DIOPHANTINE PROPERTIES OF MEASURES INVARIANT WITH RESPECT TO THE GAUSS MAP

By

LIOR FISHMAN[1] DAVID SIMMONS, AND MARIUSZ URBAŃSKI[2]

Abstract. Motivated by the work of D. Y. Kleinbock, E. Lindenstrauss, G. A. Margulis, and B. Weiss [8, 9], we explore the Diophantine properties of probability measures invariant under the Gauss map. Specifically, we prove that every such measure which has finite Lyapunov exponent is extremal, i.e., gives zero measure to the set of very well approximable numbers. We show, on the other hand, that there exist examples where the Lyapunov exponent is infinite and the invariant measure is not extremal. Finally, we construct a family of Ahlfors regular measures and prove a Khinchine-type theorem for these measures. The series whose convergence or divergence is used to determine whether or not μ-almost every point is ψ-approximable is different from the series used for Lebesgue measure, so this theorem answers in the negative a question posed by Kleinbock, Lindenstrauss, and Weiss [8].

1 Introduction

Definitions 1.1. Let $\psi : \mathbb{N} \to (0, \infty)$ be an arbitrary function. We recall that an irrational $x \in [0, 1]$ is **ψ-approximable** if there exist infinitely many $p/q \in \mathbb{Q}$ such that

$$\left| x - \frac{p}{q} \right| \leq \psi(q). \tag{1.1}$$

We also recall the following facts and definitions from the classical theory of Diophantine approximation.

- Every x is ψ-approximable when $\psi(q) = q^{-2}$.
- x is **badly approximable** if there exists $\varepsilon > 0$ such that x is not ψ-approximable when $\psi(q) = \varepsilon q^{-2}$. The set of badly approximable numbers has Hausdorff dimension 1 but Lebesgue measure 0.

[1]Supported in part by the Simons Foundation grant 245708.
[2]Supported in part by NSF Grant DMS 1001874.

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 122 (2014)
DOI 10.1007/s11854-014-0009-8

- x is **very well approximable** if there exists $c > 0$ such that x is ψ-approximable when $\psi(q) = q^{-(2+c)}$. The set of very well approximable numbers has Hausdorff dimension 1 but Lebesgue measure 0.
- x is a **Liouville** number if for all $c > 0$, the number x is ψ-approximable when $\psi(q) = q^{-c}$. The set of Liouville numbers has Hausdorff dimension 0.

1.1 Extremal measures. A measure[1] μ on $\mathbb{R}$ is said to be **extremal** if the set of very well approximable numbers is null with respect to μ. In other words, μ behaves like Lebesgue measure with respect to very well approximable numbers. This definition was introduced by D. Y. Kleinbock, E. Lindenstrauss, and B. Weiss in [8] as a generalization of the notion of an extremal manifold, which was defined by V. Sprindžuk. B. Weiss [13] proved that measures satisfying a certain decay condition, called *absolutely decaying*, are extremal.

Definition 1.2. For $\alpha > 0$, a measure μ on $\mathbb{R}$ is said to be **absolutely α-decaying** if there exists $C > 0$ such that for all $x \in \mathbb{R}$, for all $0 < r \leq 1$, and for all $0 < \varepsilon \leq 1$,

$$\mu(B(x, \varepsilon r)) \leq C\varepsilon^{\alpha}\mu(B(x, r)). \tag{1.2}$$

It is said to be **absolutely decaying** if it is absolutely α-decaying for some $\alpha > 0$.

We recall also that for $\delta > 0$, a measure μ on $\mathbb{R}$ is **Ahlfors δ-regular** if there exist positive constants C_1 and C_2 such that

$$C_1 r^{\delta} \leq \mu(B(x, r)) \leq C_2 r^{\delta}$$

for all x in the topological support of μ and for all $0 < r \leq 1$. Examples of Ahlfors regular measures include Lebesgue measure and the Hausdorff measure in the appropriate dimension on certain fractals such as the Cantor set. Clearly, any Ahlfors δ-regular measure on $\mathbb{R}$ is automatically absolutely δ-decaying.

Generalizations of Weiss's result to higher dimensions have been considered by Kleinbock, Lindenstrauss, and Weiss [8]. However, here we consider only Weiss's original result and not the higher dimensional generalizations.

Let $G : [0, 1] \to [0, 1]$ be the Gauss map, i.e.,

$$G(x) = \begin{cases} 1/x - \lfloor 1/x \rfloor & x > 0, \\ 0 & x = 0, \end{cases} \tag{1.3}$$

where $\lfloor x \rfloor$ is the integer part of x. A measure μ is **invariant** with respect to the Gauss map if $\mu \circ G^{-1} = \mu$. We consider the extremality of probability measures

[1] In this paper, all measures are assumed to be Borel and locally finite.

invariant with respect to the Gauss map. Specifically, we show that if an invariant measure μ has finite Lyapunov exponent, then μ is extremal.

Definition 1.3. If μ is a probability measure on $[0,1]$ invariant with respect to the Gauss map, the number $\chi_\mu(G) = \int \log|G'|d\mu$ is called the **Lyapunov exponent** of the measure μ with respect to the Gauss map G.

Theorem 2.1. *A probability measure μ on $[0,1]\setminus\mathbb{Q}$ which is invariant with respect to the Gauss map G and has finite Lyapunov exponent $\chi_\mu(G)$ is extremal.*

The assumption that $\chi_\mu(G)$ is finite is very reasonable and is satisfied for a large class of dynamically defined measures; see Section 4. In particular, there exist measures which satisfy this assumption but are not absolutely decaying. It is also a necessary assumption for Theorem 2.1 to hold, as seen from the following theorem.

Theorem 4.5. *There exists a measure μ invariant with respect to the Gauss map which gives full measure to the Liouville numbers. In particular, μ is not extremal.*

1.2 A question about absolutely decaying measures.

In [8], Kleinbock, Lindenstrauss, and Weiss asked the following questions.[2]

Question 1.4 ([8, Question 10.1])**.** *Let μ be an absolutely decaying measure on $\mathbb{R}$.*

(a) *Is it true that for every decreasing function $\psi : \mathbb{N} \to (0,+\infty)$, either the set of ψ-approximable numbers or its complement has μ-measure* 0*?*

(b) *Is it true that for all ψ as in* (a)*, μ-almost every point is ψ-approximable if and only if*

$$\sum_{q=1}^{\infty} q\psi(q) = \infty?^3 \tag{1.4}$$

We answer these questions in the negative by constructing a family of measures on $\mathbb{R}$ which are Ahlfors regular (and, in particular, absolutely decaying) and yet satisfy neither (a) nor (b).

To construct these measures, we fix a set $I \subseteq \mathbb{N}$ and let

$$J_I = \{x \in [0,1]\setminus\mathbb{Q} : \text{ the continued fraction entries of } x \text{ lie in } I\}.$$

[2] Actually, Kleinbock, Lindenstrauss, and Weiss's question was about friendly measures on $\mathbb{R}^d$. When restricted to one dimension, friendly measures are the same as absolutely decaying measures; see [8, Lemma 2.2].

[3] In this formula, we have replaced $\psi(q)$ by $q\psi(q)$ because of a difference in the definition of ψ-approximability.

Theorem 6.1. *Let $I \subseteq \mathbb{N}$ be infinite and h be the Hausdorff dimension of J_I. Assume that the h-dimensional Hausdorff measure $\mathcal{H}^h$ restricted to J_I is Ahlfors h-regular. Let $\mu = \mathcal{H}^h \upharpoonright_{J_I}$, and let $\psi : \mathbb{N} \to (0, +\infty)$ be such that the function $q \mapsto q^2\psi(q)$ is nonincreasing. Then μ-almost every (respectively, μ-almost no) point is ψ-approximable, assuming that the series*

$$\sum_{q=1}^{\infty} q^{2\alpha-1}\psi(q)^{\alpha} \tag{6.1}$$

diverges (respectively, converges).

We note that the convergence case of Theorem 6.1 is a theorem of Weiss [13]. He proved this theorem for any absolutely decaying measure μ and for all functions $\psi : \mathbb{N} \to (0, +\infty)$.

Note that when $I = \mathbb{N}$, μ is Lebesgue measure on $[0, 1]$ and Theorem 6.1 reduces to the classical Khinchine Theorem.

It appears that the only easy example of a set I satisfying the hypotheses of Theorem 6.1 is the set $I = \mathbb{N}$. Nevertheless, we prove the following result.

Theorem 7.1. *For each $0 < \delta \leq 1$, there exists an infinite set $I \subseteq \mathbb{N}$ such that $\mathrm{HD}(J_I) = \delta$ and $\mathcal{H}^\delta \upharpoonright_{J_I}$ is Ahlfors δ-regular.*[4]

Combining Theorems 6.1 and 7.1 in the obvious way yields the following corollary.

