EMBEDDINGS OF WEIGHTED SOBOLEV SPACES AND GENERALIZED CAFFARELLI-KOHN-NIRENBERG INEQUALITIES

By

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Abstract. We characterize all the real numbers a, b, c and $1 \leq p, q, r < \infty$ such that the weighted Sobolev space

$W^{(q,p)}_{\{a,b\}}(\mathbb{R}^N \backslash \{0\}) := \{u \in L^1_{loc}(\mathbb{R}^N \backslash \{0\}) : |x|^{a/q} u \in L^q(\mathbb{R}^N), |x|^{b/p} \nabla u \in (L^p(\mathbb{R}^N))^N\}$

is continuously embedded into

$$L^r(\mathbb{R}^N; |x|^c dx) := \{u \in L^1_{loc}(\mathbb{R}^N \backslash \{0\}) : |x|^{c/r} u \in L^r(\mathbb{R}^N)\}$$

with norm $||\cdot||_{c,r}$. It turns out that, except when $N \geq 2$ and $a = c = b - p = -N$, such an embedding is equivalent to the multiplicative inequality

$$||u||_{c,r} \leq C||\nabla u||^{\theta}_{b,p} ||u||^{1-\theta}_{a,q}$$

for some suitable $\theta \in [0, 1]$, which is often but not always unique. If $a, b, c > -N$, then $C^\infty_0(\mathbb{R}^N) \subset W^{(q,p)}_{\{a,b\}}(\mathbb{R}^N \backslash \{0\}) \cap L^r(\mathbb{R}^N; |x|^c dx)$ and such inequalities for $u \in C^\infty_0(\mathbb{R}^N)$ are the well-known Caffarelli-Kohn-Nirenberg inequalities; but their generalization to $W^{(q,p)}_{\{a,b\}}(\mathbb{R}^N \backslash \{0\})$ cannot be proved by a denseness argument. Without the assumption $a, b, c > -N$, the inequalities are essentially new, even when $u \in C^\infty_0(\mathbb{R}^N \backslash \{0\})$, although a few special cases are known, most notably the Hardy-type inequalities when $p = q$.

In a different direction, the embedding theorem easily yields a generalization when the weights $|x|^a$, $|x|^b$ and $|x|^c$ are replaced with more general weights w_a, w_b and w_c, respectively, having multiple power-like singularities at finite distance and at infinity.

1 Introduction

Suppose N is a positive integer, $d \in \mathbb{R}$, and $1 \leq s < \infty$. Let $||\cdot||_{d,s}$ denote the norm of the space $L^s(\mathbb{R}^N; |x|^d dx)$, where the $|x|^d dx$ -measure of $\{0\}$ is defined to be 0 (which must be specified if $d \leq -N$). With this definition, $u \in L^s(\mathbb{R}^N; |x|^d dx)$ if and only if $|x|^{d/s} u \in L^s(\mathbb{R}^N)$ and $||u||_{d,s} = ||\,|x|^{d/s} u||_s$, where $||\cdot||_s := ||\cdot||_{0,s}$. Throughout the paper, $\mathbb{R}^N_* := \mathbb{R}^N \backslash \{0\}$.

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 118 (2012)
DOI 10.1007/s11854-012-0035-1

Given $a, b \in \mathbb{R}$ and $1 \leq p, q < \infty$, consider the weighted Sobolev space

$$W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) := \{u \in L^1_{loc}(\mathbb{R}^N_*) : u \in L^q(\mathbb{R}^N; |x|^a dx), \nabla u \in (L^p(\mathbb{R}^N; |x|^b dx))^N\}, \tag{1.1}$$

equipped with the norm

$$||u||_{a,q} + ||\nabla u||_{b,p}. \tag{1.2}$$

Since $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ may contain functions which are not locally integrable near 0, and hence not distributions on $\mathbb{R}^N$, it is generally larger than the space $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N)$ (self-explanatory notation) which, incidentally, is not always complete.

In this paper, we characterize all the real numbers a, b, c and $1 \leq p, q, r < \infty$ such that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$, where "$\hookrightarrow$" denotes continuous embedding. This provides sufficient conditions for $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$; their necessity is not investigated.

In spite of the large literature devoted to embeddings of weighted Sobolev spaces, there seems to be little that addresses and resolves the same question in special cases. While most results allow for weights satisfying general properties, they also incorporate a number of restrictive hypotheses which are rarely necessary. Only a few are applicable to the whole (or punctured) space, and even fewer accommodate weights which, like all nontrivial power weights, exhibit singularities at 0 and infinity simultaneously. This is especially true when more than one weight (here, $a \neq b$) or more than one order of integration (i.e., $p \neq q$) is involved in the source space. In addition, the weighted spaces are often defined to be the unknown closure of some subspace of smooth (enough) functions, as the issue of denseness in (1.1) is a notorious difficulty [30]. In particular, this is the definition chosen in [17] (see also the more recent and expanded book [18]), except in the unweighted case.

Before continuing this discussion, we state the embedding theorem. In addition to the standard notation

$$p^* = \begin{cases} \infty & \text{if } p \geq N, \\ Np/(N-p) & \text{if } 1 \leq p < N, \end{cases}$$

we denote by c^0 and c^1 the two points

$$c^0 := \frac{r(a+N)}{q} - N \qquad \text{and} \qquad c^1 := \frac{r(b-p+N)}{p} - N, \tag{1.3}$$

where it is understood that a, b, p, q and r are given. The points c^0 and c^1 are distinct if and only if $(a+N)/q \neq (b-p+N)/p$. If so, and if $c \in [c^0, c^1]$, we set

$$\theta_c := \frac{c - c^0}{c^1 - c^0}, \tag{1.4}$$

so that $\theta_c \in [0, 1]$ and

$$c = \theta_c c^1 + (1-\theta_c)c^0. \tag{1.5}$$

Observe that $\theta_{c^0} = 0, \theta_{c^1} = 1$ and by (1.3), (1.4), and (1.5),

$$\frac{c+N}{r} = \theta_c \frac{b-p+N}{p} + (1-\theta_c)\frac{a+N}{q}. \tag{1.6}$$

Theorem 1.1. *Let N be a positive integer. If $N > 1$, let $a, b, c \in \mathbb{R}$ and $1 \leq p, q, r < \infty$. If $N = 1$, let $1 \leq p < \infty$ and $0 < q, r < \infty$. Then $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ (and hence $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow W^{1,(r,p)}_{\{c,b\}}(\mathbb{R}^N_*)$) if and only if $r \leq \max\{p^*, q\}$ and one of the following conditions[1] holds:*

(i) *a and $b-p$ are on the same side of $-N$ (including $-N$), $(a+N)/q \neq (b-p+N)/p$, c is in the open interval with endpoints c^0 and c^1, and $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$;*

(ii) *a and $b-p$ are strictly on opposite sides of $-N$ (hence $(a+N)/q \neq (b-p+N)/p$), c is in the open interval with endpoints c^0 and $-N$, and $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$;*

(iii) *$r = q$ and $c = a$ $(= c^0)$;*

(iv) *$p \leq r \leq p^*$, $a \leq -N$ and $b-p < -N$ or $a \geq -N$ and $b-p > -N$, $c = c^1$;*

(v) *$(\max\{p^*, q\} \geq)$ $r \geq \min\{p, q\}$, $(a+N)/q = (b-p+N)/p \neq 0$, and $c = c^1$ $(= c^0)$;*

(vi) *$a = -N$, $b = p-N$, $q < r \leq p^*$ and $c = c^1$ $(= c^0 = -N)$.*

Since r is finite, $r = p^*$ is impossible when $p \geq N$. The set of admissible values of c is an interval (possibly empty, see Remark 1), of which c^0, c^1 and $-N$ may or may not be endpoints, but are never interior points. When c^0 or c^1 are endpoints, their admissibility is decided by parts (iii) to (vi). Endpoints other than c^0, c^1 or $-N$ are always admissible, but $-N$ is never admissible when $a \neq -N$. If $a = -N$, then $-N$ is admissible only in the trivial case (iii) and the exceptional case (vi).

Apparently, aside from the trivial part (iii), only parts (v) and (vi) of Theorem 1.1 when $q = p$ (hence $a = b - p$) are known with nontrivial weights. See Opic and Kufner [22, p. 291], where the result is credited to Opic and Gurka [21].

[1]The overlap between conditions (iii), (iv) and (v) makes for a simpler and clearer statement.

Curiously, part (v) shows that if $b - p \neq -N$ and $a_q := q(b - p + N)/p - N$, the space $W^{1,(q,p)}_{\{a_q,b\}}(\mathbb{R}^N_*)$ is independent of $q \in [p, p^*], q < \infty$, with equivalent norms as q varies. When $N = 1$, part (iv) can and will be deduced from an inequality of Bradley [5]. Related, but different, work is discussed further below.

In the unweighted case $a = b = c = 0$ and if $p = q$ and $N \geq 2$ (a minor point), Theorem 1.1 gives again $W^{1,p}(\mathbb{R}^N_*) = W^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ if and only if ($r < \infty$ and) $p \leq r \leq p^*$ (Subsection 11.1). If $p \neq q$ (and still $a = b = c = 0$), Theorem 1.1 is akin to embedding theorems in [2, 3].

Remark 1. If $r \leq \min\{p^*, q\}$, then $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$ for every c between c^0 and c^1. In contrast, all the conditions of Theorem 1.1 fail (i.e., no embedding holds for any c) if $p < N$ and $r > \max\{p^*, q\}$ or if either

(i) $p < N, r = p^* > q, b - p = -N \neq a$, or

(ii) $q < r \leq p^*$, a and $b - p$ are strictly on opposite sides of $-N$ (hence θ_{-N} is defined) and $\theta_{-N}(p^{-1} - N^{-1} - q^{-1}) \geq r^{-1} - q^{-1}$.

A simple rescaling shows (Corollary 2.1) that when $(a+N)/q \neq (b-p+N)/p$, the embedding $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ is equivalent to the multiplicative inequality

$$||u||_{c,r} \leq C||\nabla u||^{\theta_c}_{b,p}||u||^{1-\theta_c}_{a,q}, \tag{1.7}$$

rather than just $||u||_{c,r} \leq C\left(||u||_{a,q} + ||\nabla u||_{b,p}\right)$.

When $a, b, c > -N$ and $u \in C^\infty_0(\mathbb{R}^N)$, (1.7) is one of the well-known Caffarelli-Kohn-Nirenberg (CKN for short) inequalities in [6]. Therefore, parts (i) and (ii) of Theorem 1.1 give necessary and sufficient conditions for the validity of the CKN inequality (1.7) when $(a+N)/q \neq (b-p+N)/p$, but without the restriction $a, b, c > -N$ and for $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$. Note that $C^\infty_0(\mathbb{R}^N) \subset W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ when $a, b > -N$, so that even in this case, (1.7) is a genuine generalization. As already pointed out, it does not follow by a denseness argument without many extra conditions ($\mathbb{R}^N_*$ replaced with $\mathbb{R}^N$, $p = q$, $a = b$ and $|x|^a$ an A_p weight, i.e. $-N < a < (p-1)N$; see [11, Theorem 1.27] or [20]). The denseness of $C^\infty_0(\mathbb{R}^N)$ is obviously meaningless when $a \leq -N$ or $b \leq -N$ while that of $C^\infty_0(\mathbb{R}^N_*)$, always contained in $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$, is generally false (see Subsection 11.3) and hence definitely not a viable approach.

Inequalities of CKN type have been discussed, beginning with the 1961 work of Il'in [12, Theorem 1.4], who proved (with c^1 given by (1.3)) $||u||_{c^1,r,G} \leq C||\nabla u||_{b,p,\Omega}$ when Ω is a fairly general open subset of $\mathbb{R}^N$, G is a bounded measurable subset of Ω, and u is C^1. There are further limitations on b, p, and r, but the result has various generalizations when higher order derivatives are involved, or when G is a bounded subset of a section of Ω by a lower-dimensional hyperplane. Results of a somewhat similar nature are proved in [17, Section 2.1.6], [18] when $\Omega = \mathbb{R}^N$ and $u \in C_0^\infty(\mathbb{R}^N)$.

When Ω is an open subset of $\mathbb{R}^N$, μ and ν are nonnegative Borel measures, $\Phi \geq 0$ is continuous and positively homogeneous of degree 1 in its second argument, and $1/r \leq \theta/p + (1-\theta)/q$, Maz'ya [16, Theorem 9] (reproduced in [17, p.127] and [18]) gives interesting *necessary and sufficient* conditions for the inequality

$$||u||_{L^r(\Omega;\mu)} \leq C \left(\int_\Omega \Phi(x, \nabla u)^p dx \right)^{\theta/p} ||u||^{1-\theta}_{L^q(\Omega;\nu)}, \tag{1.8}$$

to hold for $u \in C_0^\infty(\Omega)$. When $\Omega = \mathbb{R}^N_*$, $\mu(E) = \int_E |x|^c dx$, $\nu(E) = \int_E |x|^a dx$, and $\Phi(x, y) = |x|^{b/p}|y|$, the setting of Theorem 1.1 is recovered.

Maz'ya's conditions for (1.8) are expressed in terms of the (p, Φ)-capacity of "admissible" sets and their μ and ν measures. As early as 1960, he noted in [15] that such conditions could be used to prove the equivalence between various multiplicative inequalities (e.g., Sobolev and Nash). This kind of equivalence has since been revisited by a number of authors. For example, it follows from Bakry et al. [1] that when $a = c$, if the inequality $||u||_{a,r} \leq C||\nabla u||^\theta_{b,p}||u||^{1-\theta}_{a,q}$ holds when $q = q_0, r = r_0, \theta = \theta_0$ and (say) u is a Lipschitz continuous function with compact support, then the same inequality continues to hold for a family of other values of q, r and θ. Once again, denseness issues are an obstacle to extending this property to the spaces $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ unless $a = b = c = 0$ (unweighted case).

The connection of this work with the CKN inequalities can be found in some of the preliminary results in [6] which, possibly in generalized form, are also useful for the proof of Theorem 1.1. However, without the compactness of the supports and other key assumptions, a mere tweaking of the arguments of [6] is not possible.

In the next section, we show that (1.7) is equivalent to an embedding inequality and that the hypotheses of Theorem 1.1 are necessary. The necessity of $r \leq \max\{p^*, q\}$ and of $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$ in parts (i) and (ii) of Theorem 1.1 follows very simply from (1.7) and a remark in [6] used here in a more general framework (Theorem 2.2 (i)). A variant of it proves the necessity of $r \leq \max\{p^*, q\}$ in the remaining cases (Theorem 2.2 (ii)).

The verification of the sufficiency is demanding. The general idea is first to prove Theorem 1.1 for radially symmetric functions. Once this is done, there are two different ways to proceed. The first one is to reduce the problem to the symmetric case by a suitable radial symmetrization. This works when $1 \leq r \leq \min\{p, q\}$. The second option is to prove an independent embedding theorem for a direct complement of the subspace of radially symmetric functions. This can be done, based on ideas in [6], under assumptions about p, q and r that rule out $r < \min\{p, q\}$. This is why it is crucial that this case be settled by other arguments.

