POINTWISE ESTIMATES FOR THE FUNDAMENTAL SOLUTIONS OF A CLASS OF SINGULAR PARABOLIC PROBLEMS

By

F. RAGNEDDA, S. VERNIER PIRO, AND V. VESPRI

Abstract. We deal with the Cauchy problem associated to a class of quasilinear singular parabolic equations with L^∞ coefficients whose prototypes are the p-Laplacian ($2N/(N+1) < p < 2$) and the porous medium equation ($((N-2)/N)_+ < m < 1$). We prove existence of and sharp pointwise estimates from above and from below for the fundamental solutions. Our results can be extended to general non-negative L^1 initial data.

1 Introduction

Let us consider the homogeneous quasilinear parabolic equation

$$u_t = \operatorname{div} A(x, t, u, Du), \quad (x, t) \in \mathbb{R}^N \times [0, +\infty), \tag{1.1}$$

where the functions $A := (A_1, \ldots, A_N)$ are assumed only to be measurable in $(x, t) \in \mathbb{R}^N \times [0, +\infty)$ and continuous with respect to u and Du for almost all (x, t). For the p-Laplacian type equation, we suppose that A satisfies the structure conditions

$$\begin{aligned} A(x, t, u, \eta) \cdot \eta &\geq c_0 |\eta|^p, \\ |A(x, t, u, \eta)| &\leq c_1 |\eta|^{p-1} \end{aligned} \tag{1.2}$$

for almost all $(x, t) \in \mathbb{R}^N \times [0, +\infty)$ and $(u, \eta) \in \mathbb{R} \times \mathbb{R}^N$, where

$$\frac{2N}{N+1} < p < 2 \tag{1.3}$$

(supercritical range of the fast diffusion case) and c_0, c_1 are given positive constants. Moreover, we assume that there exists $L > 0$ such that

$$\begin{aligned} &(A(x, t, u, \eta_1) - A(x, t, u, \eta_2)) \cdot (\eta_1 - \eta_2) \geq 0, \\ &\qquad |A(x, t, u_1, \eta) - A(x, t, u_2, \eta)| \leq L\, |u_1 - u_2|(1 + |\eta|^{p-1}), \end{aligned} \tag{1.4}$$

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for almost all $(x, t) \in \mathbb{R}^N \times [0, +\infty)$ and all $u, u_i \in \mathbb{R}$ and $\eta, \eta_i \in \mathbb{R}^N$, $i = 1, 2$.

For the porous medium type equation, we follow the notation of [10, Chapter 7, Section 5]. Let

$$u_t = \operatorname{div} A(x, t, u, Du^m), \quad (x, t) \in \mathbb{R}^N \times [0, +\infty), \tag{1.5}$$

where A satisfies the conditions

$$\begin{aligned} A(x, t, u, \eta) \cdot \eta &\geq c_0'|\eta|^2, \\ |A(x, t, u, \eta)| &\leq c_1'|\eta|, \end{aligned} \tag{1.6}$$

for almost all $(x, t) \in \mathbb{R}^N \times [0, +\infty)$ and $(u, \eta) \in \mathbb{R} \times \mathbb{R}^N$ with

$$\Big(\frac{N-2}{N}\Big)_+ < m < 1. \tag{1.7}$$

We assume the monotonicity and Lipschitz conditions

$$\begin{aligned} &A(x, t, u, \eta_1) - A(x, t, u, \eta_2) \cdot (\eta_1 - \eta_2) \geq 0, \\ &|A(x, t, u_1, \eta) - A(x, t, u_2, \eta)| \leq L' \Big| |u_1|^{m-1} u_1 - |u_2|^{m-1} u_2 \Big| (1 + |\eta|). \end{aligned} \tag{1.8}$$

These conditions are sufficient for a comparison principle and to preserve the positivity of solutions. We remark that, in fact, hypotheses (1.4) and (1.8) not only imply a comparison principle for weak solution of (1.1), but also guarantee the existence of the solution; see, e.g., [22].

The aim of this paper is to estimate from above and from below solutions of (1.1) with initial datum the Dirac mass in $\mathbb{R}^N$. Our results can be extended to general non-negative L^1 initial data.

Estimates for the p-Laplacian and porous medium equations (for both the slow diffusion and fast diffusion case) have been considered by several authors; see, e.g., [30], [31], [18] and the references therein and also [1], [4], [15], [17], [24], [25]. In the literature, results similar to ours are proved only for the prototype equations using comparison functions and entropy methods. Since we seek estimates for more general operators, we are forced to use a completely different and more sophisticated approach, based on DiBenedetto's techniques, recent Harnack inequalities, and De Giorgi estimates. One difficulty we face is that whereas the equation gives local estimates of solutions, the global behaviour of a solution differs significantly from its local behavior.

Fundamental solutions (i.e., solutions with initial datum a Dirac mass) of the prototype equations are known explicitly. First, let us consider the p-Laplacian. It is known (see, e.g., [30], [31]) that there exists $C_p > 0$ such that

$$\mathcal{B}_p = t^{-N/\lambda} \Big[C_p + \gamma_p \Big(\frac{|x|}{t^{1/\lambda}} \Big)^{p/p-1} \Big]^{-(p-1)/(2-p)} \tag{1.9}$$

with

$$\gamma_p = \Big(\frac{1}{\lambda}\Big)^{1/p-1}\Big(\frac{2-p}{p}\Big) \quad \text{and} \quad \lambda = N(p-2)+p \tag{1.10}$$

is the solution of the Cauchy problem in $\mathbb{R}^N \times (t > 0)$

$$u_t = \operatorname{div}(|Du|^{p-2}Du), \quad u(x,0) = \delta(0).$$

Analogously, it is known (see, for instance, [30], [31]) that for porous medium equations, there exists $C_m > 0$ such that

$$\mathcal{B}_m = t^{-N/\kappa}\Big[C_m + \gamma_m\Big(\frac{|x|}{t^{1/\kappa}}\Big)^2\Big]^{-1/(1-m)} \tag{1.11}$$

with

$$\gamma_m = \Big(\frac{1}{\kappa}\Big)\frac{1-m}{2} \quad \text{and} \quad \kappa = N(m-1)+2$$

is the solution of the Cauchy problem in $\mathbb{R}^N \times (t > 0)$

$$u_t = \Delta(u^m), \quad u(x,0) = \delta(0).$$

These solutions are positive for all time t and their decay depends on the dimension N, whereas, in the case of a bounded domain, solutions extinguish in finite time and their decay does not depend on N; see, for instance, [12] and [30]. Hence, to capture the global behaviour, we are compelled to "stretch" the local estimates.

