ON CENTRALIZERS OF INTERVAL DIFFEOMORPHISMS IN CRITICAL (INTERMEDIATE) REGULARITY

By

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In memory of Sergio Plaza Salinas

Abstract. We extend to the critical (intermediate) regularity several results concerning rigidity for centralizers and group actions on the interval.

0 Introduction

Group action on 1-dimensional manifolds is a well-developed subject whose source is the theory of codimension-1 foliations; see [9] for a general panorama. For such an action given by a sufficiently smooth (namely C^2) diffeomorphism, the general picture is essentially well understood via the classical works of Denjoy, Sacksteder, and Kopell, among others. The interest in considering actions of lower regularity comes from different sources; see, for example, [5, 6, 11]. It appears that many interesting phenomena from both the group theoretical and the dynamical viewpoints arise in *intermediate regularity*, that is, for actions by diffeomorphisms of differentiability classes between C^1 and C^2. This is the main subject of [3, 4, 7, 8], where several relevant problems have already been settled. For technical reasons, in many cases it was necessary to avoid certain *critical regularities*, for which existing arguments do not apply. Despite this, it has been conjectured that the corresponding rigidity phenomena should still hold in these critical cases.

In this work, we confirm this intuition for centralizers and group actions on the interval by providing concrete proofs. According to the (methods of) construction of [11] for Theorem A below, [7] for Theorem B, and [3] for Theorem C, our results are optimal (in the Hölder scale). Unfortunately, one of our main arguments does not apply in the most important context, namely, that of the generalized

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Denjoy Theorem in critical regularity, although it provides important evidence for its validity.

The generalized Kopell Lemma in critical regularity. Our first result is the extension to the critical regularity of [4, Theorem B]. Actually, this may be considered our main result, as all subsequent results are based on similar ideas and only use more involved combinatorial constructions.

Theorem A. *Let* $\{I_{i_1,\ldots,i_{d+1}} : (i_1,\ldots,i_{d+1}) \in \mathbb{Z}^{d+1}\}$ *be a family of subintervals of* $[0,1]$ *that are disposed respecting the lexicographic order. Assume that* $f_1,\ldots,f_{d+1}$ *are diffeomorphisms such that*

$$f_j(I_{i_1,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_{d+1}}) = I_{i_1,\ldots,i_{j-1},1+i_j,i_{j+1},\ldots,i_{d+1}}$$

for all $1 \leq j \leq d+1$*. Then* $f_1,\ldots,f_d$ *cannot be all of class* $C^{1+1/d}$ *provided that* f_{d+1} *is of class* $C^{1+\alpha}$ *for some* $\alpha > 0$ *and commutes with* $f_1,\ldots,f_d$*.*

This result is an improvement on [4, Theorem B] in what concerns the hypothesis of regularity for $f_1,\ldots,f_d$. Nevertheless, we impose an extra regularity assumption for f_{d+1}. (In [4], f_{d+1} is required only to be C^1.) Moreover, [4, Theorem B] holds in the case of noncommuting maps (see [9, Exercise 4.1.36]), whereas here we strongly use the fact that f_{d+1} commutes with the other f_i's (although these f_i's are not assumed to commute between themselves).

The generalized Kopell Lemma in critical but different regularities. Our second result, inspired by [7], is an extension of Theorem A.

Theorem B. *Assume the hypotheses of Theorem A, except that now each* f_i *is of class* $C^{1+\alpha_i}$*, with* $0 < \alpha_i \leq 1$*, for not necessarily equal values of* α_i*. Then* $\alpha_1 + \cdots + \alpha_d < 1$*.*

No smoothing of the Farb-Franks action in the critical regularity. Finally, we extend [3, Theorem A] to the critical regularity. The details of the statement below are provided in Section 3.

Theorem C. *The Farb-Franks action of* N_d *is not topologically semiconjugated to an action by* $C^{1+\alpha}$ *diffeomorphisms for* $\alpha = 2/d(d-1)$*.*

As with Theorem A, Theorem C should have a version for different regularities for the elements in a canonical generating set, the proof of which should follow from a combination of the techniques of Sections 2 and 3 below. Moreover, a combination of the ideas of [3, Section 3] and [7, Section 3] should show that such an extended result is, in fact, optimal. The reader will certainly agree that including all of these details would have artificially overloaded this already very technical article.

About the proofs. Roughly speaking, the proof of all the results stated above proceeds as follows. Assume that g is a diffeomorphism of the interval that commutes with many diffeomorphisms $f_1, \ldots, f_k$, and let $h_n = f_{i_n} \circ \cdots \circ f_{i_1}$ be a "random composition" of n factors among these f_i's. Taking derivatives of the equality $g^k = h_n^{-1} \circ g^k \circ h_n$ yields

$$Dg^k(x) = \frac{Dh_n(x)}{Dh_n(h_n^{-1} g^k h_n(x))} \cdot Dg^k(h_n(x)) = \frac{Dh_n(x)}{Dh_n(g^k(x))} \cdot Dg^k(h_n(x)).$$

As shown in [3, 4, 7], whenever the regularity is strictly larger than the corresponding critical regularity, it is possible to estimate (uniformly in n) the value of the *distortion* of h_n, that is, an expression of type $Dh_n(x)/Dh_n(y)$ as above. This leads to showing that the derivatives of the iterates of g are uniformly bounded, which is impossible unless g is trivial. However, as was already noticed in the aforementioned works, this is no longer possible for the critical regularity because of the failure of convergence of a certain series. The main new idea consists of noticing that despite the absence of uniform control for the distortion, elementary estimates show that its growth (in n) is slow (actually, sublinear). Choosing $n = n(k)$ appropriately, we show that the growth of the derivatives of g is sublinear, which is impossible. This last issue was cleverly noticed by Polterovich and Sodin in [10]. For the convenience of the reader, we reproduce their original proof.

Lemma (Polterovich-Sodin)**.** *If $g : I \to I$ is a nontrivial C^1 diffeomorphism of a bounded interval, then there exists an infinite increasing sequence $\{k_j\}$ of positive integers such that* $\max_{x \in I} Dg^{k_j}(x) > k_j$.

Proof. Let $x_0 \in I$ be such that $g(x_0) \neq x_0$ and denote by J the open interval whose endpoints are x_0 and $g(x_0)$. Then $J, g^{-1}(J), g^{-2}(J), \ldots$ are pairwise disjoint. Therefore,

$$\sum_{k \geq 1} \left| g^{-k}(J) \right| < \infty. \tag{1}$$

If the conclusion of the lemma does not hold, then there exists k_0 such that $Dg^{-k}(y) \geq 1/k$ for all $k \geq k_0$ and all $y \in I$. Since $\left| g^{-k}(J) \right| = Dg^{-k}(y_k)|J|$ for a certain $y_k \in J$, this implies that

$$\sum_{k \geq k_0} \left| g^{-k}(J) \right| \geq \sum_{k \geq k_0} \frac{|J|}{k} = \infty,$$

which contradicts (1). □

The following result of Borichev [1], which extends prior results of Polterovich and Sodin (valid in the C^2 context) to the $C^{1+\alpha}$ category (see also [2]), is crucial to our argument.

Theorem (Borichev)**.** *For $0 < \alpha < 1$, let g be a $C^{1+\alpha}$ diffeomorphism of a closed interval I with no hyperbolic fixed points. Let C_g be the α-Hölder constant of* $\log(Dg)$*. Then for every $k \geq 0$,*

$$\max_{x \in I} Dg^k(x) \leq \exp\left(3C_g|I|^{\alpha}k^{1-\alpha}\right). \tag{2}$$

It is important to point out that in [1], Borichev's theorem is not stated in this form. However, our statement of the theorem follows readily from Borichev's original statement (and its proof). Indeed, [1, Theorem 4] claims only that for $I = [0,1]$, $\max_{x\in[0,1]} Dg^k(x) \leq \exp\left(Ak^{1-\alpha}\right)$, where $A = A(C_g)$ is a certain constant. However, a careful reading of Borichev's proof shows it suffices to choose $A = 3C_g$.

As the reader will notice, a nice quantitative version of Borichev's theorem is important in our proof. Moreover, the introduction of the factor $|I|^{\alpha}$ is also important. This factor comes from an easy renormalization argument. Indeed, if $\bar{g}$ denotes the renormalization of g to the unit interval (viz. $\bar{g} := \varphi_I \circ g \circ \varphi_I^{-1}$, where φ_I is the unique orientation-preserving affine homeomorphism sending I into $[0,1]$), then (the adapted version of) Borichev's theorem (for the unit interval) yields

$$\max_{x\in I} Dg^k(x) = \max_{y\in[0,1]} D\bar{g}^k(y) \leq \exp(3C_{\bar{g}}k^{1-\alpha}). \tag{3}$$

Since $C_{\bar{g}} = C_g|I|^{\alpha}$, (3) implies (2).

1 Proof of the critical generalized Kopell lemma via a random walk argument

To prove Theorem A, we let $g := f_{d+1}$ and consider a composition of the f_i's $h_n = f_{i_n} \circ \cdots \circ f_{i_2} \circ f_{i_1}$. Then, $g^k = h_n^{-1} \circ g^k \circ h_n$ for each $k \geq 1$, which yields

$$Dg^k(x) = \frac{Dh_n(x)}{Dh_n(h_n^{-1}g^k h_n(x))} \cdot Dg^k(h_n(x)) = \frac{Dh_n(x)}{Dh_n(g^k(x))} \cdot Dg^k(h_n(x)).$$

We restrict this equality to x in the interval I defined as the convex closure of $\bigcup_{i_{d+1}\in\mathbb{Z}} I_{0,0,\ldots,0,i_{d+1}}$. (Note that I is invariant under g.) Let C be a simultaneous $1/d$-Hölder constant for $\log(Df_i)$, where $1 \leq i \leq d$. Letting $h_j := f_{i_j} \circ \cdots \circ f_{i_1}$

whenever $0 \leq j \leq n$, and $y = y_k := g^k(x) \in I$, we obtain

$$\begin{aligned}\left|\log\left(\frac{Dh_n(x)}{Dh_n(g^k(x))}\right)\right| &= \left|\log\left(\frac{\prod_{j=1}^{n} Df_{i_j}(h_{j-1}(x))}{\prod_{j=1}^{n} Df_{i_j}(h_{j-1}(y))}\right)\right| \\ &\leq \sum_{j=1}^{n} \left|\log Df_{i_j}(h_{j-1}(x)) - \log Df_{i_j}(h_{j-1}(y))\right| \\ &\leq C \sum_{j=1}^{n} \left|h_{j-1}(x) - h_{j-1}(y)\right|^{1/d} \\ &\leq C \sum_{j=0}^{n-1} \left|h_j(I)\right|^{1/d}.\end{aligned}$$

This implies that

$$Dg^k(x) \leq \exp(CM_n) \cdot Dg^k(h_n(x)), \tag{4}$$

where $M_n = C \sum_{j=0}^{n-1} \left|h_j(I)\right|^{1/d}$. In order to control the growth of M_n, we use the first of the two properties provided by the next assertion.

