PONTRYAGIN-DE BRANGES-ROVNYAK SPACES OF SLICE HYPERHOLOMORPHIC FUNCTIONS

By

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Abstract. We study reproducing kernel Hilbert and Pontryagin spaces of slice hyperholomorphic functions. These are analogs of the Hilbert spaces of analytic functions introduced by de Branges and Rovnyak. In the first part of the paper, we focus on the case of Hilbert spaces and introduce, in particular, a version of the Hardy space. Then we define Blaschke factors and Blaschke products and consider an interpolation problem. In the second part of the paper, we turn to the case of Pontryagin spaces. We first prove some results from the theory of Pontryagin spaces in the quaternionic setting and, in particular, a theorem of Shmulyan on densely defined contractive linear relations. We then study realizations of generalized Schur functions and of generalized Carathéodory functions.

1 Introduction

Functions s analytic in the open unit disk $\mathbb{D}$ and contractive there, or equivalently such that the kernel

$$\frac{1 - s(z)s(w)^*}{1 - zw^*}$$

is positive definite in $\mathbb{D}$, play an important role in operator theory, and their study is a part of a field called Schur analysis. The present work is a continuation of [5] and deals with various aspects of Schur analysis in the case of slice hyperholomorphic functions. To review the classical case, and to present the outline of the paper, we first recall a definition. A **signature matrix** is a matrix J (say, with complex entries; in the sequel, quaternionic entries are allowed) which is both self-adjoint and unitary. We denote by $\mathrm{sq}_- J$ the multiplicity (possibly equal to 0) of the eigenvalue -1. Now let J_1 and J_2 be two signature matrices belonging to $\mathbb{C}^{N\times N}$ and $\mathbb{C}^{M\times M}$ respectively, and assume that $\mathrm{sq}_- J_1 = \mathrm{sq}_- J_2$. Functions Θ

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which are $\mathbb{C}^{M\times N}$-valued and meromorphic in $\mathbb{D}$ and such that the kernel

$$K_\Theta(z,w)=\frac{J_2-\Theta(z)J_1\Theta(w)^*}{1-zw^*} \tag{1.1}$$

has a finite number of negative squares in $\mathbb{D}$ are called **generalized Schur functions**. They have been studied by Krein and Langer in a long series of papers; see, for instance, [46, 47, 49, 48, 50]. These authors also considered the case of operator-valued functions and other classes and, in particular, kernels of the form

$$k_\varphi(z,w)=\frac{\varphi(z)J+J\varphi(w)^*}{1-zw^*}, \tag{1.2}$$

where φ is $\mathbb{C}^{N\times N}$-valued and analytic in a neighborhood of the origin, $J\in\mathbb{C}^{N\times N}$ is a signature matrix, and the counterparts of these kernels when the open unit disk is replaced by the open upper half-plane. Meromorphic functions Θ for which the kernel (1.1) has a finite number of negative squares are called **generalized Schur functions**, and meromorphic functions Θ for which the kernel (1.2) has a finite number of negative squares are called **generalized Carathéodory functions**. Associated problems (such as realization and interpolation questions) have been studied extensively.

As mentioned above, a study of Schur analysis in the setting of slice hyperholomorphic functions has been initiated recently in [5], and it is the purpose of the present paper to continue this study. The paper [5] was set in the Hilbert space framework and presented, in particular, the notions and properties of Schur multipliers, de Branges Rovnyak space, and coisometric realizations in the slice hyperholomorphic setting. In the first part of this work, we also focus on the Hilbert space case, while in the second part, we consider the case of indefinite inner product spaces. The next steps in this study are as follows.

(1) The study of the indefinite case and, in particular, the Krein-Langer factorization for generalized Schur functions and the characterization of de Branges-Rovnyak spaces associated to generalized Schur functions in the slice hyperholomorphic setting. This study is presented in [4].
(2) The study in [2] of the case of Hilbert spaces, of the classical Beurling-Lax Theorem, and interpolation for Schur multipliers.
(3) The multiplicative structure of Schur multipliers, in a way similar to the multiplicative structure of bounded analytic functions.
(4) The finite dimensional case, i.e., the analogue of the paper [9] and of part of [7] here. Some results appear in the present paper in Section 9.
(5) Next, and as already mentioned in [5], we attack the case of several quaternionic variables; see [3].

To set the present work into perspective, we recall that the theory of slice hyperholomorphic functions represents a novelty with respect to other theories of hyperholomorphic functions that can be defined in the quaternionic setting. In fact, it allows the definition of a quaternionic functional calculus and its associated S-resolvent operator. The importance of the S-resolvent operator in the context of this paper is the definition of the quaternionic version of the operator $(I - zA)^{-1}$ that appears in the realization function $s(z) = D + zC(I - zA)^{-1}B$. It turns out that when A is a quaternionic matrix and p is a quaternion, $(I - pA)^{-1}$ has to be replaced by $(I - pA)^{-\star} = (I - \bar{p}A)(|p|^2A^2 - 2\text{Re}(p)A + I)^{-1}$, which is equal to $p^{-1}S_R^{-1}(p^{-1}, A)$, where $S_R^{-1}(p^{-1}, A)$ is the right S-resolvent operator associated to the quaternionic matrix A.

Slice hyperholomorphic functions have two main formulations depending on whether the functions we consider are defined on quaternions and are quaternion-valued, in which case they are called **slice regular** (see [39, 21, 25]), or they are defined on the euclidean space $\mathbb{R}^{N+1}$ and take values in the Clifford algebra $\mathbb{R}_N$, in which case they are called **slice monogenic functions** (see [31, 32]). We also point out that there exists a non-constant coefficient differential operator whose kernel contains slice hyperholomorphic functions defined on suitable domains; see [23].

As already discussed, slice hyperholomorphicity has applications in operator theory, specifically, in the case of quaternions, it allows the definition of a quaternionic functional calculus (see, e.g., [22, 24, 27]), while slice monogenic functions admit a functional calculus for n-tuples of operators (see [30, 26, 28]). The book [33] collects some of the main results on the theory of slice hyperholomorphic functions and the related functional calculi.

Finally, we mention the papers [13, 14, 12], where Schur multipliers were introduced and studied in the quaternionic setting using the Cauchy-Kovalevskaya product and series of Fueter polynomials, and the papers [42, 54, 53], which treat various aspects of a theory of linear systems in the quaternionic setting. Our approach is quite different from the methods used there.

In Sections 2 and 3, we review some basic definitions on slice hyperholomorphic functions. In Section 4, we discuss the notion of multipliers in the case of reproducing kernel Hilbert spaces of slice hyperholomorphic functions. In Section 5, we discuss the Hardy space in the present setting and introduce Blaschke products. Interpolation in the Hardy space is studied in Section 6. Sections 7–10 are in the framework of indefinite metric spaces. A number of facts on quaternionic Pontryagin spaces, as well as a proof of a theorem of Shmulyan on relations, are proved in Section 7. Negative squares are discussed in Section 8, while Section 9 introduces generalized Schur functions and discusses their realizations. We also consider in that section the finite dimensional case. Finally, in Section 10, we briefly discuss the case of generalized Carathéodory functions.

2 Slice hyperholomorphic functions

The literature contains several different notions of quaternion-valued hyperholomorphic functions. In this paper, we consider a notion which includes power series in the quaternionic variable, the so-called slice regular or slice hyperholomorphic functions; see [33]. In order to introduce the class of slice hyperholomorphic functions, we fix some preliminary notation. We denote the algebra of real quaternions $p = x_0 + ix_1 + jx_2 + kx_3$ by $\mathbb{H}$. A quaternion can also be written as $p = \mathrm{Re}(p) + \mathrm{Im}(p)$, where $x_0 = \mathrm{Re}(p)$ and $ix_1 + jx_2 + kx_3 = \mathrm{Im}(p)$. It can also be written as $p = \mathrm{Re}(p) + I_p|\mathrm{Im}(p)|$, where $I_p = \mathrm{Im}(p)/|\mathrm{Im}(p)|$, as long as p is non-real. The element I_p belongs to the 2-sphere

$$\mathbb{S} = \{p = x_1 i + x_2 j + x_3 k \ : \ x_1^2 + x_2^2 + x_3^2 = 1\}$$

of unit purely imaginary quaternions.

Definition 2.1. Let $\Omega \subseteq \mathbb{H}$ be an open set and $f : \ \Omega \to \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and f_I be the restriction of f to the complex plane $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I. Denote by $x + Iy$ an element in $\mathbb{C}_I$.

(1) We say that f is a **left slice regular function** (or **left slice hyperholomorphic** or **slice hyperholomorphic**) if for every $I \in \mathbb{S}$,

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0.$$

(2) We say that f is a **right slice regular function** (or **right slice hyperholomorphic**) if for every $I \in \mathbb{S}$,

$$\frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy)I\right) = 0.$$

Definition 2.2. The set of all elements of the form $\mathrm{Re}(p) + J|\mathrm{Im}(p)|$, where J varies in $\mathbb{S}$ is called the 2**-sphere defined by** p and is denoted by $[p]$.

The most important feature of slice hyperholomorphic functions is that on a suitable class of open sets, they can be reconstructed from their values on a complex plane $\mathbb{C}_I$ by the so-called Representation Formula; see [21, 25, 33] and Theorem 2.4 below.

Definition 2.3. Let Ω be a domain in $\mathbb{H}$. We say that Ω is a **slice domain** (**s-domain** for short) if $\Omega \cap \mathbb{R}$ is non-empty and $\Omega \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$. We say that Ω is **axially symmetric** if for all $p \in \Omega$, the 2-sphere $[p]$ is contained in Ω.

Theorem 2.4 (Representation Formula). *Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain. If f is a left slice regular function on $\Omega \subseteq \mathbb{H}$, then for all $p = x + I_p y \in \Omega$,*

$$f(p) = f(x+I_py) = \frac{1}{2}\Big[f(z)+f(\overline{z})\Big] + \frac{1}{2}I_pI\Big[f(\overline{z}) - f(z)\Big], \tag{2.1}$$

where $z := x + Iy$, $\overline{z} := x - Iy \in \Omega \cap \mathbb{C}_I$. If f is a right slice regular function on $\Omega \subseteq \mathbb{H}$ then for all $p = x + I_p y \in \Omega$,

$$f(p) = f(x+I_py) = \frac{1}{2}\Big[f(z)+f(\overline{z})\Big] + \frac{1}{2}\Big[f(\overline{z}) - f(z)\Big]II_p. \tag{2.2}$$

The Representation Formula allows us to extend a function $f : \widetilde{\Omega} \subseteq \mathbb{C}_I \to \mathbb{H}$ defined on a domain $\widetilde{\Omega}$ symmetric with respect to the real axis and in the kernel of the corresponding Cauchy-Riemann operator to a function $f : \Omega \subseteq \mathbb{H} \to \mathbb{H}$ slice hyperholomorphic, where Ω is the smallest axially symmetric open set in $\mathbb{H}$ containing $\widetilde{\Omega}$. In the above notation, the extension is obtained by means of the *extension operator*

$$\mathrm{ext}(f)(p) := \frac{1}{2}\Big[f(z)+f(\overline{z})\Big] + \frac{1}{2}I_pI\Big[f(\overline{z}) - f(z)\Big], \quad z, \overline{z} \in \Omega \cap \mathbb{C}_I,\ p \in \Omega. \tag{2.3}$$

For example, in the case of the kernel associated to the Hardy space, the extension operator applied to the function $\sum_{n=0}^{\infty} z^n \bar{w}^n$ gives the following result.

Proposition 2.5. (see [5, Proposition 5.3]) *Let p and q be quaternionic variables. The sum of the series $\sum_{n=0}^{+\infty} p^n \bar{q}^n$ is the function $k(p,q)$ given by*

$$k(p,q) = (1-2\mathrm{Re}(q)p+|q|^2p^2)^{-1}(1-pq) = (1-\bar{p}\bar{q})(1-2\mathrm{Re}(p)\bar{q}+|p|^2\bar{q}^2)^{-1}. \tag{2.4}$$

The kernel $k(p,q)$ is defined for all p outside the 2-sphere defined by $[q^{-1}]$ (or, equivalently, for all q outside the 2-sphere $[p^{-1}]$). Moreover,

(a) *$k(p,q)$ is slice hyperholomorphic in p and right slice hyperholomorphic in $\bar{q}$;*
(b) *$\overline{k(p,q)} = k(q,p)$.*

The function $k(p,q)$ in Proposition 2.5 is positive definite and is the reproducing kernel of the slice hyperholomorphic counterpart of the Hardy space $\mathbf{H}_2(\mathbb{B})$ of functions analytic in the open unit ball $\mathbb{B}$; see [5] and Section 5 below.

Remark 2.6. The two expressions for $k(p, q)$ given in (2.4) correspond to the left slice regular reciprocal of $1 - p\bar{q}$ in the variable p and to the right slice regular reciprocal in the variable $\bar{q}$, (see the discussion in [5, Proposition 5.3]) and these two reciprocals coincide. Thus, henceforth, we often write $(1 - p\bar{q})^{-\star}$ instead of $k(p, q)$.

Remark 2.7. Note that whenever a function $k(p, q)$ is slice hyperholomorphic in p and Hermitian, it is also right slice hyperholomorphic in $\bar{q}$.

