s_c}} \le \delta$ for some $\delta \ll 1$, then there exist a unique global solution*

$$u \in C(\mathbb{R} : H^{s_c}) \cap L_t^{p+2} L_x^{\frac{2n(p+2)}{2(n-\sigma)+np}}(\mathbb{R} \times \mathbb{R}^n),$$

*and $u_\pm \in \dot{H}^{s_c}$ such that $\|u - S_\sigma(t)u_\pm\|_{\dot{H}^{s_c}} \to 0$, as $t \to \pm\infty$.*

(2) *Large data local well-posedness: there exists a unique solution*

$$u \in C((-T,T) : H^{s_c}) \cap L_t^{p+2} L_x^{\frac{2n(p+2)}{2(n-\sigma)+np}}((-T,T) \times \mathbb{R}^n)$$

*for some $T = T(u_0) > 0$.*

**Proof.** Since $\sigma \ge 2n/(2n-1)$, it follows that $2(n+\sigma)/n \ge 2(2n+1)/(2n-1)$. Thus it is easy to see that $(2+\frac{2\sigma}{n}, 2+\frac{2\sigma}{n})$ is an n-D radial Schrödinger admissible pair. By Proposition 3.9, we then get $\|S_\sigma(t)u_0\|_{L_{t,x}^{2+\frac{2\sigma}{n}}(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{L_x^2}$. Interpolating this with the trivial estimate $\|S_\sigma(t)u_0\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|u_0\|_{L_x^2}$, we get more estimates. The key point is that these Strichartz estimates are without loss of regularity.

With these estimates, the proof is quite standard; see, for example, [19]. First we show part (1). By Duhamel's principle,

$$u = \Phi_{u_0}(u) = S_\sigma(t)u_0 + \mu \int_0^t S_\sigma(t-s)(|u|^p u)(s)ds,$$

Take

$$q = \tilde{q} = p+2, \ r = \tilde{r} = \frac{2n(p+2)}{2(n-\sigma)+np}.$$

It is easy to verify that $(q,r)$, $(\tilde{q},\tilde{r})$ satisfy the conditions in Proposition 3.9 with $\gamma = 0$. Then we define the sets

$$B_1 = \{u \in L_t^\infty H_x^{s_c}(\mathbb{R}\times\mathbb{R}^n) : \ \|u\|_{L_t^\infty H_x^{s_c}} \le 2\|u_0\|_{H_x^{s_c}} + C(n)(2\eta)^{1+p}\},$$
$$B_2 = \{u \in L_t^q W_x^{s_c,r}(\mathbb{R}\times\mathbb{R}^n) : \ \|u\|_{L_t^{p+2}\dot{W}_x^{s_c,r}} \le 2\eta, \ \|u\|_{L_t^q L_x^r} \le 2C(n)\|u_0\|_{L_x^2}\},$$

with some sufficiently small $\eta > 0$ to be determined later, and consider the set $X = B_1 \cap B_2$ endowed with the metric $d(u,v) := \|u-v\|_{L_t^q L_x^r}$. It is easy to see that $(X, d)$ is complete. We show that the solution map $\Phi_{u_0}$ with the initial data condition

$$\|u_0\|_{\dot{H}^{s_c}} \leq \eta \ll 1. \tag{4.8}$$

is a contraction on $(X, d)$.

First we show that $\Phi_{u_0}$ maps $X$ into itelf. It is easy to see that since $q' = (p+2)/(p+1)$ and $r' = 2n(p+2)/(2(n+\sigma)+np)$,

$$\frac{1}{q'} = \frac{1}{q} + \frac{1}{pq}, \quad \frac{1}{r'} = \frac{1}{r} + \frac{2\sigma}{n(p+2)}.$$