Lemma 1.1. *Let $\ell : \mathbb{N}_0^d \to (0, \infty)$ and assume that*

$$\sum_{(i_1,\ldots,i_d)\in\mathbb{N}_0^d} \ell(i_1, \ldots, i_d) < \infty. \tag{5}$$

Then there exists a constant $B > 0$ such that for each $n \in \mathbb{N}$, there is a geodesic path of length n in $\mathbb{N}_0^d$, say $\{(i_1(j), \ldots, i_d(j)) : 0 \leq j \leq n\}$, satisfying

$$i_1(0) = \ldots = i_d(0) = 0,$$

$$\sum_{j=0}^{n-1} \ell(i_1(j), \ldots, i_d(j))^{1/d} \leq B\big(\log(n+1)\big)^{1-1/d} \tag{6}$$

$$\ell(i_1(n), \ldots, i_d(n)) \leq \frac{B}{(n+1)^{d-1}}. \tag{7}$$

Proof. Denote the sum in (5) by L. As in [4], we consider the Markov process on $\mathbb{N}_0^d$ with transition probabilities

$$p\big((i_1, \ldots, i_d) \mapsto (i_1, \ldots, i_{j-1}, 1 + i_j, i_{j+1}, \ldots, i_d)\big) := \frac{1 + i_j}{i_1 + \ldots + i_d + d}.$$

For this process, for every $n \geq 1$, the transition probabilities in the nth step are equidistributed along the n-sphere S_n, i.e.,

$$i_1 + \cdots + i_d = n \implies \mathbb{P}_n\big((0, \ldots, 0) \mapsto (i_1, \ldots, i_d)\big) = \frac{1}{|S_n|}.$$

Let $A_d > 0$ be such that $|S_n| \geq A_d (n+1)^{d-1}$ for all $n \geq 0$. A direct application of Hölder's inequality then yields

$$\begin{aligned}
\mathbb{E}\Big(\sum_{j=0}^{n-1} \ell(i_1(j), \ldots, i_d(j))^{\tau} \Big) &= \sum_{j=0}^{n-1} \mathbb{E}\left(\ell(i_1(j), \ldots, i_d(j))^{\tau} \right) \\
&= \sum_{j=0}^{n-1} \frac{1}{|S_j|} \sum_{(i_1,\ldots,i_d) \in S_j} \ell(i_1, \ldots, i_d)^{\tau} \\
&\leq \left(\sum_{j=0}^{n-1} \sum_{(i_1,\ldots,i_d) \in S_j} \ell(i_1, \ldots, i_d) \right)^{\tau} \left(\sum_{j=0}^{n-1} \sum_{(i_1,\ldots,i_d) \in S_j} \left(\frac{1}{|S_j|} \right)^{\frac{1}{1-\tau}} \right)^{1-\tau} \\
&\leq L^{\tau} \left(\sum_{j=0}^{n-1} |S_j| \left(\frac{1}{|S_j|} \right)^{\frac{1}{1-\tau}} \right)^{1-\tau} \\
&= L^{\tau} \left(\sum_{j=0}^{n-1} \left(\frac{1}{|S_j|} \right)^{\frac{\tau}{1-\tau}} \right)^{1-\tau} \\
&\leq \frac{L^{\tau}}{A_d^{\tau}} \left(\sum_{j=0}^{n-1} \frac{1}{(j+1)^{(d-1)\frac{\tau}{1-\tau}}} \right)^{1-\tau}
\end{aligned}$$

for every $0 < \tau < 1$. Now, for $\tau = 1/d$, we have $(d-1)\tau/(1-\tau) = 1$; hence

$$\mathbb{E}\left(\sum_{j=0}^{n-1} \ell(i_1(j), \ldots, i_d(j))^{1/d} \right) \leq \frac{L^{1/d}}{A_d^{1/d}} \left(\sum_{j=1}^{n} \frac{1}{j} \right)^{1-1/d} \leq \frac{L^{1/d}}{A_d^{1/d}} \big(\log(n+1) \big)^{1-1/d}.$$

A direct application of Chebyshev's inequality then shows that with probability larger than $2/3$,

$$\sum_{j=0}^{n-1} \ell(i_1(j), \ldots, i_d(j))^{1/d} \leq \frac{3L^{1/d}}{A_d^{1/d}} \big(\log(n+1) \big)^{1-1/d}.$$

Moreover, since $\sum_{(i_1,\ldots,i_d) \in S_n} \ell(i_1, \ldots, i_d) \leq L$ and the arrival probabilities in n steps are equidistributed along S_n with probability greater than $2/3$, it follows that

$$\ell\big(i_1(n), \ldots, i_d(n)\big) \leq \frac{3L}{|S_n|} \leq \frac{3L}{A_d (n+1)^{d-1}}.$$

Thus, letting

$$B := \max \left\{ \frac{3L^{1/d}}{A_d^{1/d}}, \frac{3L}{A_d} \right\}$$

shows that (6) and (7) hold simultaneously with probability greater than $1/3$. This ensures the existence of the desired geodesic path. □

Returning to the proof of Theorem A, we let ℓ be the function that associates to $(i_1, \ldots, i_d)$ the length of the convex closure of $\bigcup_{i_{d+1} \in \mathbb{Z}} I_{i_1,\ldots,i_d,i_{d+1}}$. (Observe that this interval coincides with $f_1^{i_1} \circ \cdots \circ f_d^{i_d}(I)$.) Let $h_n := f_{i_n} \circ \cdots \circ f_{i_1}$ be a random composition for which (6) and (7) hold, and let g_n be the restriction of g to $h_n(I)$. We claim that g_n has no hyperbolic fixed point. Indeed, since g commutes with each f_i, if g_n had a hyperbolic fixed point, it would have a sequence of hyperbolic fixed points (with the same derivative) accumulating at a limit point. However, this is clearly impossible.

We are hence under the hypothesis of Borichev's theorem, and substituting (2) into (4) yields

$$\begin{aligned}\max_{x \in I} Dg^k(x) &\leq \exp(CM_n) \cdot Dg_n^k(h_n(x)) \\ &\leq \exp\left(CB\big(\log(n+1)\big)^{1-1/d}\right) \exp(3C_{g_n}|h_n(I)|^{\alpha} k^{1-\alpha}) \\ &\leq \exp\left(CB\big(\log(n+1)\big)^{1-1/d}\right) \exp\left(\frac{3C_g B^{\alpha} k^{1-\alpha}}{(n+1)^{(d-1)\alpha}}\right).\end{aligned}$$

Taking $n = n_k$, so that $k^{1-\alpha} \sim n^{(d-1)\alpha}$, hence $\log(k) \sim \log(n_k)$, we obtain

$$\max_{x \in I} Dg^k(x) \leq \exp\left(A\big(\log(k)\big)^{1-1/d}\right),$$

where A is a constant (independent of k). However, since the last expression is of order $o(k)$, this turns out to be impossible because of the Polterovich-Sodin lemma.

2 Proof of the critical generalized Kopell lemma for different regularities via a deterministic argument

The proof of Theorem B consists of combining of the ideas of Section 1 and [7]. The case $d = 2$ is relatively straightforward. Nevertheless, for larger d, we need a slight but nontrivial modification of the concatenation argument of [7]. Solely for pedagogical reasons , we develop the cases $d = 2$, $d = 3$, and the general case $d \geq 3$ independently in order to introduce the necessary new ideas in a progressive manner (although the reader should encounter no difficulties in passing directly from the case $d = 2$ to the general case $d \geq 3$.)

For all cases, we argue by contradiction. We assume that $\alpha_1 + \ldots + \alpha_d = 1$, and again let ℓ be the function that associates to $(i_1, \ldots, i_d)$ the length of the convex closure of $\bigcup_{i_{d+1} \in \mathbb{Z}} I_{i_1,\ldots,i_d,i_{d+1}}$. Then we consider parallelepipeds $Q(n)$ in $\mathbb{N}_0^d$ whose s^{th}-side has length of order $2^{n\alpha_s}$. For such a $Q(n)$, we set

$$L_n := \sum_{(i_1,\ldots,i_d) \in Q(n)} \ell(i_1, \ldots, i_d). \tag{8}$$

Our task consists of showing that for an appropriately chosen sequence $\{Q(n)\}$ of finite multiplicity M (that is, such that no point is contained in more than M of these parallelepipeds), there is a positive constant B for which there exist (not necessarily nonempty) geodesic segments $\gamma_1^1, \gamma_1^2, \ldots, \gamma_1^{d_1}, \gamma_2^1, \gamma_2^2, \ldots, \gamma_2^{d_2}, \ldots$, with $d_n \leq d$, satisfying the following properties.

- For each $n \geq 1$ and $1 \leq k \leq d_n$, the segment γ_n^k is contained in $Q(n)$
- Each of these segments intersects the next nonempty one in the sequence above.
- For certain positive constants α, D and each $n \geq 1$, at least one of the segments $\gamma_n^1, \ldots, \gamma_n^{d_n}$ contains no fewer than $2^{n\alpha}/D$ points.
- Each γ_n^k is an unidirectional path pointing in a s-direction, with $s = s_{n,k}$, and

$$\sum_{(i_1,\ldots,i_d)\in\gamma_n^k} \ell(i_1,\ldots,i_d)^{\alpha_s} \leq B \max\{L_n^{\alpha_s}, L_{n+1}^{\alpha_s}\}. \tag{9}$$

We now explain how to use such a sequence of geodesic segments to prove Theorem B. The next paragraphs are devoted to constructing parallelepipeds as well as sequences of geodesics segments satisfying the desired properties in the corresponding cases.

First of all, note that concatenating the geodesic segments along intersecting points produces an infinite (not necessarily geodesic) path $\gamma : \mathbb{N}_0 \to \mathbb{N}_0^d$. We assume that γ starts at the origin. (If this is not the case, the same arguments apply, modulo changes in the constant B, after an initial segment $\gamma_0 \subset Q(1)$ from the origin to the initial point of γ_1^1 is added.)

For each $n \geq 1$, denote by $N = N(n)$ the entry time of γ into $Q(n+1)$ and by $s(m)$ the direction corresponding to the jump from $\gamma(m)$ to $\gamma(m+1)$. Then (9) combined with Hölder's inequality yields

$$\begin{aligned}\sum_{m=0}^{N} \ell(\gamma(m))^{\alpha_{s(m)}} &\leq B \sum_{m=1}^{n+1}\sum_{k=1}^{d} L_m^{\alpha_{s_{m,k}}} \leq B \sum_{k=1}^{d} \left(2\sum_{m=1}^{n+1} L_m\right)^{\alpha_{s_{m,k}}} (n+1)^{1-\alpha}\\ &\leq 2^{\alpha'} dBM^{\alpha'}(n+1)^{1-\alpha},\end{aligned}$$

where $\alpha := \min\{\alpha_1,\ldots,\alpha_d\}$ and $\alpha' := \max\{\alpha_1,\ldots,\alpha_d\}$. The assumption on the size of $Q(n)$ easily implies the asymptotic equivalence $n \sim \log(N)$. (This equivalence is even more transparent for the explicit choice of $Q(n)$ which we make later.) As a consequence, there exists a constant $A' > 0$ for which the previous estimate becomes

$$\sum_{m=0}^{N} \ell\big(\gamma(m)\big)^{\alpha_{s(m)}} \leq A'\big(\log(N)\big)^{1-\alpha}.$$

The path γ induces a sequence $\{h_n\}$ of maps, each h_n of which is a composition of maps taken from the set $\{f_1, f_1^{-1}, \dots, f_d, f_d^{-1}\}$, such that

$$\sum_{m=0}^{N} \left|h_m(I)\right|^{\alpha_{s(m)}} \leq A'\big(\log(N)\big)^{1-\alpha}, \tag{10}$$

where I denotes the convex closure of $\bigcup_{i_{d+1}\in\mathbb{Z}} I_{0,\dots,0,i_{d+1}}$. Let $h_m = f_{i_m} \circ \cdots \circ f_{i_1}$, and let C be a common upper bound for the α_i-Hölder constants of $\log(Df_i)$, $\log(Df_i^{-1})$, where $1 \leq i \leq d$. Given $n > 1$, let N' satisfy $N(n-1) \leq N' \leq N(n) = N$. For each x, y in I, estimate (10) yields

$$\begin{aligned}
\left|\log\left(\frac{Dh_{N'}(x)}{Dh_{N'}(y)}\right)\right| &= \left|\log\left(\frac{\prod_{m=1}^{N'} Df_{i_m}(h_{m-1}(x))}{\prod_{m=1}^{N'} Df_{i_m}(h_{m-1}(y))}\right)\right| \\
&\leq \sum_{m=1}^{N'} \big|\log Df_{i_m}(h_{m-1}(x)) - \log Df_{i_m}(h_{m-1}(y))\big| \\
&\leq C\sum_{m=1}^{N'} \left|h_{m-1}(x) - h_{m-1}(y)\right|^{\alpha_{s(m-1)}} \\
&\leq C\sum_{m=0}^{N} \left|h_m(I)\right|^{\alpha_{s(m)}} \\
&\leq CA'\big(\log(N)\big)^{1-\alpha}.
\end{aligned}$$

Moreover, by the third property of our sequence, we may choose k such that γ_n^k contains at least $2^{n\alpha}/D$ points. Since the sum of the values of ℓ along these points is at most 1, such a segment must contain a point at which the value of ℓ is at most $D/2^{n\alpha}$. In other words, we may choose N' such that $\left|h_{N'}(I)\right| \leq D/2^{n\alpha}$.