3 Slice hyperholomorphic multiplication

Recall that it is possible to introduce a binary operation called the **$\star$-product** of two left slice hyperholomorphic functions f, g such that $f \star g$ is a left slice hyperholomorphic function. Similarly, we can define the $\star$-product of two right slice hyperholomorphic functions. When considering both products in the same formula, it is often useful to distinguish between them. In this case, we denote the left slice regular product by $\star_l$ and the right slice regular product by $\star_r$. When no subscript appears, the $\star$-product is assumed to be the left $\star$-product.

Let Ω be an axially symmetric s-domain. Let $f, g : \Omega \subseteq \mathbb{H}$ be slice hyperholomorphic functions whose restrictions to the complex plane $\mathbb{C}_I$ can be written respectively as

$$f_I(z) = F(z) + G(z)J, \quad g_I(z) = H(z) + L(z)J,$$

where $J \in \mathbb{S}$, $J \perp I$. The functions F, G, H, L are holomorphic functions of the variable $z \in \Omega \cap \mathbb{C}_I$ and exist by the Splitting Lemma; see [33, p. 117]. The $\star_l$-product of f and g is defined as the unique left slice hyperholomorphic function whose restriction to the complex plane $\mathbb{C}_I$ is given by

$$(3.1) \quad (F(z) + G(z)J) \star_l (H(z) + L(z)J) := (F(z)H(z) - G(z)\overline{L(\bar{z})}) + (G(z)\overline{H(\bar{z})} + F(z)L(z))J;$$

see [33, p. 125]. Analogously, one can define the $\star_r$ product of two right slice hyperholomorphic functions. While pointwise multiplication and slice multiplication are different, they can be related, as the following result shows.

Proposition 3.1. ([33, Proposition 4.3.22]) *Let $U \subseteq \mathbb{H}$ be an axially symmetric s-domain, and let $f, g : U \to \mathbb{H}$ be slice hyperholomorphic functions such that $f(p) \neq 0$ for all $p \in U$. Then*

$$(3.2) \quad (f \star g)(p) = f(p)g(f(p)^{-1}pf(p)),$$

for all $p \in U$.

Remark 3.2. The transformation $p \to f(p)^{-1}pf(p)$ is clearly a rotation in $\mathbb{H}$, since $|p| = |f(p)^{-1}pf(p)|$; this allows us to rewrite the $\star$-product as a pointwise product. Note also that if $f \star g(p) = 0$, then either $f(p) = 0$ or $g(f(p)^{-1}pf(p)) = 0$.

Corollary 3.3. *If* $\lim_{r\to 1} |f(re^{I\theta})| = 1$ *for all* $I \in \mathbb{S}$, *then*

$$\lim_{r\to 1} |f \star g(re^{I\theta})| = |g(e^{I'\theta})|$$

where $\theta \in [0, 2\pi)$ *and* $I' \in \mathbb{S}$ *depends on* θ, I, *and* f.

Proof. Set $b = f(re^{I\theta})$ and write $b = Re^{J\alpha}$ for suitable R, J, α. By hypothesis, we can assume that $b \neq 0$ as $r \to 1$; thus b^{-1} exists. We have

$$\begin{aligned} b^{-1}re^{I\theta}b &= e^{-J\alpha}(re^{I\theta})e^{J\alpha} \\ &= r(\cos\alpha - J\sin\alpha)(\cos\theta + I\sin\theta)(\cos\alpha + J\sin\alpha) \\ &= r(\cos\theta + I\cos^2\alpha\sin\theta - JI\cos\alpha\sin\alpha\sin\theta) \\ &\quad + r(IJ\cos\alpha\sin\alpha\sin\theta - JIJ\sin^2\alpha\sin\theta) \\ &= r(\cos\theta + \cos\alpha e^{-J\alpha}I\sin\theta + e^{-J\alpha}IJ\sin\alpha\sin\theta) \\ &= r(\cos\theta + e^{-J\alpha}Ie^{J\alpha}\sin\theta) = r(\cos\theta + I'\sin\theta), \end{aligned}$$

where $I' = e^{-J\alpha}Ie^{J\alpha}$. The result now immediately follows from the equalities

$$\lim_{r\to 1} |f \star g(re^{I\theta})| = \lim_{r\to 1} |f(re^{I\theta})g(b^{-1}re^{I\theta}b)| = \lim_{r\to 1} |g(re^{I'\theta})| = |g(e^{I'\theta})|. \qquad \square$$

It is possible to construct a slice regular reciprocal of a left slice regular function f, which is denoted by $f^{-\star}$. The general construction can be found in [33]. In this paper, we need the reciprocal of a polynomial or a power series with center at the origin, which can be described in the easier way illustrated below.

Definition 3.4. Given $f(p) = \sum_{n=0}^{\infty} p^n a_n$, let

$$f^c(p) = \sum_{n=0}^{\infty} p^n \bar{a}_n, \quad f^s(p) = (f^c \star f)(p) = \sum_{n=0}^{\infty} p^n c_n, \quad c_n = \sum_{r=0}^{n} a_r \bar{a}_{n-r}.$$

If f converges in the ball of radius R centered at 0, the other series also converge in that ball. The **left slice hyperholomorphic reciprocal** of f is defined as $f^{-\star} := (f^s)^{-1}f^c$. Analogously, we define the **right slice hyperholomorphic reciprocal** of a right slice regular function $f(q) = \sum_n a_n q^n$ as $f^{-\star} := f^c(f^s)^{-1}$.

Note that the series f^s has real coefficients.

Remark 3.5. Let Ω be an axially symmetric open set. Recall that if f is left slice hyperholomorphic in $q \in \Omega$, then $\overline{f(q)}$ is right slice hyperholomorphic in $\overline{q}$. This fact follows immediately from $(\partial_x + I\partial_y)f_I(x+Iy) = 0$, since conjugation gives $\overline{f_I(x+Iy)}(\partial_x - \partial_y I) = 0$ for all $I \in \mathbb{S}$.

Lemma 3.6. *Let Ω be an axially symmetric s-domain and $f, g : \Omega \to \mathbb{H}$ be two left slice hyperholomorphic functions. Then $\overline{f \star_l g} = \overline{g} \star_r \overline{f}$, where $\star_l$, $\star_r$ are the left and right $\star$-products with respect to q and $\bar{q}$, respectively.*

Proof. Let $f_I(z) = F(z) + G(z)J$ and $g_I(z) = H(z) + L(z)J$ be the respective restrictions of f and g to the complex plane $\mathbb{C}_I$. The functions F, G, H, L are holomorphic functions of the variable $z \in \Omega \cap \mathbb{C}_I$, which exist by the Splitting Lemma. Also, J is an element in the sphere $\mathbb{S}$ orthogonal to I. The $\star_r$-product of the two right slice hyperholomorphic functions $\overline{g}$ and $\overline{f}$ in the variable $\overline{q}$ is defined as the unique right slice hyperholomorphic function whose restriction to a complex plane $\mathbb{C}_I$ is given by

$$(\overline{H(z)} - J\,\overline{L(z)}) \star_r (\overline{F(z)} - J\,\overline{G(z)}) := (\overline{H(z)}\,\overline{F(z)} - L(\bar{z})\overline{G(z)}) - J(\overline{L(z)}\,\overline{F(z)} + H(\bar{z})\overline{G(z)}).$$

Thus, a comparison with (3.1) makes clear that $\overline{f_I \star_l g_I} = \overline{g_I} \star_r \overline{f_I}$. Taking the unique right slice hyperholomorphic extension, we obtain the lemma. □

Remark 3.7. For the sake of completeness, we adapt some of the previous definitions to matrix-valued functions. We say that a real differentiable function $f : \Omega \subseteq \mathbb{H} \to \mathbb{H}^{N\times M}$ is **left** (respectively, **right**) **slice hyperholomorphic** if and only if for any linear and continuous functional Λ acting on $\mathbb{H}^{N\times M}$, the function Λf is left (respectively, right) slice hyperholomorphic in Ω. It can be shown using standard techniques that if, in particular, $\Omega = \mathbb{B}$, then f is left slice hyperholomorphic if and only if $f(p) = \sum_{n=0}^{\infty} p^n A_n$, where $A_n \in \mathbb{H}^{N\times M}$ and the series converges in $\mathbb{B}$. Let $f : \mathbb{B} \to \mathbb{H}^{N\times M}$, $g : \mathbb{B} \to \mathbb{H}^{M\times L}$ be left slice hyperholomorphic and $f(p) = \sum_{n=0}^{\infty} p^n A_n$, $g(p) = \sum_{n=0}^{\infty} p^n B_n$. The $\star$-product of f and g is defined as $f \star g := \sum_{n=0}^{\infty} p^n C_n$, where $C_n = \sum_{r=0}^{n} A_r B_{n-r}$. Analogous definitions can be given in more general cases (see [2]) and also for right slice hyperholomorphic functions.

Remark 3.8. When considering the function $\sum_{n=0}^{\infty} p^n A^n$, where $A \in \mathbb{H}^{N\times N}$ and $|p| < 1/\|A\|$ or, more generally, a bounded right linear quaternionic operator A from a quaternionic Hilbert space to itself, $(I - pA)^{-\star}$ can be constructed using the functional calculus; see [5, Proposition 2.16]. It is sufficient to construct the right slice regular inverse of $1 - pq$ with respect to q and then replace q with the operator A. Note that, for simplicity, we write $(I - pA)^{-\star}$ using the symbol $\star$ instead of $\star_r$; the discussion in Remark 2.6 justifies this abuse of notation.

4 Multipliers in reproducing kernel Hilbert spaces

In this section, we study the multiplication operators and their adjoints. We show that positivity implies slice hyperholomorphicity for a class of functions and we prove that if a kernel is positive and slice hyperholomorphic, then the corresponding reproducing kernel Hilbert space consists of slice hyperholomorphic functions.

Let us begin by recalling the following definition; cf. [11].

Definition 4.1. A quaternionic Hilbert space $\mathscr{H}$ of $\mathbb{H}^N$-valued functions defined on an open set $\Omega \subseteq \mathbb{H}$ is called a **reproducing kernel quaternionic Hilbert space** if there exists a $\mathbb{H}^{N\times N}$-valued function defined on $\Omega \times \Omega$ such that

(1) for every $q \in \Omega$ and $a \in \mathbb{H}^N$, the function $p \mapsto K(p,q)a$ belongs to $\mathscr{H}$;

(2) for every $f \in \mathscr{H}$, $q \in \Omega$, and $a \in \mathbb{H}^N$, $\langle f, K(\cdot,q)a\rangle_{\mathscr{H}} = a^* f(q)$.

The function $K(p,q)$ is called the **reproducing kernel** of the space. As observed in [11], Definition 4.1, one may impose the weaker requirement that $\mathscr{H}$ be a quaternionic pre-Hilbert space. However, the next result guarantees that a reproducing kernel quaternionic pre-Hilbert space has a unique completion as a reproducing kernel quaternionic Hilbert space. We denote this space by $\mathscr{H}(K)$.

Theorem 4.2 ([11]). *Given an $\mathbb{H}^{N\times N}$-valued function $K(p,q)$ positive on a set $\Omega \subset \mathbb{H}$, there exists a uniquely defined reproducing kernel quaternionic Hilbert space of $\mathbb{H}^N$-valued function defined on Ω and with reproducing kernel $K(p,q)$.*

Recall that $\mathscr{H}(K)$ is the completion of the linear span $\overset{\circ}{\mathscr{H}}(K)$ of functions of the form

$$p \mapsto K(p,q)a, \quad q \in \Omega,\ a \in \mathbb{H}^N, \tag{4.1}$$

with the inner product

$$\langle K(\cdot,q)a, K(\cdot,s)b\rangle_{\overset{\circ}{\mathscr{H}}(K)} := b^* K(s,q)a. \tag{4.2}$$

Proposition 4.3. *Let ϕ be a slice hyperholomorphic function defined on an axially symmetric s-domain Ω and with values in $\mathbb{H}^{N\times M}$. Let $K_1(p,q)$ and $K_2(p,q)$ be positive definite kernels in Ω, respectively $\mathbb{H}^{M\times M}$- and $\mathbb{H}^{N\times N}$-valued, and slice hyperholomorphic in the variable p.*

(1) *If the slice multiplication operator $M_\phi : \mathcal{H}(K_1) \to \mathcal{H}(K_2)$ given by $M_\phi : f \mapsto \phi \star f$ is continuous, then the adjoint operator is given by the formula*

$$M_\phi^*(K_2(\cdot,q)d) = K_1(\cdot,q) \star_r \phi^*(q)d.$$

(2) *The multiplication operator* M_ϕ *has norm less than or equal to* k *(in particular,* M_ϕ *is bounded) if and only if the function*

$$K_2(p,q) - \frac{1}{k^2}\phi(p) \star_l K_1(p,q) \star_r \phi(q)^* \tag{4.3}$$

is positive on Ω.