The proof of the embedding theorem for radially symmetric functions and, next, by radial symmetrization requires some preliminaries. It is more natural to work with the larger spaces (for simplicity, the domain $\mathbb{R}^N_*$ is not mentioned)

$$(1.9) \quad \widetilde{W}^{1,(q,p)}_{\{a,b\}} := \{u \in L^1_{loc}(\mathbb{R}^N_*) : u \in L^q(\mathbb{R}^N; |x|^a dx), \quad \partial_\rho u \in L^p(\mathbb{R}^N; |x|^b dx)\},$$

equipped with the norm

$$(1.10) \qquad ||u||_{\{a,b\},(q,p)} := ||u||_{a,q} + ||\partial_\rho u||_{b,p},$$

where $\partial_\rho u := \nabla u \cdot (x/|x|)$ is the radial derivative of u. Since $|x|^{-1}x$ is a smooth field on $\mathbb{R}^N_*$, this definition makes sense for every distribution u on $\mathbb{R}^N_*$.

When $0 < q < 1$, the definitions (1.1) and (1.9) can still be used, but (1.2) and (1.10) are only quasi-norms. The equivalence of continuity and boundedness for linear operators remains true in quasi-normed spaces. For more details about such spaces, see [4] or [24].

The spaces $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ and $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ contain the same radially symmetric functions and the induced (quasi) norms are the same, because $\nabla u = (\partial_\rho u)(x/|x|)$ when u is radially symmetric. Thus, when referring to radially symmetric functions, the ambient space $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ or $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is unimportant.

In the next section, the basic features of a related space $\widetilde{W}^{1,p}_{loc}(\mathbb{R}^N_*)$ (abbreviated $\widetilde{W}^{1,p}_{loc}$) are discussed, along with some of their implications regarding $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$. This material is directly relevant to the proof of the main results of Sections 4 and 5.

Necessary and sufficient conditions for the continuity of the embedding of the subspace of radially symmetric functions when $q, r > 0$ and $p \geq 1$ are given in Theorem 4.1. Of course, this is a (barely) disguised form of Theorem 1.1 when $N = 1$. Compared with the treatment of the same problem in [6], convenient tools (e.g., radial integration by parts) cannot be used and some estimates (e.g., of $|u(0)|$) no longer make sense. For that reason, our approach is technically completely different.

The proof of Theorem 1.1 for arbitrary N begins in Section 5, where the case $1 \leq r \leq \min\{p, q\}$ is considered. As mentioned, this is done by radial symmetrization, though not in the obvious way (Lemma 5.1). The result (Theorem 5.1) is more general and sharper than the corresponding part of Theorem 1.1, since it establishes the continuous embedding of the larger space $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ with a weaker norm into $L^r(\mathbb{R}^N; |x|^c dx)$ under the conditions already necessary for the embedding of $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$. Thus, the embedding is obtained without assuming the integrability of the first derivatives, except for just the radial one.

The case $r > \min\{p, q\}$ is split into the three parts: $p < r \leq q$ (Theorem 7.1), $r > q$ and $r \geq p$ (Theorem 8.1) and $q < r < p$ (Theorem 9.1). If $p = q$, Sections 7 and 9 can be skipped with no prejudice. A preliminary embedding lemma for functions with null radial symmetrization, essentially due to Caffarelli, Kohn and Nirenberg, is proved in Section 6 (Lemma 6.1), then rephrased in a more convenient way (Corollary 6.1). The technical steps are simple, but cannot be repeated with the larger space $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$. The proofs of Theorem 7.1 (when $p < r \leq q$) and Theorem 9.1 (when $1 \leq q < p < r$) also rely heavily on Theorem 5.1 (when $1 \leq r \leq \min\{p, q\}$, but with other parameters).

The relationship between Theorem 1.1 and the CKN inequalities does not stop with (1.7) when $(a + N)/q \neq (b - p + N)/p$. In Section 10, we show that the embedding $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ continues to be equivalent to a multiplicative inequality $||u||_{c,r} \leq C||\nabla u||^{\theta}_{b,p}||u||^{1-\theta}_{a,q}$ for some suitable $\theta \in [0, 1]$ when $(a + N)/q = (b - p + N)/p$ (Theorem 10.1), except when $N \geq 2$ and $a = c = b-p = -N$ (Theorem 10.2). Of course, θ is no longer θ_c in (1.4), which is not defined, and it may not always be unique; see Remark 6. When $\theta = 1$, this is an N-dimensional weighted Hardy inequality more general than those in the current literature ([9], [22]). The case when $u \in C^\infty_0(\mathbb{R}^N_*)$, $p = q = r = 2$, $c = (a+b)/2-1$ and $\theta = 1/2$ was recently investigated by Catrina and Costa [7].

In Section 11, three special cases are discussed and the (simple) generalization when $|x|^a$, $|x|^b$ and $|x|^c$ are replaced by weights w_a, w_b and w_c having multiple power-like singularities is sketched briefly.

1.1 Notation.

Throughout the paper, $C > 0$ denotes a constant which, as is customary, may have different values in different places. If $k \geq 1$ is a real number, $k' \leq \infty$ always denotes the Hölder conjugate of k, i.e., $k^{-1} + k'^{-1} = 1$. Also, $\zeta \in C^\infty_0(\mathbb{R}^N)$ is chosen once and for all such that $0 \leq \zeta \leq 1$ is radially symmetric,

$$\zeta(x) = \begin{cases} 1 & \text{if } |x| \leq 1/2, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Naturally, we also use the notation introduced more formally earlier. Up to and including Section 4, we frequently refer to the Kelvin transform, defined in the following remark.

Remark 2. The **Kelvin transform** is the map $x \mapsto |x|^{-2}x$ on $\mathbb{R}^N_*$. It is is an isometry from $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ onto $\widetilde{W}^{1,(q,p)}_{\{-2N-a,2p-2N-b\}}$ and from $L^r(\mathbb{R}^N; |x|^c dx)$ onto $L^r(\mathbb{R}^N; |x|^{-2N-c} dx)$ for all values of the parameters. As a result, in many proofs that split into two complementary cases, it suffices to discuss only one of them, because the other follows from this isometry.

2 Necessary conditions for continuous embedding

In this section, we prove that the conditions given in Theorem 1.1 are necessary.

Theorem 2.1. *Let* $a, b, c \in \mathbb{R}$ *and* $1 \leq p < \infty, 0 < q, r < \infty$. *Then* $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ *(hence, a fortiori,* $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$*) is not contained* $L^r(\mathbb{R}^N; |x|^c dx)$ *if*

(i) *c does not belong to the closed interval with endpoints* c^0 *and* c^1*, or*

(ii) $b - p \leq -N < a$ *or* $b - p \geq -N > a$ *and c does not belong to the interval with endpoints* c^0 *(included) and* $-N$ *(not included).*

Furthermore, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ *(hence, a fortiori,* $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$*) is not continuously*[2] *embedded into* $L^r(\mathbb{R}^N; |x|^c dx)$ *if any of the following hold:*

(iii) $(a+N)/q \neq (b-p+N)/p$, $c = c^0$ *and* $r \neq q$ *(if* $r = q$*, then* $c^0 = a$ *and the embedding is trivial);*

(iv) $(a+N)/q \neq (b-p+N)/p$, $c = c^1$ *and* $r < p$*;*

(v) $(a+N)/q = (b-p+N)/p$, $r < \min\{p, q\}$ *and* $c = c^0$ $(= c^1)$*;*

(vi) $a = -N$, $b = p - N$, $r < q$ *and* $c = c^0$ $(= c^1 = -N)$.

Proof. (i) If $c < \min\{c^0, c^1\}$, let $u(x) := |x|^{-(c+N)/r}\zeta(x)$ with ζ as in Subsection 1.1. Then, $u \notin L^r(\mathbb{R}^N; |x|^c dx)$ since $|x|^c|u(x)|^r = |x|^{-N}$ on a neighborhood of 0, but $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ since $\min\{a - (q(c+N))/r, b - p - (p(c+N))/r\} > -N$ and $\nabla\zeta$ has compact support and vanishes on a neighborhood of 0.

If $c > \max\{c^0, c^1\}$, let $u(x) := |x|^{-(c+N)/r}(1 - \zeta(x))$ and argue as above, with obvious modifications.

(ii) By Kelvin transform, it suffices to consider the case $b-p \leq -N < a$. Note that $c^1 \leq -N < c^0$ and let $c \notin (-N, c^0]$. By (i), $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \not\subseteq L^q(\mathbb{R}^N; |x|^c dx)$ if $c > c^0$. If now $c \leq -N$, then $\zeta \notin L^r(\mathbb{R}^N; |x|^c dx)$ since $\zeta = 1$ on a neighborhood of 0, but $\zeta \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ because $a > -N$ and $\nabla\zeta$ has compact support and vanishes on a neighborhood of 0.

(iii) The argument is by contradiction. If $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^{c^0} dx)$, then $||u||_{c^0,r} \leq C(||u||_{a,q} + ||\nabla u||_{b,p})$ for every $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$. By rescaling, i.e., changing $u(x)$ into $u(\lambda x)$, and since $(c^0 + N)/r = (a+N)/q$, it follows that

$$||u||_{c^0,r} \leq C(||u||_{a,q} + \lambda^{(a+N)/q-(b-p+N)/p}||\nabla u||_{b,p})$$

for the same constant C, where C is independent of λ. Because of the fact that $(a+N)/q = (b-p+N)/q$, this yields $||u||_{c^0,r} \leq C||u||_{a,q}$. In particular, if $u(x) := |x|^{-(c^0+N-1)/r}g(|x|) = |x|^{1/r-(a+N)/q}g(|x|)$ with $g \in C^\infty_0(0, \infty)$, it follows that $||g||_r \leq C||g||_{q/r-1,q}$ when $g \in C^\infty_0(0, \infty)$, $g \geq 0$, or g is the a.e. limit of a

[2] In principle at least, that does not rule out $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \subset L^r(\mathbb{R}^N; |x|^c dx)$.

nondecreasing sequence of such functions. Thus, a contradiction is obtained by choosing $g := \chi_{(n,n+1)}$ if $r > q$ and $g := t^{1/n-1/r}\chi_{(0,1)}$ if $r < q$, and by letting n tend to ∞.

(iv) The scaling used in (iii) now shows that if $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^{c^1}dx)$, then $||u||_{c^1,r} \le C||\nabla u||_{b,p}$ for some constant $C > 0$. The proof that C does not exist is slightly different when $a \neq -N$ and when $a = -N$.

Case (iv-1): $a \neq -N$. By Kelvin transform, we may assume, with no loss of generality, that $a < -N$. It suffices to prove that given $C > 0$,

$$||f||_{c^1+N-1,r} \le C||f'||_{b+N-1,p}, \tag{2.1}$$

cannot hold for every $f \in W^{1,p}_{loc}(0,\infty)$ with $f \ge 0, f = 0$ on a neighborhood of 0, and $f = M$ (constant) on a neighborhood of ∞ (if so, $u(x) = f(|x|)$ is in $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ irrespective of $b \in \mathbb{R}$ and of $p \ge 1, q > 0$).

It is well known that if $1 \le r < p$ and $C > 0$, the weighted Hardy inequality

$$\left(\int_0^\infty t^{(r(b-p+N)-p)/p}\left(\int_0^t g(\tau)d\tau\right)^r dt\right)^{1/r} \le C\left(\int_0^\infty t^{b+N-1}g(t)^p dt\right)^{1/p}$$

does not hold for every measurable $g \ge 0$ on $(0,\infty)$, because power weights never satisfy the necessary compatibility condition when $r < p$ ([17, Theorem 1, p. 47]). This is also true, but more delicate, when $0 < r < 1$ ([26], [27]). Thus, if $0 < r < p$, there is a sequence $g_n \ge 0$ such that $\int_0^\infty t^{b+N-1}g_n(t)^p dt < \infty$ and

$$\left(\int_0^\infty t^{r(b-p+N)/p-1}\left(\int_0^t g_n(\tau)d\tau\right)^r dt\right)^{1/r} > n\left(\int_0^\infty t^{b+N-1}g_n^p(t)dt\right)^{1/p}.$$

If $b - p \ge -N$, the left-hand side is even ∞ when $g_n \neq 0$, so it may be assumed that $b-p < -N$ whenever convenient (which happens to be the case when $p = 1$). The simple proof by Sinnamon and Stepanov ([27, Theorem 2.4 for $p > 1$], [27, Theorem 3.3 for $p = 1$]) reveals at once that g_n may be chosen in $L^p(0,\infty)$ and with compact support. Then $f_n(t) := \int_0^t g_n(\tau)d\tau \ge 0$ vanishes on a neighborhood of 0 and is eventually constant. Since $(r(b-p+N)-p)/p - 1 = c^1 + N - 1$, (2.1) does not hold for f_n if n is large enough.

Case (iv-2): $a = -N$. In this case, $b-p \neq -N$, since $(a+N)/q \neq (b-p+N)/p$. By the usual Kelvin transform argument (which does not affect $a = -N$), we may assume that $b - p < -N$. It suffices to show that (2.1) cannot hold for every $f \in W^{1,p}_{loc}(0,\infty)$ with $f \ge 0, f = 0$ on a neighborhood of 0 and $f(t) = Mt^{-\varepsilon}$ for some constants $M, \varepsilon > 0$ and large t (if so, $u(x) = f(|x|)$ is in $W^{1,(q,p)}_{\{-N,b\}}(\mathbb{R}^N_*)$ since $b - p < -N$).

With f_n and $g_n = f_n'$ as in Case (iv-1) above, set $h_n(t) := f_n(t)$ if $0 < t < 1$ and $h_n(t) := t^{-\varepsilon_n}f_n(t)$ if $t \ge 1$, where $\varepsilon_n > 0$ will be chosen shortly. Note that

$h_n = 0$ on a neighborhood of 0 and $h_n(t) = M_n t^{-\varepsilon_n}$ for $t > 0$ large enough, since $f_n(t) = M_n$ is constant for large t. Since (2.1) does not hold for f_n and $h_n = f_n$ on (0, 1), (2.1) does not hold for h_n either if, when n is fixed, $\varepsilon_n > 0$ is chosen so that $\int_1^\infty t^{c^1+N-1} h_n(t)^r dt$ is arbitrarily close to $\int_1^\infty t^{c^1+N-1} f_n(t)^r dt$ and $\int_1^\infty t^{b+N-1} |h_n'(t)|^p dt$ is arbitrarily close to $\int_1^\infty t^{b+N-1} |f_n'(t)|^p dt$.

That $\varepsilon_n > 0$ can be chosen so that $\int_1^\infty t^{c^1+N-1} h_n(t)^r dt$ is arbitrarily close to $\int_1^\infty t^{c^1+N-1} f_n(t)^r dt$ follows by the monotone convergence of $\int_1^\infty t^{c^1+N-1-r\varepsilon} f_n(t)^r dt$ as $\varepsilon \searrow 0$. To see that $\varepsilon > 0$ can be chosen so that $\int_1^\infty t^{b+N-1} |h_n'(t)|^p dt$ is arbitrarily close to $\int_1^\infty t^{b+N-1} |f_n'(t)|^p dt$, it suffices to use

(1) $\lim_{\varepsilon \to 0} \int_1^\infty t^{b+N-1-p\varepsilon} g_n(t)^p dt = \int_1^\infty t^{b+N-1} g_n(t)^p dt$, which is also proved by a monotone convergence argument, and

(2) $\lim_{\varepsilon \to 0} \varepsilon^p \int_1^\infty t^{-p\varepsilon+b-p+N-1} f_n(t)^p dt = 0$, which follows from the boundedness of f_n and from $b - p < -N$.