Finally, we mention that pointwise estimates for general operators have been considered only recently. T. Kuusi and G. Mingione ([19] and [20]) gave estimates based on the Wolff potential for equations of the type

$$u_t = \operatorname{div} A(x,t,u,Du) + \mu,$$

where A is a p-Laplacian-type operator with regular coefficients and μ is a Radon measure. Their powerful method seems applicable only to p-Laplacian-type operators with quite regular coefficients.

Recall that a locally bounded non-negative function $u(x,t)$ is said to be a **weak solution** of (1.1) in $\mathbb{R}^N \times \mathbb{R}^+$ if $u \in C(\mathbb{R}^+; L^2(\mathbb{R}^N)) \cap L^p(\mathbb{R}^+; W^{1,p}(\mathbb{R}^N))$, and for every subinterval $[t_1, t_2] \subset \mathbb{R}^+$,

$$\int_{\mathbb{R}^N} u\,\phi\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathbb{R}^N}(-u\phi_t + A(x,t,u,Du)\cdot D\phi)dxdt = 0$$

for all test functions $\phi \in W^{1,2}(\mathbb{R}^+; L^2(\mathbb{R}^N)) \cap L^p(\mathbb{R}^+; W^{1,p}(\mathbb{R}^N))$. We use this definition because u_t may have limited regularity and, in general, has meaning as a solution only in the sense of distributions; see, e.g., [6] and [10].

Observe that the explicit fundamental solutions given above have less regularity than is required in the our definition of a solution. In general, when the initial datum is a measure, the gradient of the solution belongs only to the Marcinkiewicz space of order $N(p-1)/(N-1)$. However, the gradient raised to the power $(p-1)$ belongs to the Marcinkiewicz space of order $N/(N-1)$ and therefore to L^1. Hence, a distributional solution is well defined. For a more refined theory, see [2] and [3] for the definition of entropy solutions and [5] and [27] for the definition of renormalised solutions.

Following the approach of [18] and [26], we define the notion of fundamental solution of (1.1). A non-negative function $u(x, t)$ is defined to be a **fundamental solution** of (1.1) if

(i) $u \in C(\mathbb{R}^+; L^1(\mathbb{R}^N))$;
(ii) for all $s > 0$, $u(x, t)$ is a weak solution of (1.1) in $\mathbb{R}^N \times [s, +\infty)$;
(iii) for all $R > 0$, $\lim_{t\to 0} \int_{B_R} u(x, t)dx = 1$;
(iv) for all $R > 0$, $\lim_{t\to 0} \int_{\mathbb{R}^N \setminus B_R} u(x, t)dx = 0$.

Here, B_R denotes the euclidean ball in $\mathbb{R}^N$ of radius R centered at the origin.

We are now able to state our main theorem concerning p-Laplacian type equations.

Theorem 1.1 (p-Laplacian type)**.** *Let u be a fundamental solution of*

$$u_t = \operatorname{div} A(x, t, u, Du), \; (x, t) \in \mathbb{R}^N \times [0, +\infty), \quad u(x, 0) = \delta(0), \; x \in \mathbb{R}^N, \tag{1.12}$$

where A satisfies (1.2)*,* (1.4)*, and p in the supercritical range* (1.3)*. There exist positive constants $\underline{\gamma}$, $\overline{\gamma}$ depending only on N, p, c_0, c_1 such that for all $x \in \mathbb{R}^N$ and $t > 0$,*

$$\underline{\gamma}\mathcal{B}_p(x, t) \le u(x, t) \le \overline{\gamma}\mathcal{B}_p(x, t), \tag{1.13}$$

where $\mathcal{B}_p$, defined in (1.9)*, is the fundamental solution of the p-Laplacian equation.*

Analogously to p-Laplacian type equation, we can introduce the definitions of weak and fundamental solutions of porous medium type equations. For brevity, we omit these definitions and refer the reader to [10], [26] and [31] for all of the necessary details.

Theorem 1.2 (porous medium type)**.** *Let u be a fundamental solution of*

$$u_t = \operatorname{div} A(x, t, u, Du^m), \; (x, t) \in \mathbb{R}^N \times [0, +\infty), \quad u(x, 0) = \delta(0), \; x \in \mathbb{R}^N, \tag{1.14}$$

where A satisfies (1.6)*,* (1.8) *and m is in the supercritical range* (1.7)*. There exist positive constants $\underline{\gamma'}$, $\overline{\gamma'}$ depending only upon N, m, c_0', c_1' such that for all $x \in \mathbb{R}^N$*

and $t > 0$,

$$\underline{\gamma}'\mathcal{B}_m(x,t) \le u(x,t) \le \overline{\gamma}'\mathcal{B}_m(x,t), \tag{1.15}$$

where $\mathcal{B}_m$ *is the Barenblatt solution defined in* (1.11).

The existence of fundamental solutions for p-Laplacian type and porous medium type equations is guaranteed by the following result, whose proof is given in the Appendix.

Theorem 1.3 (Existence of fundamental solutions)**.** *There exists at least one non-negative fundamental solution of the fast diffusion equation* (1.1) *(respectively,* (1.5)*) with A satisfying* (1.2), (1.4) *(respectively,* (1.6), (1.8)*) and p in the supercritical range* (1.3) *(respectively, m in the supercritical range* (1.7)*).*

We stress that the main results of this paper are estimates (1.13) and (1.15). These results imply that the potentials of the equation with L^∞ coefficients behave exactly like the potentials for the prototype equations. This in turn implies that the cases of the p-Laplacian and the porous medium equations are surprisingly analogous to the non-degenerate case. It is known that in that case, the potential of general quasilinear equations behaves as the heat kernel. We find it quite surprising that the proof of such a result requires neither explicit solutions (nor supersolutions and barriers) nor any kind of functional associated to these equations. The estimates on the potential are purely structural and do not depend on any gradient flow of functionals.