Denote by $g_{N'}$ the restriction of $g := f_{d+1}$ to $h_{N'}(I)$. As in Section 1, the map g (hence $g_{N'}$) cannot have hyperbolic fixed points. Therefore, taking derivatives in the equality $g^k = h_{N'}^{-1} \circ g^k \circ h_{N'}$ and using the previous estimate for $y := g^k(x) \in I$, we obtain from Borichev's theorem

$$\begin{aligned}
Dg^k(x) &= \frac{Dh_{N'}(x)}{Dh_{N'}(h_{N'}^{-1}g^k h_{N'}(x))} \cdot Dg^k(h_{N'}(x)) \\
&= \frac{Dh_{N'}(x)}{Dh_{N'}(g^k(x))} \cdot Dg_{N'}^k(h_{N'}(x)) \\
&\leq \exp\left(CA'\big(\log(N)\big)^{1-\alpha}\right) \exp\left(3C_g|h_{N'}(I)|^{\alpha_{d+1}}k^{1-\alpha_{d+1}}\right) \\
&\leq \exp\left(CA'\big(\log(N)\big)^{1-\alpha}\right) \exp\left(\frac{3C_g D^{\alpha} k^{1-\alpha_{d+1}}}{2^{n\alpha\alpha_{d+1}}}\right).
\end{aligned}$$

Take $n = n_k$, so that $k^{1-\alpha_{d+1}} \sim 2^{n\alpha\alpha_{d+1}}$; hence $n_k \sim \log(k)$. Since $\log(N(n_k)) \sim n_k$,

$$\max_{x\in I} Dg^k(x) \leq \exp\left(A\big(\log(k)\big)^{1-\alpha}\right)$$

for a certain constant A (independent of k). However, this last expression is of order $o(k)$, which is impossible by the Polterovich-Sodin lemma.

2.1 The case $d = 2$. Following [7], define the rectangles

$$Q(2n+1) := [[4^{n\alpha_1}, 4^{(n+1)\alpha_1}]] \times [[4^{n\alpha_2}, 4^{(n+2)\alpha_2}]],$$
$$Q(2n+2) := [[4^{n\alpha_1}, 4^{(n+2)\alpha_1}]] \times [[4^{(n+1)\alpha_2}, 4^{(n+2)\alpha_2}]],$$

where $[[x, y]]$ stands for the set of integers between x and y. Note that the multiplicity of the sequence $(Q(n))$ is 4.

A set of the form $Q(n) \cap \{j = \text{constant}\}$ (respectively, $Q(n) \cap \{i = \text{constant}\}$) is said to be a **horizontal segment** (respectively, **vertical segment**) in $Q(n)$. Note that the cardinality of this set $\mathcal{H}_n$ (respectively, $\mathcal{V}_n$) of horizontal (respectively, vertical) segments is at least $2^{n\alpha_2}/D_1$ (respectively, at least $2^{n\alpha_1}/D_1$), where $D_1 > 0$ is a constant (independent of n). Moreover, there exists a positive constant D_2 such that the number of points in each of these horizontal (respectively, vertical) segments is at least $D_2 2^{n\alpha_1}$ (respectively, at least $D_2 2^{n\alpha_2}$).

We say that a horizontal segment γ in $Q(2n+2)$ is **good** if

$$\sum_{(i,j)\in\gamma} \ell(i,j) \leq \frac{L_{2n+2}}{|\mathcal{H}_{2n+2}|}.$$

Clearly, there must be at least one good horizontal segment. For such a segment $\gamma = \gamma_{2n+2}$, Hölder's inequality yields

$$\sum_{(i,j)\in\gamma} \ell(i,j)^{\alpha_1} \leq \left(\frac{L_{2n+2}}{|\mathcal{H}_{2n+2}|}\right)^{\alpha_1} |\gamma|^{1-\alpha_1} \leq L_{2n+2}^{\alpha_1} \left(\frac{D_1}{2^{(2n+2)\alpha_2}}\right)^{\alpha_1} (D_2 2^{(2n+2)\alpha_1})^{1-\alpha_1}$$
$$= D_1^{\alpha_1} D_2^{\alpha_2} L_{2n+2}^{\alpha_1}.$$

Similarly, we say that a vertical segment γ in $Q(n)$ is **good** if

$$\sum_{(i,j)\in\gamma} \ell(i,j) \leq \frac{L_n}{|\mathcal{V}_n|}.$$

Again, there must exist a good vertical segment $\gamma = \gamma_{2n+1} \subset Q(2n+1)$, and for this segment,

$$\sum_{(i,j)\in\gamma} \ell(i,j)^{\alpha_2} \leq \left(\frac{L_{2n+1}}{|\mathcal{V}_{2n+1}|}\right)^{\alpha_2} |\gamma|^{1-\alpha_2} \leq L_{2n+1}^{\alpha_2} \left(\frac{D_1}{2^{(2n+1)\alpha_1}}\right)^{\alpha_2} (D_2 2^{(2n+1)\alpha_2})^{1-\alpha_2}$$
$$= D_1^{\alpha_2} D_2^{\alpha_1} L_{2n+1}^{\alpha_2}.$$

Thus, the segments $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$ satisfy (9) for $B \geq \max\{D^{\alpha_1} D_2^{\alpha_2}, D_1^{\alpha_2} D_2^{\alpha_1}\}$. Each segment γ_n intersects γ_{n+1}; and it is easy to check that between the concatenating points, γ_n contains at least $2^{n\alpha}/D$ points for a certain constant $D > 0$, where $\alpha := \min\{\alpha_1, \alpha_2\}$. Therefore, all the conditions of (2) are satisfied. This concludes the proof of Theorem B in the case $d = 2$.

2.2 The case $d = 3$. For the case $d \geq 3$, we define $Q(n) := \prod_{i=1}^{d} [[x_n^i, y_n^i]]$ inductively by

$$Q(1) := [[1, 4^d]]^d,$$
$$Q(n+1) := \cdots \times [[1 + 2^{d\alpha_m}(x_n^m - 1), y_n^m]] \times [[x_n^{m+1}, 1 + 2^{d\alpha_{m+1}}(y_n^{m+1} - 1)]] \times \cdots,$$

where for each $n \geq 1$, $m = m(n) \in \{1, \ldots, d\}$ denotes the residue class of n modulo d. (Here, "$\cdots$" mean that the corresponding factors remain untouched.) Note that the sequence $(Q(n))$ has multiplicity $d + 2$.

It is easy to verify the asymptotic equivalence $y_n^k - x_n^k \sim 2^{n\alpha_k}$. Let D_1 be a constant such that $2^{n\alpha_i}/D_1 \leq y_n^k - x_n^k \leq D_1 2^{n\alpha_i} - 1$ and fix a constant $D_2 > 0$ such that

$$y_{n+1}^m - x_{n+1}^m \geq D_2(y_n^m - x_n^m) \quad \text{and} \quad y_n^{m+1} - x_n^{m+1} \geq D_2(y_{n+1}^{m+1} - x_{n+1}^{m+1}). \tag{11}$$

We now specialize to the case $d = 3$. A plane P of the form

$$P = Q(n) \cap \{i_{m+2} = \text{constant}\}$$

is said to be an ***h*-plane**, and we denote the family of h-planes in $Q(n)$ by $\mathcal{P}_n$. Note that the cardinality of $\mathcal{P}_n$ is at least $2^{n\alpha_{m+2}}/D_1$.

Given an h-plane $P \in \mathcal{P}_n$, a **horizontal segment** (respectively, **vertical segment**) in P is a set of the form $P \cap \{i_{m+1} = \text{constant}\}$, (respectively, of the form $P \cap \{i_m = \text{constant}\}$). The cardinality of the family $\mathcal{H}_n$ (respectively, $\mathcal{V}_n$) of horizontal (respectively, vertical) segments is at least $2^{n\alpha_{m+1}}/A$ (respectively, at least$2^{n\alpha_m}/A'$), where A and A' are positive constants.

Fix $\lambda \geq 1$. We say that an h-plane P in $\mathcal{P}_n$ is **λ-good** (see (8)) if

$$\sum_{(i_1,i_2,i_3)\in P} \ell(i_1, i_2, i_3) \leq \frac{\lambda L_n}{|\mathcal{P}_n|}.$$

By Chebyshev's inequality, the fraction of h-planes that are λ-good is larger than $1 - 1/\lambda$. Similarly, we say that a horizontal segment γ of $P \in \mathcal{P}_n$ is **λ-good relative to P** if

$$\sum_{(i_1,i_2,i_3)\in\gamma} \ell(i_1, i_2, i_3) \leq \frac{\lambda \sum_{(i_1,i_2,i_3)\in P} \ell(i_1, i_2, i_3)}{|\mathcal{H}_n|}.$$

As before, the fraction of horizontal directions which are relatively λ-good is larger than $1 - 1/\lambda$. Finally, we say that a vertical segment γ in $P \in \mathcal{P}_n$ is **λ-good** if

$$\sum_{(i_1,i_2,i_3)\in\gamma} \ell(i_1,i_2,i_3) \leq \frac{\lambda \sum_{(i_1,i_2,i_3)\in P} \ell(i_1,i_2,i_3)}{|\mathcal{V}_n|}.$$

Once again, the fraction of vertical segments which are λ-good is larger than $1 - 1/\lambda$.

Now fix $\lambda \geq 2/D_2$, and let $P \in \mathcal{P}_n$ be a λ-good h-plane. By the first inequality in (11), more than half of the vertical segments of P contained in $Q(n+1)$ are λ-good. Since $\lambda \geq 2$, more than half of the h-planes in $Q(n+1)$ are λ-good. Therefore, there must exist a λ-good h-plane $P' \in \mathcal{P}_{n+1}$ such that $P \cap P'$ is a λ-good vertical segment of P. Moreover, by the second inequality in (11), more than half of the vertical segments of P' contained in $Q(n)$ are λ-good relative to P'.