(v) The main difference in the proof of (v) from the proofs of parts (iii) and (iv) is that the scaling argument used in the latter proofs is inoperative. This is because all the powers of λ cancel.

Let η denote the common value

$$\eta := \frac{a+N}{q} = \frac{b-p+N}{p} = \frac{c+N}{r}. \tag{2.2}$$

If $||u||_{c,r} \le C(||u||_{a,q} + ||\nabla u||_{b,p})$ for every $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$, the choice $u(x) := f(|x|)$ with $f \in C_0^\infty(0,\infty)$ yields $||f||_{c+N-1,r} \le C(||f'||_{b+N-1,p} + ||f||_{a+N-1,q})$. If now $g \in C_0^\infty(\mathbb{R})$, then $f(t) = t^{-\eta} g(\ln t)$ with η from (2.2) is in $C_0^\infty(0,\infty)$. By the change of variable $s := \ln t$, we obtain the unweighted inequality $||g||_r \le C(||g'||_p + ||g||_q + ||g||_p)$ for every $g \in C_0^\infty(\mathbb{R})$. With $g \neq 0$ chosen once and for all and $g(t)$ replaced by $g(\lambda t)$, $\lambda > 0$, it follows that

$$I_1 \le C(\lambda^{1/p'+1/r} I_2 + \lambda^{1/r-1/q} I_3 + \lambda^{1/r-1/p} I_4)$$

with $I_1, \ldots, I_4 > 0$ independent of λ. Since $r < \min\{p,q\}$, the right-hand side tends to 0 with λ, which is absurd.

(vi) Argue as in (v), just noticing that now $\eta = 0$ in (2.2). This produces the simpler $||g||_r \le C(||g'||_p + ||g||_q)$ when $g \in C_0^\infty(\mathbb{R})$. Then

$$I_1 \le C(\lambda^{1/p'+1/r} I_2 + \lambda^{1/r-1/q} I_3)$$

for $\lambda > 0$ by rescaling, which is absurd for $r < q$. □

We obtain as a corollary that the embedding is often characterized by a multiplicative rather than additive norm inequality; see also Section 10.

Corollary 2.1. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p < \infty, 0 < q, r < \infty$ be such that $(a+N)/q \neq (b-p+N)/p$. Then, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ if and only if c is in the closed interval with endpoints c^0 and c^1 and there exists $C > 0$ such that*

$$||u||_{c,r} \leq C||\nabla u||^{\theta_c}_{b,p}||u||^{1-\theta_c}_{a,q} \quad \text{for all } u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*), \tag{2.3}$$

where θ_c is given by (1.4). *The same property is true with $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ replaced with $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ and* (2.3) *replaced with*

$$||u||_{c,r} \leq C||\partial_\rho u||^{\theta_c}_{b,p}||u||^{1-\theta_c}_{a,q}, \quad \text{for all } u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}. \tag{2.4}$$

Proof. The sufficiency follows from the arithmetic-geometric inequality. We prove the necessity only for $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$; similar arguments work for $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$.

Suppose then that $\widetilde{W}^{1,(q,p)}_{\{a,b\}} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$. By Theorem 2.1(i), c is in the closed interval with (distinct) endpoints c^0 and c^1. Furthermore,

$$||u||_{c,r} \leq C(||u||_{a,q} + ||\partial_\rho u||_{b,p})$$

for every $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$. In this inequality, replace $u(x)$ by $u(\lambda x)$ with $\lambda > 0$ to obtain

$$\begin{aligned} ||u||_{c,r} &\leq C\lambda^{(c+N)/r-(a+N)/q}||u||_{a,q} + C\lambda^{(c+N)/r-(b-p+N)/p}||\partial_\rho u||_{b,p} \\ &= C\lambda^{\theta_c(c^1-c^0)/r}||u||_{a,q} + C\lambda^{(1-\theta_c)(c^0-c^1)/r}||\partial_\rho u||_{b,p}. \end{aligned} \tag{2.5}$$

If $c = c^0$ ($c = c^1$), then $\theta_c = 0$ ($\theta_c = 1$), so that $||u||_{c,r} \leq C||u||_{a,q}$ ($||u||_{c,r} \leq C||\partial_\rho u||_{b,p}$); i.e., (2.4) holds, by letting λ tend to 0 or to ∞. Otherwise, (2.4) follows by minimizing the right-hand side of (2.5) for $\lambda > 0$. This changes C, which however remains independent of u even though the minimizer is of course u-dependent. (If $\theta_c \neq 0$, (2.5) shows that $u = 0$ if $\partial_\rho u = 0$, so that it is not restrictive to assume $||u||_{a,q} > 0$ and $||\partial_\rho u||_{b,p} > 0$ in the minimization step.) □

The next theorem gives a different necessary condition for the continuity of the embedding $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$.

Theorem 2.2. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p < \infty, 0 < q, r < \infty$.*

(i) *If $(a+N)/q \neq (b-p+N)/p$ and $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$, then $\theta_c \in [0, 1]$ and*

$$\theta_c\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \leq \frac{1}{r} - \frac{1}{q}. \tag{2.6}$$

In particular, $r \leq \max\{p^*, q\}$.

(ii) *If* $(a+N)/q = (b-p+N)/p$, $c = c^0$ $(= c^1)$, *and* $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$, *then* $r \leq \max\{p^*, q\}$.

Proof. (i) Theorem 2.1(i) shows that $\theta_c \in [0, 1]$. The next argument is taken from [6], with a minor adjustment to fit the setting of this paper. Let $\varphi \in C^\infty_0(\mathbb{R}^N)$, $\varphi \neq 0$, be chosen once and for all. If $x_0 \in \mathbb{R}^N$ and $R := |x_0|$ is large enough, then $\varphi(\cdot + x_0) \in C^\infty_0(\mathbb{R}^N_*) \subset W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ irrespective of a, b, p and q. Using (2.3) with $u = \varphi(\cdot + x_0)$ and letting $R \to \infty$, we obtain (because $\operatorname{Supp}\varphi$ is compact) $R^{c/r}||\varphi||_r \leq CR^{b\theta_c/p+a(1-\theta_c)/q}||\nabla\varphi||^{\theta_c}_p||\varphi||^{1-\theta_c}_q$ for large R after changing C, whence $c/r \leq (b\theta_c)/p + a(1 - \theta_c)/q$. Then (2.6) follows by adding N/r and using (1.6).

If $p < N$ and $r > \max\{p^*, q\}$, then (2.6) cannot hold, since it fails when $\theta_c = 0$ and when $\theta_c = 1$. Thus $r \leq \max\{p^*, q\}$ is necessary.

(ii) Use the same method as in (i), but with the additive inequality $||\varphi||_{c^0,r} \leq C(||\varphi||_{a,q} + ||\nabla\varphi||_{b,p})$. This yields $R^{c^0/r}||\varphi||_r \leq C(R^{a/q}||\varphi||_q + R^{b/p}||\nabla\varphi||_p)$ for large $R > 0$. By (1.3), $c^0/r = a/q + N/q - N/r$ and (since $c^0 = c^1$) $b/p = a/q + N/q + 1 - N/p$, whence $R^{N/q-N/r}||\varphi||_r \leq C(||\varphi||_q + R^{N/q+1-N/p}||\nabla\varphi||_p)$. This implies that if $r > q$, then $N/q - N/r \leq N/q + 1 - N/p$, i.e., $r \leq p^*$, so that $r \leq \max\{p^*, q\}$ in all cases. □

The above proof may give the wrong impression that (2.6) arises only as a result of integrability at infinity. That this is not the case can be seen by noticing that the choice $\varphi(x|x|^{-2}+x_0)$ instead of $\varphi(x+x_0)$ also yields (2.6), while the support of $\varphi(x|x|^{-2} + x_0)$ shrinks towards 0 as $|x_0| \to \infty$.

The verification that Theorems 2.1 and 2.2 together imply that the hypotheses of Theorem 1.1 are necessary is routine and left to the reader.

3 The spaces $\widetilde{W}^{1,p}_{loc}$ and related concepts

In this section, we develop the background material needed for the proofs of the main results of the next two sections. Let ω_N denote the volume of the unit ball of $\mathbb{R}^N$. For $u \in L^p_{loc}(\mathbb{R}^N_*)$ with $p \geq 1$, define the **spherical mean** of u

$$f_u(t) := (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} u(t\sigma)d\sigma. \tag{3.1}$$

By Fubini's theorem in spherical coordinates, $f_u(t)$ is defined for a.e. $t > 0$ and $f_u \in L^p_{loc}(0, \infty)$. If $u \in \widetilde{W}^{1,p}_{loc}$, where

$$\widetilde{W}^{1,p}_{loc} := \{u \in L^p_{loc}(\mathbb{R}^N_*) : \partial_\rho u \in L^p_{loc}(\mathbb{R}^N_*)\}$$

and $\partial_\rho u := \nabla u \cdot (x/|x|)$, more is true.

Lemma 3.1. *If* $1 \leq p < \infty$ *and* $u \in \widetilde{W}^{1,p}_{loc}$, *then* $f_u \in W^{1,p}_{loc}(0,\infty)$. *Furthermore,*

$$f_u'(t) = (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} \partial_\rho u(t\sigma) d\sigma. \tag{3.2}$$

Conversely, if $f \in W^{1,p}_{loc}(0,\infty)$ *and* $u(x) := f(|x|)$, *then* $u \in \widetilde{W}^{1,p}_{loc}$ *and* $f_u = f, \partial_\rho u(x) = f'(|x|)$.

Proof. Let $u \in \widetilde{W}^{1,p}_{loc}$. If $\varphi \in C_0^\infty(0,\infty)$, set $\psi(x) := \varphi(|x|)$, so that $\psi \in C_0^\infty(\mathbb{R}^N_*)$ and $\partial_\rho \psi(x) = \varphi'(|x|)$. It follows that

$$\langle f_u', \varphi\rangle = -(N\omega_N)^{-1} \left\langle u, |x|^{1-N}\partial_\rho \psi\right\rangle = (N\omega_N)^{-1} \left\langle |x|^{1-N}\partial_\rho u, \psi\right\rangle$$

(use $\nabla \cdot (|x|^{-N}x) = 0$). Since $\partial_\rho u \in L^p_{loc}(\mathbb{R}^N_*)$, this shows that $\langle f_u', \varphi\rangle = \langle f_{\partial_\rho u}, \varphi\rangle$, that is, $f_u' = f_{\partial_\rho u} \in L^p_{loc}(0,\infty)$. Thus, $f_u \in W^{1,p}_{loc}(0,\infty)$ and (3.2) holds.

Conversely, suppose that $f \in W^{1,p}_{loc}(0,\infty)$ and set $u(x) := f(|x|)$. Then, $u \in L^p_{loc}(\mathbb{R}^N_*)$ (it is continuous) and, by [14, Theorem 4.3], $\nabla u(x) = f'(|x|)(x/|x|)$, because f is locally absolutely continuous. Thus, $u \in W^{1,p}_{loc}(\mathbb{R}^N_*) \subset \widetilde{W}^{1,p}_{loc}$ and $f'(|x|) = \nabla u(x) \cdot (x/|x|) = \partial_\rho u(x)$. That $f_u = f$ is obvious. □

Lemma 3.2. *If* $1 \leq p < \infty$ *and* $u \in \widetilde{W}^{1,p}_{loc}$, *then* $|u|^p \in \widetilde{W}^{1,1}_{loc}$ *and* $\partial_\rho(|u|^p) = p|u|^{p-1}(\operatorname{sgn} u)\partial_\rho u$, *where* $\operatorname{sgn} u := 0$ *on* $u^{-1}(0)$.

Proof. It is well known that if Ω is an open subset of $\mathbb{R}^N$ and if $u \in W^{1,1}(\Omega)$, then $|u| \in W^{1,1}(\Omega)$ with $\nabla|u| = (\operatorname{sgn} u)\nabla u$ (see for instance [31, p. 48] or [14, Theorem 2.2] for more general statements), where $\operatorname{sgn} u$ is defined to be 0 at points where $u = 0$. This is proved by showing that if $u \in L^1(\Omega)$ and $\partial_i u \in L^1(\Omega)$ for some index $1 \leq i \leq N$, then $\partial_i|u| \in L^1(\Omega)$ and $\partial_i|u| = (\operatorname{sgn} u)\partial_i u$ because the assumptions suffice to ensure the local absolute continuity of u on almost every line segment in Ω parallel to the x_i-axis. Since a radial derivative is just a directional derivative after passing to spherical coordinates, the same arguments show that if $u \in \widetilde{W}^{1,1}_{loc}$, then $|u| \in \widetilde{W}^{1,1}_{loc}$ and $\partial_\rho|u| = (\operatorname{sgn} u)\partial_\rho u$. (That the derivative of $u(\cdot,\sigma)$ is $\partial_\rho u(\cdot,\sigma)$ can be justified by a variant of the proof of Lemma 3.1.)

Another well-known result, usually proved by localization and mollification, is that if $u \in W^{1,p}(\Omega)$ and $u \geq 0$, then $u^p \in W^{1,1}(\Omega)$ and $\partial_i(u^p) = pu^{p-1}\partial_i u$. Not surprisingly, the proof actually requires only that u and $\partial_i u$ be in $L^p(\Omega)$, so that completely similar arguments show that if $u \in \widetilde{W}^{1,p}_{loc}$ and $u \geq 0$, then $u^p \in \widetilde{W}^{1,1}_{loc}$ and $\partial_\rho u^p = pu^{p-1}\partial_\rho u$. The lemma follows by combining these facts. □

Since $f_{|u|}$ is continuous on $(0,\infty)$ when $u \in \widetilde{W}^{1,1}_{loc}$, the following two subsets are well defined:

$$\widetilde{W}^{1,1}_{loc,-} := \{u \in \widetilde{W}^{1,1}_{loc} : \underline{\lim}_{t\to\infty} f_{|u|}(t) = 0\}, \tag{3.3}$$

$$\widetilde{W}^{1,1}_{loc,+} := \{u \in \widetilde{W}^{1,1}_{loc} : \underline{\lim}_{t\to 0^+} f_{|u|}(t) = 0\}. \tag{3.4}$$

The sets $\widetilde{W}^{1,1}_{loc,\pm}$ are not closed under addition and so are not vector spaces. They are transformed into one another by Kelvin transform. Various other properties are collected in the next lemma.