Then by Proposition 3.9, the fractional chain rule Lemma 4.6, and Sobolev embedding, we find that for $u \in X$,

$$\begin{aligned}
\|\Phi_{u_0}(u)\|_{L_t^\infty H_x^{s_c}(I\times\mathbb{R}^n)} &\leq \|u_0\|_{H_x^{s_c}} + C(n)\|\langle\nabla\rangle^{s_c}(|u|^p)u\|_{L_t^{q'}L_x^{r'}} \\
&\leq \|u_0\|_{H_x^{s_c}} + C(n)\|\langle\nabla\rangle^{s_c}u\|_{L_t^qL_x^r}\|u\|^p_{L_t^qL_x^{\frac{np(p+2)}{2\sigma}}} \\
&\leq \|u_0\|_{H_x^{s_c}} + C(n)(2\eta + 2C(n)\|u_0\|_{L_x^2})\||\nabla|^{s_c}u\|^p_{L_t^qL_x^r} \\
&\leq \|u_0\|_{H_x^{s_c}} + C(n)(2\eta + 2C(n)\|u_0\|_{L_x^2})(2\eta)^p.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\Phi_{u_0}(u)\|_{L_t^qL_x^r} &\leq C(d)\|u_0\|_{L_x^2} + C(d)\|(|u|^p)u\|_{L_t^{q'}L_x^{r'}} \\
&\leq C(d)\|u_0\|_{L_x^2} + 2C(d)^2\|u_0\|_{L_x^2}(2\eta)^p
\end{aligned}$$

and

$$\begin{aligned}
\||\nabla|^{s_c}\Phi_{u_0}(u)\|_{L_t^qL_x^r} &\leq \||\nabla|^{s_c}S_\sigma(t)u_0\|_{L_t^qL_x^r} + C(n)(2\eta)^{p+1} \\
&\leq C(n)\eta + C(n)(2\eta)^{p+1}.
\end{aligned}$$

Thus, choosing $\eta_0 = \eta_0(n)$ sufficiently small, we see that for $0 < \eta \leq \eta_0$, the functional $\Phi_{u_0}$ maps the set $X$ into itself. To see that $\Phi_{u_0}$ is a contraction, we repeat the computations above and get

$$\begin{aligned}
\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{L_t^qL_x^r} &\leq C(d)\|(|u|^p)u - (|v|^p)v\|_{L_t^{q'}L_x^{r'}} \\
&\leq C(d)(2\eta)^p\|u - v\|_{L_t^qL_x^r}
\end{aligned}$$

for $u, v \in X$. Thus for sufficiently small $\eta$, the map $\Phi_{u_0}$ is a contraction. By the Contraction Mapping Theorem, it follows that $\Phi_{u_0}$ has a unique fixed point in $X$. The rest of part (1) (e.g., uniqueness) follows from standard arguments; see [19].

To show part (2), we observe that since $q \neq \infty$,

$$\lim_{T\to 0} \left\| |\nabla|^{s_c} S_\sigma(t) u_0 \right\|_{L^q_{t\in[-T,T]} L^r_x} = 0.$$

Part (2) then also follows from standard fixed-point arguments. □

The following corollary, whose proof is omitted, follows easily by similar arguments and the conservation laws.

**Corollary 4.8** ($H^s$ subcritical)**.** *Assume $n \geq 2$, $2n/(2n-1) < \sigma < 2$ and $u_0$ is radial. Then for $0 < p < 2\sigma/n$, the IVP* (4.7) *is globally well-posed if $u_0 \in L^2$. Moreover, for $2\sigma/n \leq p < 2\sigma/(n-2\sigma)$, the IVP* (4.7) *is locally well-posed (globally well-posed in the defocusing case) if $u_0 \in H^{\sigma/2}$.*

Indeed, we can also prove some other subtle well-posedness results. We can also go below $L^2$, as long as $\sigma$ is close to 2; however, we do not pursue this here. On the other hand, in the $H^s$-critical case, we assumed $u_0 \in H^{s_c}$ instead of $u_0 \in \dot{H}^{s_c}$ as in the work of Cazenave and Weissler [2]. This makes the proof much simpler [19]. We will address this in subsequent works which concern the large data scattering theory for (4.7).

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