The proofs in this paper are based on several powerful tools. The two key ingredients are L^1-L^1 estimates and some quite recent Harnack estimates. The L^1-L^1 estimates are not only a kind of integral Harnack estimate, but also give a sharp quantitative estimate on the speed of the propagation of the solution. The Harnack estimates are very surprising because for our singular equations, the diffusion is so strong that it wins with respect to the time evolution. Hence an elliptic-type Harnack inequality holds. We state all these results with all due detail in the next section. We prove only Theorem 1.1 and refer the reader to the recent monograph [10] to see how to extend results to porous medium type equations.

Note that below the critical values, i.e., for $1 < p \le 2N/(N+1)$ and $0 < m \le ((N-2)/N)_+$, there are no longer smoothing effects; for this reason, the Dirac mass cannot be chosen as initial datum. In the supercritical range, i.e., $2N/(N+1) < p < 2$ and $((N-2)/N)_+ < m < 1$, the initial datum needs to have finite mass; otherwise, the solution becomes $+\infty$ at every positive time. For these results, see [30].

The case of bounded domains and equations with variable coefficients was dealt with in [29]. There, because of a lack of known boundary Harnack estimates, results were obtained only for a smaller class of equations. The degenerate case, i.e., $p > 2$ and $m > 1$, was considered in [28] for bounded domains. The case of the whole space is more difficult because it requires sharp estimates on the free boundary and is still open in the sense that analogous estimates (1.13) and (1.15) have not yet been obtained.

Estimates of the type (1.13) and (1.15) might be useful in proving the asymptotic limit of the solutions. We will consider this problem in a forthcoming paper.

As for the uniqueness of the fundamental solution, we note that this issue was addressed for the p-Laplacian equation for $p > 2$ in [18], for the porous medium equation for $N = 1$ in [16], and for arbitrary N and in a more general setting in [26]. The extension of uniqueness results to our case seems to be not at all trivial, not least because uniqueness would seem to depend on the choice of the definition of a solution (renormalised, entropy, distributional). This will be the object of future investigation.

The paper is organised as follows. In Section 2, we collect some known results which are used in our proofs. In Section 3, estimates from above are derived for the solutions of problems (1.12) and (1.14). In Section 4, we derive interior estimates from below. In Section 5, using a recent Harnack inequality [9], we obtain estimates from below in the whole space. Combining the estimates of Sections 2– 5, we prove Theorem 1.1 and Theorem 1.2. In Section 6, we prove pointwise estimates on the derivatives, and in Section 7, we sketch proofs of some miscellaneous results (estimates in bounded domains, problems with more general initial conditions, and estimates from below and above for solutions of Fokker Planck equations). In Section A, we prove the existence result for the fundamental solutions.

Throughout this paper, γ_k is a positive constant that depends only on the data; i.e., for equation (1.1), it depends only on N, p, c_0, c_1, and for equation (1.5), it depends only on N, m, c'_0, c'_1.

2 Preliminaries

Let $B_\rho(x)$ denote the euclidean ball in $\mathbb{R}^N$ centered at x of radius ρ, and set $B_\rho(0) = B_\rho$.

We need the following known results.

Theorem 2.1 (Local L^1 form of the Harnack inequality [9]). *Let*

$$u \in C_{loc}(\mathbb{R}^+; L^2_{loc}(\mathbb{R}^N)) \cap L^p_{loc}(\mathbb{R}^+; W^{1,p}_{loc}(\mathbb{R}^N))$$

be a non-negative local weak solution of (1.1)–(1.2) *in* $\mathbb{R}^N \times [0,+\infty)$*, and* $1 < p < 2$. *There exists a constant* γ *depending only on the data such that for all cylinders* $B_{2\rho}(y) \times [s,t] \subset \mathbb{R}^N \times [0,+\infty)$*,*

$$\sup_{s\le\tau\le t}\int_{B_\rho(y)} u(x,\tau)dx \le \gamma \inf_{s\le\tau\le t}\int_{B_{2\rho}(y)} u(x,\tau)dx + \gamma\Big(\frac{t-s}{\rho^\lambda}\Big)^{1/(2-p)},$$

where $\lambda = N(p-2)+p$.

Theorem 2.2 (L^1-L^∞ estimates [9])**.** *Let*

$$u \in C_{loc}(\mathbb{R}^+; L^2_{loc}(\mathbb{R}^N)) \cap L^p_{loc}(\mathbb{R}^+; W^{1,p}_{loc}(\mathbb{R}^N))$$

be a non-negative local weak solution of (1.1)–(1.2) *in* $\mathbb{R}^N \times [0,+\infty)$*, and assume* (1.3) *holds. There exists a constant* γ_1 *depending only on the data such that for all cylinders* $B_{2\rho}(y) \times [s-(t-s), s+(t-s)] \subset \mathbb{R}^N \times [0,+\infty)$*,*

$$\sup_{B_\rho(y)\times[s,t]} u(x,t) \le \frac{\gamma_1}{(t-s)^{N/\lambda}}\Big(\inf_{2s-t\le\tau\le t}\int_{B_{2\rho}(y)} u(x,\tau)dx\Big)^{p/\lambda} + \gamma_1\Big(\frac{t-s}{\rho^p}\Big)^{1/(2-p)}. \tag{2.1}$$

Note that this theorem states that if a solution is in L^1 at a certain time t_0, it remains in L^∞ for all time $s > t_0$. Also, (2.1) implies that if the solution v of (1.12) can be approximated by regular problems, then

$$v \in C([s,\infty); L^2(\mathbb{R}^N)) \cap L^p((s,\infty); W^{1,p}(\mathbb{R}^N))$$

for all $s > 0$.