Corollary 1.5. *For every $0 < \alpha \leq 1$, there exists an Ahlfors α-regular, and therefore absolutely α-decaying, measure μ such that for every function $\psi : \mathbb{N} \to (0, +\infty)$ for which the function $q \mapsto q^2\psi(q)$ is nonincreasing, μ-almost every (respectively, μ-almost no) point is ψ-approximable, assuming that the series* (6.1) *diverges (respectively, converges).*

In the case $\alpha = 1$, the measure is simply Lebesgue measure.

Remark 1.6. It appears that (when $\alpha < 1$) this is the first known example of a measure μ which is neither atomic nor absolutely continuous to Lebesgue for which a complete criterion has been given for when the set of ψ-approximable numbers is μ-null or μ-full.

Corollary 1.7. *The answer to Question 1.4 is negative (for both parts* (a) *and* (b)*).*

[4]Here and henceforth, $\mathrm{HD}(S)$ denotes the Hausdorff dimension of a set S and $\mathcal{H}^\delta(S)$ and $\mathcal{P}^\delta(S)$ denote its δ-dimensional Hausdorff and packing measure, respectively.

Proof. Fix $0 < \alpha < 1$ and let μ be the measure guaranteed by Corollary 1.5. To see that the answer to (b) is negative, we merely note the existence of a function ψ for which (1.4) converges but (6.1) diverges, for example,

$$\psi(q) = \frac{1}{q^2 \log^{1/\alpha}(q)}.$$

To see that the answer to (a) is negative, let $y \in \mathbb{R}$ be chosen at random with respect to Lebesgue measure. As noted in [8] (see the paragraph immediately following Question 10.1), the measure $\nu := \mu \circ (x \mapsto x + y)^{-1}$ satisfies (b) of Question 1.4. But then the measure $\mu + \nu$ is also Ahlfors regular, but does not satisfy (a); indeed, for the function ψ given above, μ-almost every point but ν-almost no point is ψ-approximable. □

In the process of proving Theorems 6.1 and 7.1, we establish the following criterion for determining whether $\mathcal{H}^\alpha \upharpoonright_{J_I}$ is Ahlfors regular. This improves more complicated criteria which can be found in [10].

Theorem 5.5 (*Abridged*). *Fix an infinite set $I \subseteq \mathbb{N}$ and let $h = \mathrm{HD}(J_I)$. The following are equivalent.*

(a) $\mathcal{H}^h(J_I) > 0$ *and* $\mathcal{P}^h(J_I) < \infty$.
(b1) $\mathcal{H}^h \upharpoonright_{J_I}$ *is Ahlfors h-regular.*
(c1) *For all $y \in I$ and $r \geq 1$, $\#(B(y,r) \cap I) \asymp r^h$.*

Thus, the Ahlfors regularity of J_I is equivalent to the "dual Ahlfors regularity" of the generating set I.

Note that it is possible for (c1) to be satisfied for some $h \neq \mathrm{HD}(J_I)$. In such a case, the set J_I is not Ahlfors regular.

The structure of the paper is as follows. In Section 2, we prove Theorem 2.1. In Section 3, we recall some basic definitions and theorems from the theory of conformal iterated function systems which are needed to prove Theorems 4.5, 5.5, 6.1, and 7.1. In Section 4, we give some examples of measures which satisfy the hypotheses of Theorem 2.1, and prove Theorem 4.5. In Section 5, we discuss various characterizations of Ahlfors regularity and semiregularity of J_I, and prove Theorem 5.5. In Section 6, we prove Theorem 6.1. Finally, in Section 7, we prove Theorem 7.1.

The interdependence of the sections is as follows. Section 4 depends on Sections 2 and 3; Section 5 depends on Section 3; Section 6 depends on Sections 2, 3, and 5; Section 7 depends on Sections 3 and 5.

2 Proof of Theorem 2.1

In this section, we prove the following theorem.

Theorem 2.1. *A probability measure μ on $[0, 1] \setminus \mathbb{Q}$ which is invariant with respect to the Gauss map G and has finite Lyapunov exponent $\chi_\mu(G)$ is extremal.*

2.1 A relation between continued fractions and Diophantine approximation. In the proof of Theorem 2.1, we make use of a relation between the continued fraction expansion of an irrational $x \in [0, 1]$ with its Diophantine properties.

Definition 2.2. For a function $\psi : \mathbb{N} \to (0, \infty)$, x is **ψ-well approximable** if it is $\varepsilon\psi$-approximable for every $\varepsilon > 0$.

Remark 2.3. Note that ψ-well approximability implies ψ-approximability but not conversely. For example, if x is a badly approximable number and $\psi(q) = 1/q^2$, then x is ψ-approximable but not ψ-well approximable.

Lemma 2.4. *Let $x \in [0, 1]$ be irrational, $[0; \omega_0, \omega_1, \ldots]$ the continued fraction expansion of x, and $(p_n/q_n)_{n=0}^\infty$ the convergents of x. Let $\psi : \mathbb{N} \to (0, \infty)$ be a function satisfying*

$$\psi(q) \leq \frac{1}{q^2}$$

for all q. Then x is ψ-well approximable if and only if for every $K > 0$, there exist infinitely many $n \in \mathbb{N}$ such that

$$\omega_n \geq K\phi(q_n), \tag{2.1}$$

where

$$\phi(q) := \frac{1}{q^2\psi(q)} \geq 1.$$

Remark 2.5. Lemma 2.4 can be deduced from [7, Theorem 8.5], which is proved in a similar manner, but we include the proof for completeness. The ideas of this proof can also be found in the proof of [6, Theorem 32].

Proof of Lemma 2.4. Fix an irrational $x \in [0, 1]$. We recall the following well-known facts (see, e.g., [6] or [1]):

(i) A rational approximation p/q of x such that $|x - p/q| < 1/(2q^2)$ is a convergent of x.

(ii) For every $n \in \mathbb{N}$,

$$\frac{1}{q_n(q_n + q_{n+1})} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \tag{2.2}$$

and

$$q_{n+1} = \omega_n q_n + q_{n-1}. \tag{2.3}$$

It follows from (i) that for no $0 < \varepsilon \leq 1/2$ can (1.1) be satisfied for any p/q which is not a convergent. Thus we may restrict our attention to approximations of x which are convergents. Fix $n \in \mathbb{N}$ and note that by (2.3), $q_{n+1} \asymp \omega_n q_n$.[5] Thus (2.2) implies

$$\left| x - \frac{p_n}{q_n} \right| \asymp \frac{1}{q_n^2 \omega_n};$$

and x is ψ-well approximable if and only if for every $\varepsilon > 0$, there exist infinitely many $n \in \mathbb{N}$ such that

$$\frac{1}{q_n^2 \omega_n} \geq \varepsilon \psi(q_n) = \frac{\varepsilon}{q_n^2 \phi(q_n)}.$$

(We are using the "ε" to absorb the constant coming from the asymptotic.) The lemma follows by rearranging and letting $K = 1/\varepsilon$. □

For $x \in [0, 1]$, let

$$\xi(x) = \lfloor 1/x \rfloor \tag{2.4}$$

be the first entry in the continued fraction expansion of x, so that $\xi(G^n(x)) = \omega_n$ for all n. Let

$$\eta = \log(1 + \xi). \tag{2.5}$$

Corollary 2.6. *Fix an irrational $x \in [0, 1]$ and let $[0; \omega_0, \omega_1, \ldots]$ be the continued fraction expansion of x. The following are equivalent.*

(i) *The number x is very well approximable.*

(ii) *There exists $c > 0$ such that for infinitely many $n \in \mathbb{N}$,*

$$\log(1 + \omega_n) \geq c \sum_{j=0}^{n-1} \log(1 + \omega_j), \tag{2.6}$$

or, equivalently,

$$\eta(G^n(x)) \geq c \sum_{j=0}^{n-1} \eta(G^j(x)). \tag{2.7}$$

[5] Here and henceforth, $\asymp$ denotes a multiplicative asymptotic.

Formula (2.7) is more useful than (2.6) for our ergodic theory purposes.

Proof. We first give some bounds for q_n in terms of the continued fraction entries $\omega_0, \ldots, \omega_{n-1}$. The upper bound is easy: the recursion equation (2.3) implies $q_n \leq \prod_{j=0}^{n-1}(\omega_j + 1)$. For the other direction, we divide into cases according to whether n is even or odd. If $n = 2k$,

$$q_n \geq \prod_{j=0}^{k-1}(\omega_{2j}\omega_{2j+1} + 1) \geq \prod_{j=0}^{n-1} \sqrt{\omega_j + 1};$$

while if $n = 2k + 1$,

$$q_n \geq \omega_{n-1}q_{n-1} \geq \frac{1}{\sqrt{2}} \prod_{j=0}^{n-1} \sqrt{\omega_j + 1}.$$

Let $t \geq 0$. Taking logarithms, we can rewrite the above inequalities as

$$\frac{1}{2}\sum_{j=0}^{n-1} \eta(G^j(x)) - \log\sqrt{2} \leq \log q_n \leq \sum_{j=0}^{n-1} \eta(G^j(x)). \tag{2.8}$$

The corollary now follows immediately from (2.8) together with Lemma 2.4 and the following characterization of the set VWA of very well approximable numbers:

> an irrational $x \in [0, 1]$ is very well approximable if and only if there exists $c > 0$ such that x is ψ-well approximable where $\psi(q) = q^{-(2+c)}$. □

Using (2.8) and Lemma 2.4, we also deduce the following corollary.