We may therefore fix a sequence $\{P_n\}$ of λ-good h-planes such that for each $n \geq 1$, $P_n \cap P_{n+1}$ is a λ-good vertical segment γ_n^2 of P_n; see Figure 1. Each P_n must contain a relatively 1-good horizontal segment γ_n^1. Finally, let γ_n^3 be a λ-good vertical segment of P_{n+1} contained in $Q(n)$; see Figure 2. We have thus constructed an infinite sequence of geodesic segments $\gamma_1^1, \gamma_1^2, \gamma_1^3, \gamma_2^1, \gamma_2^2, \gamma_2^3, \ldots$, each γ_j of which intersects its successor in the sequence. Moreover, since P_n is λ-good and γ_n^1 is a relatively 1-good horizontal segment in P_n,

$$\sum_{(i_1,i_2,i_3)\in\gamma_n^1} \ell(i_1,i_2,i_3) \leq \frac{\sum_{(i_1,i_2,i_3)\in P_n} \ell(i_1,i_2,i_3)}{|\mathcal{H}_n|} \leq \frac{1}{|\mathcal{H}_n|}\frac{\lambda L_n}{|\mathcal{P}_n|} \leq \frac{AD_1\lambda L_n}{2^{n\alpha_{m+1}}2^{n\alpha_{m+2}}}.$$

By Hölder's inequality, this implies that

$$\begin{aligned}\sum_{(i_1,i_2,i_3)\in\gamma_n^1} \ell(i_1,i_2,i_3)^{\alpha_m} &\leq \left(\frac{AD_1\lambda L_n}{2^{n\alpha_{m+1}}2^{n\alpha_{m+2}}}\right)^{\alpha_m} |\gamma_n^2|^{1-\alpha_m}\\ &\leq \left(\frac{AD_1\lambda L_n}{2^{n\alpha_{m+1}}2^{n\alpha_{m+2}}}\right)^{\alpha_m} (D_1 2^{n\alpha_m})^{1-\alpha_m};\end{aligned}$$

hence,

$$\sum_{(i_1,i_2,i_3)\in\gamma_n^1} \ell(i_1,i_2,i_3)^{\alpha_m} \leq (AD_1\lambda)^{\alpha_m} D_1^{1-\alpha_m} L_n^{\alpha_m}. \tag{12}$$

Similarly, since P_n is λ-good and γ_n^2 is a λ-good vertical segment of P_n,

$$\sum_{(i_1,i_2,i_3)\in\gamma_n^2} \ell(i_1,i_2,i_3) \leq \frac{\lambda\sum_{(i_1,i_2,i_3)\in P_n} \ell(i_1,i_2,i_3)}{|\mathcal{V}_n|} \leq \frac{\lambda}{|\mathcal{V}_n|}\frac{\lambda L_n}{|\mathcal{P}_n|} \leq \frac{A'D_1\lambda^2 L_n}{2^{n\alpha_m}2^{n\alpha_{m+2}}}.$$

Again, by Hölder's inequality,

$$\sum_{(i_1,i_2,i_3)\in\gamma_n^2} \ell(i_1,i_2,i_3)^{\alpha_{m+1}} \leq \left(\frac{A'D_1\lambda^2 L_n}{2^{n\alpha_m}2^{n\alpha_{m+2}}}\right)^{\alpha_{m+1}} |\gamma_n^2|^{1-\alpha_{m+1}}$$
$$\leq \left(\frac{A'D_1\lambda^2 L_n}{2^{n\alpha_m}2^{n\alpha_{m+2}}}\right)^{\alpha_{m+1}} (D_1 2^{n\alpha_{m+1}})^{1-\alpha_{m+1}};$$

hence

$$\sum_{(i_1,i_2,i_3)\in\gamma_n^k} \ell(i_1,i_2,i_3)^{\alpha_{m+1}} \leq (A'D_1\lambda^2)^{\alpha_{m+1}} D_1^{1-\alpha_{m+1}} L_n^{\alpha_{m+1}}. \tag{13}$$

Also, γ_n^3 is a λ-good vertical segment of P_{n+1}, which is a λ-good horizontal plane of $Q(n+1)$; hence

$$\sum_{(i_1,i_2,i_3)\in\gamma_n^3} \ell(i_1,i_2,i_3) \leq \frac{\lambda\sum_{(i_1,i_2,i_3)\in P_{n+1}} \ell(i_1,i_2,i_3)}{|\mathcal{V}_{n+1}|}$$
$$\leq \frac{\lambda}{|\mathcal{V}_{n+1}|}\frac{\lambda L_{n+1}}{|\mathcal{P}_{n+1}|} \leq \frac{A'D_1\lambda^2 L_{n+1}}{2^{(n+1)\alpha_{m+1}}2^{(n+1)\alpha_m}},$$

and Hölder's inequality yields

$$\sum_{(i_1,i_2,i_3)\in\gamma_n^3} \ell(i_1,i_2,i_3)^{\alpha_{m+2}} \leq \left(\frac{A'D_1\lambda^2 L_{n+1}}{2^{(n+1)\alpha_{m+1}}2^{(n+1)\alpha_m}}\right)^{\alpha_{m+2}} |\gamma_n^3|^{1-\alpha_{m+2}}$$
$$\leq \left(\frac{A'D_1\lambda^2 L_{n+1}}{2^{(n+1)\alpha_{m+1}}2^{(n+1)\alpha_m}}\right)^{\alpha_{m+2}} (D_1 2^{(n+1)\alpha_{m+2}})^{1-\alpha_{m+2}},$$

i.e.,

$$\sum_{(i_1,i_2,i_3)\in\gamma_n^3} \ell(i_1,i_2,i_3)^{\alpha_{m+2}} \leq (A'D_1\lambda^2)^{\alpha_{m+2}} D_1^{1-\alpha_{m+2}} L_{n+1}^{\alpha_{m+2}}. \tag{14}$$

By (12), (13), and (14), condition (9) holds for

$$B \geq \max_k \max\{(AD_1\lambda)^{\alpha_k} D_1^{1-\alpha_k}, (A'D_1\lambda^2)^{\alpha_k} D_1^{1-\alpha_k}\}.$$

Finally, it is easy to see that for a certain constant $D > 0$, each γ_n^1 contains at least $2^{n\alpha}/D$ points between the concatenating points, where $\alpha := \min\{\alpha_1, \alpha_2, \alpha_3\}$. This concludes verification of the properties from Section 2.

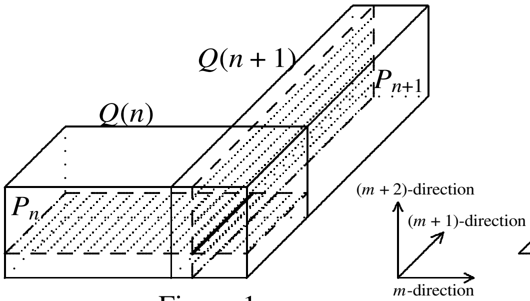

Figure 1

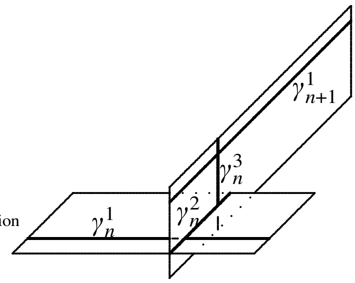

Figure 2

2.3 The general case $d \geq 3$. For the proof in the general ($d \geq 3$) case, it is best to isolate and extend inductively a concatenation argument that has already been used (in a weak form) in the cases $d = 2$ and $d = 3$.

2.4 The d-dimensional black box. Let Q be a d-dimensional parallelepiped, with $d \geq 3$, and let ℓ be a positive function defined on Q. Set

$$L_Q := \sum_{(i_1,\ldots,i_d)\in Q} \ell(i_1, \ldots, i_d).$$

For $1 \leq m \leq d$, an **m-segment** is a set of the form

$$Q \cap \{i_j = \text{constant}_j \text{ except for } j = m\}.$$

Denote the family of all m-segments by $\mathcal{S}(m)$. The elements in the family $\mathcal{S} := \bigcup_m \mathcal{S}(m)$ are said to be **unidirectional segments**. Given $\lambda \geq 1$, we say that $\gamma \in \mathcal{S}(m)$ is **λ-good** if

$$\sum_{(i_1,\ldots,i_d)\in\gamma} \ell(i_1, \ldots, i_d) \leq \frac{\lambda L_Q}{|\mathcal{S}(m)|}.$$

More generally, let us consider a d'-dimensional parallelepiped Q' and a d''-dimensional parallelepiped Q'', both contained in Q, such that $Q' \subset Q''$ and $d' < d''$. Denote by $c(Q', Q'')$ the number of (disjoint) translates of Q' that fill Q'' (that is, the number of copies of Q' contained in Q''). For example, $c(\gamma, Q) = |\mathcal{S}(m)|$ for every m-segment γ. We say that Q' is **λ-good relative to Q''** whenever

$$\sum_{(i_1,\ldots,i_d)\in Q'} \ell(i_1, \ldots, i_d) \leq \frac{\lambda}{c(Q', Q'')} \sum_{(i_1,\ldots,i_d)\in Q''} \ell(i_1, \ldots, i_d).$$

Finally, we say that $\gamma \in \mathcal{S}(m)$ is **fully λ-good** if there exists a *flag* of d'-dimensional parallelepipeds $Q^{d'}$ that are λ-good in Q, have the form

$$Q^{d'} = Q \cap \{i_j = \text{constant}_j \text{ except for } j = m, m+1, \ldots, m+d'-1\},$$

and satisfy $\gamma = Q^1 \subset Q^2 \subset \cdots \subset Q^{d-1}$.

Lemma 2.1. *Given $0 < \kappa < 1$ and $\mu \geq 1$, there exists $\lambda' = \lambda'(\mu, \kappa, d)$ (not depending on Q) such that the following holds. If γ is a fully μ-good 1-segment of Q, then for a proportion larger than κ of the points $(i_1, \ldots, i_d) \in Q$, there exists a finite sequence of λ'-good unidirectional segments $\gamma_1, \ldots, \gamma_{d'-1}$, with $d' \leq d$, such that each γ_i intersects γ_{i+1}, with γ_1 starting at a point of γ and $\gamma_{d'-1}$ ending at $(i_1, \ldots, i_d)$.*

Proof. We leave the case $d = 3$ to the reader. (It uses arguments similar to those below; cf. also Section 2.2).

Assume inductively that the claim holds in dimension d, and let us deal with the $(d + 1)$-dimensional case. Let $\gamma = Q^1 \subset \ldots \subset Q^d$ be the flag of μ-good parallelepipeds associated to γ. Fix $\lambda \geq 1$ so large that

$$\kappa < (d - 1)(1 - 1/\lambda) - (d - 2).$$

Chebyshev's inequality implies that for each $2 \leq m \leq d$, the proportion of the set of integers i in the projection of Q along the first coordinate for which $Q^m(i) := Q^m \cap \{i_1 = i\}$ is λ-good relative to Q^m is larger than $1 - 1/\lambda$. Therefore, for a proportion larger than $(d - 1)(1 - 1/\lambda) - (d - 2)$ of this set of integers i, these properties hold simultaneously, which means that the segment $Q^2(i)$ is fully λ-good in $Q^d(i)$. On each such a $Q^d(i)$, the inductive procedure yields a proportion larger than $\kappa/([(d-1)(1-1/\lambda)-(d-2)])$ of points in $Q^d(i)$ that can be reached by concatenating no more than d unidirectional λ'-good segments $\gamma_2, \ldots, \gamma_{d'}$ of $Q^d(i)$ (with γ_2 starting at a point of $\gamma_1 := Q^2(i)$), where $\lambda' = \lambda'\big(\lambda, \frac{\kappa}{[(d-1)(1-1/\lambda)-(d-2)]}, d\big)$. Note that each of these segments is $\mu\lambda'$-good in Q. Thus we have a proportion larger than

$$[(d - 1)(1 - 1/\lambda) - (d - 2)] \cdot \frac{\kappa}{[(d - 1)(1 - 1/\lambda) - (d - 2)]} = \kappa$$

of points in Q that can be reached by a sequence of d unidirectional segments that are $\mu\lambda'$-good, the first of which intersects γ. □

Proof of Theorem B (for $d \geq 3$). Consider the sequence of parallelepipeds $Q(n)$ defined at the beginning of Section 2.2. Fix $\lambda > 2(d - 1)$, and let $\lambda' := \lambda'(\lambda, 1/2, d)$ be the constant defined in the statement of Lemma 2.1. We perform a process that starts by arbitrarily choosing a fully λ-good 1-segment γ_1^1 of $Q(1)$. (Since $\lambda > 2(d - 1)$, we have $(d - 1)(1 - 1/\lambda) - (d - 2) > 1/2 > 0$, and an application of Chebyshev's inequality ensures the existence of such a segment.)