Lemma 3.3. *The following properties hold.*

(i) *If $u \in \widetilde{W}^{1,1}_{loc,-}$ ($\widetilde{W}^{1,1}_{loc,+}$), then $|u| \in \widetilde{W}^{1,1}_{loc,-}$ ($\widetilde{W}^{1,1}_{loc,+}$).*

(ii) *$u \in \widetilde{W}^{1,1}_{loc,-}$ ($\widetilde{W}^{1,1}_{loc,+}$) implies $u_S := f_u \circ |\cdot| \in \widetilde{W}^{1,1}_{loc,-}$ ($\widetilde{W}^{1,1}_{loc,+}$) and $\partial_\rho u_S(x) = f'_u(|x|)$.*

(iii) *If $u \in \widetilde{W}^{1,1}_{loc}$ and $|x|^a|u|^q \in L^1(\mathbb{R}^N)$ for some $a \in \mathbb{R}$ and some $q \geq 1$, then $u \in \widetilde{W}^{1,1}_{loc,-}$ ($\widetilde{W}^{1,1}_{loc,+}$) if $a \geq -N$ ($a \leq -N$). In particular (see* (1.9)*), if $a \geq -N$ ($a \leq -N$), then $\widetilde{W}^{1,(q,p)}_{\{a,b\}} \subset \widetilde{W}^{1,1}_{loc,-}$ ($\widetilde{W}^{1,1}_{loc,+}$).*

(iv) *If $u \in \widetilde{W}^{1,1}_{loc}$ is radially symmetric and $|x|^a|u|^q \in L^1(\mathbb{R}^N)$ for some $a \in \mathbb{R}$ and $q > 0$, then $u \in \widetilde{W}^{1,1}_{loc,-}$ ($\widetilde{W}^{1,1}_{loc,+}$) if $a \geq -N$ ($a \leq -N$). In particular, if $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is radially symmetric, then $u \in \widetilde{W}^{1,1}_{loc,-}$ ($\widetilde{W}^{1,1}_{loc,+}$) if $a \geq -N$ ($a \leq -N$).*

Proof. (i) Use Lemma 3.2 and the definitions (3.3) and (3.4).

(ii) That $u_S := f_u \circ |\cdot| \in \widetilde{W}^{1,1}_{loc}$ and $\partial_\rho u_S(x) = f'_u(|x|)$ follows from Lemma 3.1. Next, the remark that $f_{|u_S|} = |f_u| \leq f_{|u|}$ shows that if also $\underline{\lim}_{t\to\infty} f_{|u|}(t) = 0$ (or $\underline{\lim}_{t\to 0^+} f_{|u|}(t) = 0$), then $\underline{\lim}_{t\to\infty} f_{|u_S|}(t) = 0$ (or $\underline{\lim}_{t\to\infty} f_{|u_S|}(t) = 0$).

(iii) Suppose that $a \geq -N$ and, to argue by contradiction, that $u \notin \widetilde{W}^{1,1}_{loc,-}$. Then $f_{|u|}(t) \geq \ell > 0$ for $t \geq T$ and large $T > 0$. Thus, by (3.1), $\ell^q \leq (f_{|u|})^q(t) \leq f_{|u|^q}(t)$ for $t \geq T$, so that

$$\int_{|x|\geq T} |x|^a|u|^q = N\omega_N \int_T^\infty t^{a+N-1} f_{|u|^q}(t)dt \geq N\omega_N \ell^q \int_T^\infty t^{a+N-1}dt = \infty,$$

since $a \geq -N$. This contradicts the fact that $|x|^a|u|^q \in L^1(\mathbb{R}^N)$. The case $a \leq -N$ follows by Kelvin transform; the "in particular" part is obvious.

(iv) If u is radially symmetric, then $f_{|u|^q} = (f_{|u|})^q$ for every $q > 0$, so that the argument by contradiction in the proof of (iii) works for $q > 0$, not just for $q \geq 1$. The "in particular" part is clear once we show that $u \in \widetilde{W}^{1,1}_{loc}$. To see this, note that $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ implies $\partial_\rho u \in L^p_{loc}(\mathbb{R}^N_*)$, which, by radial symmetry, implies $\nabla u \in L^p_{loc}(\mathbb{R}^N_*)$. Thus, $u \in W^{1,p}_{loc}(\mathbb{R}^N_*)$ ([17, p. 7]). That $W^{1,p}_{loc}(\mathbb{R}^N_*) \subset \widetilde{W}^{1,1}_{loc}$ is obvious. □

If $u \in L^1_{loc}(\mathbb{R}^N_*)$ is radially symmetric, then $u(x) = f_u(|x|)$. This justifies referring to the function u_S in Lemma 3.3(ii) as the "radial symmetrization" of u.

Lemma 3.4. *Let $a, b \in \mathbb{R}$ and $1 \leq p, q < \infty$. If $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$, then*

(i) $|u| \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ *and* $|| \, |u| \, ||_{a,q} = ||u||_{a,q}$, $||\partial_\rho |u| \, ||_{b,p} = ||\partial_\rho u||_{b,p}$ *(if u is also radially symmetric, this remains true when $0 < q < 1$);*

(ii) $u_S \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ *and* $||u_S||_{a,q} \leq ||u||_{a,q}$, $||\partial_\rho u_S||_{b,p} \leq ||\partial_\rho u||_{b,p}$.

Proof. (i) This follows from $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}} \subset \widetilde{W}^{1,1}_{loc}$ (see Lemma 3.3 (iv) if u is radially symmetric and $0 < q < 1$), so that $\partial_\rho |u| = (\operatorname{sgn} u)\partial_\rho u$ by Lemma 3.2.

(ii) Since $u_S(x) = f_u(|x|)$ and f_u in (3.1) is continuous, u_S is continuous, and so $u_S \in L^1_{loc}(\mathbb{R}^N_*)$. By (3.1), $|u_S(x)|^q \leq (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} |u(|x|\sigma)|^q d\sigma$ since $q \geq 1$ and, by (3.2) and Lemma 3.3(ii), $|\partial_\rho u_S(x)|^p \leq (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} |\partial_\rho u(|x|\sigma)|^p d\sigma$ almost everywhere. Therefore, $||u_S||_{a,q} \leq ||u||_{a,q}$ and $||\partial_\rho u_S||_{b,p} \leq ||\partial_\rho u||_{b,p}$. □

We complete this section with an inequality (Theorem 3.1) which is the basic tool for the proof of Lemmas 4.3 and 4.4 in the next section.

Lemma 3.5. *Let $f \in W^{1,1}_{loc}(0, \infty)$, $f \geq 0$ and $\gamma \in \mathbb{R}$.*

(i) *If $\gamma \geq 1 - N$ and $\underline{\lim}_{t\to\infty} f(t) = 0$, then*

$$0 \leq t^{N-1+\gamma} f(t) \leq \int_t^\infty \tau^{N-1+\gamma} |f'(\tau)| d\tau \leq \infty, \quad \text{for all } t > 0. \tag{3.5}$$

(ii) *If $\gamma \leq 1 - N$ and $\underline{\lim}_{t\to 0^+} f(t) = 0$, then*

$$0 \leq t^{N-1+\gamma} f(t) \leq \int_0^t \tau^{N-1+\gamma} |f'(\tau)| d\tau \leq \infty, \quad \text{for all } t > 0. \tag{3.6}$$

Proof. (i) Given $t > 0$, let $T > t$ and write $f(t) = f(T) - \int_t^T f'(\tau) d\tau$. Since $\gamma \geq 1 - N$ implies $t^{N-1+\gamma} \leq \tau^{N-1+\gamma}$ when $t \leq \tau$, this yields

$$t^{N-1+\gamma} f(t) \leq t^{N-1+\gamma} f(T) + \int_t^T \tau^{N-1+\gamma} |f'(\tau)| d\tau \leq t^{N-1+\gamma} f(T) + \int_t^\infty \tau^{N-1+\gamma} |f'(\tau)| d\tau.$$

Thus (3.5) follows from $f \geq 0$ and $\underline{\lim}_{T\to\infty} f(T) = 0$.

(ii) Given $t > 0$, let $0 < \varepsilon < t$ and write $f(t) = f(\varepsilon) + \int_\varepsilon^t f'(\tau) d\tau$. Since $\gamma \leq 1 - N$ implies $t^{N-1+\gamma} \leq \tau^{N-1+\gamma}$ when $t \geq \tau$, this yields

$$t^{N-1+\gamma} f(t) \leq t^{N-1+\gamma} f(\varepsilon) + \int_\varepsilon^t \tau^{N-1+\gamma} |f'(\tau)| d\tau \leq t^{N-1+\gamma} f(\varepsilon) + \int_0^t \tau^{N-1+\gamma} |f'(\tau)| d\tau.$$

Thus (3.6) follows from $f \geq 0$ and $\underline{\lim}_{\varepsilon\to 0} f(\varepsilon) = 0$. □

In Theorem 3.1 below, the norm notation is used only for convenience, since all the norms may actually be infinite. In practice, this simply means that in the inequalities, the finiteness of the right-hand side implies the finiteness of the left-hand side, which therefore need not be assumed separately. An alternate proof can be based on the case $q = \infty$ of [17, Theorem 2, p.40] and Kelvin transform, but the direct argument used below is more explicit and not longer.

Theorem 3.1. *Let $\gamma \in \mathbb{R}$ and $1 \leq p < \infty$. There exists a constant $C > 0$ such that if $u \in \widetilde{W}^{1,1}_{loc}$ is radially symmetric and either $\gamma > 1 - N$ and $u \in \widetilde{W}^{1,1}_{loc,-}$ or $\gamma < 1 - N$ and $u \in \widetilde{W}^{1,1}_{loc,+}$, then $|| \, |x|^{N-1+\gamma} u||_\infty \leq C|| \, |x|^{\gamma+N/p'} \partial_\rho u||_p$. Furthermore, if $p = 1$, this inequality remains true when $\gamma = 1 - N$.*

Proof. Suppose first that $p = 1$ and $\gamma \geq 1 - N$, and let $u \in \widetilde{W}^{1,1}_{loc,-}$. By Lemmas 3.3(i) and 3.2, we may assume that $u \geq 0$ with no loss of generality, since $|| \, |x|^\gamma \partial_\rho u||_1$ and $|| \, |x|^{N-1+\gamma} u||_\infty$ are unchanged when u is replaced by $|u|$.

By Lemma 3.1, $u(x) = f_u(|x|)$, where $f_u \in W^{1,1}_{loc}(0, \infty)$ satisfies $f_u \geq 0$ and $\underline{\lim}_{t\to\infty} f_{|u|}(t) = 0$, by (3.3). Thus, $|| \, |x|^{N-1+\gamma} u||_\infty = \sup_{t>0} t^{N-1+\gamma} f_u(t)$ and $\left\| \, |x|^\gamma \partial_\rho u \right\|_1 = \int_0^\infty \tau^{N-1+\gamma} |f_u'(\tau)| d\tau$ since $f_u'(|x|) = \partial_\rho u(x)$ (use $u = u_S$ and Lemma 3.3(ii)). Hence, it suffices to show that $t^{N-1+\gamma} f_u(t) \leq \int_0^\infty \tau^{N-1+\gamma} |f_u'(\tau)| d\tau \leq \infty$ for every $t > 0$, which follows at once from (3.5) for $f = f_u$. If $\gamma \leq 1 - N$ and $u \in \widetilde{W}^{1,1}_{loc,+}$, use (3.6) instead of (3.5).

Now let $1 < p < \infty$. Once again, assume, with no loss of generality, that $u \geq 0$, so that $u(x) = f_u(|x|)$ with $f_u \in W^{1,1}_{loc}(0, \infty)$ and $f_u \geq 0$. It suffices to prove

$$t^{N-1+\gamma} f_u(t) \leq C \left(\int_0^\infty |f_u'(\tau)|^p \tau^{pN+p\gamma-1} d\tau \right)^{1/p}, \tag{3.7}$$

for every $t > 0$. We merely show how the proof in the case $p = 1$ above can be modified to yield this inequality.

Suppose $\gamma > 1 - N$ and let $u \in \widetilde{W}^{1,1}_{loc,-}$. Inequality (3.5) with $\gamma = 1 - N$ (which is allowed in Lemma 3.5) and $f = f_u$ yields $f_u(t) \leq \int_t^\infty |f_u'(\tau)| d\tau$ for every $t > 0$. Write $|f_u'(\tau)| = \left(|f_u'(\tau)| \tau^{N-1+\gamma+1/p'} \right) \tau^{1-N-\gamma-1/p'}$ and, since $\gamma > 1 - N$, use Hölder's inequality to get $f_u(t) \leq Ct^{1-N-\gamma} \left(\int_t^\infty |f_u'(\tau)|^p \tau^{pN+p\gamma-1} d\tau \right)^{1/p}$ with $C := [p'(\gamma + N - 1)]^{-1/p'}$, which is stronger than (3.7). If $\gamma < 1 - N$ and $u \in \widetilde{W}^{1,1}_{loc,+}$, follow the same procedure, but start with the inequality (3.6). □

4 Embedding theorem for radially symmetric functions

In this section, we give necessary and sufficient conditions for the continuity of the embedding of the subspace of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions into

$L^r(\mathbb{R}^N; |x|^c dx)$. In principle, this can of course be done by reduction to the half-line, which is reflected in the proofs, but we have found no expository or technical advantage in doing so explicitly. Our first task is to make sure that the cut-off operation is continuous. As a preamble, we need the following result.

Lemma 4.1. *Let* Ω *be a bounded open annulus centered at* $0 \notin \overline{\Omega}$, $a, b \in \mathbb{R}$, *and* $1 \leq p < \infty, 0 < q < \infty$. *There exists a constant* $C > 0$ *such that* $||u||_{p,\Omega} \leq C||u||_{\{a,b\},(q,p)}$ *for every radially symmetric* $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$.

Proof. Let $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ be radially symmetric. We already pointed out in the Introduction that $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ and $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ have the same radially symmetric functions, with the same induced (quasi) norms. Since $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ implies $\nabla u \in L^p_{loc}(\mathbb{R}^N_*)$, it follows that $u \in W^{1,p}_{loc}(\mathbb{R}^N_*)$ (this was already used in the proof of Lemma 3.3(iv)) and hence that $u \in W^{1,p}(\Omega)$. Thus, it suffices to prove that $||v||_{p,\Omega} \leq C(||v||_{q,\Omega} + ||\nabla v||_{p,\Omega})$ for every $v \in W^{1,p}(\Omega)$.

This is common knowledge when $q \geq 1$; but since only $q > 0$ is assumed, we give a proof. To argue by contradiction, assume that there exists a sequence $\{v_n\} \subset W^{1,p}(\Omega)$ such that $||v_n||_{p,\Omega} = 1$ and $\lim_{n\to\infty} ||v_n||_{q,\Omega} + ||\nabla v_n||_{p,\Omega} = 0$. Since $\{v_n\}$ is bounded in $W^{1,p}(\Omega)$ and the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact (even when $p = 1$), there exist $v \in L^p(\Omega)$ and a subsequence, still denoted by $\{v_n\}$, such that $v_n \to v$ in $L^p(\Omega)$ and $v_n \to v$ a.e. on Ω. Obviously, $||v||_p = 1$.