Theorem 2.3 (Harnack inequality [10, Theorem 2.2, p. 190])**.** *Let*

$$u \in C_{loc}(\mathbb{R}^+; L^2_{loc}(\mathbb{R}^N)) \cap L^p_{loc}(\mathbb{R}^+; W^{1,p}_{loc}(\mathbb{R}^N))$$

be a non-negative local weak solution of (1.1)–(1.2) *in* $\mathbb{R}^N \times [0,+\infty)$ *and p be in the supercritical range* (1.3). *There exist positive constants* $\delta, \gamma, \varepsilon$ *such that for all* $P_0 = (x_0,t_0) \in \mathbb{R}^N \times [0,+\infty)$ *for which*

$$B_{8\rho}(x_0) \times \Big\{t_0 - u(P_0)^{2-p}(8\rho)^p < t < t_0 + u(P_0)^{2-p}(8\rho)^p\Big\} \subset \mathbb{R}^N \times [0,+\infty),$$

the inequalities

$$\gamma^{-1}\sup_{B_\rho(x_0)} u(\cdot,\sigma) \le u(P_0) \le \gamma \inf_{B_\rho(x_0)} u(\cdot,\tau)$$

hold for all pairs of time levels σ, τ *in the range*

$$t_0 - \varepsilon u(P_0)^{2-p}\rho^p \le \sigma, \tau \le t_0 + \varepsilon u(P_0)^{2-p}\rho^p.$$

We now recall the property of non-negative local weak solutions of (1.1)–(1.2), known as "expansion of positivity". (For a detailed discussion of this argument, see [10] and [8]; for a proof of the expansion of positivity exactly in this form, see [13].)

Theorem 2.4 (Expansion of positivity). *Let*

$$u \in C_{loc}(\mathbb{R}^+; L^2_{loc}(\mathbb{R}^N)) \cap L^p_{loc}(\mathbb{R}^+; W^{1,p}_{loc}(\mathbb{R}^N))$$

be a non-negative local weak solution of (1.1)–(1.2) *in* $\mathbb{R}^N \times [0, +\infty)$ *and* p *be in the supercritical range* (1.3). *Assume that for some* $(y, s) \in \mathbb{R}^N \times [0, +\infty)$ *and some* $\rho > 0$, $|[u(\cdot, t) \geq M] \cap B_\rho(y)| \geq \alpha |B_\rho|$ *for all times* $0 < s - \varepsilon M^{2-p}\rho^p \leq t \leq s$ *for some* $M > 0$ *and* $\alpha, \varepsilon \in (0, 1)$. *Then there exists* $\xi \in (0, 1)$, *which can be determined, a priori, quantitatively only in terms of the data and the numbers* ε *and* α *and is independent of* M *and* ρ *and is such that* $u(x, t) \geq \xi M$, $x \in B_{2\rho}(y)$ *for all times* $s - \varepsilon M^{2-p}\rho^p/2 < t \leq s$.

Remark. The results of this section hold for porous medium type equations; see [10, Appendix B]. The result of Theorem 2.4 can be extended to porous media type equations by modifying the De Giorgi estimates in a suitable way, i.e. replacing the intrinsic geometry related to the p-Laplacian equation, based on cylinders with height equal to $M^{2-p}\rho^p$, by the geometry related to the porous medium equation, based on cylinders with height equal to $M^{1-m}\rho^2$.

3 Estimates from above

Lemma 3.1 (Local L^1 estimates). *Let* u *be a fundamental solution of* (1.12) *under conditions* (1.2) *and* (1.4) *with* $1 < p < 2$. *Let* $\gamma_2 := (2\gamma)^{p-2}$, *where* γ *is the constant defined in Theorem 2.1. Let*

$$T = \gamma_2 R^\lambda, \tag{3.1}$$

where λ *is as defined* (1.10). *There exists* $\gamma_3 > 0$ *depending on* p *and* $N > 0$ *such that for all* x *satisfying* $|x| < R$ *and all* $0 \leq t \leq T$,

$$\int_{B_{2R}(x)} u(y, t)dy \geq \gamma_3.$$

Proof. Apply the L^1 -estimates of Theorem 2.1, with $\rho = R$, $s = \varepsilon$, and $t = T$. Noting that $\lim_{\varepsilon \to 0} \int_{B_R(x)} u(y, \varepsilon)dy = 1$ and passing to the limit for $\varepsilon \to 0$, we see that for every $T > 0$,

$$1 \leq \sup_{0 \leq t \leq T} \int_{B_R(x)} u(y, t)dy \leq \gamma \inf_{0 \leq t \leq T} \int_{B_{2R}(x)} u(y, t)dy + \gamma \left(\frac{T}{R^\lambda}\right)^{1/(2-p)}.$$

Since $\gamma_2 := (2\gamma)^{p-2}$, (3.1)) yields $\int_{B_{2R}(x)} u(y,t)dy \geq 1/2\gamma > 0$, as desired. □

In order to obtain a pointwise estimate in the case $|x| < R$, we prove the following.

Lemma 3.2 (Local L^∞ estimates)**.** *Let u be a fundamental solution of* (1.12) *and p be in the supercritical range* (1.3)*. Assume conditions* (1.2) *and* (1.4)*. There exists $\gamma_4 > 0$ such that $u(x,T) \leq \gamma_4 T^{-N/\lambda}$ for all (x,T) such that $T > 0$ and $|x| < R = (T/\gamma_2)^{1/\lambda}$.*

Proof. Applying Theorem 2.2 with $s = (T+\varepsilon)/2$, $t = T$, and $\rho = R$, we see that for all $T > 0$,

$$u(x,T) \leq \sup_{y\in B_R(x)} u(y,T) \leq \gamma_1 2^{N/\lambda}(T-\varepsilon)^{-N/\lambda}\Big(\inf_{\varepsilon\leq t\leq T}\int_{B_{2R}(x)} u(y,t)dy\Big)^{p/\lambda} + \gamma_1 2^{-1/(2-p)}\Big(\frac{T-\varepsilon}{R^p}\Big)^{1/(2-p)}. \tag{3.2}$$

Recalling that $\lim_{\varepsilon\to 0}\big(\int_{B_{2R}(x)} u(y,\varepsilon)dy\big)^{p/\lambda} \leq 1$ and letting $\varepsilon \to 0$ in (3.2), we obtain

$$u(x,T) \leq \gamma_1 2^{\frac{N}{\lambda}}(T)^{-\frac{N}{\lambda}} + \gamma_1 2^{-1/(2-p)}\Big(\frac{T}{R^p}\Big)^{1/(2-p)}.$$