Assume now that there is a concatenating sequence of unidirectional segments $\gamma_1^1, \ldots, \gamma_1^{d_1'}, \ldots, \gamma_{n-1}^1, \ldots, \gamma_{n-1}^{d_{n-1}'}, \gamma_n^1$, with each $d_j' \leq d$, such that

- for each $1 \leq n' \leq n-1$ and $2 \leq k \leq d$, the segment $\gamma^k_{n'}$ is λ'-good in $Q(n') \cap Q(n'+1)$;
- for each $1 \leq n' \leq n$, the segment $\gamma^1_{n'}$ is a fully λ'-good $m(n')$-segment in $Q(n')$.

We would like to extend this sequence by choosing $\gamma^2_n, \ldots, \gamma_n^{d'_n}$ and γ^1_{n+1} appropriately. To do so, we first invoke Lemma 2.1, which ensures that more than half of the points of $Q(n) \cap Q(n+1)$ can be reached starting at a point of γ^1_n by concatenating λ'-good segments $\gamma^2_n, \ldots, \gamma^{d'}_n$, with $d' \leq d$. On the other hand, since $(d-1)(1-1/\lambda) - (d-2) > 1/2 > 0$, an application of Chebyshev's inequality ensures that more than half of the points of $Q(n) \cap Q(n+1)$ lie in a fully λ-good $m(n+1)$-segment γ^1_{n+1} of $Q(n+1)$. These two sets must intersect, and this fact allows us to define the desired concatenating segments.

Checking the properties from Section 2 now mimics the cases $d=2$ and $d=3$. Indeed, let $A > 0$ be a constant such that for every s-segment γ in Q,

$$2^{n(1-\alpha_s)}/A \leq c(\gamma, Q) \leq A2^{n(1-\alpha_s)}.$$

Let $\gamma = \gamma^k_n$, $k \neq 1$, and let $s := s_{n,k}$ be its direction. Hölder's inequality yields

$$\begin{aligned}\sum_{(i_1,\ldots,i_d)\in\gamma^k_n} \ell(i_1,\ldots,i_d)^{\alpha_s} &\leq \left(\sum_{(i_1,\ldots,i_d)\in\gamma^k_n} \ell(i_1,\ldots,i_d)\right)^{\alpha_s} |\gamma^k_n|^{1-\alpha_s} \\ &\leq \left(\frac{\lambda' L_{Q(n)\cap Q(n+1)}}{c(\gamma^k_n, Q_n)}\right)^{\alpha_s} (A'2^{n\alpha_s})^{1-\alpha_s} \\ &\leq (\lambda' A)^{\alpha_s}(A')^{1-\alpha_s} L_n^{\alpha_s}.\end{aligned}$$

In the case γ^1_n, a similar argument applies, so that (9) holds for

$$B \geq \max_k \{ (\lambda'' A)^{\alpha_k} (A')^{1-\alpha_k} \},$$

where $\lambda'' = \max\{\lambda, \lambda'\}$. Finally, it is easy to see that for a certain constant $D > 0$, each segment γ^1_n contains at least $2^{n\alpha}/D$ points, where $\alpha := \min\{\alpha_1, \ldots, \alpha_d\}$. □

3 Proof of the non-smoothability of the Farb-Franks action in critical regularity

We now deal with the group N_d of $(d+1)\times(d+1)$ lower-triangular matrices with integer entries, all of whose diagonal entries equal 1. For $i > j$, we denote by $f_{i,j}$ the element represented by a matrix, all of whose nondiagonal entries are zero except for (i,j)-entry, which is 1. Observe that the $f_{i,j}$'s generate N_d.

Let us briefly recall the Farb-Franks action of N_d on $[0, 1]$. First, note that N_d acts linearly on $\mathbb{Z}^{d+1}$ and that the affine hyperplane $1 \times \mathbb{Z}^d$ is invariant under this action. The induced action on $\mathbb{Z}^d$ produces an action on the interval as follows. Let $\{I_{i_1,\ldots,i_d} : (i_1, \ldots, i_d) \in \mathbb{Z}^d\}$ be a family of intervals such that the sum $\sum_{i_1,\ldots,i_d} |I_{i_1,\ldots,i_d}|$ is finite, say 1 after normalization. We join these intervals lexicographically on the closed interval $[0, 1]$ and identify $f \in N_d$ with the (unique) homeomorphism of $[0, 1]$ that sends the interval $I_{i_1,\ldots,i_d}$ affinely into $I_{f(i_1,\ldots,i_d)}$. Here, $f(i_1, \ldots, i_d)$ stands for the action of $f \in N_d$ on $\mathbb{Z}^d \sim \{1\} \times \mathbb{Z}^d$.

Let $\alpha = \alpha(d) := 2/d(d-1)$. As is shown in [3], for every $\varepsilon > 0$, this action is conjugated to an action by $C^{1+\alpha-\varepsilon}$ diffeomorphisms but cannot be (semi-) conjugated to an action by $C^{1+\alpha+\varepsilon}$ diffeomorphisms. Our aim is to extend the last result to the critical regularity $C^{1+\alpha}$. To this end, we follow a strategy similar to that of the generalized Kopell Lemma.

Assume that a topological conjugacy exists and, for simplicity, continue to denote by $I_{i_1,\ldots,i_d}$ the image of the corresponding interval under this conjugacy. Let I be the convex closure of $\bigcup_{i_d\in\mathbb{Z}} I_{0,\ldots,0,i_d}$. Note that the element $g := f_{d+1,1}$ lies in the center of N_d and fixes the interval I. Moreover, every element in N_d sends I either into itself or an interval disjoint from itself. Consider the isomorphic copy $N^*_{d-1} \subset N_d$ of N_{d-1} formed by all elements whose last row and column coincide with those of the identity. The orbit of I under N_d coincides with that under N^*_{d-1}. Moreover, the stabilizer of I under the N^*_{d-1}-action corresponds to the subgroup formed by the elements whose first column coincides with those of the identity. Since this subgroup is naturally isomorphic to N_{d-2}, the orbit-graph of I is identified with a coset space N_{d-1}/N_{d-2} and has $\mathbb{Z}^{d-1}$ as set of vertices; see [3, Figure 2] for an illustration in the case $d = 3$.

3.1 From sublinear distortion to the proof of Theorem C.

As in previous sections, we decompose (part of) the orbit of I (which is identified with $\mathbb{Z}^{d-1}$) into parallelepipeds. Following [3, Section 2.4], we define $Q(n)$ by induction. We first let $Q(0) := [1, 1 + 4^{d+1}]^{d-1}$. Next, assuming that

$$Q(n) := [x_{1,n}, y_{1,n}] \times \cdots \times [x_{d-1,n}, y_{d-1,n}]$$

has been already defined, we let $i(n) \in \{1, \ldots, d-1\}$ be the residue class (modulo $d-1$) of n and set

$$Q(n{+}1) := \cdots \times [1{+}4^{i(n)}(x_{i(n),n}{-}1),\, y_{i(n),n}] \times [x_{i(n)+1,n},\, 1{+}4^{i(n)+1}(y_{i(n)+1,n}{-}1)] \times \cdots,$$

where "$\cdots$" mean that the corresponding factors remain untouched. Note that all $x_{i,n}$, $y_{i,n}$, and $y_{i,n} - x_{i,n}$, are asymptotically equivalent to $4^{in/d-1}$.

For each $(i_1,\ldots,i_{d-1}) \in \mathbb{Z}^{d-1}$, let $\ell(i_1,\ldots,i_{d-1})$ be the length of the interval $I_{i_1,\ldots,i_{d-1}}$ defined as the convex closure of $\bigcup_{i_d\in\mathbb{Z}} I_{i_1,\ldots,i_{d-1},i_d}$. Also set

$$L_n := \sum_{(i_1,\ldots,i_{d-1})\in Q(n)} \ell(i_1,\ldots,i_{d-1}).$$

Our task now consists of showing that there exists a sequence of paths (segments) $\gamma_0, \gamma_1^1, \ldots, \gamma_1^{k_1}, \ldots, \gamma_n^1, \ldots, \gamma_n^{k_n}, \ldots$, with each $k_j \leq K_d$ for a certain constant K_d, such that

- each γ_n^k is contained in $Q(n)$, whereas γ_0 is contained in $Q(0)$;
- for each n, k, there exists a generator $f_{i,j}$ of N_{d-1}^* such that two consecutive points in γ_n^k differ by the action of either $f_{i,j}$ or its inverse;
- there exists a constant $D > 0$ such that for each n, at least one of the γ_n^k has no fewer than $4^{n/d-1}/D$ points;
- there exists a constant $B > 0$ such that for all n, k,

$$\sum_{(i_1,\ldots,i_{d-1})\in\gamma_n^k} \ell(i_1,\ldots,i_{d-1})^\alpha \leq BL_n^\alpha. \tag{15}$$

Assume that this task has been accomplished. We next explain how to complete the proof of Theorem C along the lines of the arguments given for Theorem B. (Showing the existence of the desired sequences of parallelepipeds and segments is postponed to the next two sections.)

Concatenation of the segments above produces an infinite path $\gamma : \mathbb{N}_0 \to \mathbb{N}_0^d$, which we may assume starts at the origin. (If not, we add an extra initial segment and change the constant B slightly.) For each $m \geq 0$, let f_m be the element of the form $f_{i,j}^{\pm 1}$ that moves the m^{th} point of γ to the $(m+1)^{\text{th}}$ point and let $h_m := f_m \circ \cdots \circ f_1$, with $h_0 := Id$. For each $n \geq 1$, denote by $N = N(n)$ the entry-time of γ into $Q(n+1)$. Because of the asymptotics of the lengths of the sides of $Q(n)$, $n \sim \log(N)$. By (15) and Hölder's inequality, for a certain constant $A' > 0$,

$$\begin{aligned}\sum_{m=0}^{N} \big|h_m(I)\big|^\alpha \leq B\sum_{m=0}^{n}\sum_{k=1}^{K_d} L_m^\alpha \leq B\sum_{k=1}^{K_d}\left(\sum_{m=0}^{n} L_m\right)^\alpha (n+1)^{1-\alpha} \\ \leq BK_d(d+2)^\alpha(n+1)^{1-\alpha} \leq A'\big(\log(N)\big)^{1-\alpha},\end{aligned}$$

where the factor $(d+2)$ comes from the multiplicity of the sequence $(Q(n))$.