Now, since $|v_n|^q \to 0$ in $L^1(\Omega)$, there exists a subsequence $\{v_{n_k}\}$ such that $|v_{n_k}|^q \to 0$ a.e. on Ω. Thus, $v_{n_k} \to 0$ a.e. on Ω, so that $v = 0$, which contradicts $||v||_p = 1$. □

With the help of Lemma 4.1, we can now prove that truncation has the expected properties in the subspace of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions.

Lemma 4.2. *Let* $a, b \in \mathbb{R}$, $1 \leq p < \infty, 0 < q < \infty$, *and* $\varphi \in C^\infty(\mathbb{R}^N)$ *be radially symmetric, constant on a neighborhood of* 0 *and constant outside a ball with center* 0. *Then multiplication by* φ *is continuous on the subspace of radially symmetric functions of* $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$.

Proof. If $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$, then $||\varphi u||_{a,q} \leq ||\varphi||_\infty ||u||_{a,q}$ and

$$\partial_\rho(\varphi u) = \varphi \partial_\rho u + (\partial_\rho \varphi)u.$$

Clearly, $||\varphi \partial_\rho u||_{b,p} \leq ||\varphi||_\infty ||\partial_\rho u||_{b,p}$. To evaluate $||(\partial_\rho \varphi)u||_{b,p}$ when u is radially symmetric, note that Supp $\partial_\rho \varphi$ is contained in a bounded open annulus Ω centered at $0 \notin \overline{\Omega}$. Thus $||(\partial_\rho \varphi)u||_{b,p} \leq C||\partial_\rho \varphi||_\infty ||u||_{\{a,b\},(q,p)}$ by Lemma 4.1, since $|x|^b$ is bounded on Ω. Altogether, this yields $||\varphi u||_{\{a,b\},(q,p)} \leq C||u||_{\{a,b\},(q,p)}$. □

Radial symmetry is unimportant in Lemmas 4.1 and 4.2 if $q \geq p$ or if $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is replaced with $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$, but it does matter if $q < p$.

We first address the embedding when a and $b-p$ are on the same side of $-N$.

Lemma 4.3. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p < \infty, 0 < q, r < \infty$ be given. If a and $b-p$ are on the same side of $-N$ (including $-N$), the subspace of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ in the following two cases (recall the definition of c^0 and c^1 in* (1.3)*):*

- (i) *$(a+N)/q \neq (b-p+N)/p$, $r \leq q$, and c is in the open interval with endpoints c^0 and c^1.*
- (ii) *$(a+N)/q \neq (b-p+N)/p$, $b-p \neq -N$ if $p > 1$, $r > q$, and c is in the half-open interval with endpoints $c^* := (1-q/r)c^1 + qc^0/r$ (included) and c^1 (not included).*

Proof. By Kelvin transform, we may assume that $a \geq -N$ and $b-p \geq -N$ and, by Lemma 3.4, that $u \geq 0$. By Lemma 4.2 and with ζ as in Subsection 1.1, it suffices to show that $||(1-\zeta)u||_{c,r} \leq C||(1-\zeta)u||_{\{a,b\},(q,p)}$ and that $||\zeta u||_{c,r} \leq C||\zeta u||_{\{a,b\},(q,p)}$ for some constant $C > 0$ independent of u.

(i) The assumption $0 < r \leq q$ is retained.

Case (i-1): $b-p > -N$ or $p = 1$ and $b-1 \geq -N$. We first prove $||v||_{c,r} \leq C||v||_{\{a,b\},(q,p)}$ when $v := (1-\zeta)u$ (≥ 0). Given $\xi \in \mathbb{R}$ and $c \in \mathbb{R}$, we can write $|x|^c v^r = |x|^{-\xi}(|x|^{c+\xi}v^r)$. Since $\operatorname{Supp} v \subset \mathbb{R}^N \backslash B(0,1/2)$ and by Hölder's inequality, $||v||^r_{c,r} = \int_{\mathbb{R}^N} |x|^c v^r \leq (\int_{\mathbb{R}^N\backslash B(0,1/2)} |x|^{-k'\xi})^{1/k'}(\int_{\mathbb{R}^N} |x|^{k(c+\xi)}v^{kr})^{1/k}$, where $k > 1$ is arbitrary.

If $k'\xi > N$, then $M_{k,\xi} := (\int_{\mathbb{R}^N\backslash B(0,1/2)} |x|^{-k'\xi})^{1/k'} < \infty$ and it suffices to find a majorization of $\int_{\mathbb{R}^N} |x|^{k(c+\xi)}v^{kr}$. Split $|x|^{k(c+\xi)}v^{kr} = (|x|^{k(c+\xi)-a}v^{kr-q})|x|^a v^q$, so that if $kr-q > 0$, then

$$\int_{\mathbb{R}^N} |x|^{k(c+\xi)}v^{kr} \leq \left\| |x|^{k(c+\xi)-a}v^{kr-q}\right\|_\infty \int_{\mathbb{R}^N} |x|^a v^q = \left\| |x|^{(k(c+\xi)-a)/(kr-q)}v\right\|_\infty^{kr-q} ||v||^q_{a,q}.$$

The next task is to majorize $\||x|^{(k(c+\xi)-a)/(kr-q)}v\|_\infty$. This can be done by using Theorem 3.1, as we now explain. Suppose, in addition to our other assumptions, that k and ξ are chosen so that $(k(c+\xi)-a)/(kr-q) = (b-p+N)/p$. By Lemma 3.3(iii), $v \in \widetilde{W}^{1,1}_{loc,-}$, since $a \geq -N$. Next, if $\gamma := b/p - N/p'$, then $\gamma > 1-N$ if $p > 1$ since $b-p > -N$, and $\gamma \geq 1-N$ if $p = 1$ since $b-1 \geq -N$. Thus $\||x|^{(b-p+N)/p}v\|_\infty \leq C||\partial_\rho v||_{b,p} < \infty$ by Theorem 3.1. To summarize,

$$||v||_{c,r} \leq M^{1/r}_{k,\xi}C^{1-q/(kr)}||\partial_\rho v||^{1-q/(kr)}_{b,p}||v||^{q/(kr)}_{a,q}, \tag{4.1}$$

if k and $\xi \in \mathbb{R}$ can be found such that $k'\xi > N$, $kr-q > 0$ (hence $k > 1$ since $r \leq q$) and $(k(c+\xi)-a)/(kr-q) = (b-p+N)/p$. By introducing $s := kr-q > 0$, so that $k = (s+q)/r$, it follows that $(k(c+\xi) - a)/(kr - q) = (b - p + N)/p$ if and only if $\xi = (arp + rs(b - p + N))/(p(s+q)) - c$, and then $k'\xi > N$ if and only if

$$c < \frac{arp + rs(b - p + N) - Nps - Npq + Npr}{p(s+q)} = \frac{c^1 s + c^0 q}{s+q}. \tag{4.2}$$

Thus, this inequality holding for some $s > 0$ ensures that (4.1) holds with $k := (s+q)/r > 1$ and $\xi = (arp+rs(b-p+N))/(p(s+q)) - c$. The right-hand side of (4.2) is a monotone function of $s > 0$ with limits c^0 and c^1 as s tends to 0 and ∞, respectively. Therefore, $s > 0$ can be chosen so that (4.2) holds if and only if $c < \max\{c^0, c^1\}$, and then, since $v = (1-\zeta)u$ in (4.1), the arithmetic-geometric inequality yields $||(1-\zeta)u||_{c,r} \le C||(1-\zeta)u||_{\{a,b\},(q,p)}$, with $C > 0$ independent of u.

If now $v := \zeta u$, then once again $v \in \widetilde{W}^{1,1}_{loc,-}$, because v has bounded support. The same procedure, but with $k'\xi > N$ replaced with $k'\xi < N$, shows that

$$||v||_{c,r} = ||\zeta u||_{c,r} \le C||u||_{\{a,b\},(q,p)}$$

if $c > \min\{c^0, c^1\}$. Hence, both $||(1-\zeta)u||_{c,r} \le C||u||_{\{a,b\},(q,p)}$ and $||\zeta u||_{c,r} \le C||u||_{\{a,b\},(q,p)}$ hold when c is in the open interval with endpoints c^0 and c^1.

Case (i-2): $b-p = -N$ (and[3] $p > 1$). In this case, $(a+N)/q \neq (b-p+N)/p = 0$ and $a \ge -N$ imply $a > -N$ and $-N = c^1 < c < c^0$. If $v := (1-\zeta)u$, then $||v||_{c,r} \le C||v||_{a,q} \le C||v||_{\{a,b\},(q,p)}$ by Hölder's inequality (use $|x|^c|v|^r = |x|^{c-ar/q}(|x|^{ar/q}(1-\zeta)^r|u|^r)$, $\mathrm{Supp}\,(1-\zeta) \subset \mathbb{R}^N \backslash B(0, 1/2)$ and $(cq-ar)/(q-r) < -N$, i.e., $c < c^0$ if $r < q$, or $c < a = c^0$ if $r = q$).

Next, choose $\hat{b} > b$ (so that $\hat{b} - p > -N$) such that

$$\hat{c}^1 := \frac{r(\hat{b} - p + N)}{p} - N < c$$

and use Case (i-1) with b replaced with $\hat{b}$ (which changes c^1 into $\hat{c}^1$, but does not change c^0) and u replaced with ζu. This yields

$$||\zeta u||_{c,r} \le C||\zeta u||_{\{a,\hat{b}\},(q,p)} \le C||\zeta u||_{\{a,b\},(q,p)},$$

where the second inequality follows from the fact that $\hat{b} > b$ and $\mathrm{Supp}\,\zeta \subset \overline{B}(0, 1)$ (so that $||\nabla(\zeta u)||_{\hat{b},p} \le ||\nabla(\zeta u)||_{b,p}$).

(ii) The assumption $0 < q < r$ is retained.

By Lemmas 3.3(iv) and 4.2, u, ζu and $(1-\zeta)u$ are in $\widetilde{W}^{1,1}_{loc}$ (even $\widetilde{W}^{1,1}_{loc,-}$, since $a \ge -N$ and ζu has bounded support), because of radial symmetry, even when

[3]The argument also works when $p = 1$.

$q < 1$. Since $b-p \geq -N$ and $b-p \neq -N$ when $p > 1$, it follows that $b-p > -N$ if $p > 1$.

The general procedure is the same as in Case (i-1), with the following difference. To prove (4.1) with $v := (1-\zeta)u$ (≥ 0), k and $\xi \in \mathbb{R}$ must be found so that $k'\xi > N$, $k > 1$ and $(k(c+\xi)-a)/(kr-q) = (b-p+N)/p$. With the same change of variable $k := (s+q)/r$ as before, $k > 1$ amounts to $s > r-q$, so that (4.1) holds for some ξ if and only if $c < \max\{c^*, c^1\}$ (the supremum of the right-hand-side of (4.2) when $s > r-q$).

Likewise, as in Case (i-1), (4.1) holds with $v = \zeta u$ if $c > \min\{c^*, c^1\}$. This proves (ii) when $b-p > -N$, or $p = 1$ and $b-1 \geq -N$, and when c is in the *open* interval with endpoints c^* and c^1. Thus, it only remains to discuss the case $c = c^*$.

This can be done by proving inequality (4.1) for $v = u$ radially symmetric, with $k = 1$ and $\xi = 0$ (no need to split u). Specifically, since $r > q$ (unlike in part (i)), we can write

$$||u||^r_{c^*,r} = \int_{\mathbb{R}^N} |x|^{c^*}|u|^r = \int_{\mathbb{R}^N} |x|^{a+(r-q)((b-p+N)/p)}|u|^r \leq \||x|^{(b-p+N)/p}u\|^{r-q}_\infty ||u||^q_{a,q}$$

and notice that $\||x|^{(b-p+N)/p}u\|_\infty \leq C||\partial_\rho u||_{b,p}$ using, as before, Theorem 3.1 with $\gamma := b/p - N/p'$. This requires that $b-p > -N$ if $p > 1$, but $b-1 = -N$ is allowed if $p = 1$. □

Lemma 4.3(ii) is not optimal; however, before improving it (in Lemma 4.6 below), we prove a similar result when a and $b-p$ are on opposite sides of $-N$.

Lemma 4.4. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p < \infty$, $0 < q, r < \infty$. If a and $b-p$ are strictly on opposite sides of $-N$, the subspace of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions can be continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ in the following two cases:*

(i) *$r \leq q$ and c is in the open interval with endpoints c^0 and $-N$,*

(ii) *$q < r$, $1 - q/r < \theta_{-N}$, and[4] c is in the half-open interval with endpoints $c^* := (1-q/r)c^1 + qc^0/r$ (included) and $-N$ (not included).*

Proof. Since a and $b-p$ are strictly on opposite sides of $-N$, we may assume that $b-p < -N < a$ by the usual Kelvin transform argument.

(i) By (1.3), $c^1 < -N < c^0$. Let $c \in (-N, c^0)$ be given. As in the proof of Lemma 4.3, it suffices to show that $||(1-\zeta)u||_{c,r} \leq C||(1-\zeta)u||_{\{a,b\},(q,p)}$ and that $||\zeta u||_{c,r} \leq C||\zeta u||_{\{a,b\},(q,p)}$ when u is radially symmetric.

Since $\operatorname{Supp}(1-\zeta) \subset \mathbb{R}^N \backslash B(0, 1/2)$, it follows that $(1-\zeta)u \in \widetilde{W}^{1,1}_{loc,+}$. As a result, the argument of the proof of Case (i-1) of Lemma 4.3, which is based on

[4]Since $-N$ is between c_0 and c_1 when a and $b-p$ are on opposite sides of $-N$, it follows that $\theta_{-N} \in (0, 1)$; see (1.4).

Theorem 3.1, can be repeated verbatim, now with $\gamma := b/p - N/p' < 1 - N$. This shows that $||(1-\zeta)u||_{c,r} \leq C||(1-\zeta)u||_{\{a,b\},(q,p)}$, since $c < \max\{c^0, c^1\} = c^0$.

The inequality $||\zeta u||_{c,r} \leq C||\zeta u||_{\{a,b\},(q,p)}$ cannot be obtained as in Case (i-1) of Lemma 4.3, because $b - p < -N$ but $\zeta u \notin \widetilde{W}^{1,1}_{loc,+}$, so that Theorem 3.1 is not applicable. However, it can be proved with the trick used in Case (i-2) of that lemma. Since $-N < c < c^0$, Lemma 4.3(i) can be used with b replaced with $p - N > b$, because $a \neq -N$ and c^1 becomes $-N$ when b is replaced with $p - N$ while c^0 is unchanged. Thus $||\zeta u||_{c,r} \leq C||\zeta u||_{\{a,p-N\},(q,p)}$, while

$$||\zeta u||_{\{a,p-N\},(q,p)} \leq ||\zeta u||_{\{a,b\},(q,p)},$$

since $p - N > b$ and $\operatorname{Supp} \zeta \subset \overline{B}(0,1)$.

(ii) Observe that $c^1 < c^* < c^0$ because $q < r$ and $c^1 < c^0$ (recall that $b - p < -N < a$), while $1 - q/r < \theta_{-N}$ ensures that $-N < c^*$.