Taking into account $R = (T/\gamma_2)^{1/\lambda}$, we obtain

$$u(x,T) \leq \gamma_1 2^{N/\lambda} T^{-N/\lambda} + (\gamma_2)^{p/(p-2)\lambda} 2^{-1/(2-p)}\gamma_1\Big(\frac{T}{T^{p/\lambda}}\Big)^{1/(2-p)}$$

and finally $u(x,T) \leq \gamma_4 T^{-N/\lambda}$, where

$$\gamma_4 = \gamma_1 2^{N/\lambda}\big(1 + \frac{\gamma_2}{2^\lambda}\big)^{p/\lambda} + \gamma_1 2^{-1/(2-p)}\gamma_2^{p/(p-2)\lambda}.$$

□

To complete the estimate from above, we have to estimate $u(x,T)$ when $|x| \geq R = (T/\gamma_2)^{1/\lambda}$. We do this in the following lemma.

Lemma 3.3. *Let u be a fundamental solution of* (1.12) *under conditions* (1.2)– (1.4)*. Let $T > 0$. For all x such that*

$$|x| \geq \Big(\frac{T}{\gamma_2}\Big)^{1/\lambda}, \tag{3.3}$$

the inequality

$$u(x,T) \leq \gamma_1 2^{-1/(2-p)}\Big(\frac{T}{|x|^p}\Big)^{1/(2-p)} \tag{3.4}$$

hold.

Proof. Let $R = |x|/2$ and apply Theorem 2.2 with $s = (T - \varepsilon)/2$, $t = T$, and $\rho = R$ to get

$$u(x,T) \le \sup_{y \in B_R(x)} u(y,T) \le \gamma_1 2^{N/\lambda}(T-\varepsilon)^{-N/\lambda}\Big(\inf_{\varepsilon \le t \le T} \int_{B_{2R}(x)} u(y,t)dy\Big)^{p/\lambda} + \gamma_1 2^{-1/(2-p)}\Big(\frac{T-\varepsilon}{R^p}\Big)^{1/(2-p)}. \tag{3.5}$$

Taking into account that δ has its mass concentrated at the origin, which by hypothesis (3.3), lies outside the ball $B_R(x)$, we obtain $\lim_{\varepsilon \to 0}\big(\int_{B_R(x)} u(y,\varepsilon)dy\big)^{p/\lambda} = 0$. Therefore, letting $\varepsilon \to 0$ and arguing as in the previous lemma, we get (3.4). □

Theorem 3.1. *Let u be a fundamental solution of* (1.12) *under conditions* (1.2)–(1.4). *Then for all* $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$, *there exists* $\overline{\gamma} = \overline{\gamma}(N,p) > 0$ *such that* $u(x,t) \le \overline{\gamma}\mathcal{B}_p(x,t)$.

Proof. Apply Lemmas 3.2 and 3.3. □

Remark. The techniques applied in the case of the Dirac mass apply more easily to the case of an initial datum $u_0 \in L^1$ without passing to the limit for $t \to 0$. Also, as mentioned in Section 2, these results can be proved for the porous medium case following the same arguments used for the p-Laplacian type case.

4 Interior estimate from below

In this section, we present a bound from below in a domain

$$\mathcal{P} = \Big\{(x,t) : t > 0, x \in B_{R_t}\Big\}, \text{ where } R_t = \Big(\frac{t}{\gamma_2}\Big)^{1/\lambda}.$$

In the next theorem, we prove that $\underline{\gamma}\mathcal{B}_p(x,t) \le u(x,t)$ for all $(x,t) \in \mathcal{P}$.

Theorem 4.1. *Let u be a fundamental solution of* (1.12) *under conditions* (1.2)–(1.4). *Then there exists a positive γ_5 such that for each* $(x,t) \in \mathcal{P}$,

$$u(x,t) \ge \gamma_5 t^{-N/\lambda}. \tag{4.1}$$

Proof. First, decompose the ball B_{2R_t} into the two complementary sets

$$A_1 = \Big\{(x,t) : u < \frac{\varepsilon_0}{R_t^N} \cap B_{2R_t}\Big\}, \quad A_2 = \Big\{(x,t) : u \ge \frac{\varepsilon_0}{R_t^N} \cap B_{2R_t}\Big\},$$

where ε_0 is a positive constant to be determined later. Lemma 3.1 implies

$$\gamma_3 \le \int_{B_{2R_t}} u(x,t)dx = \int_{A_1} u(x,t)dx + \int_{A_2} u(x,t)dx;$$

therefore,

$$\begin{aligned}\int_{A_2} u(x,t)dx &\geq \gamma_3 - \int_{A_1} u(x,t)dx \geq \gamma_3 - \frac{\varepsilon_0}{(R_t)^N}|A_1| \\ &\geq \gamma_3 - \frac{\varepsilon_0}{(R_t)^N}(2R_t)^N \omega_N = \frac{\gamma_3}{2},\end{aligned} \tag{4.2}$$

where $\varepsilon_0 = \gamma_3/\omega_N 2^{N+1}$. Now by Lemma 3.2,

$$\int_{A_2} u(x,t)dx \leq \gamma_4 t^{-N/\lambda}|A_2| = \gamma_4\gamma_2^{-N}R_t^{-N}|A_2|. \tag{4.3}$$

Substituting (4.3) into (4.2) yields

$$\gamma_4\gamma_2^{-N}R_t^{-N}|A_2| \geq \gamma_3/2 \tag{4.4}$$

and then $|A_2| \geq \gamma_6 R_t^N$ and $\gamma_6 = \gamma_3\gamma_2^N/2\gamma_4$, which means

$$\left|\left\{u(x,t) \geq \frac{\varepsilon_0}{R_t^N}\right\} \cap B_{2R_t}\right| \geq \gamma_6 R_t^N.$$

Let $M = \varepsilon_0/R_t^N$. Then for all $s \in [t/2, t]$,

$$|\{u(x,s) \geq M\} \cap B_{R_t}| \geq \gamma_6 R_t^N.$$

Thus we can apply Theorem 2.4 to obtain

$$u(x,t) \geq \xi M. \tag{4.5}$$

By the definitions of M and R_t, estimate (4.1) holds for all $(x,t) \in \mathcal{P}$. □

Remark. Observe that thanks to the corresponding results, this lemma can also be extended to the porous medium case.