Proof. We compute the adjoint of the operator $M_\phi : \mathcal{H}(K_1) \to \mathcal{H}(K_2)$:

$$\begin{aligned} c^*(M_\phi^*(K_2(\cdot,q)d))(p) &= \langle M_\phi^*(K_2(\cdot,q)d), K_1(\cdot,p)c\rangle_{\mathcal{H}(K_1)} \\ &= \langle K_2(\cdot,q)d, \phi \star_l K_1(\cdot,p)c\rangle_{\mathcal{H}(K_2)} \\ &= \langle \phi \star_l K_1(\cdot,p)c, K_2(\cdot,q)d\rangle^*_{\mathcal{H}(K_2)} \\ &= (d^*(\phi(q) \star_l K_1(q,p))c)^* \\ &= c^*(\phi(q) \star_l K_1(q,p))^* d. \end{aligned}$$

Now observe that by Lemma 3.6, we have $(\phi(q) \star_l K_1(q,p))^* = K_1(p,q) \star_r \phi^*(q)$, and so $M_\phi^*(K_2(\cdot,q)d) = K_1(\cdot,q) \star_r \phi^*(q)d$. The positivity of (4.3) follows from the positivity of the operator $k^2 - M_\phi M_\phi^*$.

Conversely, the standard argument shows that $\|M_\phi\| \le k$ if (4.3) is positive. □

Example 4.4. Let us consider the case in which the kernel K is of the form

$$K(p,q) = \sum_{n=0}^{\infty} p^n \overline{q}^n \alpha_n, \quad \alpha_n \in \mathbb{R} \text{ for all } n \in \mathbb{N}.$$

In this case,

$$\phi(p) \star_l K(p,q) = \sum_{n=0}^{\infty} p^n \phi(p) \overline{q}^n \alpha_n$$

and

$$(\phi(p) \star_l K(p,q))^* = \sum_{n=0}^{\infty} q^n \phi(p)^* \overline{p}^n \alpha_n,$$

from which we obtain

$$\phi(q) \star_l (\phi(p) \star_l K(p,q))^* = \phi(q) \star_l \sum_{n=0}^{\infty} q^n \phi(p)^* \overline{p}^n \alpha_n = \phi(q) \star_l K(q,p) \star_r \phi(p)^*.$$

Recall that in [5], a Schur function S is defined to be a $\mathbb{H}^{N\times M}$-valued function, slice hyperholomorphic in $\mathbb{B}$, whose kernel

$$k_S(p,q) = \sum_{n=0}^{\infty} p^n (I_N - S(p)S(q)^*) \overline{q}^n = (I_N - S(p)S(q)^*) \star (1 - p\overline{q})^{-\star} \tag{4.4}$$

is positive on $\mathbb{B}$. We show, in Theorem 4.6 below, the converse for a subclass of slice hyperholomorphic functions, i.e. that for a subclass of slice hyperholomorphic functions, positivity forces hyperholomorphicity. This subclass, denoted by $\mathcal{N}$, corresponds to those functions f such that $f : \mathbb{B} \cap \mathbb{C}_I \to \mathbb{C}_I$ for all $I \in \mathbb{S}$. For functions $f \in \mathcal{N}$, the pointwise multiplication of f with a monomial of the form p^n is well-defined and commutative. In fact, f takes the complex plane $\mathbb{C}_{I_p}$ to itself and thus behaves, on each plane, like a complex-valued function.

We need the following preliminary result.

Proposition 4.5. ([11, Proposition 9.3]) *Suppose Ω is an arbitrary set. Let $K_1 : \Omega \to \mathbb{H}^{N\times N}$ and $K_2 : \Omega \to \mathbb{H}^{M\times M}$ be positive functions and $\phi : \Omega \to \mathbb{H}^{N\times M}$. The pointwise multiplication operator by ϕ has norm less than or equal to k (and, in particular, is bounded) if and only if the function*

$$K_2(p, q) - \frac{1}{k^2}\phi(p)K_1(p, q)\phi(q)^* \tag{4.5}$$

is positive on Ω.

Theorem 4.6. *Let $S : \mathbb{B} \to \mathbb{H}^{N\times M}$ be a function such that for every $I \in \mathbb{S}$, $S : \mathbb{B} \cap \mathbb{C}_I \to \mathbb{C}_I^{N\times M}$. Then the following are equivalent.*

1. *The function $\sum_{n=0}^{\infty} p^n(I_N - S(p)S(q)^*)\bar{q}^n$ is positive on $\mathbb{B}$.*
2. *The operator M_S is a contraction from $\mathbf{H}_2^M(\mathbb{B})$ to $\mathbf{H}_2^N(\mathbb{B})$.*
3. *S is a Schur function belonging to $\mathcal{N}(\mathbb{B})$.*

Proof. The equivalence between (1) and (2) follows as in [1, Theorem 2.6.3], and its proof is based on Proposition 4.5. Indeed, in (4.5) set

$$K_1(p, q) = I_M(1 - p\bar{q})^{-\star}, \quad K_2(p, q) = I_N(1 - p\bar{q})^{-\star}.$$

Then

$$\begin{aligned} I_N(1 - p\bar{q})^{-\star} - \frac{1}{k^2}S(p)(1 - p\bar{q})^{-\star}S(q)^* \\ = I_N\sum_{n=0}^{\infty} p^n\bar{q}^n - \frac{1}{k^2}S(p)(\sum_{n=0}^{\infty} p^n\bar{q}^n)S(q)^*. \end{aligned} \tag{4.6}$$

Now observe that, by hypothesis, $S(p)$ commutes with p^n since S takes the complex plane $\mathbb{C}_{I_p}$ to itself; similarly, $S(q)^*$ commutes with $\bar{q}^n$. So we conclude that (4.6) equals

$$\sum_{n=0}^{\infty} p^n(I_N - \frac{1}{k^2}S(p)S(q)^*)\bar{q}^n.$$

Thus, if (1) holds, then by Proposition 4.5, M_S is a contraction from $\mathbf{H}_2^M(\mathbb{B})$ to $\mathbf{H}_2^N(\mathbb{B})$. Conversely, if (2) holds, then again Proposition 4.5 allows us to conclude that (1) holds.

The implication (3) implies (2) follows from the fact that S is a Schur function.

To see that (2) implies (3), notice that S is slice hyperholomorphic, since $Sc \in \mathbf{H}_2^N(\mathbb{B})$ for every $c \in \mathbb{H}^M$. Moreover, S is contractive, since M_S^* acts as

$$M_S^*\left((1-p\overline{q})^{-\star}d\right) = (1-p\overline{q})^{-\star}S(q)^*d$$

and M_S^* is also a contraction. □

Definition 4.7. A subset Ω of $\mathbb{B}$ is called a **set of uniqueness** if every slice hyperholomorphic function on $\mathbb{B}$ which vanishes on Ω is identically zero on $\mathbb{B}$.

Example 4.8. Every open subset Ω of $\mathbb{B}\cap\mathbb{C}_I$ is a set of uniqueness. More generally, every subset Ω of $\mathbb{B}\cap\mathbb{C}_I$, $I\in\mathbb{S}$, having an accumulation point in $\mathbb{C}_I$ is a set of uniqueness.

Theorem 4.9. *Let Ω be a set of uniqueness in $\mathbb{B}$ and S a function defined on Ω such that $S:\Omega\cap\mathbb{C}_I\to\mathbb{C}_I^{N\times M}$ for every $I\in\mathbb{S}$. Then S can be extended slice hyperholomorphically to a Schur function in $\mathcal{N}(\mathbb{B})$ if and only if the kernel*

$$\sum_{n=0}^{\infty} p^n(I_N - S(p)S(q)^*)\overline{q}^n \tag{4.7}$$

is positive on Ω.

Proof. If S can be extended hyperholomorphically to a Schur function, then the kernel (4.7) is positive definite on Ω.

To prove the converse, define the right linear quaternionic operator T as

$$T\left((1-p\overline{q})^{-\star}d\right) = (1-p\overline{q})^{-\star}S(q)^*d, \quad q\in\Omega$$

and reason as in the proof of Theorem 4.6. By assumption, the kernel

$$\sum_{n=0}^{\infty} p^n(I_N - S(p)S(q)^*)\overline{q}^n$$

is positive; thus T is well-defined and contractive. Its domain is dense, since Ω is a set of uniqueness. So T extends to a contraction from $\mathbf{H}_2^M$ to $\mathbf{H}_2^N$. Its adjoint is a contraction; and for every $q\in\Omega$ and $F\in\mathbf{H}_2^N$,

$$\begin{aligned}\langle T^*F,(1-p\overline{q})^{-\star}d\rangle &= \langle F, T\left((1-p\overline{q})^{-\star}d\right)\rangle\\ &= \langle F,(1-p\overline{q})^{-\star}S(q)^*d\rangle\\ &= d^*S(q)F(q).\end{aligned}$$

Since we obtained a function equal to $S(q)F(q)$ on Ω, the choice $F = 1$ shows that $S = T^*1$ is the restriction to Ω of a Schur function. $\square$

To conclude this section, we show that if a kernel $K(p, q)$ is positive and slice hyperholomorphic in p, then its corresponding reproducing kernel Hilbert space consists of slice hyperholomorphic functions.

Theorem 4.10. *Let $K(p, q)$ be an $\mathbb{H}^{N\times N}$-valued function defined on an open set $\Omega \subset \mathbb{H}$ and let $\mathscr{H}(K)$ be the associated reproducing kernel quaternionic Hilbert space. Assume that for all $q \in \Omega$, the function $p \mapsto K(p, q)$ is slice hyperholomorphic. Then the entries of the elements of $\mathscr{H}(K)$ are also slice hyperholomorphic.*

Proof. It suffices to consider the case of $\mathbb{H}$-valued functions, as the matrix case is similar. For any $f \in \mathscr{H}(K)$, $p, q \in \Omega$, and sufficiently small $\varepsilon \in \mathbb{R} \setminus \{0\}$,

$$\frac{1}{\varepsilon}(K(p, q+\varepsilon) - K(p, q)) = \frac{1}{\varepsilon}\overline{(K(q+\varepsilon, p) - K(q, p))}.$$

Let $(u + Iv, x + Iy) \in \mathbb{C}_I \times \mathbb{C}_I$. Then

$$\frac{\partial K(p, q)}{\partial x} = \frac{\partial \overline{K(q, p)}}{\partial u}.$$

Similarly,

$$\frac{1}{\varepsilon}(K(p, q+I\varepsilon) - K(p, q)) = \frac{1}{\varepsilon}\overline{(K(q+I\varepsilon, p) - K(q, p))},$$

from which we deduce

$$\frac{\partial K(p, q)}{\partial y} = \frac{\partial \overline{K(q, p)}}{\partial v}.$$

The two families

$$\left\{\frac{1}{\varepsilon}(K(p, q+\varepsilon) - K(p, q))\right\}_{\varepsilon\in\mathbb{R}\setminus\{0\}}, \qquad \left\{\frac{1}{\varepsilon}(K(p, q+I\varepsilon) - K(p, q))\right\}_{\varepsilon\in\mathbb{R}\setminus\{0\}},$$

are uniformly bounded in norm and therefore have weakly convergent subsequences which converge to $\partial K(p, q)/\partial x$ and $\partial K(p, q)/\partial y$, respectively. Moreover,

$$\frac{1}{\varepsilon}(f(p+\varepsilon) - f(p)) = \langle f(\cdot), \frac{1}{\varepsilon}(K(\cdot, p+\varepsilon) - K(\cdot, p))\rangle_{\mathscr{H}(K)}$$

and

$$\frac{1}{\varepsilon}(f(p+I\varepsilon) - f(p)) = \langle f(\cdot), \frac{1}{\varepsilon}(K(\cdot, p+I\varepsilon) - K(\cdot, p))\rangle_{\mathscr{H}(K)}.$$

Thus, we can write

$$\frac{\partial f}{\partial u}(p) = \langle f(\cdot), \frac{\partial K(\cdot, p)}{\partial x}\rangle_{\mathscr{H}(K)},$$

and

$$\frac{\partial f}{\partial v}(p) = \langle f(\cdot), \frac{\partial K(\cdot,p)}{\partial y}\rangle_{\mathscr{H}(K)}.$$

To show that f is slice hyperholomorphic, we consider its restriction to an arbitrary complex plane $\mathbb{C}_I$ and show that it is in the kernel of the corresponding Cauchy-Riemann operator:

$$\begin{aligned}\frac{\partial f}{\partial u} + I\frac{\partial f}{\partial v} &= \left\langle f, \frac{\partial K(\cdot,q)}{\partial x}\right\rangle_{\mathscr{H}(K)} + I\left\langle f(\cdot), \frac{\partial K(\cdot,q)}{\partial y}\right\rangle_{\mathscr{H}(K)} \\ &= \left\langle f, \frac{\partial K(\cdot,q)}{\partial x} - \frac{\partial K(\cdot,q)}{\partial y}I\right\rangle_{\mathscr{H}(K)} \\ &= \left\langle f, \overline{\frac{\partial K(q,\cdot)}{\partial u} + I\frac{\partial K(q,\cdot)}{\partial v}}\right\rangle_{\mathscr{H}(K)} = 0,\end{aligned}$$

since the kernel $K(q,p)$ is slice hyperholomorphic in the first variable q. □

5 Blaschke products

The space $\mathbf{H}_2(\mathbb{B})$ was introduced in [5] as the space of power series

$$f(p) = \sum_{n=0}^{\infty} p^n f_n,$$

where the coefficients f_n lie in $\mathbb{H}$ and satisfy

$$\|f\|_{\mathbf{H}_2(\mathbb{B})} \stackrel{\text{def.}}{=} \sqrt{\sum_{n=0}^{\infty} |f_n|^2} < \infty. \tag{5.1}$$

$\mathbf{H}_2(\mathbb{B})$ endowed with the inner product

$$[f,g]_2 = \sum_{n=0}^{\infty} \overline{g_n} f_n, \text{ where } g(p) = \sum_{n=0}^{\infty} p^n g_n,$$

is the right quaternionic reproducing kernel Hilbert space with reproducing kernel

$$k(p,q) = \sum_{n=0}^{\infty} p^n \overline{q}^n = (1 - p\overline{q})^{-\star}.$$

The norm (5.1) admits another expression.