Corollary 2.7. *For irrational* $x \in [0, 1]$, *the following are equivalent:*
(i) *x is a Liouville number;*
(ii) *for all $c > 0$, there exist infinitely many $n \in \mathbb{N}$ for which* (2.7) *holds.*

Remark 2.8. The well-known fact that an irrational $x \in [0, 1]$ is badly approximable if and only if its continued fraction entries are bounded is also a corollary of Lemma 2.4.

Proof of Theorem 2.1. First of all, note that it suffices to consider the case where μ is ergodic with respect to G, since if χ_μ is finite, then χ_ν is finite for almost all measures ν in the ergodic decomposition of μ. Since the set VWA is invariant with respect to the Gauss map, it follows that every ergodic measure must give either 0 or full measure to VWA.

Let μ be an ergodic invariant measure whose Lyapunov exponent is finite. Let η be as in (2.4). Since $\eta(x) \asymp -2\log(x) = \log|G'(x)|$, it follows that $\int \eta \mathrm{d}\mu$ is also finite. On the other hand, η is a strictly positive function, and so

$$0 < \int \eta \mathrm{d}\mu < \infty. \tag{2.9}$$

We claim that μ is extremal. Suppose, to the contrary, that μ-almost every number $x \in [0, 1]$ is very well approximable. It then follows from Corollary 2.6, (2.9), and the Birkhoff Ergodic Theorem that for μ-almost all such numbers x,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \eta(G^j(x)) \le \frac{1}{c_x} \limsup_{n\to\infty} \frac{1}{n} \eta(G^n(x)) = 0,$$

where $c_x > 0$ comes from Corollary 2.6. Invoking the Birkhoff Ergodic Theorem again, we conclude that $\int \eta \mathrm{d}\mu \le 0$. This contradiction finishes the proof. □

3 Iterated function systems and conformal measures

Our main example of a measure invariant with respect to the Gauss map is the unique invariant probability measure absolutely continuous to a conformal measure associated with an iterated function system consisting of inverse branches of the Gauss map. In this section, we recall some definitions and main theorems. All theorems in this section, except for those in Subsection 3.5, were first proved in [10] and then, in a more general context, in [11].

3.1 IFSs and continued fractions. For each $i \in \mathbb{N}$, the mapping $g_i : [0, 1] \to [0, 1]$ defined by $g_i(x) = 1/(i + x)$ is an inverse branch of the Gauss map G. For each $I \subseteq \mathbb{N}$, the collection of maps $\mathcal{S}_I = \{g_i\}_{i\in I}$ is a conformal iterated function system **IFS**; see [10] or [11] for the definition.

Given $\omega = \omega_0\omega_1\omega_2 \ldots \omega_{n-1} \in \mathbb{N}^n$, let

$$g_\omega := g_{\omega_0} \circ g_{\omega_1} \ldots \circ g_{\omega_{n-1}} : [0, 1] \to [0, 1],$$

so that

$$g_\omega(x) = \cfrac{1}{\omega_0 + \cfrac{1}{\omega_1 + \cfrac{1}{\ddots + \cfrac{1}{\omega_{n-1} + x}}}}.$$

In particular, $g_\omega(0) = [0;\omega_0,\omega_1,\ldots,\omega_{n-1}]$. The set $J_I = \bigcap_{n\in\mathbb{N}}\bigcup_{\omega\in I^{\mathbb{N}}} g_\omega([0,1])$. is called the **limit set** of the IFS $\mathcal{S}_I$. It coincides with the set of irrational numbers in $[0,1]$ whose continued fraction entries all lie in I. If I is infinite, the set $\overline{J_I}\setminus J_I$ consists of the set of rational numbers whose continued fraction entries all lie in I. Moreover, J_I is forward invariant under the Gauss map G, i.e., $G(J_I) = J_I$.

3.2 A formula for the Hausdorff dimension of J_I. Fix $I \subseteq \mathbb{N}$. A famous formula of R. Bowen relates the Hausdorff dimension of J_I to an invariant of the IFS $\mathcal{S}_I$.

Given $t \geq 0$, the limit

$$P_I(t) := \lim_{n\to\infty}\frac{1}{n}\log\sum_{\omega\in I^n}\|g'_\omega\|^t_\infty)$$

exists. This limit is called the **topological pressure** of the IFS $\mathcal{S}_I$ at t.

Theorem 3.1 (Bowen's formula [11, Theorem 4.2.13])**.** *For all $I\subseteq\mathbb{N}$,*

$$\mathrm{HD}(J_I) = \inf\{t\geq 0 : P_I(t)\leq 0\}.$$

In particular, $\mathrm{HD}(J_I)$ *is the unique zero of P_I if such a zero exists.*

We also need the following result.

Theorem 3.2 ([11, Theorem 2.1.5])**.** *Given $t \geq 0$, for each set $I\subseteq\mathbb{N}$,*

$$P_I(t) = \lim_{N\to\infty} P_{I\cap\{1,\ldots,N\}}(t).$$

3.3 Conformal measures. Conformal measures are an important tool for understanding the geometry of the limit set J_I. In many cases, they coincide with either the normalized Hausdorff measure or the normalized packing measure.

Definition 3.3. Fix $t\geq 0$ and $I\subseteq\mathbb{N}$. A probability measure m on $[0,1]$ is called **t-conformal** with respect to the iterated function system $\mathcal{S}_I$ if $m(J_I) = 1$ and

$$m(g_i(A)) = \int_A |g'_i|^t \mathrm{d}m$$

for every Borel set $A\subseteq[0,1]$ and for every $i\in I$.

Definition 3.4. Fix $I\subseteq\mathbb{N}$. The system $\mathcal{S}_I$ is said to be **regular** if there exists $t\geq 0$ such that $P_I(t) = 0$.

Proposition 3.5 ([11, Theorem 4.2.9])**.** *Fix $I\subseteq\mathbb{N}$. The following are equivalent:*

(a) *the IFS $\mathcal{S}_I$ is regular;*

(b) *there exists a measure m and $t \geq 0$ such that m is t-conformal.*

Furthermore, when these conditions are met, m and t are both unique and $P(t) = 0$.

Corollary 3.6. *Let $t \geq 0$ and $I \subseteq \mathbb{N}$. Then, if $P(t) = 0$, there exists a measure m which is t-conformal.*

Proof. Since $P(t) = 0$, the IFS $\mathcal{S}_I$ is regular, so, by Proposition 3.5, there exist a measure m and $t' \geq 0$ such that m is t'-conformal and $P(t') = 0$. But since P is strictly decreasing ([11, Proposition 4.2.8(b)]), $t = t'$, so m is t-conformal. □

Proposition 3.7. *Let $I \subseteq \mathbb{N}$, and suppose that the IFS $\mathcal{S}_I$ is regular. Let m_I be the unique conformal measure and $h = \mathrm{HD}(J_I)$, so that m_I is h-conformal. Then there exists a unique G-invariant Borel probability measure μ_I on J_I, absolutely continuous with respect to m_I. This measure is ergodic and equivalent to m_I. The logarithm of the corresponding Radon-Nikodym derivative is a bounded function on J_I.*

Proof. See [11, Theorem 2.4.3 and Corollary 2.7.5(c)]. □

Proposition 3.8.

(a) *If the h_I-dimensional Hausdorff measure of J_I is positive (it is always finite), then the system S_I is regular, and*

$$m_I = \frac{\mathcal{H}^{h_I} \upharpoonright_{J_I}}{\mathcal{H}^{h_I}(J_I)}.$$

(b) *If the h_I-dimensional packing measure of J_I is finite (it is always positive), then the system S_I is regular, and*

$$m_I = \frac{\mathcal{P}^{h_I} \upharpoonright_{J_I}}{\mathcal{P}^{h_I}(J_I)}.$$

Proof. An argument analogous to the proof of the change of variables formula demonstrates that both of the above expressions are h_I-conformal. The result therefore follows from Proposition 3.5. □

3.4 Regularity properties of the IFS $\mathcal{S}_I$. Fix $I \subseteq \mathbb{N}$. In this subsection, we discuss properties of the IFS $\mathcal{S}_I$ which are stronger than regularity.

Let $\theta_I := \inf\{t \geq 0 : P_I(t) < +\infty\}$.