5 Estimate from below. Proof of Theorem 1.1.

To complete the proof of Theorem 1.1, we need an estimate from below that holds for all $(x,t) \notin \mathcal{P}$. Since $\mathcal{B}_p(x,t)$ behaves as $(t/|x|^p)^{1/(2-p)}$ outside $\mathcal{P}$, the following result suffices.

Theorem 5.1. *Let u be a fundamental solution of* (1.1) *under conditions* (1.2)–(1.4). *Then there exists $\gamma_7 > 0$ such that*

$$u(x,t) \geq \gamma_7\left(\frac{t}{|x|^p}\right)^{1/(2-p)} \tag{5.1}$$

for all $(x,t) \notin \mathcal{P}$.

Proof. Suppose $(x,t) \notin \mathcal{P}$ and let $\gamma_8 = 8^{p/(p-2)}/2^{1/(2-p)}$. Estimate (5.1) is clearly satisfied with $\gamma_7 = \gamma_8$ if $u(x,t) \geq \gamma_8(t/|x|^p)^{1/(2-p)}$. If, on the other hand, $u(x,t) < \gamma_8(t/|x|^p)^{1/(2-p)}$, the Harnack inequality of Theorem 2.3 applied with $P_0 = (x,t)$ and $\rho = |x|$ gives $u(x,t) \geq \gamma^{-1}u(0,t)$. By Theorem 4.1,

$$u(x,t) \geq \gamma^{-1}\gamma_5 t^{-N/\lambda}. \tag{5.2}$$

Now $t < \gamma_2|x|^\lambda$ since $(x,t) \notin \mathcal{P}$, Therefore, by (5.2),

$$u(x,t) \geq \gamma^{-1}\gamma_5\gamma_2^{p/\lambda(p-2)}\left(\frac{t}{|x|^p}\right)^{1/(2-p)}.$$

The result follows with γ_7 equal to the smaller of γ_8 and $\gamma^{-1}\gamma_5\gamma_2^{p/\lambda(p-2)}$. □

Remark. As observed in the previous sections, the results for the p-Laplacian type equations can be extended to the porous medium case. This extension yields Theorem 1.2.

6 Pointwise estimates for the derivatives

Determining pointwise estimates for the derivatives is a natural application of the results of the previous sections.

Let us first consider the case of the p-Laplacian type equations. The classical energy estimates (see [10, Chapter 3]) imply that the L^p norm of the derivatives can be estimated with the L^2 norm of u. For the prototype equations, following the argument introduced by DiBenedetto-Friedman in [7], sharp pointwise estimates for the derivatives can be deduced using the sharp estimates on the L^p norm of Du. In the case of general operators with regular coefficients, it is necessary to follow the argument introduced by Kuusi and Mingione in [19] and [20].

Porous medium type equations are certainly more interesting. DiBenedetto, Kwong and Vespri [11] proved sharp estimates for the derivatives of any order in space and time; see also [10, Chapter 6, Section 18]). Here, working in $\mathbb{R}^N$ rather than in a bounded domain, we are able to improve this result. We follow the techniques introduced in [11].

Let $(x_0,t_0) \in \mathbb{R}^N \times (0,+\infty)$. The change of variables

$$y = \frac{x - x_0}{\sqrt{t_0}\,u(x_0,t_0)^{(m-1)/2}}, \qquad \tau = \frac{t-t_0}{t_0}, \qquad v = \frac{u}{u(x_0,t_0)} \tag{6.1}$$

transforms our operator A into an operator that we denote by $\tilde{A}$.

Theorem 6.1. *Let u be a fundamental solution of* (1.14) *under conditions* (1.6)–(1.8)*. Assume that $\tilde{A}$, defined by* (6.1) *is uniformly analytic with respect to* *every* (x_0, t_0). *There exists a constant* C, *depending only on* c_1', c_2', L', m, N, *and independent of* u, *such that for all* $x_0 \in \mathbb{R}^N$, $t_0 > 0$, *and every multi-index* α,

$$|D^\alpha u(x_0, t_0)| \le \frac{C^{|\alpha|+1}|\alpha|!}{t_0^{|\alpha|/2}} u(x_0, t_0)^{1+(1-m)|\alpha|/2}. \tag{6.2}$$

Moreover, for every positive integer k,

$$\left|\frac{\partial^k}{\partial t^k} u(x_0, t_0)\right| \le \frac{C^{2k+1}(k!)^2}{t_0^k} u(x_0, t_0). \tag{6.3}$$

Proof. We only sketch the proof, following Theorem 2.1 in [11], to which we refer the reader for more details. Let

$$Q = \{|x - x_0| < 8\varepsilon\sqrt{t_0}u(x_0, t_0)^{(m-1)/2} \times (t_0 - \varepsilon^2 t_0, t_0 + \varepsilon^2 t_0)\},$$

with ε so small that Q is strictly included in our domain. The change of variables (6.1) maps Q into $Q^\epsilon := B_{8\epsilon} \times]-\epsilon^2, \epsilon^2[$, and v satisfies

$$v_\tau = \operatorname{div}(\tilde{A}(y, \tau, v, Dv)), \quad v(0, 0) = 1$$

for $(y, \tau) \in Q^\epsilon$. By assumption, $\tilde{A}$ is analytic. Moreover, in light of the sharp estimate (1.15), we can choose ε independent of (x_0, t_0) and such that $c_0^{-1} \le v(y, \tau) \le c_0$ for all (y, τ) in Q_ϵ. Therefore, v is a solution of a quasi-linear non-degenerate parabolic equation with analytic coefficients. By classical results due to Friedman [14] (see also [21]), the solution is uniformly analytic in the space variables and uniformly of Gevrey order 2 in the time variable.