Now, for every $x \in I$, the equality $g^k = h_m^{-1} \circ g^k \circ h_m$ yields

$$Dg^k(x) = \frac{Dh_m(x)}{Dh_m(y)} \cdot Dg^k(h_m(x)), \tag{16}$$

where $y := y_k = g^k(x)$. Since $y \in I$, we have for each N' such that $N(n-1) \leq N' \leq N(n) = N$,

$$\left|\log\left(\frac{Dh_{N'}(x)}{Dh_{N'}(y)}\right)\right| = \left|\log\left(\frac{\prod_{m=1}^{N'} Df_{i_m}(h_{m-1}(x))}{\prod_{m=1}^{N'} Df_{i_m}(h_{m-1}(y))}\right)\right|$$
$$\leq \sum_{m=1}^{N'} \left|\log Df_{i_m}(h_{m-1}(x)) - \log Df_{i_m}(h_{m-1}(y))\right|$$
$$\leq C\sum_{m=1}^{N'} \left|h_{m-1}(x) - h_{m-1}(y)\right|^{\alpha}$$
$$\leq C\sum_{m=0}^{N} \left|h_m(I)\right|^{\alpha}$$
$$\leq CA'\big(\log(N)\big)^{1-\alpha},$$

where $i > j$ and C is a common upper bound for the α_i-Hölder constants of $\log(Df_{i,j}), \log(Df_{i,j}^{-1})$.

Moreover, since at least one of the γ_n^k's is assumed to have no fewer than $4^{n/d-1}/D$ points, we may choose such an N' so that $\left|h_{N'}(I)\right| \leq D/4^{n/d-1}$. Borichev's theorem then yields

$$Dg^k(h_{N'}(x)) \leq \exp\left(3C_g\left|h_{N'}(I)\right|^{\alpha}k^{1-\alpha}\right) \leq \exp\left(\frac{3D^{\alpha}C_g k^{1-\alpha}}{4^{n\alpha/(d-1)}}\right),$$

which, in light of (16) and the previous estimate, implies

$$Dg^k(x) \leq \exp(CA'(\log(N))^{1-\alpha})\exp\left(\frac{3D^{\alpha}C_g k^{1-\alpha}}{4^{n\alpha/(d-1)}}\right).$$

Choose $n = n_k$ such that $k^{1-\alpha} \sim 4^{n\alpha/(d-1)}$, so that $n \sim \log(N) \sim \log(k)$. Then there exist a constant $A > 0$ such that $\max_{x\in I} Dg^k(x) \leq \exp\left(A\big(\log(k)\big)^{1-\alpha}\right)$. However, this last expression is of order $o(k)$, which is impossible by the Polterovich-Sodin lemma.

3.2 The case $d = 3$. Again for purely pedagogical reasons, we first deal with the case $d = 3$, although the reader should have no problem in passing directly to the general case treated in the next section. Observe that for $d = 3$, the critical value of α is $1/3$. In parallel with Section 2.2, let us introduce some terminology.

A **horizontal set** in $Q(2n+1)$ is a subset $P = P_r$ of the form

$$Q(2n+1) \cap \{(i,j) : i \in [x_{1,2n+1}, y_{1,2n+1}],$$
$$j \in [x_{2,2n+1} + (r-1)y_{1,2n+1}, x_{2,2n+1} + ry_{1,2n+1}[\},$$

where $r \in \{1, 2 \ldots, r_{2n+1}\}$ and $r_{2n+1} \sim (y_{2,2n+1} - x_{2,2n+1})/y_{1,2n+1}$ is the smallest possible index such that $Q(2n+1)$ is the union of the P_r's. Given $\lambda \geq 1$, such a set is said to be **λ-good** whenever $r < r_{2n+1}$ and

$$\sum_{(i,j) \in P} \ell(i, j) \leq \frac{\lambda L_{2n+1}}{r_{2n+1}}. \tag{17}$$

A **horizontal segment** in $Q(2n+1)$ is a subset of the form

$$Q(2n+1) \cap \{(i, j) : j = \text{constant}\}.$$

Such a segment γ is said to be **λ-good relative to** the horizontal set P containing it whenever

$$\sum_{(i,j) \in \gamma} \ell(i, j) \leq \frac{\lambda}{y_{1,2n+1}} \sum_{(i,j) \in P} \ell(i, j). \tag{18}$$

A **vertical set in** $Q(2n) \cap Q(2n+1)$ is a set of type

$$P_{2n}^{2n+1}(k) := Q(2n) \cap Q(2n+1) \cap \{(i, j) : i = k\}.$$

A **vertical segment in** $Q(2n) \cap Q(2n+1)$ is a set of the form

$$\gamma_{2n}^{2n+1}(k, r) := P_r \cap P_{2n}^{2n+1}(k).$$

This segment is **λ-good relative to** the vertical set $P_{2n}^{2n+1}(k)$ in $Q(2n) \cap Q(2n+1)$ containing it whenever

$$\sum_{(i,j) \in \gamma_{2n}^{2n+1}(k,r)} \ell(i, j) \leq \frac{\lambda}{r_{2n+1}} \sum_{(i,j) \in P_{2n}^{2n+1}(k)} .\ell(i, j) \tag{19}$$

A **vertical set in** $Q(2n+1) \cap Q(2n+2)$ is a set of the form

$$P_{2n+1}^{2n+2}(k) := Q(2n+1) \cap Q(2n+2) \cap \{(i, j) : i = k\}.$$

A **vertical segment in** $Q(2n+1) \cap Q(2n+2)$ is a set of the form

$$\gamma_{2n+1}^{2n+2}(k, r) := P_r \cap P_{2n+1}^{2n+2}(k).$$

This segment is **λ-good relative to** the horizontal set P_r in $Q(2n+1)$ containing it whenever

$$\sum_{(i,j) \in \gamma_{2n+1}^{2n+2}(k,r)} \ell(i, j) \leq \frac{\lambda}{1 + y_{1,2n+1} - x_{1,2n+1}} \sum_{(i,j) \in P_r} \ell(i, j). \tag{20}$$

Finally, a **vertical set in** $Q(2n+2)$ is a set of the form

$$P(k) := Q(2n+2) \cap \{(i,j) : i = k\}.$$

Such a set P is **λ-good** provided that

$$\sum_{(i,j)\in P} \ell(i,j) \leq \frac{\lambda L_{2n+2}}{1 + y_{1,2n+2} - x_{1,2n+2}}. \tag{21}$$

Now, for each $k \in [x_{1,2n+2}, y_{1,2n+2}]$, we decompose

$$\{k\} \times [x_{2,2n+2}, y_{2,2n+2}] \sim [x_{2,2n+2}, y_{2,2n+2}]$$

into k paths, each of which has consecutive points at distance k. The resulting paths are said to be **vertical segments in** $Q(2n+2)$.[1] Such a vertical segment γ is said to be **λ-good relative to** the vertical set $P = P(i)$ in $Q(2n+2)$ containing it if

$$\sum_{(i,j)\in\gamma} \ell(i,j) \leq \frac{\lambda}{i} \sum_{(i,j)\in P} \ell(i,j). \tag{22}$$

(Note that vertical segments in $Q(2n+2)$ naturally arise from the action of $f_{3,2}$.)

Assume as given a λ-good vertical set $P = P(k)$ in $Q(2n)$ and a 1-good vertical segment γ^1_{2n} relative to P. For at least half of the $r \in \{1, 2, \ldots, r_{2n+1} - 1\}$, the vertical segment $\gamma^{2n+1}_{2n}(k, r)$ is 2-good relative to $P^{2n+1}_{2n}(k)$. Similarly, at least half of the horizontal sets in $Q(2n+1)$ are 2-good. Consequently, there must be some $r \in \{1, 2, \ldots, r_{2n+1} - 1\}$ such that the corresponding vertical segment $\gamma^{2n+1}_{2n}(k, r) \subset Q(2n) \cap Q(2n+1)$ and horizontal set $P_r \subset Q(2n+1)$ are 2-good. Let γ^1_{2n+1} be a 1-good horizontal segment in P_r. The segments γ^1_{2n} and γ^1_{2n+1} do not necessarily intersect, but using the vertical segment $\gamma^2_{2n} := \gamma^{2n+1}_{2n}(k, r)$, we can concatenate them.

Assume now as given r such that P_r is a λ-good horizontal set in $Q(2n+1)$ together with a 1-good horizontal segment γ^1_{2n+1} relative to P_r. For more than half of the $k' \in [x_{1,2n+2}, y_{1,2n+2}]$, the vertical segment $\gamma^{2n+2}_{2n+1}(k', r)$ is 2-good relative to P_r. Similarly, for more than half of these k', the vertical set $P(k') \subset Q(2n+2)$ is 2-good. Take k' lying simultaneously in both sets, and choose any vertical segment γ^1_{2n+2} that is 1-good relative to $P(k')$. Again, the segments γ^1_{2n+1} and γ^1_{2n+2} do not necessarily intersect, but using $\gamma^2_{2n+1} := \gamma^{2n+2}_{2n+1}(k', r)$, we can concatenate them.

Thus, starting with any vertical segment γ_0 which is 1-good relative to a 1-good vertical set in Q_0, we can produce a concatenating sequence $\gamma_0, \gamma^1_1, \gamma^2_1, \gamma^1_2, \gamma^2_2, \ldots$.

[1] Here, rather surprisingly, there is no need for the intricate decomposition of [3, Section 2.5].

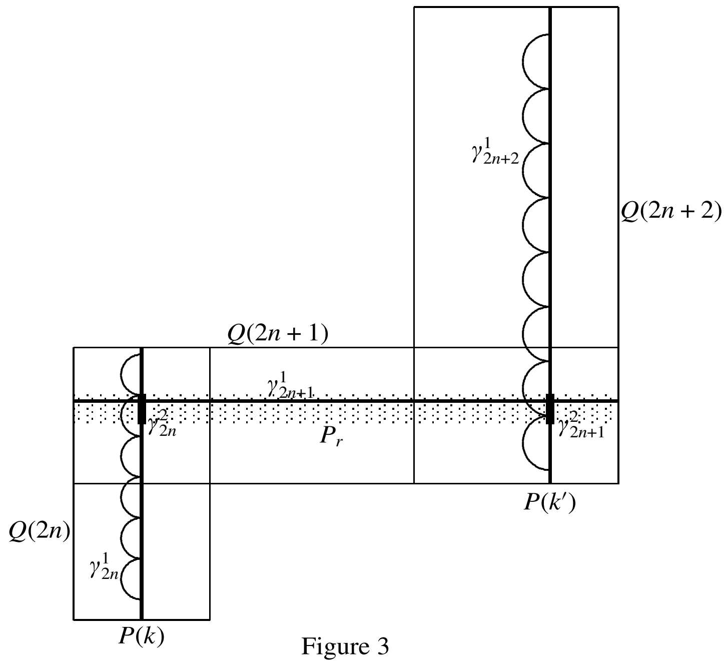

Figure 3

We claim that this induces a sequence of segments good enough in the sense that it satisfies the properties of Section 3.1. This finishes the proof of Theorem C for the case $d = 3$. Indeed, the first two properties are clear from the construction, whereas the third is easily seen to hold for γ_n^1. To check the fourth property, viz. (15), we use throughout the asymptotics of $x_{i,n}, y_{i,n}, y_{i,n} - x_{i,n}$ (which are all of order $4^{in/(d-1)}$). Recall also that $\alpha = 1/3$.