Proposition 3.9.

$$\theta_I = \inf\left\{t \geq 0 : \sum_{i\in I} \|g_i'\|_\infty^t < +\infty\right\} = \inf\left\{t \geq 0 : \sum_{i\in I} i^{-2t} < +\infty\right\}.$$

Proof. The first equality is [11, Proposition 4.2.8(a)]. The second equality follows from the fact that $\|g_i'\|_\infty = i^{-2}$ for all $i \in \mathbb{N}$. □

Definition 3.10. Fix $I \subseteq \mathbb{N}$. The system $\mathcal{S}_I$ is said to be **strongly regular** if there exists $t \geq 0$ such that $0 < P_I(t) < +\infty$ and is called **cofinitely regular** (or **hereditarily regular**) if $P_I(\theta_I) = +\infty$.

Proposition 3.11 ([11, Theorem 4.3.5])**.**

(a) *Every cofinitely regular system is strongly regular, and every strongly regular system is regular.*

(b) *For each strongly regular system $\mathcal{S}_I$, $\mathrm{HD}(J_I) > \theta_I$.*

Proposition 3.12 ([11, Theorem 4.3.4])**.** *Let $I \subseteq \mathbb{N}$. The system $\mathcal{S}_I$ is cofinitely regular if and only if the series $\sum_{i\in I} \|g_i'\|_\infty^{\theta_I} = \sum_{i\in I} i^{-2\theta_I}$ diverges.*

Recall from Definition 1.3 that the Lyaponov exponent of a probability measure on $[0, 1]$ invariant wirh respect to the Gauss map G is given by $\chi_\mu(G) = \int \log |G'| \mathrm{d}\mu$.

Proposition 3.13. *Fix $I \subseteq \mathbb{N}$. If the system $\mathcal{S}_I$ is strongly regular, then $\chi_{\mu_I}(G) < +\infty$.*

Proof. By Proposition 3.11(b), $h_I > \theta_I$. Fix $\theta_I < t < h_I$. Then by the definition of θ_I, the series $\sum_{i\in I} i^{-2t}$ converges. Now we can estimate the Lyapunov exponent of μ_I as follows.

$$\begin{aligned}\int \log |G'| \mathrm{d}\mu_I \asymp \int \eta \mathrm{d}m_I &= \sum_{i\in I} \log(1+i) m_I(g_i([0, 1])) \\ &\asymp \sum_{i\in I} \log(1+i) i^{-2h_I} \lesssim \sum_{i\in I} i^{-2t} < \infty.\end{aligned}$$

□

3.5 Two lemmas.

The lemmas given in this subsection are used several times throughout the remainder of the paper.

For each set $I \subseteq \mathbb{N}$ and $t \geq 0$, let $\lambda_t(I) = e^{P_I(t)}$.

Lemma 3.14 ([5, Lemma 4.3])**.** *Fix $\delta > 0$. Let $i \geq 2$, and let I be a finite subset of $\mathbb{N} \setminus \{i\}$. Then*

$$\lambda_\delta(I) + \left(\frac{1}{i+1}\right)^{2\delta} \leq \lambda_\delta(I \cup \{i\}) \leq \lambda_\delta(I) + \left(\frac{2}{i+2}\right)^{2\delta}. \tag{3.1}$$

Remark 3.15. Applying Theorem 3.2 to this lemma, we conclude that (3.1) holds for all sets $I \subseteq \mathbb{N} \setminus \{i\}$.

Recall that we have defined $\xi(x) = \lfloor 1/x \rfloor$ to be the first entry of the continued fraction expansion of x. For any $\omega \in \mathbb{N}^n$, let

$$S_\omega := g_\omega([0,1]) = \{x \in [0,1] : \xi(G^j(x)) = \omega_j \text{ for all } j = 0, \ldots, n-1\},$$

i.e., S_ω is the set of all numbers whose continued fraction expansions begin with the sequence $\omega_0, \ldots, \omega_{n-1}$. Furthermore, for each $k \in \mathbb{N}$ let $S^+_{\omega,k} = \bigcup_{i \le k} S_{\omega i}$.

Lemma 3.16. *Fix $I \subseteq \mathbb{N}$, and suppose that the IFS $\mathcal{S}_I$ is regular. Let h_I be the Hausdorff dimension of J_I and m_I the unique h_I-conformal measure of $\mathcal{S}_I$. Then*

$$\frac{m_I(S_{\omega i})}{m_I(S_\omega)} \ge \frac{1}{4^{h_I}} i^{-2h_I} \tag{3.2}$$

and

$$\frac{m_I(S^+_{\omega,k})}{m_I(S_\omega)} \le 1 - \frac{1}{4^{h_I}} \sum_{\substack{i \in I \\ i > k}} i^{-2h_I}. \tag{3.3}$$

Proof. It is clear that (3.3) follows from (3.2). To prove (3.2), note that since m_I is h_I-conformal,

$$\frac{m_I(S_{\omega i})}{m_I(S_\omega)} = \frac{m_I(g_{\omega i}([0,1]))}{m_I(g_\omega([0,1]))} = \frac{\int |(g'_{\omega i}(x)|^{h_I} \mathrm{d}m_I(x)}{\int |g'_\omega(x)|^{h_I} \mathrm{d}m_I(x)}$$
$$\ge \frac{\min_{[0,1]} |g'_\omega|^{h_I}}{\max_{[0,1]} |g'_\omega|^{h_I}} \min_{[0,1]} |g'_i|^{h_I}.$$

On the other hand (see [10, line -10 of p. 4997] or by direct computation),

$$\frac{\max_{[0,1]} |g'_\omega|}{\min_{[0,1]} |g'_\omega|} \le 4,$$

which yields (3.2). □

4 Extremality of conformal measures

Fix $I \subseteq \mathbb{N}$ and suppose that the IFS $\mathcal{S}_I$ is regular. In this section, we discuss the extremality of the measures m_I and μ_I defined in Section 3. Since m_I and μ_I are absolutely continuous with respect to one another, m_I is extremal if and only if μ_I is.

By Theorem 2.1, if $\chi_{\mu_I} < \infty$, then μ_I is extremal. By Proposition 3.13, if $\mathcal{S}_I$ is strongly regular, then $\chi_{\mu_I} < \infty$. The following proposition gives very general sufficient conditions for $\mathcal{S}_I$ to be strongly regular.

Proposition 4.1. *Let $I \subseteq \mathbb{N}$, Each of the following four conditions entails strong regularity of the iterated function system $\mathcal{S}_I$, and thus the extremality of the measures m_I and μ_I.*

(a) *I is finite.*
(b) *The series $\sum_{a\in I} a^{-2\theta_I}$ diverges.*
(c) *The Hausdorff dimension of the limit set of the IFS is strictly greater than 1/2.*
(d) *$1, 2 \in I$.*

Proof. Item (a) follows directly from the definition (since $0 < P_I(0) < \infty$). Item (b) follows from Proposition 3.11(a) and Proposition 3.12. Item (c) follows from [11, Theorem 4.3.10] along with the observation that $\theta_I \leq \theta_{\mathbb{N}} = 1/2$. Item (d) follows from item (c) and the fact, proved in [3], that $h_{\{1,2\}} = \mathrm{HD}(J_{\{1,2\}}) > 1/2$.□

Remark 4.2. In case (a), the extremality of μ_I is obvious since μ_I's topological support J_I is contained in the set of badly approximable numbers; see Remark 2.8.

Remark 4.3. The main result of [12], namely, the extremality part of Theorem 4.5 of that paper, can be deduced from Proposition 4.1(b).

Example 4.4. Fix $a \geq 2$ and let I be the geometric series $I = \{a, a^2, \ldots\}$. Then condition (b) of Proposition 4.1 is satisfied. Thus the measure μ_I is extremal. On the other hand, μ_I is not absolutely decaying (see below), so the extremality of μ_I does not follow from Weiss's theorem [13].

Proof that μ_I is not absolutely decaying. Fix $n \in \mathbb{N}$, and let $x_n = a^{-n} \in \overline{J_I}$. Then

$$B\left(x_n, \frac{1}{a^n} - \frac{1}{a^n+1}\right) \cap J_I = B\left(x_n, \frac{1}{a^n} - \frac{1}{a^{n+1}}\right) \cap J_I.$$

If μ were absolutely α-decaying, we would therefore have

$$\begin{aligned} 1 = \frac{\mu\left[B\left(x_n, \frac{1}{a^n} - \frac{1}{a^n+1}\right)\right]}{\mu\left[B\left(x_n, \frac{1}{a^n} - \frac{1}{a^{n+1}}\right)\right]} &\leq C\left(\frac{\frac{1}{a^n} - \frac{1}{a^n+1}}{\frac{1}{a^n} - \frac{1}{a^{n+1}}}\right)^\alpha \\ &\asymp \left(\frac{1/a^{2n}}{1/a^n}\right)^\alpha = \frac{1}{a^{n\alpha}}, \end{aligned}$$

which, for sufficiently large n, is a contradiction. □

The remainder of this section is devoted to proving the following theorem.

Theorem 4.5. *There exists a measure μ invariant with respect to the Gauss map which gives full measure to the Liouville numbers. In particular, μ is not extremal.*

The measure μ is of the form μ_I for some $I \subseteq \mathbb{N}$ defining a regular system $\mathcal{S}_I$.