Returning to our solution u, we obtain estimates (6.2) and (6.3). For further computations see [11]. □

Remark. Assume that the operator $\tilde{A}$ is uniformly $C^{k,\alpha}$. Reasoning as before, we can prove that the solution is $C^{k+1,\alpha}$ in the space variables and $C^{(k+1)/2,\alpha/2}$ in the time variable, and the derivatives (up to the order $k+1$ for the space variables and up to the order $(k+1)/2$ for the time variable) satisfy estimates (6.2) and (6.3).

7 Miscellaneous applications

In this section, we sketch how to apply the previous results to the case of bounded domains with more general initial conditions and to the case of the Fokker-Planck equation.

7.1 Bounded domain. Consider the problem

$$\begin{aligned} u_t &= \operatorname{div} A(x,t,u,Du), \quad (x,t) \in B_R \times (t \geq 0), \\ u(x,t) &= 0, \quad (x,t) \in B_R \times (t \geq 0), \\ u(x,0) &= \delta(0) \quad x \in B_R. \end{aligned}$$

Repeating the interior estimates from below shows that the solution decays as $t^{-N/\lambda}$ for all $0 \leq t \leq \gamma_2 R^\lambda$ with $\gamma_2 := (1/2\gamma)^{2-p}$ as introduced in Lemma 3.1. Roughly speaking, we can say that it is only after time $t = \gamma_2 R^\lambda$ that the solution realises that B_R is a bounded domain, changes its rate of decay, and vanishes at a certain time T^*. This remark can be extended also to the case of a regular bounded domain $\Omega \subset \mathbb{R}^N$ that contains a large ball B_R.

7.2 More general initial conditions. Let us consider the initial datum $u(x,0) = u_0(x) \geq 0$ with $u_0 \in L^1(R^N)$ and $\int_{\mathbb{R}^N} u_0(x)dx > 0$. As already noticed, the estimates from above can be studied as in Section 3. For the estimates from below, we follow [28, Section 4, step 1], to which we refer the reader for more details. We deduce the existence of λ, ν,R, and x_0 such that

$$|\{u_0 \geq \lambda\} \cap B_R(x_0)| \geq \nu |B_R(x_0)|.$$

Applying energy estimates, we prove the existence of $\tilde{t}$, $\sigma > 0$, and β such that for all $t \in (0, \tilde{t})$,

$$|B_{\sigma R}(x_0) \cap (u \geq \lambda\beta)| \geq \frac{\nu}{8} |B_{\sigma R}(x_0)|.$$

We are essentially in the same position to apply Theorem 2.4 as we were at the end of the proof of Theorem 4.1. Therefore, arguing exactly as in Section 5. we deduce that for each $t_0 > 0$, there exists a positive constant $\underline{\gamma}$ depending only on $N, p, c_0, c_1, \lambda, \nu, R, \beta, \sigma, t_0$ and the distance of x_0 from the origin, such that for all $x \in \mathbb{R}^N$ and $t > t_0$,

$$\underline{\gamma}\mathcal{B}_p(x,t) \leq u(x,t).$$

7.3 Fokker-Planck equation. Let us consider the problem

$$\begin{aligned} w_t &= \operatorname{div}(A(x,t,w,Dw) + \operatorname{div}(xw), \quad (x,t) \in \mathbb{R}^N \times (t > 0), \\ w(x,0) &= \delta(0), \quad x \in \mathbb{R}^N, \end{aligned} \tag{7.1}$$

where the operator A satisfies conditions (1.2)–(1.4) (respectively, conditions (1.6)–(1.8). As proved by Carrillo-Toscani [4] (see also [30] and references therein), (7.1) can be transformed into equation (1.1) by the change of variables $w(x,t) = \alpha(t)^N u(\alpha(t)x, \beta(t))$, where $\alpha(t) = e^t$ and $\beta(t) = (e^{kt} - 1)/k$.

From the estimates on the p-Laplacian type equation (respectively, porous medium type equation), we can deduce sharp estimates on the solutions of the Fokker Planck equation; i.e., for every $s_0 > 0$, there exist positive constants k_1, k_2 (respectively, $\tilde{k}_1$ and $\tilde{k}_2$), depending on s_0, such that for all $t > s_0$,

$$k_1 \mathcal{B}_p(x, 1) \le u(x, t) \le k_2 \mathcal{B}_p(x, 1)$$

(respectively,

$$\tilde{k}_1 \mathcal{B}_m(x, 1) \le u(x, t) \le \tilde{k}_2 \mathcal{B}_m(x, 1)).$$

Remark. Observe that in contrast to other results in the literature, we deduce the estimates for the Fokker-Planck equation from estimates for the p-Laplacian type equation and porous medium type equations, and not vice-versa.

A Existence of fundamental solutions

In this Appendix, we prove Theorem 1.3. We prove only the existence of the fundamental solution for the p-Laplacian case and omit the proof for the porous medium case, since it is completely analogous.

Define $\psi : \mathbb{N} \times \mathbb{R}^N \to \mathbb{N}$ by

$$\psi_k(x_1, \dots, x_n) = \begin{cases} k^N & \text{if } -1/2k \le x_j \le 1/2k, \quad j = 1, \dots, N, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\psi_k \to \delta(0)$ in the sense of measures as $k \to +\infty$. Let u_k be the solution of the Cauchy problem

$$\text{(A.1)} \qquad \begin{aligned} (u_k)_t &= \operatorname{div} A(x, t, u_k, Du_k), \quad (x, t) \in \mathbb{R}^N \times [0, +\infty), \\ u_k(x, 0) &= \psi_k(x), \quad x \in \mathbb{R}^N. \end{aligned}$$

As already noted (see, e,g., [10, chap. 7]), there exists a unique solution of (A.1) under conditions (1.2)–(1.4). Arguing as in Lemmas 3.2 and 3.3, we see that as $\psi_k \to \delta$,

$$\text{(A.2)} \qquad u_k(x, t) \le \tilde{\gamma} \mathcal{B}_p(x, t),$$

in $(\mathbb{R}^N \times \mathbb{R}^+) \setminus \big([-1/k, 1/k]^N \times [0, \gamma_2^\lambda (1/k)^\lambda]\big)$, where $\mathcal{B}_p$ is defined in (1.9) and $\tilde{\gamma}$ is a positive constant that does not depend on k.