For γ_{2n}^1, the adapted version of (21) and (22) together with Hölder's inequality yield

$$\begin{aligned}\sum_{(i,j)\in\gamma_{2n}^1} \ell(i,j)^\alpha &\leq \left(\frac{2}{i}\sum_{(i,j)\in P(k)} \ell(i,j)\right)^\alpha |\gamma_{2n}^1|^{1-\alpha} \\ &\leq \left(\frac{2}{i}\cdot\frac{2L_{2n}}{1+y_{1,2n}-x_{1,2n}}\right)^\alpha C\left(\frac{y_{2,2n}-x_{2,2n}}{i}\right)^{1-\alpha} \\ &\leq \frac{C'(y_{2,2n}-x_{2,2n})^{1-\alpha}}{x_{1,2n}(y_{1,2n}-x_{1,2n})^\alpha}\cdot L_{2n}^\alpha \leq B\frac{4^{2n(1-\alpha)}}{4^n 4^{n\alpha}} L_{2n}^\alpha \\ &= B\frac{4^{4n/3}}{4^n 4^{n/3}} L_{2n}^\alpha = BL_{2n}^\alpha.\end{aligned}$$

For γ_{2n}^2, (19) and Hölder's inequality yield

$$
\begin{aligned}
\sum_{(i,j)\in\gamma_{2n}^2} \ell(i,j)^\alpha &\le \left(\frac{2}{r_{2n+1}} \sum_{(i,j)\in P_{2n}^{2n+1}(k)} \ell(i,j)\right)^\alpha |\gamma_{2n}^2|^{1-\alpha} \\
&\le \left(\frac{2Cy_{1,2n+1}}{y_{2,2n+1}-x_{2,2n+1}} \cdot \frac{2L_{2n}}{1+y_{1,2n}-x_{1,2n}}\right)^\alpha y_{1,2n+1}^{1-\alpha} \\
&\le C' \frac{4^n}{(4^{(2n+1)}4^n)^\alpha} \cdot L_{2n}^\alpha \\
&\le B \frac{4^n}{4^{3n\alpha}} L_{2n}^\alpha \\
&= BL_{2n}^\alpha.
\end{aligned}
$$

For γ_{2n+1}^1, the appropriate versions (17) and (18) and Hölder's inequality yield

$$
\begin{aligned}
\sum_{(i,j)\in\gamma_{2n+1}^1} \ell(i,j)^\alpha &\le \left(\frac{2}{y_{1,2n+1}} \sum_{(i,j)\in P_r} \ell(i,j)\right)^\alpha |\gamma_{2n+1}^1|^{1-\alpha} \\
&\le \left(\frac{2}{y_{1,2n+1}} \cdot \frac{2L_{2n+1}}{r_{2n+1}}\right)^\alpha C(1+y_{1,2n+1}-x_{1,2n+1})^{1-\alpha} \\
&\le C' \frac{(1+y_{1,2n+1}-x_{1,2n+1})^{(1-\alpha)} y_{1,2n+1}^\alpha}{y_{1,2n+1}^\alpha (1+y_{2,2n+1}-x_{2,2n+1})^\alpha} \cdot L_{2n+1}^\alpha \\
&\le B \frac{4^{n(1-\alpha)}}{4^{(2n+1)\alpha}} L_{2n+1}^\alpha \\
&= BL_{2n+1}^\alpha.
\end{aligned}
$$

Finally, for γ_{2n+1}^2, using (20), we obtain

$$
\begin{aligned}
\sum_{(i,j)\in\gamma_{2n+1}^2} \ell(i,j)^\alpha &\le \left(\frac{2}{1+y_{1,2n+1}-x_{1,2n+1}} \sum_{(i,j)\in P_r} \ell(i,j)\right)^\alpha |\gamma_{2n+1}^2|^{1-\alpha} \\
&\le \left(\frac{CL_{2n+1}}{(1+y_{1,2n+1}-x_{1,2n+1})r_{2n+1}}\right)^\alpha y_{1,2n+1}^{1-\alpha} \\
&\le C' \frac{y_{1,2n+1}^\alpha y_{1,2n+1}^{1-\alpha}}{(1+y_{2,2n+1}-x_{2,2n+1})^\alpha (1+y_{1,2n+1}-x_{1,2n+1})^\alpha} \cdot L_{2n+1}^\alpha \\
&\le B \frac{4^n}{4^{2n\alpha}4^{n\alpha}} L_{2n+1}^\alpha \\
&= BL_{2n+1}^\alpha.
\end{aligned}
$$

3.3 The general case. As with Theorem B, we prove Theorem C by induction.

The vertical subdivision procedure. Given $d \geq 3$, let $Q := \prod_{k=1}^{d-1} [x_k, y_k]$ be a parallelepiped in $\mathbb{Z}^{d-1}$, where $x_1, y_1, \ldots, x_{d-1}, y_{d-1}$ are integers. For $A \geq 1$, we say that Q is A**-round** if

$$\begin{aligned} \frac{(1+y_1-x_1)^i}{A} &\leq x_i < y_i \leq A(1+y_1-x_1)^i, \\ \frac{(1+y_1-x_1)^i}{A} &\leq 1+y_i-x_i \leq A(1+y_1-x_1)^i. \end{aligned} \tag{23}$$

Cutting along the last coordinate, we can divide every A-round parallelepiped Q into disjoint parallelepipeds $Q_1, \ldots, Q_{M_1}$, each of which, except possibly the last one, has $[(d-1)]^{\mathrm{st}}$-side of length $y_{d-2} - 1$. By (23),

$$\frac{1+y_1-x_1}{A^2} \leq \frac{1+y_{d-1}-x_{d-1}}{y_{d-2}-1} \leq M_1 \leq 1+\frac{1+y_{d-1}-x_{d-1}}{y_{d-2}-1} \leq 1+A^2(1+y_1-x_1).$$

Similarly, we can subdivide each Q_{m_1} satisfying $m_1 < M_1$ into disjoint parallelepipeds $Q_{m_1,1}, Q_{m_1,2}, \ldots, Q_{m_1,M_2}$, each of which, except possibly the last one, has (d − 1)st-side of length $y_{d-3} - 1$. Again, (23) implies that

$$\frac{1+y_1-x_1}{A^2} \leq M_2 \leq 1+A^2(1+y_1-x_1).$$

In general, for $k \leq d-2$, each parallelepiped $Q_{m_1,\ldots,m_{k-1}}$ satisfying $m_j \neq M_j$ for all $j \leq k-1$ can be divided into $Q_{m_1,\ldots,m_{k-1},1}, Q_{m_1,\ldots,m_{k-1},2}, \ldots, Q_{m_1,\ldots,m_{k-1},M_k}$, where each small parallepiped, except possibly the last one, has (d − 1)$^{\mathrm{st}}$-side of length y_{d-k-2}. Moreover, (23) implies that

$$\frac{1+y_1-x_1}{A^2} \leq M_k \leq 1+A^2(1+y_1-x_1). \tag{24}$$

Here, for $k=0$, we interpret $Q_{m_1,\ldots,m_k}$ as Q.

A **level** in Q is a set of the form $H_i \cap Q$, where

$$H_i := \{(i_1, \ldots, i_{d-2}, i_{d-1}) \in \mathbb{Z}^{d-1} : i_{d-1} = i\}.$$

To each level is associated a unique sequence

$$H_i \cap Q \subset Q_{m_1,\ldots,m_{d-2}} \subset Q_{m_1,\ldots,m_{d-3}} \subset \ldots \subset Q_{m_1} \subset Q. \tag{25}$$

We say that the level is **admissible** if each of the m_i's above differs from the corresponding M_i. A level that is not admissible is called **non-admissible**. Using

(23) and (24), it is easy to check that for a certain constant $A' = A'(A, d)$, the proportion of non-admissible levels is no larger than

$$\frac{1}{(1+y_{d-1}-x_{d-1})}\Big[(y_{d-2}-1)+(y_{d-3}-1)M_1+(y_{d-4}-1)M_1M_2+\ldots\Big] \leq \frac{A'}{1+y_1-x_1}.$$

A **vertical section** in Q is a set of the form $V_{j_1,\ldots,j_{d-2}} \cap Q$, where

$$V_{j_1,\ldots,j_{d-2}} := \{(j_1, \ldots, j_{d-2}, i) : i \in \mathbb{Z}\}.$$

Very good points and levels. Assume now that we are given a positive function ℓ defined on $\mathbb{Z}^{d-1}$. For each parallelepiped $Q' \subset \mathbb{Z}^{d-1}$, let

$$L_{Q'} := \sum_{(i_1,\ldots,i_{d-1})\in Q'} \ell(i_1, \ldots, i_{d-1})$$

and $\langle \ell_{Q'} \rangle = L_{Q'}/\mathrm{card}\,(Q')$.

Given $\lambda \geq 1$, we say that a level $H_i \cap Q$ with associated sequence (25) is **fully λ-good** if $\langle \ell_{Q_{m_1,\ldots,m_k}} \rangle \leq \lambda \langle \ell_Q \rangle$ for all $k \geq 1$. Note that the fraction of levels which are fully λ-good is larger than $(1-(d-2)/\lambda)$. Analogously, we say that the point $p := (j_1, \ldots, j_{d-2}, i_{d-1}) = V_{j_1,\ldots,j_{d-2}} \cap H_{i_{d-1}}$ is **fully λ-good** whenever

$$\langle \ell_{Q_{m_1,\ldots,m_k} \cap V_{j_1,\ldots,j_{d-2}}} \rangle \leq \lambda \langle \ell_Q \rangle \tag{26}$$

for all $k \geq 1$. For each $\lambda' \geq 1$, the proportion of fully $\lambda\lambda'$-good points in any fully λ-good level is larger than $(1-(d-2)/\lambda-(d-2)\lambda')$.

Reaching points from very good points along good vertical sections. A **segment** in Q is a sequence of points for which there exists a generator $f_{i,j} \in N_d$ such that each point in the sequence is obtained from its predecessor by the action of either $f_{i,j}$ or its inverse. Such a segment γ is said to be **horizontal** if the generator is $f_{2,1}$ and γ contains $(1+y_1-x_1)$ points. The segment is said to be **vertical** if the generator is one of $f_{d,1}, \ldots, f_{d,d-1}$ (with no hypothesis on the number of points).

Given $\lambda \geq 1$, we say that a segment γ in Q is **λ-good** if $\langle \ell_\gamma \rangle \leq \lambda \langle \ell_Q \rangle$.