Proof. Fix $0 < \delta \leq 1/2$. Then define a sequence of finite subsets $I_N \subseteq \mathbb{N}$ recursively in the following manner.

1. Let $I_0 = \varnothing$.
2. Suppose that the set I_{N-1} has been defined. Let $M_{N-1} = \max(I_{N-1})$. (By convention let $\max(\varnothing) = 0$.)
3. Choose $m_N \in \mathbb{N}$ so large that
$$\log(1 + m_N) \geq N4^N \log(1 + M_{N-1}),$$
$$\left(\frac{2}{m_N + 2}\right)^{2\delta} \leq 2^{-N}.$$
4. Let $R_N \subseteq \{m_N, \ldots\}$ be a finite set satisfying
$$1 - 2^{-(N-1)} \leq \lambda_\delta(I_{N-1} \cup R_N) < 1 - 2^{-N}. \tag{4.1}$$
(The existence of such a set is verified below.)
5. Let $I_N = I_{N-1} \cup R_N$ and then go back to step 2.

We now check that in step 4, it is always possible to find a set R_N which satisfies (4.1). We first claim that

$$\lambda_\delta(I_{N-1}) < 1 - 2^{-N} < \lambda_\delta(I_{N-1} \cup \{m_N, \ldots\}). \tag{4.2}$$

Indeed, the left inequality follows from the induction hypothesis (or by direct computation in the case $N = 1$). The right hand side follows from Lemma 3.14 and the fact that the series $\sum_{i=m_N}^{\infty} (i+1)^{-2\delta}$ diverges (since $\delta \leq 1/2$).

It follows from (4.2) that there exists $K \in \{m_N, \ldots\}$ such that

$$\lambda_\delta(I_{N-1} \cup \{m_N, \ldots, K\}) < 1 - 2^{-N} \leq \lambda_\delta(I_{N-1} \cup \{m_N, \ldots, K+1\}).$$

Let $R_N = \{m_N, \ldots, K\}$. By Lemma 3.14, we have

$$\begin{aligned}\lambda_\delta(I_{N-1} \cup \{m_N, \ldots, K\}) &\geq \lambda_\delta(I_{N-1} \cup \{m_N, \ldots, K+1\}) - \left(\frac{2}{(K+1)+2}\right)^{2\delta} \\ &\geq 1 - 2^{-N} - 2^{-N} = 1 - 2^{-(N-1)},\end{aligned}$$

which proves (4.1))

Let $I = \bigcup_N I_N$. By Theorem 3.2, $\lambda_\delta(I) = 1$, and thus $P_I(\delta) = 0$. By Corollary 3.6 and Proposition 3.7, $\mathrm{HD}(J_I) = \delta$, and there exist a δ-conformal measure m_I and an absolutely continuous G-invariant measure μ_I.

To complete the proof, we need to show that m_I, and thus μ_I, gives full measure to the set of Liouville numbers. To this end, fix $N \in \mathbb{N}$. By Lemma 3.14,

$$1 - \lambda_\delta(I \cap \{1, \ldots, M_N\}) \le \sum_{\substack{i \in I \\ i > M_N}} \left(\frac{2}{2+i}\right)^{2\delta} \le 4^\delta \sum_{\substack{i \in I \\ i > M_N}} \left(\frac{1}{1+i}\right)^{2\delta}. \tag{4.3}$$

Fix $\omega = (\omega_j)_{j=0}^{n-1} \in \mathbb{N}^n$. It then follows from (3.3) and (4.3) that

$$\frac{m_I(S^+_{\omega,M_N})}{m_I(S_\omega)} \le 1 - \frac{1}{16^\delta}(1 - \lambda_\delta(I \cap \{1, \ldots, M_N\})),$$

where S_ω and $S^+_{\omega,k}$ are defined before Lemma 3.16. Invoking (4.1) gives

$$\frac{m_I(S^+_{\omega,M_N})}{m_I(S_\omega)} \le 1 - c2^{-N}, \tag{4.4}$$

where $c = 1/16^\delta$. Now for each $n \in \mathbb{N}$, let

$$S_{n,N} = \{x \in [0,1] : \xi(G^j(x)) \le M_N \text{ for all } j = 0, \ldots, n-1\}.$$

Formula (4.4) yields

$$\frac{m_I(S_{n+1,N})}{m_I(S_{n,N})} \le 1 - c2^{-N}.$$

Iterating yields $m_I(S_{n,N}) \le (1 - c2^{-N})^n$. Letting $n = 4^N$, we see that $m_I(S_{4^N,N}) \le e^{-c2^N}$, and thus $\sum_{N=0}^\infty m_I(S_{4^N,N}) < \infty$. Thus, by the Borel-Cantelli lemma, m_I-almost every point $x \in J_I$ lies in only finitely many sets of the form $S_{4^N,N}$. We now show that every such x is a Liouville number. By Corollary 2.7, it suffices to prove that for all $c > 0$,

$$\eta(G^n(x)) \ge c \sum_{j=0}^{n-1} \eta(G^j(x)) \tag{4.5}$$

for infinitely many $n \in \mathbb{N}$, where η is defined as in 2.5. Indeed, for all but finitely many $N \in \mathbb{N}$, $x \notin S_{4^N,N}$, and so there exists $n \le 4^N$ such that $\xi(G^n(x)) > M_N$. Without loss of generality, we may assume that n is minimal with respect to this property, i.e., $\xi(G^j(x)) \le M_N$ for all $j < n$. Now, since I does not contain any numbers between M_N and m_{N+1}, it follows that $\xi(G^n(x)) \ge m_{N+1}$, and thus

$$\begin{aligned}\eta(G^n(x)) &\ge \log(1 + m_{N+1}) = N4^N \log(1 + M_N) \\ &\ge N \sum_{j=0}^{n-1} \log(1 + \xi(G^j(x))) = N \sum_{j=0}^{n-1} \eta(G^j(x)).\end{aligned}$$

This proves that (4.5) has infinitely many solutions. The proof of Theorem 4.5 is complete. □

5 Combinatorial characterizations of Ahlfors regularity

In this section, we prove Theorem 5.5, which gives a combinatorial characterization of Ahlfors regularity of J_I. We begin by recalling the following theorems.

Theorem 5.1 ([10, Theorem 4.1])**.** *Let $I \subseteq \mathbb{N}$, and suppose that the IFS $\mathcal{S}_I$ is regular. Let $h = \mathrm{HD}(J_I)$ and m_I be an h-conformal measure. Then the following are equivalent:*
(a) $\mathcal{H}^h(J_I) > 0$*;*
(b)

$$\sup_{k_1 < k_2} \frac{(k_1 k_2)^h}{(k_2 - k_1)^h} \sum_{\substack{i \in I \\ k_1 \le i \le k_2}} i^{-2h} < \infty; \tag{5.1}$$

(c) *m_I is Ahlfors h-lower regular, i.e.,*

$$m_I(B(x,r)) \lesssim r^h \text{ for all } x \in J_I \text{ and all } r \le 1.$$

Proof. The equivalence of (a) and (b) is proved in [10, Theorem 4.1]. The implication (a)⇒(c) follows from the last line of the proof of the implication (c)⇒(a) of [11, Theorem 4.5.3] (just before the mass distribution principle is applied). Finally, the implication (c)⇒(a) is the mass distribution principle; see [2, p. 55]. □

Theorem 5.2 ([10, Theorem 5.1])**.** *Let $I \subseteq \mathbb{N}$ be infinite, and suppose that the IFS $\mathcal{S}_I$ is regular. Let $h = \mathrm{HD}(J_I)$, and let m_I be an h-conformal measure. Then the following are equivalent:*
(a) $\mathcal{P}^h(J_I) < \infty$*;*
(b)

$$\inf_{\substack{k_1 < k_2 \\ B\left(\frac{2k_1k_2}{k_1+k_2}, 1\right) \cap I \neq \varnothing}} \frac{(k_1 k_2)^h}{(k_2 - k_1)^h} \sum_{\substack{i \in I \\ k_1 \le i \le k_2}} i^{-2h} > 0 \quad \text{and} \quad \inf_{k \ge 1} k^h \sum_{\substack{i \in I \\ i \ge k}} i^{-2h} > 0.$$

(c) *m_I is Ahlfors h-upper regular, i.e.,*

$$m_I(B(x,r)) \gtrsim r^h \text{ for all } x \in J_I \text{ and } r \le 1.$$

Note that the assumption that I is infinite is necessary; indeed, every finite IFS satisfies (a) and (c) but not (b).

Proof. The equivalence of (a) and (b) is proved in [10, Theorem 5.1]. The implication (a)⇒(c) follows from the last line of the proof of the implication (c)⇒(a) of [11, Theorem 4.5.5] (just before the mass distribution principle for packing measure is applied). Finally, the implication (c)⇒(a) is the mass distribution principle for packing measure. □

We can add new equivalences to Theorems 5.1 and 5.2.