For any positive integer r, define the cube $Q_r = \{(x, t) \in [-r, r]^N \times [1/2r, r^p]\}$. Applying DiBenedetto's regularity results (see, e.g., [6]), we conclude that the functions u_k, $k \ge 1$, are equi-Hölder continuous in each Q_r. By the Ascoli-Arzelà

Theorem, there exists a subsequence $\{u_{1,j}\}_{j=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ that converges uniformly in Q_1. We can extract a subsequence $\{u_{2,j}\}_{j=1}^{\infty}$ from $\{u_{1,j}\}_{j=1}^{\infty}$ that converges in Q_2. Continuing in this fashion, for each n, we can extract a subsequence $\{u_{n+1,j}\}_{j=1}^{\infty}$ from $\{u_{n,j}\}_{j=1}^{\infty}$ that converges in Q_{n+1}.

Let $v_j = u_{j,j}$ and consider the sequence $\{v_j\}_{j=1}^{\infty}$. By construction, $\{v_j\}_{j=1}^{\infty}$ converges to a function v in $(\mathbb{R}^N \times \mathbb{R}^+)$ uniformly in every compact set. Since the operator is monotone, by Minty's Lemma [23], v is a solution of (1.1) and belongs to $C([s,\infty); L^2(\mathbb{R}^N)) \cap L^p((s,\infty); W^{1,p}(\mathbb{R}^N))$ for every $s > 0$.

Let us check the initial condition. By construction, v is non-negative and satisfies (A.2). The support of v at time $t = 0$ is at most only the origin $\{0\}$. We claim that $\lim_{t\to 0}\int_{\mathbb{R}^N} v(x,t)dx = 1$. To prove the claim, let ζ be a cut-off function such that $0 \le \zeta \le 1$ and

$$\zeta(x) = \begin{cases} 1 & \text{if } s \in B_2, \\ 0 & \text{if } x \notin B_3, \end{cases}$$

where $B_a = \{|x| \le a\}$. Applying ζ as a test function in (A.1) yields

$$\int_{\mathbb{R}^N} \zeta u_k(x,t)dx - \int_{\mathbb{R}^N} \zeta u_k(x,0)dx = -\int_0^t \int_{\mathbb{R}^N} A(x,t,u_k,Du_k)\cdot D\zeta dx d\tau.$$

This, in turn, implies

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \zeta u_k(x,t)dx - 1 \right| &\le \int_0^t \int_{\mathbb{R}^N} |A(x,t,u_k,Du_k)\cdot D\zeta| dx d\tau \\ &\le C \int_0^t \int_{A_{2,3}} |Du_k|^{p-1} dx d\tau \\ &\le C \int_0^t \left(\int_{A_{2,3}} |Du_k|^p dx \right)^{(p-1)/p} d\tau, \end{aligned} \tag{A.3}$$

where $A_{2,3} = \{x \in \mathbb{R}^N : 2 \le |x| \le 3\}$. In (A.3), C is a positive constant that depends only upon the data and may change from line to line. Let θ be another cut-off function satisfying $0 \le \theta \le 1$ and

$$\theta(x) = \begin{cases} 1 & \text{if } x \in A_{2,3}, \\ 0 & \text{if } x \in \mathcal{A}_{1,4} = B_1 \setminus B_4. \end{cases}$$

Applying $\theta^p u_k$ as a test function in (A.1) and using the energy estimates (see [6, chap. 2, § 3]), we obtain

$$\int_0^t \int_{A_{2,3}} |Du_k|^p dx d\tau \le C \int_0^t \int_{\mathcal{A}_{1,4}} |u_k|^p dx d\tau \le C\tilde{\gamma}^p \int_0^t \int_{\mathcal{A}_{1,4}} |\mathcal{B}_p(x,t)|^p dx d\tau,$$

where we have used (A.2) in the last inequality. Then $\int_0^t \int_{A_{2,3}} |Du_k|^p dx d\tau \leq Ct$.

By (A.3), there exists a positive constant C such that

$$1 - Ct \leq \int_{\mathbb{R}^N} \zeta u_k(x,t) dx \leq 1 + Ct. \tag{A.4}$$

Let us now estimate $\int_{\mathbb{R}^N} (1-\zeta) u_k(x,t) dx$. By (A.2),

$$\int_{\mathbb{R}^N} |(1-\zeta) u_k(x,t)| dx \leq \tilde{\gamma} \int_{\mathbb{R}^N \setminus B_3} \mathcal{B}_p(x,t) dx \leq C t^{1/(2-p)} \int_{\mathbb{R}^N \setminus B_3} |x|^{-p/(2-p)} dx.$$

Therefore,

$$\int_{\mathbb{R}^N} |(1-\zeta) u_k(x,t)| dx \leq C t^{1/(2-p)}. \tag{A.5}$$

Estimates (A.4) and (A.5) yield the existence of a continuous function $g(t)$ such that $g(0) = 0$ and

$$1 - g(t) \leq \int_{\mathbb{R}^N} u_k(x,t) dx \leq 1 + g(t)$$

for all $t > 0$ and $k \geq 1$. By construction, the previous inequality also holds for the solution v, and therefore,

$$\lim_{t \to 0} \int_{\mathbb{R}^N} v(x,t) dx = 1.$$

To conclude, observe that for each $R > 0$,

$$\begin{aligned} \lim_{t\to 0} \int_{\mathbb{R}^N \setminus B_R} v(x,t) dx &\leq \tilde{\gamma} \lim_{t \to 0} \int_{\mathbb{R}^N \setminus B_R} \mathcal{B}_p(x,t) dx \\ &\leq C \lim_{t\to 0} t^{1/(2-p)} \int_{\mathbb{R}^N \setminus B_R} |x|^{-p/(2-p)} dx = 0. \end{aligned}$$

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F. Ragnedda
Dipartimento di Matematica e Informatica,
Viale Merello 92
09123 Cagliari, Italy
email: ragnedda@unica.it

S. Vernier Piro
Dipartimento di Matematica e Informatica
Viale Merello 92
09123 Cagliari, Italy
email: svernier@unica.it

V. Vespri
Dipartimento di Matematica "U. Dini"
viale Morgagni 67/a
50134 Firenze, Italy
email: vespri@math.unifi.it

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