Lemma 3.1. *For all $0 < \kappa < 1$ and $\mu \geq 1$, there exist constants $\lambda = \lambda_1(\kappa, \mu, A, d)$ and $D' > 0$ such that if $p := (j_1, \ldots, j_{d-2}, i) = V_{j_1,\ldots,j_{d-2}} \cap H_i$ is a fully μ-good point in a A-round parallelepiped Q such that the level $H_i \cap Q$ is admissible, then at least a proportion κ of the points in $V_{j_1,\ldots,j_{d-2}} \cap Q$ can be reached from p by concatenating no more than $d-2$ vertical segments that are λ-good and have no more than $D'(1+y_{1,n}-x_{1,n})$ points.*

Proof. All points in $Q_{m_1,\ldots,m_{d-2}} \cap V_{j_1,\ldots,j_{d-2}}$ can be reached starting from p (via the segment $\gamma := Q_{m_1,\ldots,m_{d-2}} \cap V_{j_1,\ldots,j_{d-2}}$), using $f_{d,1}^{\pm 1}$. From (26) with $k = d-2$, (23), and (24), it follows that

$$\sum_{(i_1,\ldots,i_{d-1})\in\gamma} \ell(i_1,\ldots,i_{d-1}) = L_{Q_{m_1,\ldots,m_{d-2}} \cap V_{j_1,\ldots,j_{d-2}}}$$

$$\leq \frac{\mu L_Q}{(M_{d-2}-1)\cdots(M_1-1)\prod_{j=1}^{d-2}(1+y_j-x_j)}$$

$$\leq \frac{\mu 2^{d-2}A^{3d-6}}{(1+y_1-x_1)^{d-2+\frac{(d-2)(d-1)}{2}}} L_Q = \frac{\mu 2^{d-2}A^{3d-6}}{(1+y_1-x_1)^{\frac{1}{\alpha(d)}-1}} L_Q.$$

Hence $\langle \ell_\gamma \rangle \leq \mu 2^{d-2} A^{d-4} \langle \ell_Q \rangle$. Now the action of $f_{d,2}$ divides

$$Q_{m_1,\ldots,m_{d-3}} \cap V_{j_1,\ldots,j_{d-2}}$$

into j_1 segments. Given $\lambda' \geq 1$, for a proportion larger than $(1 - 1/\lambda')$ of these segments γ,

$$\sum_{(i_1,\ldots,i_{d-1})\in\gamma} \ell(i_1,\ldots,i_{d-1}) \leq \frac{\lambda'}{j_1} L_{Q_{m_1,\ldots,m_{d-3}} \cap V_{j_1,\ldots,j_{d-2}}}$$

$$\leq \frac{A\lambda'}{(1+y_1-x_1)} \cdot \frac{\mu L_Q}{(M_{d-3}-1)\cdots(M_1-1)\prod_{j=1}^{d-2}(1+y_j-x_j)}$$

$$\leq \frac{\mu 2^{d-3}A^{3d-7}\lambda'}{(1+y_1-x_1)^{\frac{1}{\alpha(d)}-1}} L_Q.$$

By concatenating these segments with the previous ones, we can reach from p a proportion larger than $(1-2/\lambda')$ of the points of $Q_{m_1,\ldots,m_{d-3}} \cap V_{j_1,\ldots,j_{d-2}}$.[2] Similarly, the action of $f_{d,3}$ divides $Q_{m_1,\ldots,m_{d-4}}$ into j_2 paths; from these, a proportion larger than $(1 - 2/\lambda')$ is λ''-good for $\lambda'' := \mu 2^{d-4} A^{3d-8} \lambda'$. By concatenating these paths to the preceding ones, we can reach from p a proportion larger than $(1 - 2/\lambda')$ of the points in $Q_{m_1,\ldots,m_{d-4}} \cap V_{j_1,\ldots,j_{d-2}}$.

Continuing this procedure and choosing appropriately λ' yields the concatenation property. Moreover, it is clear from the construction that the claim concerning the cardinality of each of the λ-good segments holds for a certain constant $D' = D'(A)$. We leave the details to the reader. □

Concatenating sequences along finitely many parallelepipeds. Let $\mathcal{F}_d$ be the family of finite sequences $Q^1, \ldots, Q^{d-1}$ of parallelepipeds in $\mathbb{Z}^{d-1}$ such

[2]The extra factor 2 comes from the fact that the number of segments under consideration may differ by 1 from the number of points in $Q_{m_1,\ldots,m_{d-3}}$.

that if $Q^j = \prod_{i=1}^{d-1}[x_{i,j}, y_{i,j}]$, then

$$Q^{j+1} = \cdots \times [x', y_{j,j}] \times [x_{j+1,j}, y'] \times \cdots ,$$

where $x' > x_{j,j}$, $y' > y_{j+1,j}$. (Here, "$\cdots$" means that the corresponding entries remain untouched). Given $A \geq 1$, we denote by $\mathcal{F}_{d,A}$ the subfamily of all sequences of A-round parallelepipeds.

Given $\mu \geq 1$, we say that an horizontal segment

$$\gamma := \{(i, j_2, \ldots, j_{d-1}) : i \in [x_{1,1}, y_{1,1}]\} \in Q^1$$

is **fully μ-good with respect to $Q^1, \ldots, Q^{d-1}$** if

- the level $H_{j_{d-1}} \cap Q^{d-1}$ is admissible and fully μ-good in Q^{d-1},
- the level $\{(i_1, \ldots, i_{d-3}, j_{d-2}, j_{d-1}) : i_k \in \mathbb{Z}\} \cap Q^{d-2}$ is admissible and fully μ-good in the parallelepiped $Q^{d-2} \cap \{(i_1, \ldots, i_{d-2}, j_{d-1}) : i_k \in \mathbb{Z}\}$ (where the last intersection is understood as being contained in $\mathbb{Z}^{d-2} \sim \mathbb{Z}^{d-2} \times \{j_{d-1}\}$),
-

$$\vdots,$$

- the level $\{(i_1, i_2, j_3, \ldots, j_{d-2}, j_{d-1}) : i_k \in \mathbb{Z}\} \cap Q^3$ is admissible and fully μ-good in the parallelepiped $Q^3 \cap \{(i_1, i_2, i_3, j_4, \ldots, j_{d-1}) : i_k \in \mathbb{Z}\}$ (the last intersection understood as being contained in $\mathbb{Z}^3 \sim \mathbb{Z}^3 \times \{(j_4, \ldots, j_{d-1})\}$).

Note that there exists a constant $A'' = A''(A, d)$ such that a fraction larger than $(1 - (d-3)/\mu - A''/(1 + y_1 - x_1))$ of horizontal segments in Q^1 are fully λ-good.

Given a sequence $Q^1, \ldots, Q^{d-1}$ in $\mathcal{F}_d$, a **concatenating sequence from Q^1 to Q^{d-1}** is a sequence of segments $\gamma^1, \ldots, \gamma^k$ such that

- each γ^i is a segment in one of the Q^j's,
- each γ^i intersects γ^{i+1},
- the segment γ^1 is horizontal in Q^1, whereas γ^k is vertical in Q^{d-1}.

We say that such a sequence is **λ-good** for $\lambda \geq 1$ if each of its segments is λ-good in one of the Q^j's containing it.

Lemma 3.2. *Given $A > 0$, $\mu \geq 1$, and $0 < \kappa < 1$, there exists constant $\lambda = \lambda_2(\kappa, \mu, A, d)$ such that the following holds. Let $Q^1, \ldots, Q^{d-1}$ be a sequence in $\mathcal{F}_d(A)$ and $\gamma := \{(i, j_2, \ldots, j_{d-1}) : i \in [x_{1,1}, y_{1,1}]\}$ a fully μ-good horizontal segment for this sequence. Then a proportion of at least κ of the points in Q^{d-1} can be reached via a λ-good concatenating sequence from Q^1 to Q^{d-1} that starts with $\gamma^1 := \gamma$ and is formed by no more than K_d segments, where $K_d \geq 1$ is a constant.*

Proof. We proceed by induction. The argument for $d = 3$ is similar to that for the general case. It also corresponds to a more accurate quantitative version of that given in the previous section. For this reason, we leave the proof for the case $d = 3$ as an exercise to the reader.

Assume that the claim holds for $d' < d$ and consider a sequence $Q^1, \ldots, Q^d$ in $\mathcal{F}_{d+1}(A)$. The inductive hypothesis applies to the sequence

$$Q^1 \cap H_{i_d}, \ldots, Q^{d-1} \cap H_{i_d},$$

where each of these intersections is understood to be a parallelepiped in $\mathbb{Z}^{d-1}$. Indeed, the definitions above been made so that γ is also fully μ-good with respect to this sequence. Accordingly, fixing $0 < \kappa' < 1$, then starting with $\gamma^1 := \gamma$, and using no more than K_d segments that are $\lambda_2(\kappa', \mu, A, d)$-good, we can reach a proportion larger than κ' of the points in $H_{i_d} \cap Q^{d-1}$. By (23), this correspond to a proportion larger than $\kappa' + 1/A^2 - 1$ of the points in $H_{i_d} \cap Q^d$. This last level is fully μ-good in Q^d; hence, a proportion larger than $(1-(d-2)(1-\kappa'))$ of its points are fully $\mu/(1-\kappa')$-good. By Lemma 3.1, every such a point can reach a proportion of at least κ' of the points in its vertical set in Q^d by a concatenation of no more than d vertical segments that are $\lambda_1(\kappa, \mu/(1-\kappa'), A, d+1)$-good. Therefore, the concatenation of these two sequences of segments reaches a proportion of points in Q^d larger than $1 - [(1-(\kappa'+1/A^2-1)) + (1-\kappa')] = 2\kappa' + 1/A^2 - 2$. The proof for $K_{d+1} := K_d + d$ and

$$\lambda_2(\kappa, \mu, A, d+1) := \max\{\lambda_2(\kappa', \mu, A, d), \lambda_1(\kappa, \mu/(1-\kappa'), A, d+1)\}$$

is completed by choosing κ' appropriately (very close to 1). □

Proof of Theorem C. We return to the sequence of parallelepipeds $\{Q(n)\}$ introduced at the beginning of Section 3.1. By the asymptotics of the lengths of their sides, there exists a constant $A = A_d$ such that for each $l \geq 0$, the finite sequence $Q^{l(d-1)+1}, Q^{l(d-1)+2}, \ldots, Q^{l(d-1)+d-1}$ belongs to the family $\mathcal{F}_d(A)$.

Fix $\mu \geq 1$ such that for all n at least as large as a certain fixed N_0,

$$1 - \frac{d-3}{\mu} - \frac{A''(A,d)}{1 + y_{1,n(d-1)} - x_{1,n(d-1)+1}} > \frac{1}{2}.$$

Then more than half of the horizontal segments of $Q(n(d-1)+1)$ are fully μ-good. We may therefore fix $\lambda' \geq 1$ so that horizontal segments are not only fully μ-good, but also λ'-good in $Q(n(d-1)+1)$ in a proportion larger than $1/2$. By Lemma 3.2, starting with any fully μ-good horizontal segment in $Q((d-1)N_0+1)$, we can find an infinite concatenating sequence of λ-good segments for

$$\lambda := \max\{\lambda', \lambda_2(1/2, \mu, A_d, d)\}.$$

Moreover, according to Lemma 3.1, each of these segments contained in $Q(n)$ has no more than $D'(1+y_{1,n}-x_{1,n})$ points.

Modulo slightly changing the constant λ above, we may actually assume that the sequence begins at $Q(0)$. We claim that the sequence that remains after cutting along concatenation points satisfies the properties from Section 3.1. The first two properties are obvious, while the third follows easily from the fact that the segments lying in a parallelepiped $Q(n)$ must concatenate between $Q(n-1)$ and $Q(n+1)$, and these two last parallelepipeds are at distance comparable to the length of one of the sides of $Q(n)$. To conclude the proof, we need to check the appropriate version of (15). To do so, we simply observe that if γ is λ-good in $Q(n)$ and contains at least $4^{n/(d-1)}/D$ points, then by Hölder's inequality,

$$\begin{aligned}\sum_{(i_1,\ldots,i_{d-1})\in\gamma} \ell(i_1,\ldots,i_{d-1})^\alpha &\leq \left(\sum_{(i_1,\ldots,i_{d-1})\in\gamma} \ell(i_1,\ldots,i_{d-1})\right)^\alpha |\gamma|^{1-\alpha} \\ &\leq \left(\frac{\lambda L_Q}{(1+y_{1,n}-x_{1,n})^{\frac{1}{\alpha}-1}}\right)^\alpha \big(D'(1+y_{1,n}-x_{1,n})\big)^{1-\alpha} \\ &= BL_Q^\alpha,\end{aligned}$$

where the last equality defines B. □

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