Theorem 5.3. (a)–(c) *of Theorem 5.1 are equivalent to the following condition:*

(d) (i) *for all* $y \in \mathbb{N}$ *and for all* $1 \leq r \leq y/2$, $\#(B(y,r) \cap I) \lesssim r^h$ *and*
(ii) *for all* $k \in \mathbb{N}$, $\sum_{\substack{i \in I \\ i > k}} i^{-2h} \lesssim k^{-h}$.

Theorem 5.4. (a)–(c) *of Theorem 5.2 are equivalent to the following condition:*

(d) (i) *for all* $y \in I$ *and for all* $1 \leq r \leq y/2$, $\#(B(y,r) \cap I) \gtrsim r^h$, *and*
(ii) *for all* $k \in \mathbb{N}$, $\sum_{\substack{i \in I \\ i > k}} i^{-2h} \gtrsim k^{-h}$.

Proof of Theorems 5.3 and 5.4. By way of illustration, we show that Theorem 5.3(d) implies Theorem 5.1(b). The proof of the other implications are left to the reader.

Fix $k_1 < k_2$. If $I \cap [k_1, k_2] = \varnothing$, then the pair (k_1, k_2) does not contribute to the supremum (5.1). Thus, suppose that $I \cap [k_1, k_2] \neq \varnothing$ and fix $y \in I \cap [k_1, k_2]$. Let $r = \max(k_2 - y, y - k_1)$. If $r \leq y/2$, then

$$\frac{2}{3}k_2 \leq y \leq 2k_1,$$
$$r \leq k_2 - k_1 \leq 2r,$$

and thus by Theorem 5.3(d)(i),

$$\begin{aligned}\frac{(k_1k_2)^h}{(k_2-k_1)^h} \sum_{\substack{i \in I \\ k_1 \leq i \leq k_2}} i^{-2h} &\asymp \frac{(y^2)^h}{r^h} \sum_{\substack{i \in I \\ k_1 \leq i \leq k_2}} y^{-2h} \\ &= r^{-h}\#(I \cap [k_1, k_2]) \\ &\leq r^{-h}\#(I \cap B(y,r)) \lesssim r^{-h}r^h = 1.\end{aligned}$$

On the other hand, suppose that $r \geq y/2$. Then

$$k_2 - k_1 \geq r \geq \frac{k_1}{2},$$
$$k_2 \geq \frac{3}{2}k_1,$$
$$k_2 - k_1 \geq \frac{k_2}{3},$$

and thus by Theorem 5.3(d)(ii),

$$\frac{(k_1k_2)^h}{(k_2-k_1)^h}\sum_{\substack{i\in I\\ k_1\leq i\leq k_2}} i^{-2h} \leq \frac{(k_1k_2)^h}{(k_2/3)^h}\sum_{\substack{i\in I\\ i\geq k_1}} i^{-2h}$$
$$\asymp k_1^h \sum_{\substack{i\in I\\ i\geq k_1}} i^{-2h} \lesssim k_1^h k_1^{-h} = 1$$

Thus, either way, we have

$$\frac{(k_1k_2)^h}{(k_2-k_1)^h}\sum_{\substack{i\in I\\ k_1\leq i\leq k_2}} i^{-2h} \lesssim 1,$$

which is equivalent to (5.1). □

If we restrict our attention to sets I which satisfy both the conditions of Theorem 5.1 and those of Theorem 5.2, we get even more characterizations.

Theorem 5.5. *Let $I \subseteq \mathbb{N}$ be infinite and $h = h_I = \mathrm{HD}(J_I)$. Then* (a)–(c3) *below are equivalent and imply* (d)–(e)*:*

(a) $\mathcal{H}^h(J_I) > 0$ *and* $\mathcal{P}^h(J_I) < \infty$*;*

(b1) $\mathcal{H}^h \upharpoonright_{J_I}$ *is Ahlfors h-regular;*

(b2) $\mathcal{P}^h \upharpoonright_{J_I}$ *is Ahlfors h-regular;*

(b3) *the IFS $\mathcal{S}_I$ is regular, and m_I is Ahlfors h-regular;*

(b4) *the IFS $\mathcal{S}_I$ is regular, and μ_I is Ahlfors h-regular;*

(c1) *for all $y \in I$ and $r \geq 1$,*

$$\#(B(y,r)\cap I) \asymp r^h; \tag{5.2}$$

(c2) (i) (5.2)) *holds for all $y \in I$ and $1 \leq r \leq y/2$, and*
(ii) *there exists $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $[k, mk]\cap I \neq \varnothing$;*

(c3) (i) (5.2) *holds for all $y \in I$ and $1 \leq r \leq y/2$, and*
(ii) *for all $k \in \mathbb{N}$,*

$$\sum_{\substack{i\in I\\ i>k}} i^{-2h} \asymp k^{-h}; \tag{5.3}$$

(d) $\theta_I = h/2$*;*

(e) *the IFS $\mathcal{S}_I$ is cofinitely regular.*

Proof. Let us first assume that $\mathcal{S}_I$ is regular. Then the equivalence of (a), (b3), and (c3) follows directly from Theorems 5.3 and 5.4. The equivalence of (b3) and (b4) follows from Proposition 3.7. To see that (b1) and (b3) are equivalent, note that by Proposition 3.8(a), if the equivalence fails, then $\mathcal{H}^h(J_I) = 0$. But in this case, clearly (b1) and (a) are both false, so since (a) is equivalent to (b3), it follows that (b1)⇔(b3). A similar argument yields the equivalence of (b2) and (b3).

We next show that (c1)⇔(c2)⇔(c3)⇒(d), (e). In these proofs we do not assume regularity of $\mathcal{S}_I$.

Proof of (c3)⇒(c2). Suppose that (c3) holds. Let C be the implied constant of (5.3), and let $m = \lceil C^{2/h} \rceil + 1$, where $\lceil \cdot \rceil$ denotes the ceiling function. Then for any $k \in \mathbb{N}$,

$$\sum_{\substack{i \in I \\ i > mk}} i^{-2h} \leq C(mk)^{-h} < C^{-1}k^{-h} \leq \sum_{\substack{i \in I \\ i > k}} i^{-2h},$$

which demonstrates that $[k, mk] \cap I \neq \varnothing$.

Proof of (c2)⇒(c1), (c3), (d), (e). Suppose that (c2) holds. We claim that

$$\#(I \cap [k, 3mk]) \asymp k^h \tag{5.4}$$

for all $k \in \mathbb{N}$. Indeed, the upper bound can be achieved by covering $I \cap [k, 3mk]$ by finitely many sets of the form $B(y, y/2)$, where $y \in I \cap [k, 3mk]$, and applying (5.2). The lower bound follows from choosing a point $y \in I \cap [2k, 2mk]$ and applying (5.2) to the set $B(y, y/2)$.

From (5.4), we calculate that for all $t \geq 0$ and $k \in \mathbb{N}$,

$$\sum_{\substack{i \in I \\ i > k}} i^{-2t} \asymp \sum_{n \in \mathbb{N}} \sum_{\substack{i \in I \\ (3m)^n k < i \leq (3m)^{n+1} k}} i^{-2t} \asymp \sum_{n \in \mathbb{N}} [(3m)^n k]^h [(3m)^n k]^{-2t},$$

which diverges if $t \leq h/2$ and is otherwise asymptotic to $k^{h-2t} < \infty$. Specializing to the case $t = h$ yields (c3). Applying Proposition 3.9 yields (d). Finally, Proposition 3.7 yields (e).

To prove (c1), fix $y \in I$ and $r \geq 1$. If $r \leq y/2$, then we obtain (5.2) for free. Thus, suppose $r > y/2$. Let $N = \lceil \log_{3m}(y + r) \rceil$. Then

$$B(y, r) \subseteq \bigcup_{n=0}^{N} \left[(3m)^n, (3m)^{n+1}\right].$$

On the other hand, for each $n \leq N$ we have, from (5.4),

$$\#\left(I \cap \left[(3m)^n, (3m)^{n+1}\right]\right) \asymp [(3m)^n]^h,$$

and summing yields

$$\#(B(y, r) \cap I) \lesssim [(3m)^N]^h \asymp r^h.$$

To get the lower bound, note that

$$\#(B(y, r) \cap I) \geq \#(B(y, y/2) \cap I) \asymp (y/2)^h \asymp y^h.$$

This bound is good enough unless $r \geq y$. In case $r \geq y$, let $k = \lfloor r/(3m) \rfloor$; then (5.4) yields the bound.

Proof of (c1)⇒(c2). The proof is similar to the proof of (c3)⇒(c2).

This completes the proof of the theorem in the case where $\mathcal{S}_I$ is regular.

Suppose now that $\mathcal{S}_I$ is not regular. Then (b3) and (b4) are clearly false. Applying parts (a) and (b) of Proposition 3.8 yields that (a), (b1), and (b2) are false. Applying part (a) of Proposition 3.11 yields that (d) is false. Since (c1), (c2), and (c3) are equivalent and imply (d), and the proof of this does not depend on the regularity of $\mathcal{S}_I$, we see that (c1)-(c3) are also false. This completes the proof of the theorem. □

6 Proof of Theorem 6.1

In this section, we prove the following theorem.