Theorem 5.1. *The norm in* $\mathbf{H}_2(\mathbb{B})$ *can be written as*

$$\sup_{0<r<1,\ I\in\mathbb{S}} \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{I\theta})|^2\, d\theta\right]^{1/2} = \sup_{0<r<1} \left[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{I\theta})|^2\, d\theta\right]^{1/2}.$$

Proof. The equality is clear from the power series expansion of f with center at 0 and the Parseval identity. Thus the norm can be defined as in the classical complex case by computing the integral on a chosen complex plane. □

Let us prove some results associated to the Blaschke factors in the slice hyperholomorphic setting.

Definition 5.2. Let $a \in \mathbb{H}$, $|a| < 1$. The function

$$B_a(p) = (1 - p\bar{a})^{-\star} \star (a - p)\frac{\bar{a}}{|a|} \tag{5.2}$$

is called the **Blaschke factor** at a.

Lemma 5.3. *Let $a \in \mathbb{B}$. Then B_a is a slice hyperholomorphic function in $\mathbb{B}$. Furthermore,*

$$B_a(\overline{a})\overline{a} = \overline{a}B_a(\overline{a}). \tag{5.3}$$

Proof. Indeed, $B_a(p)$ is slice hyperholomorphic by its definition. Now

$$\begin{aligned} B_a(p) &= \Big(\sum_{n=0}^{\infty} p^n\overline{a}^n\Big) \star (a - p)\frac{\overline{a}}{|a|} \\ &= \sum_{n=0}^{\infty}(p^n\overline{a}^n a - p^{n+1}\overline{a}^n)\frac{\overline{a}}{|a|} \\ &= |a| + \sum_{n=0}^{\infty} p^{n+1}\overline{a}^{n+1}(|a| - \frac{1}{|a|}), \end{aligned} \tag{5.4}$$

from which (5.3) follows. □

Remark 5.4. Set $\lambda(p) = 1 - p\bar{a}$. Then

$$(1 - p\bar{a})^{-\star} = (\lambda^c(p) \star \lambda(p))^{-1}\lambda^c(p).$$

Applying formula (3.2) to the products $\lambda^c(p) \star \lambda(p)$ and $\lambda^c(p) \star (a - p)$, we can rewrite (5.2) as

$$\begin{aligned} B_a(p) &= (\lambda^c(p) \star \lambda(p))^{-1}\lambda^c(p) \star (a - p)\frac{\bar{a}}{|a|} = (\lambda^c(p)\lambda(\tilde{p}))^{-1}\lambda^c(p)(a - \tilde{p})\frac{\bar{a}}{|a|} \\ &= \lambda(\tilde{p})^{-1}(a - \tilde{p})\frac{\bar{a}}{|a|} = (1 - \tilde{p}\bar{a})^{-1}(a - \tilde{p})\frac{\bar{a}}{|a|}, \end{aligned} \tag{5.5}$$

where $\tilde{p} = \lambda^c(p)^{-1}p\lambda^c(p)$. Formula (5.5) represents the Blaschke factor $B_a(p)$ in terms of pointwise multiplication only.

Theorem 5.5. *Let $a \in \mathbb{H}$, $|a| < 1$. Then*
1. *B_a takes the unit ball $\mathbb{B}$ to itself;*
2. *B_a takes the boundary of the unit ball to itself;*
3. *B_a has a unique zero for $p = a$.*

Proof. By Remark 5.4 we can write $B_a(p) = (1 - \tilde{p}\bar{a})^{-1}(a - \tilde{p})\bar{a}/|a|$. Let us show that $|p| = |\tilde{p}| < 1$ implies $|B_a(p)|^2 < 1$. The latter inequality is equivalent to

$$|a - \tilde{p}|^2 < |1 - \tilde{p}\bar{a}|^2,$$

which is also equivalent to

$$|a|^2 + |p|^2 < 1 + |a|^2|p|^2. \tag{5.6}$$

Inequality (5.6) can be written as $(|p|^2 - 1)(1 - |a|^2) < 0$ and holds when $|p| < 1$. For $|p| = 1$, we set $p = e^{I\theta}$, so that $\tilde{p} = e^{I'\theta}$, by the proof of Corollary 3.3. Then

$$|B_a(e^{I\theta})| = |1 - e^{I'\theta}\bar{a}|^{-1}|a - e^{I'\theta}|\frac{|\bar{a}|}{|a|} = |e^{-I'\theta} - \bar{a}|^{-1}|a - e^{I'\theta}| = 1.$$

Finally, from (5.5) it follows that $B_a(p)$ has only one zero that comes from the factor $a - \tilde{p}$. Moreover $B_a(a) = (1 - \tilde{a}\bar{a})^{-1}(a - \tilde{a})\bar{a}/|a|$, where

$$\tilde{a} = (1 - a^2)^{-1}a(1 - a^2) = a,$$

and thus $B_a(a) = 0$. □

Theorem 5.6. *Let $\{a_j\} \subset \mathbb{B}$, $j = 1, 2, \ldots$, be a sequence of nonzero quaternions such that $[a_i] \neq [a_j]$ if $i \neq j$ and assume that $\sum_{j\geq 1}(1 - |a_j|) < \infty$. Then the function*

$$B(p) := \Pi^{\star}_{j\geq 1}(1 - p\bar{a}_j)^{-\star} \star (a_j - p)\frac{\bar{a}_j}{|a_j|}, \tag{5.7}$$

where $\Pi^{\star}$ denotes the $\star$-product, converges uniformly on compact subsets of $\mathbb{B}$.

Proof. Let $\alpha_j(p) := B_{a_j}(p) - 1$. Remark 5.4 yields the chain of equalities

$$\begin{aligned}\alpha_j(p) =& B_{a_j}(p) - 1 = (1 - \tilde{p}\bar{a}_j)^{-1}(a_j - \tilde{p})\frac{\bar{a}_j}{|a_j|} - 1\\ =&(1 - \tilde{p}\bar{a}_j)^{-1}\left[(a_j - \tilde{p})\frac{\bar{a}_j}{|a_j|} - (1 - \tilde{p}\bar{a}_j)\right]\\ =&(1 - \tilde{p}\bar{a}_j)^{-1}\left[(|a_j| - 1)\left(1 + \tilde{p}\frac{\bar{a}_j}{|a_j|}\right)\right].\end{aligned}$$

Thus, if $|p| < 1$ then $|\alpha_j(p)| \leq 2(1 - |p|)^{-1}(1 - |a_j|)$, since $|\tilde{p}| = |p|$, and since $\sum_{j=1}^{\infty}(1 - |a_j|) < \infty$, it follows that $\sum_{j=1}^{\infty}|\alpha_j(p)|$ converges in $\mathbb{B}$. □

Definition 5.7. The function $B(p)$ defined in (5.7) is called a **Blaschke product**.

Remark 5.8. In the complex case, the sequence of complex numbers $\{a_j\}$ turns out to be the sequence of zeroes of the Blaschke product. The quaternionic case is different, and we discuss it in the following results. In order to illustrate the differences, let us consider the simpler case of a polynomial

$$P(p) = (p - a_1) \star \ldots \star (p - a_n)$$

and assume that $[a_i] \neq [a_j]$ for all $i, j = 1, \ldots, n$. Then it can be verified that $p = a_1$ is a zero of the polynomial $P(p)$, while the other zeroes lie on the spheres $[a_j]$ defined by a_j for $j = 2, \ldots, n$. Note that in case all the elements a_j, $j = 1, \ldots, n$, lie on a common sphere and $a_{i+1} \neq \bar{a}_i$, the only zero of the polynomial is a_1 and it has multiplicity n; see [16, Lemme 2] or [52, Lemma 2.2.1]. Moreover, whenever a polynomial or, more generally, a slice hyperholomorphic function f has two zeroes belonging to the same 2-sphere, then *all* the elements of the sphere are zeroes of f. Thus the zeroes of a slice hyperholomorphic function are either isolated points or isolated spheres; see [33].

Assume that the slice hyperholomorphic function f has zero set

$$Z = \{a_1, a_2, \ldots\} \cup \{[c_1], [c_2], \ldots\}.$$

Then it is possible to construct a suitable Blaschke product having Z_f as zero set. Let us begin with the case in which the zeros are isolated points. We require of the following remark.

Remark 5.9. Direct computations show the following equality of polynomials:

$$(1 - pa) \star (a - p)\frac{\bar{a}}{|a|} = \left((a - p)\frac{\bar{a}}{|a|}\right) \star (1 - pa) = (a - p) \star (1 - pa)\frac{\bar{a}}{|a|}.$$

Proposition 5.10. *Let* $Z = \{a_1, a_2, \ldots\}$ *be a sequence of elements in* $\mathbb{B}$, $a_j \neq 0$ *for all* $j = 1, 2, \ldots$ *such that* $[a_i] \neq [a_j]$ *if* $i \neq j$, *and assume that* $\sum_{j\geq 1}(1 - |a_j|) < \infty$. *Then there exists a Blaschke product* $B(p)$ *whose zero set is* Z.

Proof. Let us prove the statement by induction. By hypothesis, the zero set of the required Blaschke product consists of isolated points, all of them belonging to different spheres. We have already proved that if $n = 1$, then $B_1(p) := B_{a_1}(p)$ has a_1 as its unique zero.

Let us now assume that the statement holds for $a_1, \dots, a_k$, so there exists a Blaschke product $B_k(p)$ vanishing at the given points. We construct a Blaschke product vanishing at $a_1, \dots, a_k, a_{k+1}$. Observe that it is possible to choose an element a'_{k+1} belonging to the sphere $[a_{k+1}]$ such that $B_k(p) \star B_{a'_{k+1}}(p)$ has zeros $a_1, \dots, a_{k+1}$. In fact, consider the product

$$B_{k+1}(p) := B_k(p) \star (1 - p\bar{a}'_{k+1})^{-\star} \star (a'_{k+1} - p)\frac{\bar{a}'_{k+1}}{|a'_{k+1}|}$$

and rewrite it using Remark 5.9 in the form

$$B_{k+1}(p) := B_k(p) \star (a'_{k+1} - p) \star (1 - pa'_{k+1})(1 - 2\mathrm{Re}(\bar{a}'_{k+1})p + |\bar{a}'_{k+1}|^2 p^2)^{-1}\frac{\bar{a}'_{k+1}}{|a'_{k+1}|}.$$

We now observe that the zeros of $B_{k+1}(p)$ belonging to the ball $\mathbb{B}$ come from the zeros of the product $\tilde{B}(p) := B_k(p) \star (a'_{k+1} - p)$. Also observe that

$$\tilde{B}(a_{k+1}) = B_k(a_{k+1})(a'_{k+1} - B_k(a_{k+1})^{-1} a_{k+1} B_k(a_{k+1})),$$

so in order for a_{k+1} to be a zero of $\tilde{B}$, and thus also of B_{k+1}, it suffices to choose

$$a'_{k+1} = B_k(a_{k+1})^{-1} a_{k+1} B_k(a_{k+1}).$$

The convergence of the Blaschke product follows as in Theorem 5.6. □

From now on, whenever we write $Z = \{(a, \mu)\}$, we mean that Z consists of the point a repeated μ times. Let us now prove the analog of Theorem 5.5(3) in the case a has multiplicity μ.

Lemma 5.11. *Let $Z = \{(a, \mu)\}$ with $a \in \mathbb{B}$ and $a \neq 0$. The Blaschke product*

$$\begin{aligned} B(p) &:= \Big((1 - p\bar{a})^{-\star} \star (a - p)\frac{\bar{a}}{|a|}\Big)^{\star\mu} \\ &:= \underbrace{\Big((1 - p\bar{a})^{-\star} \star (a - p)\tfrac{\bar{a}}{|a|} \quad \dots \quad (1 - p\bar{a})^{-\star} \star (a - p)\tfrac{\bar{a}}{|a|}\Big)}_{\mu \text{ times}} \end{aligned}$$

has zero set Z.