Let then $c \in (-N, c^*)$. Once again, use the fact that $(1-\zeta)u \in \widetilde{W}^{1,1}_{loc,+}$ since $\operatorname{Supp}(1-\zeta) \subset \mathbb{R}^N \backslash B(0,1/2)$, and Theorem 3.1 with $\gamma := b/p - N/p' < 1 - N$. The argument of the proof Lemma 4.3(ii) (with obvious modifications) then yields $||(1-\zeta)u||_{c,r} \leq C||(1-\zeta)u||_{\{a,b\},(q,p)}$ since $c < c^* = \max\{c^*, c^1\}$.

If $c = c^*$, the same argument works with $k = 1, \xi = 0$. To see this, let $v := (1-\zeta)u \in \widetilde{W}^{1,1}_{loc,+}$ and write

$$||v||^r_{c^*,r} = \int_{\mathbb{R}^N} |x|^{c^*}|v|^r = \int_{\mathbb{R}^N} |x|^{(a+(r-q))(b-p+N)/p}|v|^r \leq \left\| |x|^{(b-p+N)/p} v \right\|^{r-q}_\infty ||v||^q_{a,q}.$$

Then, use Theorem 3.1 with $\gamma := b/p - N/p' < 1 - N$ to get

$$\left\| |x|^{(b-p+N)/p} v \right\|_\infty \leq C||\partial_\rho v||_{b,p}.$$

The proof of $||\zeta u||_{c,r} \leq C||\zeta u||_{\{a,b\},(q,p)}$ when $c \in (-N, c^*]$ proceeds as in (i) above, with minor modifications. If $\hat{b} > b$, then $\hat{c}^1 := (r(\hat{b} - p + N))/p - N > c^1$, and so $\hat{c}^* := (1 - q/r)\hat{c}^1 + qc^0/r > c^*$. Note also that $\hat{c}^1$ is arbitrarily close to $-N$ if $\hat{b}$ is close enough to $p - N$. As a result, c is in the open interval with endpoints $\hat{c}^*$ and $\hat{c}^1$ (even when $c = c^*$), provided that $\hat{b} > p - N$ is close to $p - N$, while a and $\hat{b} - p$ are both on the right of $-N$. Thus, Lemma 4.3(ii) is applicable with b replaced with $\hat{b}$. (Because of the requirement in Lemma 4.3(ii) that $\hat{b} - p \neq -N$, unlike in (i), $\hat{b} = p - N$ cannot be chosen if $p > 1$.) □

We now prove optimal variants of Lemmas 4.3 and 4.4. To do this, we need a complement to Lemma 3.4(i) in the radially symmetric case.

Lemma 4.5. *Let $a, b \in \mathbb{R}$, $1 \leq p < \infty$, and $0 < q < \infty$. If $1 \leq \xi \leq q/p' + 1$ and $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is radially symmetric, then $|u|^\xi \in \widetilde{W}^{1,(q_\xi,p_\xi)}_{\{a,b_\xi\}}$, where*

$$p_\xi := \frac{pq}{p(\xi - 1) + q} \geq 1, \quad q_\xi := \frac{q}{\xi} > 0, \quad b_\xi := \left(\frac{a(\xi-1)}{q} + \frac{b}{p}\right)p_\xi. \tag{4.3}$$

Furthermore, $|u|^{\xi}$ (is radially symmetric and) satisfies

$$(4.4)\qquad \| \, |u|^{\xi} \|_{a,q_{\xi}} = \|u\|_{a,q}^{\xi}, \quad \|\partial_{\rho}(|u|^{\xi})\|_{b_{\xi},p_{\xi}} \leq \xi \|u\|_{a,q}^{\xi-1} \|\partial_{\rho} u\|_{b,p}.$$

Proof. If $\xi = 1$, then $q_{\xi} = q, p_{\xi} = p$ and $b_{\xi} = b$. This is the case covered by Lemma 3.4, which also shows that it is not restrictive to assume that $u \geq 0$. From now on, assume $\xi > 1$. The assumption $\xi \leq q/p' + 1$ ensures that $p_{\xi} \geq 1$ in (4.3).

That u^{ξ} is radially symmetric, $u^{\xi} \in L^{q_{\xi}}(\mathbb{R}^N; |x|^a dx)$, and $\| \, |u|^{\xi} \|_{a,q_{\xi}} = \|u\|_{a,q}^{\xi}$ is obvious. It remains to prove that $u^{\xi} \in L^1_{loc}(\mathbb{R}^N_*)$, that $\partial_{\rho}(u^{\xi}) \in L^{p_{\xi}}(\mathbb{R}^N; |x|^{b_{\xi}} dx)$, and the second inequality in (4.4).

By Lemma 3.3(iv), $u \in \widetilde{W}^{1,1}_{loc,\pm} \subset \widetilde{W}^{1,1}_{loc}$ (depending upon whether $a \geq -N$ or $a \leq -N$). Thus, from Lemma 3.1, $u(x) = f_u(|x|)$ with $f_u \in W^{1,1}_{loc}(0, \infty), f_u \geq 0$ and $\partial_{\rho} u(x) = f_u'(|x|)$. Clearly, since $\xi > 1$, $f_u^{\xi} \in W^{1,1}_{loc}(0, \infty)$ and $(f_u^{\xi})' = \xi f_u^{\xi-1} f_u'$. Hence, once again by Lemma 3.1, $u^{\xi}(x) = f_u^{\xi}(|x|)$ is in $\widetilde{W}^{1,1}_{loc} \subset L^1_{loc}(\mathbb{R}^N_*)$ and $\partial_{\rho}(u^{\xi})(x) = \xi f_u^{\xi-1}(|x|) f_u'(|x|)$, i.e., $\partial_{\rho}(u^{\xi}) = \xi u^{\xi-1} \partial_{\rho} u$.

In general, if $\mu, \nu > 0$, multiplication maps $L^{\mu} \times L^{\nu}$ into $L^{\mu\nu/(\mu+\nu)}$. Moreover, $\|vw\|_{\mu\nu/(\mu+\nu)} \leq \|v\|_{\mu} \|w\|_{\nu}$. This requires neither that $\mu \geq 1$ nor that $\nu \geq 1$ (just use $|v|^{\mu\nu/(\mu+\nu)} \in L^{1+\mu/\nu}$ and $|w|^{\mu\nu/(\mu+\nu)} \in L^{1+\nu/\mu}$ and Hölder's inequality). Now, $|x|^{(a(\xi-1))/q} |u|^{\xi-1} \in L^{q/(\xi-1)}(\mathbb{R}^N)$ since $|x|^a |u|^q \in L^1(\mathbb{R}^N)$ and $\xi > 1$, and $|x|^{b/p} \partial_{\rho} u \in L^p(\mathbb{R}^N)$. Therefore, $|x|^{b_{\xi}/p_{\xi}} u^{\xi-1} \partial_{\rho} u \in L^{p_{\xi}}(\mathbb{R}^N)$ with p_{ξ} and b_{ξ} given by (4.3) and

$$\left\| |x|^{b_{\xi}/p_{\xi}} u^{\xi-1} \partial_{\rho} u \right\|_{p_{\xi}} \leq \left\| |x|^{(a(\xi-1))/q} |u|^{\xi-1} \right\|_{q/(\xi-1)} \| \, |x|^{b/p} \partial_{\rho} u \|_p = \|u\|_{a,q}^{\xi-1} \|\partial_{\rho} u\|_{b,p}.$$

From the above, this implies $\partial_{\rho}(u^{\xi}) \in L^{p_{\xi}}(\mathbb{R}^N; |x|^{b_{\xi}} dx)$ with

$$\|\partial_{\rho}(u^{\xi})\|_{b_{\xi},p_{\xi}} \leq \xi \|u\|_{a,q}^{\xi-1} \|\partial_{\rho} u\|_{b,p}$$

□

.

Remark 3. Lemma 4.5 holds in the case $1 \leq \xi \leq \min\{q, q/p' + 1\}$ without the radial symmetry assumption. Indeed, if $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$, then

$$|u|^{\xi} \in L^{q_{\xi}}(\mathbb{R}^N; |x|^a dx) \subset L^1_{loc}(\mathbb{R}^N_*)$$

since $q_{\xi} \geq 1$ and $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}} \subset \widetilde{W}^{1,1}_{loc}$ implies that $|u|^{\xi}$ is locally absolutely continuous on almost every ray through the origin (see Section 3), with

$$\partial_{\rho}(|u|^{\xi}) = \xi |u|^{\xi-1} (\operatorname{sgn} u) \partial_{\rho} u \in L^{p_{\xi}}(\mathbb{R}^N; |x|^{b_{\xi}} dx) \subset L^1_{loc}(\mathbb{R}^N_*).$$

This is used elsewhere.

Lemma 4.6. *Let $a, b, c \in \mathbb{R}$, $1 \le p < \infty$, $0 < q, r < \infty$, and*

$$\breve{\theta} := \Big(1 - \frac{q}{r}\Big)\Big(\frac{q}{p'} + 1\Big)^{-1} < 1 \quad (\le 0 \text{ if } r \le q).$$

The subspace of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ in the following two cases:

(i) *a and $b - p$ are on the same side of $-N$ (including $-N$), $(a + N)/q \neq (b - p + N)/p$, c is in the open interval with endpoints c^0 and c^1, and $\theta_c \ge \breve{\theta}$ (vacuously true if $r \le q$);*

(ii) *a and $b - p$ are strictly on opposite sides of $-N$, c is in the open interval with endpoints c^0 and $-N$, and $\theta_c \ge \breve{\theta}$ (the empty set if $\breve{\theta} \ge \theta_{-N}$).*

Proof. If $r \le q$ (so that $\breve{\theta} \le 0$) or if $r > q$ and $p = 1$ (so that $\breve{\theta} = 1 - q/r$), (i) follows from Lemma 4.3 (where $b - p \neq -N$ is not required in part (ii) when $p = 1$), and (ii) follows from Lemma 4.4. From now on, $r > q$ (so that $\breve{\theta} \in (0, 1)$) and $p > 1$. For convenience, we set $\breve{\xi} := q/p' + 1 > 1$. In particular, the interval $(1, \breve{\xi}]$ is not empty, a fact used implicitly below.

(i) Let $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ be radially symmetric. If $1 \le \xi \le \breve{\xi}$, then by Lemma 4.5, $|u|^\xi \in \widetilde{W}^{1,(q_\xi,p_\xi)}_{\{a,b_\xi\}}$ with $q_\xi > 0$, $p_\xi \ge 1$ and b_ξ given by (4.3). A routine verification shows that a and $b_\xi - p_\xi$ are on the same side of $-N$ (since the same thing is true of a and $b - p$) and that $(a+N)/q_\xi \neq (b_\xi - p_\xi + N)/p_\xi$ (since $(a+N)/q \neq (b-p+N)/p$).

Furthermore, $b_\xi - p_\xi \neq -N$ if $\xi > 1$ (which need not be true if $\xi = 1$, since $b - p \neq -N$ is *not* assumed). Indeed, $b_\xi - p_\xi = -N$ amounts to

$$\frac{a + N}{q}(\xi - 1) + \frac{b - p + N}{p} = 0.$$

Since a and $b - p$ are on the same side of $-N$, this can only happen when $a + N = b - p + N = 0$ when $\xi > 1$, which contradicts the assumption $(a + N)/q \neq (b - p + N)/p$.

Accordingly, from Lemma 4.3(ii) with b, p, q and r replaced with b_ξ, p_ξ, q_ξ and s, respectively, $W^{1,(q_\xi,p_\xi)}_{\{a,b_\xi\}}(\mathbb{R}^N_*) \hookrightarrow L^s(\mathbb{R}^N; |x|^c dx)$ whenever $1 < \xi \le \breve{\xi}$, $0 < q_\xi < s$, and c is in the half-open interval with endpoints $a + ((s - q_\xi)(b_\xi - p_\xi + N))/p_\xi$ (included; this corresponds to c^* with the parameters b_ξ, p_ξ, q_ξ, s) and $(s(b_\xi - p_\xi + N))/p_\xi - N$ (not included; this corresponds to c^1 with the parameters b_ξ, p_ξ, q_ξ, s). Since $r > q$ and $q_\xi = q/\xi$, the condition $0 < q_\xi < s$ holds when $s = r/\xi$. If so, the embedding inequality $|||u|^\xi||_{c,r/\xi} \le C_\xi(|||u|^\xi||_{a,q_\xi} + ||\partial_\rho(|u|^\xi)||_{b_\xi,p_\xi})$ reads (use (4.4))

$$||u||^\xi_{c,r} \le C_\xi(||u||^\xi_{a,q} + ||u||^{\xi-1}_{a,q}||\partial_\rho u||_{b,p}) \le C_\xi(||u||_{a,q} + ||\partial_\rho u||_{b,p})^\xi,$$

so that $||u||_{c,r} \leq C_{\breve{\xi}}^{\xi-1}||u||_{\{a,b\},(q,p)}$. Above, c is in the half-open interval J_ξ with (distinct) endpoints $e_1(\xi) := a+((r-q)(b_\xi-p_\xi+N))/(\xi p_\xi)$ (included) and $e_2(\xi) := (r(b_\xi - p_\xi + N))/(\xi p_\xi) - N$ (not included) and $1 < \xi \leq \breve{\xi}$. Thus, whenever $c \in J := \cup_{\xi\in(1,\breve{\xi}]} J_\xi$,

$$||u||_{c,r} \leq C(||u||_{a,q} + ||\partial_\rho u||_{b,p}), \tag{4.5}$$

for some constant $C > 0$ independent of the radially symmetric $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ (specifically, $C = C_{\breve{\xi}}^{\xi-1}$ for any ξ such that $c \in J_\xi$).

Since the distinct endpoints of J_ξ depend continuously upon ξ, the lower (upper) endpoint $e_-(\xi)$ ($e_+(\xi)$) is either $e_1(\xi)$ for every ξ or $e_2(\xi)$ for every ξ. Hence, $e_\pm$ are continuous functions of ξ which are never equal on $(1, \breve{\xi}]$. Using this remark, it is an easy exercise to show that J contains the open interval with endpoints $\inf e_-$ and $\sup e_+$.

If $(a+N)/q > (b-p+N)/p$, then $e_1 > e_2$ and both e_1 and e_2 are increasing functions of ξ, so that J contains $(e_2(1), e_1(\breve{\xi}))$. In addition, since it contains $e_1(\breve{\xi}) \in J_{\breve{\xi}}$, J contains (and, in fact, coincides with) $(e_2(1), e_1(\breve{\xi})]$.

If $(a+N)/q < (b-p+N)/p$, then $e_2 > e_1$ and both e_1 and e_2 are decreasing functions of ξ, so that J contains the open interval $(e_1(\breve{\xi}), e_2(1))$. Once again, it also contains $e_1(\breve{\xi})$. Therefore, in all cases, J coincides with the half-open interval with endpoints $e_1(\breve{\xi}) = \breve{\theta}c^1 + (1-\breve{\theta})c^0$ (included) and $e_2(1) = c^1$ (not included). For every c in that interval, (4.5) holds for some constant C independent of the radially symmetric $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$. Clearly, J is equally characterized as the set of those c in the open interval with endpoints c^0 and c^1 such that $\theta_c \geq \breve{\theta}$.