Theorem 6.1. *Let $I \subseteq \mathbb{N}$ be infinite and h be the Hausdorff dimension of J_I. Assume that the h-dimensional Hausdorff measure $\mathcal{H}^h$ restricted to J_I is Ahlfors h-regular. Let $\mu = \mathcal{H}^h \upharpoonright_{J_I}$, and let $\psi : \mathbb{N} \to (0, +\infty)$ be such that the function $q \mapsto q^2\psi(q)$ is nonincreasing. Then μ-almost every (respectively, μ-almost no) point is ψ-approximable, assuming that the series*

$$\sum_{q=1}^{\infty} q^{2\alpha-1}\psi(q)^\alpha \tag{6.1}$$

diverges (respectively, converges).

Proof. As noted in the Introduction, the convergence case follows from Weiss's theorem [13].

Fix a function $\psi : \mathbb{N} \to (0, \infty)$ such that that the series (6.1) diverges and that the function $q \mapsto q^2\psi(q)$ is nonincreasing. By Proposition 3.8(a), we have $m_I \asymp \mathcal{H}^h \upharpoonright_{J_I}$; so to prove the theorem, it suffices to show that m_I-almost every number is ψ-approximable. In fact, we prove the (slightly) stronger statement that m_I-almost every number is ψ-well approximable.

By (b1)⇒(e) of Theorem 5.5, the iterated function system $\mathcal{S}_I = \{g_a\}_{a\in I}$ is cofinitely regular. Thus, the Lyapunov exponent of μ_I is finite (Proposition 3.13 and Proposition 3.11(a)); in particular, $0 < \int \eta \mathrm{d}\mu_I < \infty$ (see (2.9)), where η is defined as in Definition (2.5). Hence, by the Birkhoff Ergodic Theorem,

$$\frac{1}{n}\sum_{j=0}^{n-1} \eta(G^j(x)) \underset{n}{\to} E := \int \eta \mathrm{d}\mu_I$$

for μ_I-almost every $x \in [0, 1]$. Combining the above equation with (2.8) gives

$$\frac{E}{2} \leq \liminf_{n\to\infty} \frac{1}{n} \log(q_n) \leq \limsup_{n\to\infty} \frac{1}{n} \log(q_n) \leq E. \tag{6.2}$$

Let $x \in [0, 1]$ be a point at which (6.2) holds but which is not ψ-well approximable. By Lemma 2.4, there exists $K > 0$ such that for all $n \in \mathbb{N}$, (2.1) fails to hold. Combining (6.2), the negation of (2.1), and the fact that $\phi(q) = 1/q^2\psi(q)$ is nondecreasing yields

$$\omega_n = \xi(G^n(x)) \leq K\phi(\gamma^n) \tag{6.3}$$

for all n sufficiently large, where $\gamma = 1 + \lceil e^E \rceil$. By increasing K, we can ensure that (6.3) holds for all $n \in \mathbb{N}$.

Thus, we are done once we show that the set of x for which there exists K such that (6.3) holds for all $n \in \mathbb{N}$ is a null set. Given $n \in \mathbb{N}$ and $K > 0$, let

$$S_{\psi,n,K} = \{x \in J_I : \text{(6.3) holds for } n, K\}$$

and $S^+_{\psi,n,K} = \bigcap_{j=0}^{n-1} S_{\psi,j,K}$. To complete the proof of Theorem 6.1, we now show that

$$m_I\left(S^+_{\psi,\infty,K}\right) = 0 \text{ for all } K > 0. \tag{6.4}$$

Fix $K > 0$. For each $n \in \mathbb{N}$, let $k_n = K\phi(\gamma^n)$. In the notation of Lemma 3.16, we have

$$S_{\psi,n,K} = \bigcup_{\omega\in A_n} S_\omega, \quad S_{\psi,n+1,K} = \bigcup_{\omega\in A_n} S^+_{\omega,k_n},$$

where $A_n = \prod_{j=0}^{n-1}\{1, \ldots, k_j\}$.

It therefore follows from (3.3) that

$$\frac{m_I(S^+_{\psi,n+1})}{m_I(S^+_{\psi,n})} \leq 1 - \frac{1}{4^{h_I}} \sum_{\substack{i\in I\\ i>k_n}} i^{-2h_I}.$$

On the other hand, by the implication (b1)⇒(c3) of Theorem 5.5, we have

$$\sum_{\substack{i\in I\\ i>k_n}} i^{-2h} \asymp k_n^{-h} \asymp \phi(\gamma^n)^{-h}. \tag{6.5}$$

Thus, for some constant $K_2 > 0$ depending on K,

$$\frac{m_I(S^+_{\psi,n+1,K})}{m_I(S^+_{\psi,n,K})} \leq 1 - K_2\phi(\gamma^n)^{-h};$$

and hence

$$m_I(S^+_{\psi,\infty,K}) \leq \prod_{n=0}^{\infty} \left(1 - K_2 \phi(\gamma^n)^{-h}\right),$$

which vanishes if the series

$$\sum_{n=0}^{\infty} \phi(\gamma^n)^{-h} \tag{6.6}$$

diverges. Now, by Cauchy's condensation test, (6.6) diverges if and only if (6.1) diverges. This establishes (6.4), completing the proof of Theorem 6.1. □

7 Proof of Theorem 7.1

In this section, we prove the following theorem.

Theorem 7.1. *For every* $0 < \delta \leq 1$*, there exists an infinite set* $I \subseteq \mathbb{N}$ *such that* $\mathrm{HD}(J_I) = \delta$ *and* $\mathcal{H}^\delta \upharpoonright_{J_I}$ *is Ahlfors* δ*-regular.*

Fix $0 < \delta \leq 1$. If $\delta = 1$, we let $I = \mathbb{N}$; the conclusion of the theorem then holds since $\mathcal{H}^\delta \upharpoonright_{J_I}$ is simply Lebesgue measure. Thus, we may assume that $\delta < 1$. By the implication (c1)⇒(b1) of Theorem 5.5, to prove Theorem 7.1 it suffices to find a set I satisfying

$$\mathrm{HD}(J_I) = \delta \tag{7.1}$$

and

$$\#(B(y,r) \cap I) \asymp r^\delta. \tag{7.2}$$

We begin by finding a set I_0 which satisfies (7.2) but not (7.1). We then construct a set R which satisfies (7.1) but not (7.2). Finally, we combine I_0 and R into a single set I_δ which satisfies both (7.1) and (7.2).

7.1 Constructing I_0.

Lemma 7.2. *There exists a set* $I_0 \subseteq \mathbb{N}$ *satisfying* (7.2).

Proof. Let I_0 be the set of all sums of the form $1 + \sum_{n\in\mathbb{N}} a_n \lfloor 2^{n/\delta} \rfloor$, where $a_n = 0$ or 1 for all $n \in \mathbb{N}$, with only finitely many 1s. It is readily verified that I_0 satisfies (7.2). □

7.2 Constructing R. We define a sequence of subsets $R_N \subseteq \mathbb{N}$ by induction on N.

1. Let $R_1 = \{1\}$.
2. Suppose that $R_{N-1} \subseteq \{1, \ldots, N-1\}$ has been defined for some $N \geq 2$. Define

$$R_n = \begin{cases} R_{N-1} \cup \{N\} & \text{if } \lambda_\delta(R_{N-1} \cup \{N\}) < 1, \\ R_{N-1} & \text{otherwise.} \end{cases}$$

Observation 7.3. For all $N \in \mathbb{N}$, $\lambda_\delta(R_N) < 1$.

Proof. The base case follows either from direct computation or from Bowen's formula (Theorem 3.1); the inductive step follows from the construction of R_N. □

Claim 7.4. *$R := \bigcup_N R_N$ is not cofinite.*

Proof. By Theorem 3.2 and by the previous observation, $\lambda_\delta(R) \leq 1$. Combining with Bowen's formula, we see that $\mathrm{HD}(J_R) \leq \delta < 1 = \mathrm{HD}(J_\mathbb{N})$. In particular, $R \neq \mathbb{N}$.