Proof. Since

$$(1 - p\bar{a})^{-\star} \star (a - p)\frac{\bar{a}}{|a|} = (1 - 2\mathrm{Re}(a)p + p^2|a|^2)^{-1}(1 - pa) \star (a - p)\frac{\bar{a}}{|a|},$$

using the fact that $1 - 2\mathrm{Re}(a)p + p^2|a|^2$ has real coefficients, we can write

$$B(p) = (1 - 2p\mathrm{Re}(a) + p^2|a|^2)^{-\mu}\underbrace{(1 - pa) \star (a - p)\frac{\bar{a}}{|a|} \cdots (1 - pa) \star (a - p)\frac{\bar{a}}{|a|}}_{\mu \text{ times}}.$$

Thanks to Remark 5.9, we obtain

$$B(p) = (1 - 2p\mathrm{Re}(a) + p^2|a|^2)^{-\mu}\left((a-p)\frac{\bar{a}}{|a|}\right)^{\star\mu} \star (1-pa)^{\mu}.$$

Thus $B(p)$ has a unique zero in $\mathbb{B}$ at $p = a$ of multiplicity μ. Note that the zero on the sphere $[1/a]$ which (as can be proved) coincides with $1/a$, has to be excluded since $B(p)$ is not defined there; moreover, $1/a \notin \mathbb{B}$. □

Proposition 5.12. *Let $Z = \{(a_1, \mu_1), (a_2, \mu_2), \ldots\}$ be a sequence of points $a_j \in \mathbb{B}$ with respective multiplicities $\mu_j \geq 1$, $a_j \neq 0$ for $j = 1, 2, \ldots$. Let a_j be such that $[a_i] \neq [a_j]$ if $i \neq j$ and $\sum_{j\geq 1} \mu_j(1 - |a_j|) < \infty$. Then there exists a Blaschke product of the form*

$$B(p) = \prod_{j\geq 1}^{\star} (B_{a'_j}(p))^{\star\mu_j}$$

having zero set Z. Here, $a'_1 = a_1$, and $a'_j \in [a_j]$ are suitably chosen elements, $j = 2, 3, \ldots$.

Proof. We prove the assertion by induction on the number of distinct zeros. If there is just one zero a_1 with multiplicity μ_1, then the statement follows by Lemma 5.11.

Let us next assume that the proposition holds in the case of k different zeros a_i with respective multiplicities μ_i, and prove that it holds in the case of $k + 1$ different zeros. Let $B_k(p)$ be the Blaschke product having zeros at $Z = \{(a_1, \mu_1), \ldots, (a_k, \mu_k)\}$ and consider $B_{k+1}(p) := B_k(p) \star (B_{a'_{k+1}}(p))^{\star\mu_k}$, where a'_{k+1} is chosen such that $B_k(p) \star B_{a'_{k+1}}(p)$ has a zero at $p = a_{k+1}$. Then all the other zeros of B_{k+1} must belong to the sphere $[a_{k+1}]$. Moreover, they must coincide with a_{k+1}. Otherwise, the Blaschke product $(B_{a'_{k+1}}(p))^{\star\mu_k}$ vanishes at two different points on the same sphere, and thus on the entire sphere. In particular, any two conjugate elements on the sphere are zeros of the product and so

$$\begin{aligned} B_a(p) \star B_{\bar{a}}(p) &= (1 - p\bar{a})^{-\star} \star (a - p)\frac{\bar{a}}{|a|} \star (1 - pa)^{-\star} \star (\bar{a} - p)\frac{a}{|a|} \\ &= (1 - 2\mathrm{Re}(a)p + p^2|a|^2)^{-1}(|a|^2 - 2\mathrm{Re}(a)p + p^2). \end{aligned}$$

However, it is immediate that the product $(B_{a'_{k+1}}(p))^{\star\mu_k}$ does not contain factors of the above form; thus all its zeros coincide with a_{k+1} as stated. The convergence of the Blaschke product now follows as in Theorem 5.6. □

If a Blaschke product of two factors has an entire sphere of zeros then, as discussed in the proof of the previous result, it has a specific form. We are thus led to the following definition.

Definition 5.13. Let $a \in \mathbb{H}$, $|a| < 1$. The function

$$B_{[a]}(p) = (1 - 2\mathrm{Re}(a)p + p^2|a|^2)^{-1}(|a|^2 - 2\mathrm{Re}(a)p + p^2) \tag{5.8}$$

is called a **Blaschke factor at the sphere** $[a]$.

Remark 5.14. Note that the definition of $B_{[a]}(p)$ does not depend on the choice of the point a that identifies the 2-sphere. Indeed, all the elements in the sphere $[a]$ have the same real part and module. It is easy to verify that the Blaschke factor $B_{[a]}(p)$ vanishes on the sphere $[a]$.

The following result is immediate.

Proposition 5.15. *A Blaschke product having zeros at the set of spheres*

$$Z = \{([c_1], \nu_1), ([c_2], \nu_2), \ldots\},$$

where $c_j \in \mathbb{B}$, the sphere $[c_j]$ is a zero of multiplicity ν_j, $j = 1, 2, \ldots$, and $\sum_{j\geq 1} \nu_j(1 - |c_j|) < \infty$, is given by $\prod_{j\geq 1}(B_{[c_j]}(p))^{\nu_j}$.

Proof. All the factors $B_{[c_j]}(p)$ have real coefficients and thus belong to the class $\mathcal{N}$ (see Section 4), so we can use the pointwise product. The fact that the zeros are the given spheres follows by taking the zeros of each factor. The convergence of the infinite product follows as in Theorem 5.6. □

Theorem 5.16. *A Blaschke product having zeros at the set*

$$Z = \{(a_1, \mu_1), (a_2, \mu_2), \ldots, ([c_1], \nu_1), ([c_2], \nu_2), \ldots\}$$

where $a_j \in \mathbb{B}$, a_j have respective multiplicities $\mu_j \geq 1$, $a_j \neq 0$ for $j = 1, 2, \ldots$, $[a_i] \neq [a_j]$ if $i \neq j$, $c_i \in \mathbb{B}$, the spheres $[c_j]$ have respective multiplicities $\nu_j \geq 1$, $j = 1, 2, \ldots$, $[c_i] \neq [c_j]$ if $i \neq j$ and $\sum_{i,j\geq 1} \big(\mu_i(1 - |a_i|) + \nu_j(1 - |c_j|)\big) < \infty$, is given by $\prod_{i\geq 1}(B_{[c_i]}(p))^{\nu_i} \prod^{\star}_{j\geq 1}(B_{a'_j}(p))^{\star\mu_j}$, where $a'_1 = a_1$ and $a'_j \in [a_j]$ are suitably chosen elements, $j = 2, 3, \ldots$.

Proof. The theorem follows from Propositions 5.10 and 5.12.

Theorem 5.17. *Let B_a be a Blaschke factor. The operator $M_a : f \mapsto B_a \star f$ is an isometry from $\mathbf{H}_2(\mathbb{B})$ into itself.*

Proof. We first consider $f(p) = p^u h$ and $g(p) = p^v k$, where $u, v \in \mathbb{N}_0$ and $h, k \in \mathbb{H}$, and show that

$$[B_a \star f, B_a \star g]_2 = \delta_{uv}\overline{k}h. \tag{5.9}$$

Using (5.4), and with f and g as above, we have

$$(B_a \star f)(p) = p^u h|a| + \sum_{n=0}^{\infty} p^{n+1+u}\overline{a}^{n+1}(|a| - \frac{1}{|a|})h$$

and

$$(B_a \star g)(p) = p^{v} k|a| + \sum_{n=0}^{\infty} p^{n+1+v}\overline{a}^{n+1}(|a| - \frac{1}{|a|})k. \tag{5.10}$$

If $u = v$, then

$$[B_a \star f, B_a \star g]_2 = \overline{k}h\left(|a|^2 + \sum_{n=0}^{\infty} |a|^{2n+2}(|a| - \frac{1}{|a|})^2\right) = \overline{k}h = [f, g]_2.$$

To compute $[f, g]_2$, we assume that $u < v$. Then, in view of (5.10), we have

$$[p^u h|a|, B_a \star g]_2 = 0.$$

Thus,

$$\begin{aligned}
[B_a \star f, B_a \star g]_2 &= [\sum_{n=0}^{\infty} p^{n+1+u}\overline{a}^{n+1}\left(|a| - \frac{1}{|a|}\right)h, p^{v}|a|k]_2 + \\
&\quad + [\sum_{n=0}^{\infty} p^{n+1+u}\overline{a}^{n+1}\left(|a| - \frac{1}{|a|}\right)h, \sum_{m=0}^{\infty} p^{m+1+v}\overline{a}^{m+1}\left(|a| - \frac{1}{|a|}\right)k]_2 \\
&= |a|\overline{k}\overline{a}^{v-u}\left(|a| - \frac{1}{|a|}\right)h + \\
&\quad + [\sum_{m=0}^{\infty} p^{m+1+v}\overline{a}^{m+1+v-u}\left(|a| - \frac{1}{|a|}\right)h, \sum_{m=0}^{\infty} p^{m+1+v}\overline{a}^{m+1}\left(|a| - \frac{1}{|a|}\right)k]_2 \\
&= |a|\overline{k}\overline{a}^{v-u}\left(|a| - \frac{1}{|a|}\right)h + \overline{k}\left(|a| - \frac{1}{|a|}\right)^2\overline{a}^{v-u}\frac{|a|^2}{1-|a|^2}h \\
&= 0 = [f, g]_2.
\end{aligned}$$

The case $v < u$ is handled by symmetry of the inner product. Hence, (5.9) holds for polynomials. By continuity and a corollary of Runge's theorem (see [29]), it holds for all $f \in \mathbf{H}_2(\mathbb{B})$. □

Similar results hold for bicomplex numbers; see [8].

Finally we note the following. In the classical case, Blaschke factors have counterparts in the matrix-valued case. More precisely, a rational matrix-valued function which takes unitary values (with respect to a possibly indefinite metric) on the unit circle and has McMillan degree 1 is called a **Blaschke-Potapov**

factor of the first kind if its unique pole is outside $\mathbb{D}$, a **Blaschke-Potapov factor of the second kind** the second kind if the pole lies inside $\mathbb{D}$, and a **Blaschke-Potapov factor of the third kind** if the pole lies on the unit circle. These terms originate with the work of Potapov [55]; also see [36]. The results in this section are a stepping stone toward the study of these notions in the slice-hyperholomorphic setting.

6 Homogeneous interpolation in the Hardy space

In this section, we consider homogeneous interpolation in the space $\mathbf{H}_2(\mathbb{B})$. We first briefly discuss the Beurling-Lax theorem to set the problem in perspective. Beurling's theorem (see [17], [57]) characterizes the closed, shift-invariant, nontrivial subspaces of the Hardy space of the disk $\mathbf{H}_2(\mathbb{D})$. These are spaces of the form $\mathcal{M} = j\mathbf{H}_2(\mathbb{D})$, where the function j is inner, i.e., analytic and bounded by 1 in modulus in the open unit disk, and with non-tangential limits everywhere of modulus 1. Thus $\mathcal{M}$ has reproducing kernel $j(z)\overline{j(w)}/(1-z\overline{w})$, and its orthogonal complement has reproducing kernel

$$\frac{1-j(z)\overline{j(w)}}{1-z\overline{w}}. \tag{6.1}$$

In the vector-valued version of Beurling's theorem (that is, the Beurling-Lax Theorem), j is now operator-valued and takes isometric boundary values on the unit circle; see, for instance, [56, Theorem A, p. 98]. In another direction, reproducing kernel Hilbert spaces of vector-valued functions with reproducing kernel of the form (6.1) (where now j need not take isometric values on the boundary, but is merely contractive in the open unit disk) were characterized by de Branges and Rovnyak in [19] in the scalar case; see also the papers of Guyker [40, 41]. In the case of vector-valued functions, they have been characterized in [6, Theorem 3.1.2] in the setting of Pontryagin spaces. The counterpart of the result of [6] in the setting of slice hyperholomorphic functions is presented in the preprint [4].

The problem considered in this section is set in the scalar case and gives a characterization of the family of subspaces of $\mathbf{H}_2(\mathbb{B})$ which are invariant under slice multiplication by the variable; see Problem 6.1 and Theorem 6.2. This amounts to the simplest Beurling-Lax type theorem in the present setting. We note that the computations made in Section 9 below (see, in particular, Theorem 9.4) help to obtain finite dimensional matrix-valued versions of Theorem 6.2. In the study of the infinite dimensional case, one encounters problems such as the counterpart of singular inner functions in the slice holomorphic case. These problems will be considered in [2].

Problem 6.1. Given N points $a_1, \ldots, a_N \in \mathbb{B}$, and M spheres $[c_1], \ldots, [c_M]$ in $\mathbb{B}$ such that the spheres $[a_1], \ldots [a_N], [c_1], \ldots, [c_M]$ are pairwise disjoint, find all $f \in \mathbf{H}_2(\mathbb{B})$ such that

$$f(a_i) = 0, \quad i = 1, \ldots, N, \tag{6.2}$$

and

$$f([c_j]) = 0, \quad j = 1, \ldots, M. \tag{6.3}$$

Theorem 6.2. *There is a Blaschke product B such that the solutions of Problem 6.1 are the functions $f = B \star g$, when g runs through $\mathbf{H}_2(\mathbb{B})$.*

Iterative proof. We proceed in three steps. As a preliminary computation, we consider in the first step the case $N = 1$ and $M = 0$. The problem itself is solved by considering the interpolation at the spheres first.