(ii) First, since a and $b-p$ are on opposite sides of $-N$, it is obvious that $(a+N)/q \neq (b-p+N)/p$. The proof proceeds as in part (i), but extra technicalities arise from the fact that the points a and $b_\xi - p_\xi$ (see (4.3)) need not remain on opposite sides of $-N$ for all $\xi \in [1, \breve{\xi}]$.

Nonetheless, since $b_\xi - p_\xi$ is a strictly monotone function of ξ equal to $b-p$ when $\xi = 1$ and since a and $b-p$ are strictly on opposite sides of $-N$, there are only two possibilities:

(a) a and $b_\xi - p_\xi$ are strictly on opposite sides of $-N$ when $\xi = \breve{\xi}$ (which amounts to a and $b/p + a/p' - 1$ being strictly on opposite sides of $-N$) and hence the same holds for every $\xi \in [1, \breve{\xi}]$, or

(b) 1. $b_{\xi_0} - p_{\xi_0} = -N$ for some unique $\xi_0 \in (1, \breve{\xi})$,
2. $b_{\breve{\xi}} - p_{\breve{\xi}} = -N$.

In both these cases, a and $b_\xi - p_\xi$ are on the same side of $-N$ for every $\xi \in [\xi_0, \breve{\xi}]$.

Case (a). Replace q, r, b, p with $q_{\breve{\xi}} = q/\breve{\xi}$, $r_{\breve{\xi}} = r/\breve{\xi}$, $b_{\breve{\xi}} = a/p' + b/p$,

$p_{\breve{\xi}} = 1$, respectively, in Lemma 4.4(ii) and use the conclusion of that Lemma with u replaced with $|u|^{\breve{\xi}}$. This is justified by Lemma 4.5. However, it is crucial to notice that, due to the change of parameters, the condition "$1 - q/r < \theta_{-N}$" in Lemma 4.4 does not involve θ_{-N} but, instead, the number $\breve{\theta}_{-N}$ given by the same formula (1.4) when c^0 and c^1 are replaced with $\breve{c}^0$ and $\breve{c}^1$ defined by (1.3) with the new parameters $q_{\breve{\xi}}, r_{\breve{\xi}}, b_{\breve{\xi}}, p_{\breve{\xi}}$. Thus, $\breve{c}^0 = c^0$, but

$$\breve{c}^1 = r\breve{\xi}^{-1}\Big(\frac{a+N}{p'} + \frac{b-p+N}{p}\Big) - N,$$

so that $\breve{c}^1 - \breve{c}^0 = r\breve{\xi}^{-1}((b-p+N)/p - (a+N)/q)$. With this remark, it is readily checked that $\breve{\theta}_{-N} = \breve{\xi}\theta_{-N}$, so that the condition $1 - q_{\breve{\xi}}/r_{\breve{\xi}} < \breve{\theta}_{-N}$ becomes $\breve{\theta} := (1-q/r)(q/p'+1)^{-1} < \theta_{-N}$.

In summary, the continuity of the embedding is ensured if $\breve{\theta} < \theta_{-N}$ and c is in the half-open interval with endpoints $\breve{c} := (1-q/r)\breve{c}^1 + q\breve{c}^0/r = \breve{\theta}c^1 + (1-\breve{\theta})c^0$ (included) and $-N$ (not included), which, since $r > q$, coincides with the set of c in the open interval with endpoints c^0 and $-N$ such that $\theta_c \geq \breve{\theta}$.

Case (b1). Since a and $b_\xi - p_\xi$ are on the same side of $-N$ for $\xi \in [\xi_0, \breve{\xi}]$ and since $b_\xi - p_\xi \neq -N$ if $\xi \in (\xi_0, \breve{\xi}]$, Lemma 4.3(ii) with u replaced with $|u|^\xi$ and q, r, b, p replaced with $q/\xi, r/\xi, b_\xi, p_\xi$, respectively, yields that the subspace of $W^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ for every $c \in J := \cup_{\xi\in(\xi_0,\breve{\xi}]} J_\xi$, where J_ξ is the half-open interval with endpoints $e_1(\xi) := a + ((r-q)(b_\xi - p_\xi + N))/\xi p_\xi$ (included) and $e_2(\xi) := (r(b_\xi - p_\xi + N))/\xi p_\xi - N$ (not included). Both endpoints are distinct (because $(a+N)/q \neq (b-p+N)/p$) and on the same side of $-N$ when $\xi > \xi_0$. By arguing as in the proof of (i) above, J is found to be the half-open interval with endpoints $e_1(\breve{\xi}) = \breve{c} = \breve{\theta}c^1 + (1-\breve{\theta})c^0$ (included) and $e_2(\xi_0) = -N$ (not included), exactly as in (a). Therefore, the final argument is also the same.

Case (b2). Since a and $b_{\breve{\xi}} - p_{\breve{\xi}}$ are on the same side of $-N$ *and* since $p_{\breve{\xi}} = 1$, it suffices to use Lemma 4.3(ii) with u replaced with $|u|^{\breve{\xi}}$ and q, r, b, p replaced by $q/\breve{\xi}, r/\breve{\xi}, b_{\breve{\xi}} = b/p + a/p', p_{\breve{\xi}} = 1$, respectively. □

It is informative that even if $q, r \geq 1$ in Lemma 4.6, the proof involves Lemmas 4.3(ii) and 4.4(ii) when $q, r > 0$ ($q, r \geq 1$ is not enough). The next theorem gives necessary and sufficient conditions for the continuous embedding of the subspace of radially symmetric functions.

Theorem 4.1. *Let $a, b, c \in \mathbb{R}$, $1 \leq p < \infty$ and $0 < q, r < \infty$ and set $\breve{\theta} := (1-q/r)(q/p'+1)^{-1}$. The subspace of $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ of radially symmetric functions is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ (and hence into $\widetilde{W}^{1,(r,p)}_{\{c,b\}}$) if and only if one of the following conditions holds.*

(i) *a and $b-p$ are on the same side of $-N$ (including $-N$), $(a+N)/q \neq (b-p+N)/p$, c is in the open interval with endpoints c^0 and c^1, and $\theta_c \geq \breve{\theta}$ (vacuously true if $q \geq r$).*

(ii) *a and $b-p$ are strictly on opposite sides of $-N$, c is in the open interval with endpoints c^0 and $-N$ and $\theta_c \geq \breve{\theta}$ (empty set if $\breve{\theta} \geq \theta_{-N}$).*

(iii) *$r \geq p, a \leq -N$ and $b-p < -N$ or $a \geq -N$ and $b-p > -N, c = c^1$. Furthermore, there is a constant $C > 0$ such that*

$$||u||_{c,r} \leq C||\partial_\rho u||_{b,p}, \tag{4.6}$$

for every radially symmetric function $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$.

(iv) *$r = q$ and $c = c^0$ $(= a)$, or $p \neq q$, $\min\{p,q\} \leq r \leq \max\{p,q\}$, $(a+N)/q = (b-p+N)/p \neq 0$ and $c = c^0$ $(= c^1)$. Furthermore, there is a constant $C > 0$ such that*

$$||u||_{c,r} \leq C||\partial_\rho u||^{\theta}_{b,p}||u||^{1-\theta}_{a,q}, \tag{4.7}$$

for every radially symmetric function $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$, where $\theta = 0$ if $r = q$ and $c = a$ and $\theta = p(r-q)/(r(p-q))$ otherwise.

(v) *$a = -N, b = p-N, r > q$ and $c = c^0$ $(= c^1 = -N)$. Furthermore, there is a constant $C > 0$ such that*

$$||u||_{-N,r} \leq C||\partial_\rho u||^{\breve{\theta}}_{p-N,p}||u||^{1-\breve{\theta}}_{-N,q}, \tag{4.8}$$

for every radially symmetric function $u \in \widetilde{W}^{1,(q,p)}_{\{-N,p-N\}}$.

Proof. The theorem is (as it should be) equivalent to Theorem 1.1 when $N = 1$ (in particular, $p^* = \infty$ regardless of p and $p^{-1} - N^{-1} - q^{-1} = -(p'^{-1} + q^{-1})$) and a, b, c are replaced with $a+N-1, b+N-1$ and $c+N-1$, respectively. Since the hypotheses of Theorem 1.1 are necessary (Section 2), the necessity follows.

The sufficiency of parts (i) and (ii) has already been proved in Lemma 4.6. To complete the proof, we show that parts (iii), (iv) or (v) are also sufficient.

(iii) By Kelvin transform, we may assume that $a \leq -N$ and $b-p < -N$. In particular, $u \in \widetilde{W}^{1,1}_{loc,+}$ by Lemma 3.3(iv). By Lemma 3.3(i), we may also assume with no loss of generality that $u \geq 0$. Then $u(x) = f_u(|x|)$ with $f_u \in W^{1,1}_{loc}(0,\infty)$, $f_u \geq 0$ and $\underline{\lim}_{t\to 0^+} f_u(t) = 0$, so that $f_u(t) \leq \int_0^t |f_u'(\tau)|d\tau$ by (3.6) with $\gamma = 1-N$ and $f = f_u$.

On the other hand, since $r \geq p$,

$$\left(\int_0^\infty t^{r(b-p+N)/p-1}\left(\int_0^t |f_u'(\tau)|d\tau\right)^r dt\right)^{1/r} \leq C\left(\int_0^\infty t^{b+N-1}|f_u'(t)|^p dt\right)^{1/p},$$

by a weighted Hardy inequality of Bradley ([5, Theorem 1], [17, p. 40]) inspired by Muckenhoupt [19] when $r = p$. This yields (4.6), since Lemma 3.1 implies $c = c^1 = r(b-p+N)/p - N$ and $\partial_\rho u(x) = f_u'(|x|)$.

(iv) This is trivial if $r = q$ and $c = a$. From now on, take $p \neq q$ and r between p and q (both included), so that $r = \mu p + (1-\mu)q$, where $\mu = (r-q)/(p-q) \in [0, 1]$, whence $\mu(b-p) + (1-\mu)a = c^0 = c$ (use $b - p = p(a+N)/q - N$). Thus, if u is measurable, $\int_{\mathbb{R}^N} |x|^c|u|^r = \int_{\mathbb{R}^N} (|x|^{b-p}|u|^p)^\mu (|x|^a|u|^q)^{1-\mu}$ and, by Hölder's inequality,

$$||u||_{c,r}^r \leq ||u||_{b-p,p}^{\mu p} ||u||_{a,q}^{(1-\mu)q}. \tag{4.9}$$

Since $(a+N)/q = (b-p+N)/p \neq 0$, both a and $b-p$ are on the same side of $-N$ and neither equals $-N$. Therefore, when $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is radially symmetric, $||u||_{b-p,p} \leq C||\partial_\rho u||_{b,p}$ by (iii) with $r = p$ (hence $c = b-p$). By substitution into (4.9), $||u||_{c,r} \leq C||\partial_\rho u||_{b,p}^{\mu p/r} ||u||_{a,q}^{(1-\mu)q/r} = C||\partial_\rho u||_{b,p}^{\theta} ||u||_{a,q}^{1-\theta}$ with $\theta = \mu p/r = p(r-q)/(r(p-q))$. This proves (4.7) and hence the embedding property as well.

(v) It follows from part (i) of the theorem if $p = 1$ and from part (ii) if $p > 1$ that if $r > q$, $N = 1$, and $a = b = c = 0$, the subspace of even functions in the unweighted space $W^{1,(q,p)}(\mathbb{R}_*)$ is continuously embedded into $L^r(\mathbb{R})$. This readily implies the same result in this one-dimensional setting without the evenness assumption, i.e., $W^{1,(q,p)}(\mathbb{R}_*) \hookrightarrow L^r(\mathbb{R})$, and then

$$||g||_r \leq C||g'||_p^{\breve{\theta}} ||g||_q^{1-\breve{\theta}}, \tag{4.10}$$

for $g \in W^{1,(q,p)}(\mathbb{R}_*)$ by the usual rescaling argument. In particular, (4.10) holds with $g \in W^{1,(q,p)}(\mathbb{R})$ (if $g \in C_0^\infty(\mathbb{R})$ and $q \geq 1$, this also follows from [6]).

Now, on the one hand, as in (iii), if $u \in \widetilde{W}^{1,(q,p)}_{\{-N,p-N\}}$ is radially symmetric, then $u(x) = f_u(|x|)$ with $f_u \in W^{1,1}_{loc}(0,\infty)$ and $\partial_\rho u(x) = f_u'(|x|)$, so that

$$||u||_{-N,q}^q = N\omega_N \int_0^\infty t^{-1}|f_u(t)|^q dt < \infty$$

and

$$||\partial_\rho u||_{p-N,p}^p = N\omega_N \int_0^\infty t^{p-1}|f_u'(t)|^p dt < \infty.$$

On the other hand, it is easy to check that if $g(s) := f_u(e^s)$, then $g \in W^{1,(q,p)}(\mathbb{R})$ with $||g||_q^q = \int_0^\infty t^{-1}|f_u(t)|^q dt$ and $||g'||_p^p = \int_0^\infty t^{p-1}|f_u'(t)|^p dt$. Therefore, (4.10) may be rewritten as $||u||_{-N,r} \leq C||\partial_\rho u||_{p-N,p}^{\breve{\theta}} ||u||_{-N,q}^{1-\breve{\theta}}$. □

Remark 4. Since $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ and $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ contain the same radially symmetric functions and the induced norms are the same, Theorem 4.1 is also true when $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is replaced by $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$.

Remark 5. In Theorem 4.1(ii), the admissible interval is empty if $\breve{\theta} \geq \theta_{-N}$, which can only happen if $r > q$. However, a careful examination of the proofs reveals that the subspace of (radially symmetric) functions with support in a ball $\overline{B}$ centered at 0 is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ if $c > -N$ when $b - p < -N < a$ and if $c \geq \breve{c}$ (even if $\breve{c} = -N$) when $a < -N < b - p$. For functions with support in $\mathbb{R}^N \backslash B$, the conditions ($c \leq \breve{c}$ if $b - p < -N < a$ and $c < -N$ if $a < -N < b - p$) follow by Kelvin transform. Details are left to the reader.

5 Embedding theorem when $1 \leq r \leq \min\{p, q\}$

We now extend Theorem 4.1 to the non-symmetric case when $1 \leq r \leq \min\{p, q\}$. To do this, we need the following refinement of Lemma 3.4(ii).

Lemma 5.1. *Let $a, b \in \mathbb{R}$ and $1 \leq r \leq p, q < \infty$. If $u \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$, then $v := [(|u|^r)_S]^{\frac{1}{r}} \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$. Furthermore, $||v||_{a,q} \leq ||u||_{a,q}$ and $||\partial_\rho v||_{b,p} \leq ||\partial_\rho u||_{b,p}$, so that $||v||_{\{a,b\},(q,p)} \leq ||u||_{\{a,b\},(q,p)}$.*

Proof. By Lemma 3.4(i), it is not restrictive to assume that $u \geq 0$. Since $v(x) = [f_{u^r}(|x|)]^{\frac{1}{r}}$ with $f_{u^r}(t) := (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} u^r(t\sigma)d\sigma$, it follows from $r \leq q$ and Hölder's inequality that

$$(v(x))^q \leq (N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} u^q(|x|\sigma)d\sigma.$$

Thus $||v||_{a,q} \leq ||u||_{a,q}$ is clear.