Thus if we suppose, by way of contradiction, that R is cofinite, then $\mathbb{N} \setminus R$ has a maximal element M. Moreover, $M \geq 2$, since $1 \in R$. But then, by the construction of R_M, $\lambda_\delta(R_{M-1} \cup \{M\}) \geq 1$; and so, by Lemma 3.14,

$$\lambda_\delta(R_{M-1}) \geq 1 - \left(\frac{2}{2+M}\right)^{2\delta}. \tag{7.3}$$

On the other hand, by Observation 7.3,

$$\lambda_\delta(R_{M-1} \cup \{M+1, \ldots, N\}) = \lambda_\delta(R_N) < 1$$

for every $N \in \mathbb{N}$. So, applying Lemma 3.14, we see that

$$\begin{aligned} \left(\frac{2}{2+M}\right)^{2\delta} &> \lambda_\delta(R_{M-1} \cup \{M+1, \ldots, N\}) - \lambda_\delta(R_{M-1}) \\ &\geq \sum_{i=M+1}^{N} \left(\frac{1}{1+i}\right)^{2\delta} > \int_{x=M+1}^{N+1} \left(\frac{1}{1+x}\right)^{2\delta} \mathrm{d}x. \end{aligned}$$

Since N is arbitrary, we can take the limit as N approaches ∞, obtaining

$$\int_{x=M+1}^{\infty} \left(\frac{1}{1+x}\right)^{2\delta} \mathrm{d}x < \left(\frac{2}{2+M}\right)^{2\delta}.$$

If $\delta \leq 1/2$, the left hand integral diverges, a contradiction. If $\delta > 1/2$, the left hand integral converges, and

$$\frac{(M+2)^{1-2\delta}}{2\delta - 1} < \left(\frac{2}{2+M}\right)^{2\delta}.$$

Rearranging yields

$$M + 2 < 2^{2\delta}(2\delta - 1) \le 2^2(2 - 1) = 4,$$

which contradicts $M \ge 2$. □

Observation 7.5. It follows from (7.3), Observation 7.3, and Theorem 3.2 that $\mathrm{HD}(J_R) = \delta$.

7.3 Combining I_0 and R. Fix $N_1 \in \mathbb{N} \setminus R$ large; how large is to be determined later.[6] By the construction of R_{N_1}, we have (7.3) with $M = N_1$, and so

$$1 - \left(\frac{2}{2 + N_1}\right)^{2\delta} \le \lambda_\delta(R_{N_1-1}) < 1.$$

Now let I_0 be as in Lemma 7.2, and let $I_+ := 2I_0$ and $I_- := 2I_0 - 1$. It is evident that every set $I_\delta \subseteq \mathbb{N}$ satisfying

$$I_- \subseteq_* I_\delta \subseteq_* I_+ \cup I_- \tag{7.4}$$

satisfies (7.2), where $A \subseteq_* B$ means $\#(A \setminus B) < \infty$. We construct such a set recursively. By the implication (c1)⇒(c3) of Theorem 5.5[7], we have

$$\sum_{i \in I_-} \left(\frac{2}{2 + i}\right)^{2\delta} \asymp \sum_{i \in I_-} i^{-2\delta} < \infty;$$

thus we may choose N_2 so large that

$$\sum_{i \in I_- \setminus \{1,\ldots,N_2\}} \left(\frac{2}{2 + i}\right)^{2\delta} < 1 - \lambda_\delta(R_{N_1-1}). \tag{7.5}$$

We now construct a sequence of sets $(I_N)_{N \ge N_1-1}$ recursively in the following manner.

1. Let $I_{N_1-1} = R_{N_1-1} \cup (I_- \setminus \{1, \ldots, N_2\})$.
2. Suppose that I_{N-1} has been defined for some $N \ge N_1$. If $N \notin I_+ \cup \{N_1\}$, let $I_N = I_{N-1}$.
3. If $N \in I_+ \cup \{N_1\}$, and $\lambda_\delta(I_{N-1} \cup \{N\}) < 1$, let $I_N = I_{N-1} \cup \{N\}$.
4. Otherwise, let $I_N = I_{N-1}$.

Observation 7.6. For all $N \ge N_1 - 1$, $\lambda_\delta(I_N) < 1$.

[6]Specifically, N_1 should so large that (7.7) cannot hold whenever $M \ge N_1$.
[7]Note that the implication holds even when $h \ne \mathrm{HD}(J_I)$.

Proof. The base case of induction follows from Lemma 3.14 together with (7.7). The induction step follows from the construction of I_N. □

Claim 7.7. *Case 4 occurs infinitely many times.*

Proof. As $N_1 \notin R$, Case 4 occurs at least once, namely at $N = N_1$. Suppose, by contradiction, that Case 4 occurs only finitely often. Then there is some maximal value M at which it occurs. In particular $\lambda_\delta(I_{M-1} \cup \{M\}) \geq 1$. Applying Lemma 3.14 gives

$$\lambda_\delta(I_{M-1}) \geq 1 - \left(\frac{2}{2+M}\right)^{2\delta}. \tag{7.6}$$

On the other hand, by the above observation and by the maximality of M,

$$\lambda_\delta(I_{M-1} \cup (I_+ \cap \{M+1, \ldots, N\})) < 1$$

for all $N \in \mathbb{N}$. Combining these last two formulas and then applying Lemma 3.14, we see that

$$\begin{aligned}\left(\frac{2}{2+M}\right)^{2\delta} &> \lambda_\delta(I_{M-1} \cup (I_+ \cap \{M+1, \ldots, N\})) - \lambda_\delta(I_{M-1}) \\ &\geq \sum_{\substack{i=M+1 \\ i \in I_+}}^{N} \left(\frac{1}{1+i}\right)^{2\delta}.\end{aligned}$$

Since N is arbitrary, we can take the limit as N approaches ∞, and this yields

$$M^{-2\delta} \asymp \left(\frac{2}{2+M}\right)^{2\delta} > \sum_{\substack{i=M+1 \\ i \in I_+}}^{\infty} \left(\frac{1}{1+i}\right)^{2\delta} \asymp M^{-\delta}. \tag{7.7}$$

Since $\delta > 0$, this is a contradiction for sufficiently large M. Thus, the proof is completed by letting N_1 be large enough so that (7.7) cannot hold for $M \geq N_1$. □

Now let $I = I_\delta = \bigcup_{N \geq N_1 - 1} I_N$. As mentioned earlier, it is clear that I satisfies (7.2) since it satisfies (7.4). Thus to complete the proof of Theorem 7.1, it suffices to prove (7.1). To this end, let $(M_k)_k$ be an increasing sequence of points at which Case 4 occurs. For each $k \in \mathbb{N}$, (7.6) holds with $M = M_k$, i.e.,

$$1 - \left(\frac{2}{2+M_k}\right)^{2\delta} \leq \lambda_\delta(I_{M_k - 1}) < 1.$$

Taking the limit as k approaches ∞, we see that $\lambda_\delta(I) = 1$. Thus by Bowen's formula (Theorem 3.1), $\mathrm{HD}(J_I) = \delta$. This completes the proof of Theorem 7.1.

Acknowledgments. The authors thank D. Y. Kleinbock and B. Weiss for reading the manuscript and making helpful suggestions.

References

[1] Y. Bugeaud, *Approximation by Algebraic Numbers*, Cambridge University Press, Cambridge, 2004.

[2] K. J. Falconer, *Fractal Geometry. Mathematical Fundations and Applications*, John Wiley & Sons, Ltd., Chichester, 1990.

[3] I. J. Good, *The fractional dimensional theory of continued fractions*, Proc. Cambridge Philos. Soc. **37** (1941), 199–228.

[4] D. Hensley, *Continued Fractions*, World Scientific, Hackensack, NJ, 2006.

[5] M. Kesseböhmer and S. Zhu, *Dimension sets for infinite IFSs: the Texan conjecture*, J. Number Theory **116** (2006), 230–246.

[6] A. Y. Khinchine, *Continued Fractions*, The University of Chicago Press, Chicago, Ill.-London, 1964

[7] D. Y. Kleinbock, *Metric Diophantine approximation and dynamical systems*, unpublished lecture notes, http://people.brandeis.edu/˜kleinboc/203b/lectures2010.pdf.

[8] D. Y. Kleinbock, E. Lindenstrauss, and B. Weiss, *On fractal measures and Diophantine approximation*, Selecta Math. **10** (2004), 479–523.

[9] D. Y. Kleinbock and G. A. Margulis, *Flows on homogeneous spaces and Diophantine approximation on manifolds*, Ann. of Math. (2) **148** (1998), 339–360.

[10] R. D. Mauldin and M. Urbański, *Conformal iterated function systems with applications to the geometry of continued fractions*, Trans. Amer. Math. Soc. **351** (1999), 4995–5025.

[11] R. D. Mauldin and M. Urbański, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*, Cambridge University Press, Cambridge, 2003.

[12] M. Urbański, *Diophantine approximation for conformal measures of one-dimensional iterated function systems*, Compos. Math. **141** (2005), 869–886.

[13] B. Weiss, *Almost no points on a Cantor set are very well approximable*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **457** (2001), 949–952.

Lior Fishman
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NORTH TEXAS
1155 UNION CIRCLE #311430
DENTON, TX 76203-5017, USA
email: Lior.Fishman@unt.edu

David Simmons
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NORTH TEXAS
1155 UNION CIRCLE #311430
DENTON, TX 76203-5017, USA
email: DavidSimmons@my.unt.edu

Mariusz Urbański
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NORTH TEXAS
1155 UNION CIRCLE #311430
DENTON, TX 76203-5017, USA
email: urbanski@unt.edu

(Received June 28, 2012 and in revised form January 14, 2013)