STEP 1. *We solve the problem for $M = 0$ and $N = 1$.*

Let B_{a_1} be the Blaschke factor (5.2) at a_1. By (3.2), we have $(B_{a_1} \star f)(a_1) = 0$ for all $f \in \mathbf{H}_2(\mathbb{B})$. Furthermore, by Theorem 5.17, $\|B_{a_1} \star f\|_2 = \|f\|_2$. Thus, for $N = 1$, the set $\mathcal{M}$ of solutions of Problem 6.1 contains $B_{a_1} \star \mathbf{H}_2(\mathbb{B})$.

We now prove that $\mathcal{M} \subseteq B_{a_1} \star \mathbf{H}_2(\mathbb{B})$. Let $f \in \mathcal{M}$. By the reproducing kernel property, f is orthogonal to $(1 - p\overline{a_1})^{-\star}$. The range ran $\sqrt{I - M_{a_1}M_{a_1}^*}$ is equal to the span of $(1 - p\overline{a_1})^{-\star}$; see [5]. In view of Theorem 5.17, we have $\sqrt{I - M_{a_1}M_{a_1}^*} = I - M_{a_1}M_{a_1}^*$, and thus $\mathbf{H}_2(\mathbb{B}) = (I - M_{a_1}M_{a_1}^*)\mathbf{H}_2(\mathbb{B}) \oplus (M_{a_1}M_{a_1}^*)\mathbf{H}_2(\mathbb{B})$. Therefore, $f \in (M_{a_1}M_{a_1}^*)\mathbf{H}_2(\mathbb{B})$. Hence $\mathcal{M} = B_{a_1} \star \mathbf{H}_2(\mathbb{B})$.

With this preliminary computation in hand, we solve the interpolation problem by first considering the interpolation at the spheres $[c_1], \ldots, [c_M]$.

STEP 2. *Consider the sphere $[c_j]$, and let $B_{[c_j]}$ be the corresponding Blaschke factor given by* (5.8)*, $j = 1, 2, \ldots, M$. An element $f \in \mathbf{H}_2(\mathbb{B})$ vanishes on the spheres $[c_1], \ldots, [c_M]$ if and only it can be written as*

$$f = B_{[c_1]}B_{[c_2]} \cdots B_{[c_M]}g, \tag{6.4}$$

where $g \in \mathbf{H}_2(\mathbb{B})$.

Note that in (6.4), we have pointwise products since the Blaschke factors on spheres have real coefficients. By [33, Corollary 4.3.7, p. 123], f vanishes on the entire sphere $[c_1]$ if and only if $f(c_1) = f(\overline{c_1}) = 0$. By STEP 1, the first condition means that $f = B_{c_1} \star g$ for some $g \in \mathbf{H}_2(\mathbb{B})$. By (3.2), the second condition is equivalent to

$$B_{c_1}(\overline{c_1})g((B_{c_1}(\overline{c_1}))^{-1}\overline{c_1}B_{c_1}(\overline{c_1})) = 0. \tag{6.5}$$

Since $B_{c_1}(\overline{c_1}) \neq 0$, and because of (5.3), (6.5) is equivalent to $g(\overline{c_1}) = 0$. Thus, once more using STEP 1, we have $g(p) = B_{\overline{c_1}} \star h$ for some $h \in \mathbf{H}_2(\mathbb{B})$. Therefore,

$$f = B_{c_1} \star B_{\overline{c_1}} \star h = B_{[c_1]}h.$$

This argument can be iterated for the spheres $[c_2], \ldots, [c_M]$ since $B_{[c_2]}(c_1) \neq 0$ (which in turn follows from the fact that the spheres do not intersect).

We now turn to the conditions (6.2). The function f is of the form (6.4), and thus the condition $f(a_1) = 0$ becomes

$$\left(B_{[c_1]}B_{[c_2]} \cdots B_{[c_M]}\right)(a_1)g(a_1) = 0.$$

Hence, by STEP 1, $g = B_{a_1} \star g_1$ for some $g_1 \in \mathbf{H}_2(\mathbb{B})$. Now suppose $f \in \mathbf{H}_2(\mathbb{B})$ satisfies $f(a_2) = 0$. By the previous argument, f is of the form

$$\left(B_{[c_1]}B_{[c_2]} \cdots B_{[c_M]}\right) B_{a_1} \star g_1$$

for some $g_1 \in \mathbf{H}_2(\mathbb{B})$. The condition $f(a_2) = 0$ and formula (3.2) give

$$g(a_2') = 0, \quad \text{where} \quad a_2' = X^{-1}a_2X,$$

with

$$X = \left(B_{[c_1]}B_{[c_2]} \cdots B_{[c_M]}B_{a_1}\right)(a_2).$$

Hence f is a solution if and only if $g_2 = B_{a_2'} \star g_2$ for some $g_1 \in \mathbf{H}_2(\mathbb{B})$. Iterating this argument, we obtain the set of functions $f \in \mathbf{H}_2(\mathbb{B})$ which vanish at the points $a_1, \ldots, a_N$. □

Remark 6.3. It is also possible to avoid iteration and to give a "global" proof of Theorem 6.2. In other words, one can construct directly from the interpolation data an N-dimensional backward shift invariant subspace, whose structure leads to the function B.

7 Quaternionic Pontryagin spaces

Quaternionic Pontryagin spaces have been studied in [11]. In this section, we review the main definitions and prove, in the setting of quaternionic spaces, an important result due to Shmulyan in the complex setting; see [58] and [6, Theorem 1.4.2]. Consider a right vector space $\mathscr{P}$ on the quaternions endowed with a $\mathbb{H}$-valued Hermitian form $[\cdot, \cdot]$, meaning that $[va, wb] = \overline{b}[v, w]a$ for all $a, b \in \mathbb{H}$ and $v, w \in \mathscr{P}$. $\mathscr{P}$ is called a **(right, quaternionic) Pontryagin space** if it admits a decomposition

$$\mathscr{P} = \mathscr{P}_+ + \mathscr{P}_-, \tag{7.1}$$

where $\mathscr{P}_+$ and $\mathscr{P}_-$ are subspaces such that

(1) $(\mathscr{P}_+, [\cdot, \cdot])$ is a (right, quaternionic) Hilbert space,
(2) $(\mathscr{P}_-, -[\cdot, \cdot])$ is a finite dimensional (right, quaternionic) Hilbert space,
(3) the sum (7.1) is direct and orthogonal, i.e., $\mathscr{P}_+ \cap \mathscr{P}_- = \{0\}$ and $[v_+, v_-] = 0$ for all $v_+ \in \mathscr{P}_+$ and $v_- \in \mathscr{P}_-$.

The space $\mathscr{P}$ endowed with the form

$$\langle v, w\rangle = [v_+, v_-] - [w_+, w_-], \quad v = v_+ + v_-, \ w = w_+ + w_-$$

is a (right quaternionic) Hilbert space. The decomposition (7.1) is called a **fundamental decomposition**. It is not unique (except for the case where one of the components reduces to $\{0\}$), but all the corresponding Hilbert space topologies are equivalent; see [11, Theorem 12.3]. The number $\kappa = \dim \mathscr{P}_-$ is called the **index** of the Pontryagin space $\mathscr{P}$. It is the same for all the decompositions; see [11, Proposition 12.6]. The reader should be aware that in some of the literature on the complex-valued case, in particular in [18], [44], the convention is the opposite and it is the space $\mathscr{P}_+$ that is required to be finite dimensional.

Example 7.1. Let $J \in \mathbb{H}^{N\times N}$ be a signature matrix. The space $\mathbb{H}^N$ endowed with the Hermitian form $[v, w]_J = w^* J v$ is a right quaternionic Pontryagin space, which we denote by $\mathbb{H}^N_J$.

Before turning to Shmulyan's theorem, we recall the following definitions. Given two right quaternionic Pontryagin spaces $(\mathscr{P}_1, [\cdot, \cdot]_1)$ and $(\mathscr{P}_2, [\cdot, \cdot]_2)$ a **linear relation** between $\mathscr{P}_1$ and $\mathscr{P}_2$ is a right linear subspace, say R, of the product $\mathscr{P}_1 \times \mathscr{P}_2$. The **domain** of the relation is the set of elements $v_1 \in \mathscr{P}_1$ such that there exists a (not necessarily unique) $v_2 \in \mathscr{P}_2$ satisfying $(v_1, v_2) \in R$. The relation is called **contractive** if $[v_1, v_1]_1 \leq [v_2, v_2]_2$ for all $(v_1, v_2) \in R$. One example of a relation is the graph of an operator. A relation is the **graph** of an operator if and only it has no elements of the form $(0, v_2)$ with $v_2 \neq 0$.

Theorem 7.2. *A densely defined contractive relation between quaternionic Pontryagin spaces of the same index extends to the graph of a contraction from $\mathscr{P}_1$ into $\mathscr{P}_2$.*

Proof. Following the strategy of [6, pp. 29–30], we divide the proof into a number of steps. Recall that a **strictly negative subspace** is a linear subspace V such that $[v, v] < 0$ for every non-zero element of V.

STEP 1. *The domain of the relation contains a maximum negative subspace.*

Indeed, every dense linear subspace of a right quaternionic Pontryagin space of index $\kappa > 0$ contains a κ dimensional strictly negative subspace; see [11, Theorem 12.8 p. 470]. We denote such a subspace of the domain of R by $\mathscr{V}_-$.

STEP 2. *The relation R restricted to $\mathscr{V}_-$ has zero kernel, and the image of $\mathscr{V}_-$ is a strictly negative subspace of $\mathscr{P}_2$ of dimension κ.*

Let $(v_1, v_2) \in R$ with $v_1 \in \mathscr{V}_-$. Since R is contractive, $[v_2, v_2]_2 \leq [v_1, v_1]_1 \leq 0$, and the second inequality is strict when $v_1 \neq 0$. Thus, the image of $\mathscr{V}_-$ is a strictly negative subspace of $\mathscr{P}_2$. Next, let $(v, w) \in R$ and $(\widetilde{v}, w)$ with $v, \widetilde{v} \in \mathscr{V}_-$ and $w \in \mathscr{P}_2$. Then $(v - \widetilde{v}, 0) \in R$. Since R is contractive, $[0, 0]_2 \leq [v - \widetilde{v}, v - \widetilde{v}]_1$. This forces $v = \widetilde{v}$, since $\mathscr{V}_-$ is strictly negative, and proves the second step.

STEP 3. *R is the graph of a densely defined contraction.*

Choose $\mathscr{V}_-$ as in the first two steps, and take a basis $v_1, \ldots, v_\kappa$ of $\mathscr{V}_-$. There exist uniquely defined vectors $w_1 \ldots, w_\kappa \in \mathscr{P}_2$ such that $(v_i, w_i) \in R$ for $i = 1, \ldots, \kappa$. Let $\mathscr{W}_-$ be the linear span of $w_1, \ldots, w_\kappa$. By STEP 2 and since the spaces $\mathscr{P}_1$ and $\mathscr{P}_2$ have the same negative index,

$$\dim \mathscr{V}_- = \dim \mathscr{W}_- = \mathrm{ind}_- \mathscr{P}_1 = \mathrm{ind}_- \mathscr{P}_2,$$

and there exists fundamental decompositions $\mathscr{P}_1 = \mathscr{V}_- + \mathscr{V}_+$ and $\mathscr{P}_2 = \mathscr{W}_- + \mathscr{W}_+$, where $(\mathscr{V}_+, [\cdot, \cdot]_1)$ and $(\mathscr{W}_+, [\cdot, \cdot]_2)$ are right quaternionic Hilbert spaces. Now let $(0, w) \in R$. We need to show that $w = 0$. Still following [6, p. 30], we write $w = w_- + w_+$, where $w_- \in \mathscr{W}_-$ and $w_+ \in \mathscr{W}_+$. Let $w_- = \sum_{n=1}^{\kappa} w_j q_j$, where each $q_j \in \mathbb{H}$, and set $v_- = \sum_{n=1}^{\kappa} v_j q_j$. Then $(v_-, w_-) \in R$, and

$$(0, w) = (v_-, w_-) + (-v_-, w_+).$$

It follows that $(-v_-, w_+) \in R$. Now, since R is contractive, $[w_+, w_+]_2 \leq [v_-, v_-]_1$, and so $[w_+, w_+]_2 \leq 0$. Thus $w_+ = 0$. It follows that $(0, w_-) \in R$, and so $w_- = 0$ because R is one-to-one on $\mathscr{V}_-$, as follows from STEP 2.

STEP 4. *R extends to the graph of an everywhere defined contraction.*

In the complex case, this is [6, Theorem 1.4.1]. Following the arguments there, we consider the orthogonal projection from $\mathcal{P}_2$ onto $\mathscr{W}_-$. Let T be the densely defined contraction with graph the relation R. There exist $\mathbb{H}$-valued right linear functionals $c_1, \ldots, c_\kappa$, defined on the domain of R such that

$$Tv = \sum_{n=1}^{\kappa} f_n c_n(v) + w_+,$$

where $w_+ \in \mathscr{W}_+$ satisfies $[f_n, w_+]_2 = 0$ for $n = 1, 2, \ldots, \kappa$. Assume that c_1 is unbounded on its domain, let v_+ be such that $c_1(v_+) = 1$, and let v_n be vectors in $\mathscr{V}_+$ such that $c_1(v_n) = 1$ and $\lim_{n\to\infty}[v_+ - v_n, v_+ - v_n]_1 = 0$. Then v_+ belongs to the closure of $\ker c_1$, and so the closure of $\ker c_1$ equals $\mathscr{V}_+$. Thus $\ker c_1$ contains a strictly negative subspace, say $\mathscr{K}_-$, of dimension κ. For $v \in \mathscr{K}_-$, we have $Tv = \sum_{n=2}^{n} f_n c_n(v)$. This contradicts STEP 2 and completes the proof. □

8 Negative squares

The notion of kernels with a finite number of negative squares extends the notion of positive definite kernels. For this notion in the quaternionic case, we refer the reader to [11, Section 11]. Recall that an $\mathbb{H}^{N\times N}$ Hermitian matrix A has only real (right) eigenvalues. We denote by $\mathrm{sq}_-(A)$ the number of strictly negative eigenvalues.

Definition 8.1. Let $\kappa \in \mathbb{N}_0$. An $\mathbb{H}^{N\times N}$-valued function $K(z,\omega)$ defined for z, ω in the set $\Omega \subseteq \mathbb{H}$ is said to have κ negative squares if the following hold.

(i) $K(z,w)$ is Hermitian, i.e.,

$$K(z,w) = K(w,z)^*, \quad \text{for all } z, w \in \Omega; \tag{8.1}$$

(ii) for every $N \in \mathbb{N}$ and every choice of $z_1,\ldots,z_N \in \Omega$ and $c_1,\ldots,c_N \in \mathbb{H}^N$, the $N\times N$ Hermitian matrix with ℓ, j entry equal to $c_\ell^* K(z_\ell, z_j)c_j$ has at most κ strictly negative eigenvalues;

(iii) for some choice of $N \in \mathbb{N}$, $z_1,\ldots,z_N \in \Omega$ and $c_1,\ldots,c_N \in \mathbb{H}^N$, the $N\times N$ Hermitian matrix with ℓ, j entry equal to $c_\ell^* K(z_\ell, z_j)c_j$ has exactly κ strictly negative eigenvalues.

We usually call such $K(z,w)$'s a *kernel* rather than *function*. When $\kappa = 0$, the kernel $K(z,w)$ is positive definite.

In the following theorem, we recall the following quaternionic counterparts of results well known in the complex case. We first give a definition. A positive definite $\mathbb{H}^{N\times N}$-valued function $Q(z,w)$ is said to be of **finite rank** if it can be factored as $Q(z,w) = N(z,w)^* N(z,w)$, where $N(z,w)$ is $\mathbb{H}^{N\times M}$-valued for some $M \in \mathbb{N}$. The smallest such M is called the **rank** of Q.

Theorem 8.2. (a) *Let $K(z,w)$ be an Hermitian $\mathbb{H}^{N\times N}$-valued function (see* (8.1)*) for z, w in some set $\Omega \subseteq \mathbb{H}$. Then K has κ negative squares if and only if it can be written as a difference $K(z,w) = K_+(z,w) - K_-(z,w)$, where both K_+ and K_- are positive definite in Ω and K_- is of finite rank.*

(b) *There is a one-to-one correspondence between right quaternionic reproducing kernel Pontryagin spaces of index κ, of $\mathbb{H}^N$-valued functions on a set Ω, and $\mathbb{H}^{N\times N}$-valued functions with κ negative squares in Ω.*

For a proof of these facts, see [11, Theorems 11.5, and 13.1].

Theorem 8.3. *Let $K(p,q)$ be a $\mathbb{H}^{N\times N}$-valued function with κ negative squares in an open non-empty subset Ω of $\mathbb{H}$. Then there exists a unique right quaternionic reproducing kernel Pontryagin space $\mathscr{P}$ consisting of $\mathbb{H}^N$-valued function slice hyperholomorphic in Ω and with reproducing kernel $K(p,q)$.*

Proof. The fact that there exists a unique Pontryagin space $\mathcal{P}$ associated to K follows as in [11, Theorem 13.1]. We have to show that the elements in $\mathscr{P}$ are slice hyperholomorphic. Let $\overset{\circ}{\mathscr{P}}(K)$ be the linear span of the functions of the form $p \mapsto K(p,q)a$ where $q \in \Omega$ and $a \in \mathbb{H}^N$. Since K has κ negative squares, $\overset{\circ}{\mathscr{P}}(K)$ has a maximal strictly negative subspace $\mathscr{N}_-$ of dimension κ. By [11, Proposition 10.3], it is possible to write $\overset{\circ}{\mathscr{P}}(K) = \mathscr{N}_- + \mathscr{N}_-^{[\perp]}$, where $\mathscr{N}_-^{[\perp]}$ is a quaternionic pre-Hilbert space. The space $\mathscr{N}_-^{[\perp]}$ has a unique completion, denoted by $\mathscr{N}_+$. Let us define $\mathscr{P} := \mathscr{N}_+ + \mathscr{N}_-$, with the inner product $[f,f] := [f_+, f_+]_{\mathscr{N}_+} + [f_-, f_-]_{\mathscr{N}_-}$, where $f = f_+ + f_-$, $f_\pm \in \mathscr{N}_\pm$. If $f_1, \ldots, f_\kappa$ is an orthonormal basis of $\mathscr{N}_-$, then

$$K(p,q) - \sum_{j=1}^{\kappa} f_j(p) f_j(q)^* \tag{8.2}$$

is a reproducing kernel for $\mathscr{N}_+$. The functions $f_j(p)$ are clearly slice hyperholomorphic in p since they belong to $\overset{\circ}{\mathscr{P}}(K)$, and so are the products $f_j(p)f_j(q)^*$ as well as the kernel (8.2). Therefore, the elements in $\mathscr{N}_+$ are slice hyperholomorphic, and so are the elements in $\mathscr{P}$. □

9 Generalized Schur functions

Definition 9.1. Let J_1 and J_2 be two signature matrices, respectively in $\mathbb{H}^{N\times N}$ and $\mathbb{H}^{M\times M}$, and assume that $\mathrm{sq}_- J_1 = \mathrm{sq}_- J_2$. A $\mathbb{H}^{N\times M}$-valued function Θ, slice hyperholomorphic in a neighborhood $\mathcal{V}$ of the origin, is called a **generalized Schur function** if the kernel

$$K_\Theta(p,q) = \sum_{\ell=0}^{\infty} p^\ell (J_2 - \Theta(p) J_1 \Theta(q)^*) \overline{q}^\ell$$

has a finite number, say κ, of negative squares in $\mathcal{V}$.

We denote the class of such functions by $\mathscr{S}_\kappa(J_2, J_1)$ This class was introduced in [5] for the case $N = M = 1$, $\kappa = 0$ and $J_1 = J_2 = 1$. A pair of operators (C, A) between appropriate spaces is called **observable** if

$$\bigcap_{n=0}^{\infty} \ker CA^n = \{0\}\,. \tag{9.1}$$

In the next result, we make use of the multiplication of operator valued slice hyperholomorphic functions. This multiplication is studied in detail in [2].

Theorem 9.2. *Let Θ be slice hyperholomorphic in a neighborhood of the origin. Then, it is in $\mathscr{S}_\kappa(J_2, J_1)$ if and only if it can written in the form*

$$\Theta(p) = D + pC \star (I_{\mathscr{P}} - pA)^{-\star}B,$$

where $\mathscr{P}$ is a right quaternionic Pontryagin space of index κ, the pair (C, A) is observable, and the operator matrix satisfies

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_{\mathscr{P}} & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} I_{\mathscr{P}} & 0 \\ 0 & J_1 \end{pmatrix}. \tag{9.2}$$

Proof. Let $\mathscr{P}(\Theta)$ be the right quaternionic reproducing kernel Pontryagin space with reproducing kernel $K_\Theta(p, q)$. Following the proof of [6, Theorem 2.2.1], we use the same densely defined linear relation as was used in [5], but this time in $(\mathscr{P}(\Theta) \oplus \mathbb{H}^M_{J_2}) \times (\mathscr{P}(\Theta) \oplus \mathbb{H}^N_{J_1})$. More precisely, now

$$R = \left\{ \begin{pmatrix} K_\Theta(p, q)\overline{q}u \\ \overline{q}v \end{pmatrix}, \begin{pmatrix} (K_\Theta(p, q) - K_\Theta(p, 0))u + K_\Theta(p, 0)\overline{q}v \\ (\Theta(q)^* - \Theta(0)^*)u + \Theta(0)^*\overline{q}v \end{pmatrix} \right\}.$$

Since $\mathrm{sq}_-(J_1) = \mathrm{sq}_-(J_2)$, these Pontryagin spaces have same negative index The proof then follows from Shmulyan's result. The arguments are similar to those in [5] and are thus omitted. $\square$

We now characterize finite dimensional $\mathscr{P}(s)$ spaces. We begin with a preliminary proposition.

Proposition 9.3. *Let*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & J_1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} H & 0 \\ 0 & J_2 \end{pmatrix} \tag{9.3}$$

and

$$s(p) = D + pC \star (I - pA)^{-\star}B. \tag{9.4}$$

Then

$$J_2 - s(p)J_1 s(q)^* = C \star (I - pA)^{-\star} \star (H - pH\overline{q}) \star_r \left((I - qA)^{-\star}\right)^* \star_r C^*.$$

Proof. We rewrite the matrix identity (9.3) as

$$\begin{aligned} J_2 - DJ_1D^* &= CHC^* \\ BJ_1B^* &= H - AHA^* \\ AHC^* &= -BJ_1D^*. \end{aligned}$$

Let $s(p)$ be given by (9.4), and consider the function $J_2 - s(p)J_1s(q)^*$, which is slice hyperholomorphic in p and $\overline{q}$ on the left and on the right, respectively. Now

$$J_2 - s(p)J_1s(q)^* = J_2 - (D + pC\star(I - pA)^{-\star}B)J_1(D + qC\star(I - qA)^{-\star}B)^*.$$

In order to preserve the hyperholomorphicity in p, $\overline{q}$ we take, accordingly, the $\star$-product in p and the $\star_r$-product in $\overline{q}$ and obtain

$$\begin{aligned} &J_2 - s(p)J_1s(q)^* \\ &\quad = J_2 - (D + pC\star(I - pA)^{-\star}B)J_1(D^* + B^*\star_r((I - qA)^{-\star})^*\star_r C^*\bar{q}) \\ &\quad = J_2 - DJ_1D^* - pC\star(I - pA)^{-\star}BJD^* - DJ_1B^*\star_r((I - qA)^{-\star})^*\star_r C^*\bar{q} \\ &\qquad - pC\star(I - pA)^{-\star}BJ_1B^*\star_r((I - qA)^{-\star})^*\star_r C^*\bar{q}. \end{aligned}$$

Using the relations implied by (9.3) and the identities (9.6), we obtain

$$\begin{aligned} &J_2 - s(p)J_1s(q)^* \\ &= CHC^* + pC\star(I - pA)^{-\star}AHC^* + CHA^*\star_r((I - qA)^{-\star})^*\star_r C^*\bar{q} \\ &\quad - pC\star(I - pA)^{-\star}(H - AHA^*)\star_r((I - qA)^{-\star})^*\star_r C^*\bar{q} \\ &= C\star(I - pA)^{-\star}\star\Big[(I - pA)HC^* + pAHC^* + (I - pA)HA^*\star_r((I - qA)^{-\star})^*\star_r C^*\bar{q} \\ &\quad - p(H - AHA^*)\star_r((I - qA)^{-\star})^*\star_r C^*\bar{q}\Big] \\ &= C\star(I - pA)^{-\star}\star\Big[(I - pA)H(I - qA)^* + pAH(I - qA)^* + (I - pA)HA^*\bar{q} \\ &\quad - p(H - AHA^*)\bar{q}\Big]\star_r((I - qA)^{-\star})^*\star_r C^* \\ &= C\star(I - pA)^{-\star}\star\Big[H - HA^*\bar{q} - pAH + pAHA^*\bar{q} + pAH \\ &- pAHA^*\bar{q} + HA^*\bar{q} - pAHA^*\bar{q} - pH\bar{q} + pAHA^*\bar{q}\Big]\star_r((I - qA)^{-\star})^*\star_r C^* \\ &= C\star(I - pA)^{-\star}\star(H - pH\bar{q})\star_r((I - qA)^{-\star})^*\star_r C^*. \end{aligned}$$

We can also write, equivalently.

$$\begin{aligned} J_2 - s(p)J_1s(q)^* &= C\star(I - pA)^{-\star}H\star(1 - p\bar{q})\star_r(I - qA)^{-\star}\star_r C^* \\ &= C\star(I - pA)^{-\star}\star(1 - p\bar{q})\star_r H((I - qA)^{-\star})^*\star_r C^*, \end{aligned}$$

or

$$J_2 - s(p)J_1s(q)^* = (C\star(I-pA)^{-\star})\star(H-pH\bar{q})\star_r(C\star(I-qA)^{-\star})^*.$$

□

Specializing Theorem 9.2 to the finite dimensional case we obtain the following result.

Theorem 9.4. *Let s be a generalized Schur function. The associated right reproducing kernel Pontryagin space $\mathscr{P}(s)$ is finite dimensional if and only there exists a finite dimensional right Pontryagin space $\mathscr{P}$ such that*

$$s(p) = D + pC\star(I-pA)^{-\star}B,$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathscr{P}\oplus\mathbb{H}^M_{J_2} \longrightarrow \mathscr{P}\oplus\mathbb{H}^N_{J_1}$$

is coisometric, i.e.,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I_{\mathscr{P}} & 0 \\ 0 & J_2 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} I_{\mathscr{P}} & 0 \\ 0 & J_1 \end{pmatrix}. \tag{9.5}$$

Proof. One half of the theorem follows from the Propositioin 9.3, while the other half is a special case of Theorem 9.2. □

Here we focus on the case $M = N$ and $\mathscr{P}(s)$ finite dimensional.

Definition 9.5. Let $J \in \mathbb{H}^{N\times N}$ be a signature function. The $\mathbb{H}^{N\times N}$-valued generalized function s belongs to $\mathfrak{U}_\kappa(J)$ if the space $\mathscr{P}(s)$ is finite dimensional and $\mathrm{sq}_-(s) = \kappa$.

Theorem 9.6. *$s \in \mathfrak{U}_\kappa(J)$ and is slice hyperholomorphic in a neighborhood of the origin and only if it admits a realization*

$$s(p) = D + pC\star(I-pA)^{-\star}B$$

where A, B, C and D are matrices such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix}$$

for some Hermitian matrix $H \in \mathbb{H}^{N\times N}$.

Proof. First of all, observe that the definition of (left) slice hyperholomorphic product immediately yields

$$pC \star_l f(p) = C \star_l pf(p) = C \star_l f(p) \star_l p \tag{9.6}$$

in the case $f(p)$ is a slice hyperholomorphic function, and C is a matrix. Analogous equalities holds for the right slice hyperholomorphic product. It is also useful to recall (cf. Section 3) that if f, g are left slice hyperholomorphic functions, then $(f \star_l g)^* = g^* \star_r f^*$ and $f \star_l C = fC$; analogously, if h is right slice hyperholomorphic, $C \star_r h = Ch$. □

In the positive case, the formulas above give the slice-hyperholomorphic analogs of Blaschke factors of the first, second and third kind and of finite Blasckhe-Potapov products. For the complex-valued counterparts of these notions, we refer to [9], [7]. These last papers also suggest factorization results, which will be considered elsewhere.

10 Generalized Carathéodory functions

We conclude this paper with a brief study the counterparts of the kernels (1.2).

Definition 10.1. Let $J \in \mathbb{H}^{N\times N}$ be a signature matrix. A $\mathbb{C}^{N\times N}$-valued function φ, slice hyperholomorphic in a neighborhood $\mathcal{V}$ of the origin, is called a **generalized Carathéodory function** if the kernel

$$k_\varphi(p,q) = \sum_{\ell=0}^{\infty} p^\ell(\varphi(p)J + J\varphi(q)^*)\overline{q}^\ell$$

has a finite number, say κ, of negative squares in $\mathcal{V}$.

We denote the class of such functions by $\mathscr{C}_\kappa(J)$. In the case of analytic functions, and for $N = 1$ and $\kappa = 0$, these functions appear in the work of Herglotz [43], [37]. Still for analytic functions, these classes were introduced and studied by Krein and Langer, also in the operator-valued case [45]. We now give a realization theorem for such functions which is the counterpart in the present setting of a result of Krein and Langer [45]. As for the realization of generalized Schur functions, we build a densely defined relation and apply Shmulyan's theorem (Theorem 7.2 above). We follow the arguments in [15, Theorem 5.2]. For the notion of observability in the statement of Theorem 10.2, see (9.1). The observability of the pair (C, V) is equivalent to the condition

$$C \star (I - pV)^{-\star} f \equiv 0 \quad \Rightarrow \quad f = 0. \tag{10.1}$$

Theorem 10.2. *An* $\mathbb{H}^{N\times N}$*-valued function* φ *slice, hyperholomorphic in a neighborhood* $\mathcal{V}$ *of the origin belongs to* $\mathscr{C}_\kappa(J)$ *if and only if it can be written as*

$$\varphi(p) = \frac{1}{2}C \star (I_{\mathscr{P}} + pV) \star (I_{\mathscr{P}} - pV)^{-\star}C^*J + \frac{\varphi(0) - J\varphi(0)^*J}{2}, \tag{10.2}$$

where $\mathscr{P}$ *is a right quaternionic Pontryagin space of index* κ*,* V *is a co-isometry in* $\mathscr{P}$*,* C *is a bounded operator from* $\mathscr{P}$ *into* $\mathbb{H}^N$*, and the pair* (C, V) *is observable.*

Proof. Let $\mathscr{L}(\varphi)$ denote the reproducing kernel right quaternionic Pontryagin space of functions slice hyperholomorphic in $\mathcal{V}$ with reproducing kernel $k_\varphi(p, q)$, and proceed in a number of steps. In the sequel, for the sake of simplicity, I denotes the identity without specification of the space on which it is defined.

STEP 1. *The linear relation consisting of the pairs* $(F, G) \in \mathscr{L}(\varphi) \times \mathscr{L}(\varphi)$ *with*

$$F(p) = \sum_{j=1}^{n} k_\varphi(p, p_j)\overline{p_j}b_j \quad \text{and} \quad G(p) = \sum_{j=1}^{n} k_\varphi(p, p_j)b_j - k_\varphi(p, 0)\left(\sum_{\ell=1}^{n} b_\ell\right),$$

where n *varies in* $\mathbb{N}$*,* $p_1, \ldots, p_n \in \mathcal{V} \subset \mathbb{H}$*, and* $b_1, \ldots, b_n \in \mathbb{H}^N$*, is isometric. Here* $\overline{p_j}b_j$ *means multiplication on the right by* p_j *on all the components of* b_j*.*

We need to check that

$$[F, F]_{\mathscr{L}(\varphi)} = [G, G]_{\mathscr{L}(\varphi)}. \tag{10.3}$$

We have

$$\begin{aligned}
[F, F]_{\mathscr{L}(\varphi)} &= \left[\sum_{j=1}^{n} k_\varphi(p, p_j)\overline{p_j}b_j, \sum_{k=1}^{n} k_\varphi(p, p_k)\overline{p_k}b_k\right]_{\mathscr{L}(\varphi)} \\
&= \sum_{j,k=1}^{n} b_k^* p_k k_\varphi(p_k, p_j)\overline{p_j}b_j \\
&= \sum_{\ell=1}^{\infty}\sum_{j,k=1}^{n} b_k^* p_k^{\ell+1}(\varphi(p_k)J + J\varphi(p_j)^*)\overline{p_j}^{\ell+1}b_j.
\end{aligned}$$

The inner product $[G, G]_{\mathscr{L}(\varphi)}$ is a sum of four terms, the first of which is

$$\sum_{j,k=1}^{n} b_k^* k_\varphi(p_k, p_j)b_j = \sum_{\ell=1}^{\infty}\sum_{j,k=1}^{n} b_k^* p_k^{\ell}(\varphi(p_k)J + J\varphi(p_j)^*)\overline{p_j}^{\ell}b_j.$$

Let $b = \sum_{\ell=1}^{n} b_\ell$. The second and third terms are

$$\begin{aligned} -\left(\sum_{k=1}^{n} b_k^* k_\varphi(p_k, 0)\right) b &= -\sum_{k=1}^{n} b_k^*(\varphi(p_k)J + J\varphi(0)^*)b \\ &= -\left(\sum_{k=1}^{n} b_k^*\varphi(p_k)J\right) b - b^* J\varphi(0)^* b, \end{aligned}$$

and

$$\begin{aligned} -b^*\left(\sum_{k=1}^{n} k_\varphi(0, p_j)b_j\right) &= \sum_{k=1}^{n} b^*(\varphi(0)J + J\varphi(p_j)^*)b_k \\ &= -b^*\varphi(0)Jb - b^*\left(\sum_{j=1}^{n} J\varphi(p_j)^* b_j\right), \end{aligned}$$

respectively, and the fourth term is $b^* k_\varphi(0, 0)b = b^*(\varphi(0)J + J\varphi(0)^*)b$. Equation (10.3) follows, since

$$\begin{aligned} [F, F]_{\mathscr{L}(\varphi)} - \sum_{j,k=1}^{n} b_k^* k_\varphi(p_k, p_j)b_j &= \sum_{\ell=1}^{\infty}\sum_{j,k=1}^{n} b_k^* p_k^{\ell+1}(\varphi(p_k)J + J\varphi(p_j)^*)\overline{p_j}^{\ell+1} b_j - \\ &\quad - \sum_{\ell=1}^{\infty}\sum_{j,k=1}^{n} b_k^* p_k^{\ell}(\varphi(p_k)J + J\varphi(p_j)^*)\overline{p_j}^{\ell} b_j \\ &= \sum_{j,k=1}^{n} b_k^*(\varphi(p_k)J + J\varphi(p_j)^*)b_j. \end{aligned}$$

The domain of R is dense. Thus, by Shmulyan's theorem (Theorem 7.2 above), R is the graph of a densely defined isometry which extends as an isometry to all of $\mathscr{L}(\varphi)$. We denote this extension by by T.

STEP 2. *We compute the adjoint of the operator T.*

Let $f \in \mathscr{L}(\varphi)$, $h \in \mathbb{H}^N$ and $p \in \mathcal{V}$. Then

$$\begin{aligned} h^* p\left((T^* f)(p)\right) &= [T^* f, k_\varphi(\cdot, p)\overline{p}h]_{\mathscr{L}(\varphi)} \\ &= [f\ ,\ T(k_\varphi(\cdot, p)h)]_{\mathscr{L}(\varphi)} \\ &= [f\ ,\ k_\varphi(\cdot, p)h - k_\varphi(\cdot, 0)h]_{\mathscr{L}(\varphi)} \\ &= h^*\left(f(p) - f(0)\right), \end{aligned}$$

and hence (with $f(p) = \sum_{\ell=0}^{\infty} p^\ell f_\ell$)

$$(T^* f)(p) = \begin{cases} p^{-1}(f(p) - f(0)), & p \neq 0, \\ f_1, & p = 0. \end{cases}$$

STEP 3. *Formula* (10.2) *holds.*

We first note that $f_\ell = CR_0^\ell f$, and so

$$f(p) = \sum_{\ell=0}^{\infty} p^\ell CR_0^\ell f = C \star (I - pR_0)^{-\star} f.$$

Applying this formula to the function $C^*1 = k_\varphi(\cdot, 0)$, we obtain

$$\varphi(p)J + J\varphi(0)^* = C \star (I - pR_0)^{-\star} C^* 1 \quad \text{and} \quad \varphi(0)J + J\varphi(0)^* = CC^* 1.$$

Multiplying the second equality by $1/2$ and subtracting it from the first, we obtain

$$\varphi(p)J + \frac{1}{2}(J\varphi(0)^* - \varphi(0)J) = \frac{1}{2} C \star (I - pR_0)^{-\star} \star (I + pR_0)C^*.$$

STEP 4. *Conversely, every function of the form* (10.2) *is in* $\mathscr{C}_\kappa(J)$.

From (10.2), we obtain

$$\varphi(p)J + J\varphi(q)^*J = C \star (I - pV)^{-\star} \star (1 - p\overline{q}) \star_r ((I - qV)^{-\star})^* \star_r C^*; \tag{10.4}$$

and so the reproducing kernel of $\mathscr{L}(\varphi)$ can be written as

$$k_\varphi(p, q) = C \star (I - pV)^{-\star}((I - qV)^{-\star})^* \star_r C^*,$$

since, in light of (10.4), the right side of the above equation satisfies

$$k_\varphi(p, q) - pk_\varphi(p, q)\overline{q} = \varphi(p)J + J\varphi(q)^*J.$$

In view of (10.1), $\mathscr{L}(\varphi)$ consists of the functions of the form

$$f(p) = C \star (I - pV)^{-\star}\xi, \quad \xi \in \mathscr{P},$$

with the inner product

$$[f, g]_{\mathscr{L}(\varphi)} = [\xi, \eta]_{\mathscr{P}} \quad (\text{with} \quad g(p) = C \star (I - pV)^{-\star}\eta),$$

and so the kernel k_φ has exactly κ negative squares. □

Corollary 10.3. *If* $J = I_N$ *and* $\kappa = 0$, *then* φ *has a slice hyperholomorphic extension to the whole unit ball of* $\mathbb{H}$.

Proof. This follows from (10.2), since V is then contractive. □

When $J = I_N$, and in the complex variable setting, generalized Carathéodory functions admit another representation, namely,

$$\varphi(z) = g(z)\varphi_0(z)g(1/\overline{z})^*, \tag{10.5}$$

where φ_0 is a Carathéodory function (i.e., the corresponding kernel is positive definite) and g is analytic and invertible in the open unit disk; see [38, 35, 34]. We note that in the rational case, generalized Carathéodory functions are called **generalized positive functions**, and play an important role in linear system theory. We refer to [10] for a survey of the literature and a constructive proof of the factorization (10.5) (in the half-line case) in the scalar rational case.

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