We now show that $\partial_\rho v \in L^p(\mathbb{R}^N; |x|^b dx)$ and prove the desired estimate. Formally, if $h := (f_{u^r})^{1/r}$, then $h' = (1/r)(f_{u^r})^{-1/r'} f'_{u^r}$ but, by the de la Vallée Poussin criterion ([29], [14, Lemma 1.2], [25, Corollary 8]), this formula holds and $h \in W^{1,1}_{loc}(0, \infty)$ if and only if $f_{u^r} \in W^{1,1}_{loc}(0, \infty)$ and $(f_{u^r})^{-1/r'} f'_{u^r} \in L^1_{loc}(0, \infty)$, with the understanding that $(f_{u^r})^{-1/r'} f'_{u^r} = 0$ when $f'_{u^r} = 0$, irrespective of whether $(f_{u^r})^{-1/r'}$ is defined. Since $f'_{u^r} = 0$ a.e. on $(f_{u^r})^{-1}(0)$, this amounts to defining $(f_{u^r})^{-1/r'} f'_{u^r} = 0$ on $(f_{u^r})^{-1}(0)$. That $(f_{u^r})^{-1/r'} f'_{u^r} \in L^1_{loc}(0, \infty)$ is verified below.

First, $u \in \widetilde{W}^{1,r}_{loc}$, since $r \leq p, q$. By Lemma 3.2, $u^r \in \widetilde{W}^{1,1}_{loc}$ (so that $f_{u^r} \in W^{1,1}_{loc}(0, \infty)$) and $\partial_\rho(u^r) = ru^{r-1}\partial_\rho u$. Upon replacing u by u^r in (3.2) and by Hölder's inequality, it follows that

$$\begin{aligned}|f'_{u^r}| &\leq r(f_{u^r})^{1/r'} \left((N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} |\partial_\rho u|^r d\sigma\right)^{1/r} \\ &\leq r(f_{u^r})^{1r'} \left((N\omega_N)^{-1} \int_{\mathbb{S}^{N-1}} |\partial_\rho u|^p d\sigma\right)^{1/p}.\end{aligned}$$

Since $(f_{u^r})^{-1/r'} f'_{u^r} = 0$ on $(f_{u^r})^{-1}(0)$, this yields

$$(f_{u^r})^{-1/r'}|f'_{u^r}| \leq r\left((N\omega_N)^{-1}\int_{\mathbb{S}^{N-1}} |\partial_\rho u|^p d\sigma\right)^{1/p} \in L^p_{loc}(0,\infty) \subset L^1_{loc}(0,\infty).$$

From the above, $h \in W^{1,1}_{loc}(0,\infty)$, $h' = (1/r)(f_{u^r})^{-1/r'} f'_{u^r}$ and, in addition,

$$|h'(t)| \leq \left((N\omega_N)^{-1}\int_{\mathbb{S}^{N-1}} |\partial_\rho u(t\sigma)|^p d\sigma\right)^{1/p}.$$

Since $\partial_\rho v(x) = h'(|x|)$ by Lemma 3.1, $|\partial_\rho v(x)|^p \leq (N\omega_N)^{-1}\int_{\mathbb{S}^{N-1}} |\partial_\rho u(|x|\sigma)|^p d\sigma$, so that $||\partial_\rho v||_{b,p} \leq ||\partial_\rho u||_{b,p}$. □

Theorem 5.1. *Suppose that $a, b, c \in \mathbb{R}$ and that $1 \leq r \leq p, q < \infty$. Then $\widetilde{W}^{1,(q,p)}_{\{a,b\}} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ (and hence $\widetilde{W}^{1,(q,p)}_{\{a,b\}} \hookrightarrow \widetilde{W}^{1,(r,p)}_{\{c,b\}}$) in the following cases:*

(i) *a and $b-p$ are on the same side of $-N$ (including $-N$), $(a+N)/q \neq (b-p+N)/p$ and c is in the open interval with endpoints c^0 and c^1;*

(ii) *a and $b-p$ are strictly on opposite sides of $-N$ (in this case, $(a+N)/q \neq (b-p+N)/p$) and c is in the open interval with endpoints c^0 and $-N$;*

(iii) *$r = q \leq p$, and $c = a$;*

(iv) *$r = p \leq q$, $a \leq -N$, $b-p < -N$ or $a \geq -N$, $b-p > -N$, and $c = b-p$.*

Proof. (i)-(ii) Set $v := [(|u|^r)_S]^{1/r}$. By Lemma 5.1, $v \in \widetilde{W}^{1,(q,p)}_{\{a,b\}}$ and

$$||v||_{\{a,b\},(q,p)} \leq ||u||_{\{a,b\},(q,p)}.$$

Thus, since v is radially symmetric, it follows from Theorem 4.1(i) and (ii) (where $\theta_c \geq \breve{\theta}$ holds, since $\breve{\theta} \leq 0$) that $||v||_{c,r} \leq C||u||_{\{a,b\},(q,p)}$, where $C > 0$ is independent of u. The conclusion follows from the remark that $||v||_{c,r} = ||u||_{c,r}$.

(iii) is trivial.

(iv) Argue as in (i)-(ii) above, now using Theorem 4.1(iii) with $r = p$. □

When $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ is replaced with the smaller space $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$, Theorem 5.1 coincides with Theorem 1.1 when $1 \leq r \leq p, q < \infty$. Indeed, $r \leq \min\{p,q\}$ implies $r \leq \min\{p^*, q\}$, so that $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$ holds for every c in the closed interval with endpoints c^0 and c^1; see Remark 1.

6 The Caffarelli-Kohn-Nirenberg Lemma and application

The reduction to the radially symmetric case in the previous section cannot be used when $r > \min\{p,q\}$. Consistent with the strategy outlined in the Introduction, this section is devoted to the formulation and proof of an embedding property for a direct complement of the subspace of radially symmetric functions.

It turns out to be necessary to confine attention to the space $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ (as opposed to $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$). This is because integrability conditions on all the first order partial derivatives are implicitly required. Although phrased differently and under less general conditions, Lemma 6.1 below is already contained in [6].

Lemma 6.1 (CKN Lemma). *Let $a, b, c \in \mathbb{R}$ and $1 \leq p, q, r < \infty$, and suppose that there exist $\delta \leq b/p$ and $\theta \in [0, 1]$ such that*

(i) $c/r = \theta\delta + (1-\theta)a/q$,

(ii) $(c+N)/r = \theta(b-p+N)/p + (1-\theta)(a+N)/q$,

(iii) $\theta r/p + (1-\theta)r/q \geq 1$.

Then

$$W_0 := \left\{u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) : u_S = 0\right\} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx), \tag{6.1}$$

and there exists a constant $C > 0$ such that

$$||u||_{c,r} \leq C||\nabla u||^{\theta}_{b,p}||u||^{(1-\theta)}_{a,q} \quad \text{for all } u \in W_0. \tag{6.2}$$

Proof. Of course, it suffices to prove (6.2). For $\tau > 0$, let Ω_τ denote the annulus $\{x \in \mathbb{R}^N : \tau < |x| < 2\tau\}$. Under the conditions (i) and (ii) of the lemma[5], it is shown in [6, pp. 262-263] that the *unweighted* inequality

$$\int_{\Omega_1} |u|^r \leq C\left(\int_{\Omega_1} |\nabla u|^p\right)^{\theta r/p}\left(\int_{\Omega_1} |u|^q\right)^{(1-\theta)r/q}, \tag{6.3}$$

holds for some constant C and every $u \in C^\infty_0(\mathbb{R}^N)$ such that $\int_{\Omega_1} u = 0$. The proof relies on the Gagliardo-Nirenberg and Sobolev inequalities. (Since a, b, c are not involved in (6.3), what matters is the relation $r^{-1} = \theta(p^{-1} - N^{-1} + \gamma) + (1-\theta)/q$ with $\gamma \geq 0$; that $\gamma = N^{-1}(b/p - \delta)$ from (i) and (ii) combined, is not relevant at this stage.)

From the geometric properties of Ω_1, proving the denseness of $C^\infty_0(\mathbb{R}^N)$ in the unweighted space $W^{1,(q,p)}(\Omega_1) := \{u \in L^q(\Omega_1) : \nabla u \in L^p(\Omega_1)\}$ is routine (see [3], [23] for more general results) and it is trivial that denseness remains true if, in both spaces, attention is confined to functions with mean 0 on Ω_1. Thus, (6.3) continues to hold for $u \in W^{1,(q,p)}(\Omega_1)$ such that $\int_{\Omega_1} u = 0$ and hence for $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ such that $\int_{\Omega_1} u = 0$ since, irrespective of a and b, the restrictions to Ω_1 of functions in $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ are obviously in $W^{1,(q,p)}(\Omega_1)$.

[5] None of the other assumptions in [6] is involved.

If $x \in \Omega_1$, then $|x|^a$, $|x|^b$ are bounded below and $|x|^c$ is bounded above. Thus, after changing C, (6.3) yields

$$\int_{\Omega_1} |x|^c |u|^r \leq C \left(\int_{\Omega_1} |x|^b |\nabla u|^p \right)^{\theta r/p} \left(\int_{\Omega_1} |x|^a |u|^q \right)^{(1-\theta)r/q}$$

for every $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ such that $\int_{\Omega_1} u = 0$. After rescaling, this and (ii) imply that with the same C, independent of τ,

$$\int_{\Omega_\tau} |x|^c |u|^r \leq C \left(\int_{\Omega_\tau} |x|^b |\nabla u|^p \right)^{\theta r/p} \left(\int_{\Omega_\tau} |x|^a |u|^q \right)^{(1-\theta)r/q}, \tag{6.4}$$

for every $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ such that $\int_{\Omega_\tau} u = 0$. In particular, (6.4) holds for every $\tau > 0$ and every $u \in W_0$ defined in (6.1).

It has also been observed, in [6, p. 268], that if $k \in \mathbb{Z}$ and $A_k, B_k \geq 0$ and if $\alpha, \beta \geq 0$ satisfy $\alpha + \beta \geq 1$, then

$$\sum_{k\in\mathbb{Z}} A_k^\alpha B_k^\beta \leq \left(\sum_{k\in\mathbb{Z}} A_k \right)^\alpha \left(\sum_{k\in\mathbb{Z}} B_k \right)^\beta, \tag{6.5}$$

where the first (second) factor on the right is 1 when $\alpha = 0$ ($\beta = 0$). Thus, when condition (iii) holds, (6.2) follows from (6.5) and (6.4) with $\tau = 2^k$, $k \in \mathbb{Z}$.

There is a clearer and more convenient formulation of Lemma 6.1.

Corollary 6.1. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p, q, r < \infty$.*

(i) *If $(a+N)/q \neq (b-p+N)/p$ and c is in the closed interval with endpoints c^0 and c^1, then $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if the following conditions hold (with θ_c given by* (1.4)*);*
   - (i-1) *Either $r = q$ and $c = c^0$ $(= a)$, or $c \neq c^0$ and $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$.*
   - (i-2) $\theta_c r/p + (1-\theta_c)r/q \geq 1$.

(ii) *If $(a+N)/q = (b-p+N)/p$ and $c = c^0$ $(= c^1)$, then $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if $\min\{p, q\} \leq r \leq \max\{p^*, q\}$. Furthermore, there is a constant $C > 0$ such that for every $u \in W_0$,*

$$||u||_{c,r} \leq C||\nabla u||_{b,p} \quad \text{if } p \leq r \leq p^*, \tag{6.6}$$

$$||u||_{c,r} \leq C||\nabla u||^{\theta}_{b,p} ||u||^{1-\theta}_{a,q} \text{ if } r = q \text{ or if } p \neq q \text{ and } \min\{p,q\} \leq r \leq \max\{p,q\}, \tag{6.7}$$

where $\theta := p(r-q)/(r(p-q))$ if $p \neq q$ and $\theta = 0$ if $p = q = r$.

Proof. (i) Suppose $(a+N)/q \neq (b-p+N)/p$. By (1.6), condition (ii) of Lemma 6.1 holds if and only if $\theta = \theta_c$. If $r = q$ and $c = c^0 = a$, so that $\theta_{c^0} = 0$, condition (i) of Lemma 6.1 holds with any δ. On the other hand, if $c \neq c^0$, then $\theta_c \in (0, 1]$ and condition (i) of Lemma 6.1 holds with

$$\delta = \frac{b-p+N}{p} + \frac{1-\theta_c}{\theta_c}\frac{N}{q} - \frac{1}{\theta_c}\frac{N}{r}.$$

Hence, $\delta \leq b/p$ (as required in Lemma 6.1) if and only if $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$. Thus, $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if also $(\theta_c r)/p + ((1-\theta_c)r)/q \geq 1$.

(ii) Suppose $(a+N)/q = (b-p+N)/p$ and let $c = c^0 = c^1$. Then condition (ii) of Lemma 6.1 holds with any $\theta \in [0, 1]$. Thus, it only remains to show that if $\min\{p, q\} \leq r \leq \max\{p^*, q\}$, then $\delta \leq b/p$ and $\theta \in [0, 1]$ can be chosen such that $c/r = \theta\delta + (1-\theta)a/q$ and that $\theta r/p + (1-\theta)r/q \geq 1$. If so, all the requirements of Lemma 6.1 are satisfied, whence $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$.

Observe that $\min\{p, q\} \leq r \leq \max\{p^*, q\}$ if and only if either $p \leq r \leq p^*$ or $p \neq q$ and $\min\{p, q\} \leq r \leq \max\{p, q\}$ (possibly both). If $p \leq r \leq p^*$, we may choose $\delta = c/r = b/p + N/p - N/r - 1 \leq b/p$ (since $r \leq p^*$) and $\theta = 1$, so that $\theta r/p + (1-\theta)r/q = r/p \geq 1$. Then (6.6) follows from (6.2).

If now $p \neq q$ and $\min\{p, q\} \leq r \leq \max\{p, q\}$, let θ be defined by $1/r = \theta/p + (1-\theta)/q$, i.e., $\theta = p(q-r)/(r(q-p))$. Obviously, $\theta r/p + (1-\theta)r/q = 1$, but it must be checked that $c/r = \theta\delta + (1-\theta)a/q$ for some $\delta \leq b/p$. Since $c = c^0$ and $(a+N)/q = (b-p+N)/p$, a straightforward verification shows that $c/r = \theta\delta + (1-\theta)a/q$ with $\delta = b/p - 1$. Thus, (6.7) follows from (6.2). Of course, (6.7) remains true with $\theta = 0$ if $p = q = r$, for then $c = c^0 = a$. □

While Corollary 6.1 gives sufficient conditions for $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$, necessary and sufficient conditions for $W_{rad} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ are listed in Theorem 4.1, where W_{rad} is the subspace of radially symmetric functions in $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$. Thus, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ can be inferred from the remark that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) = W_{rad} \oplus W_0$ together with the following obvious lemma.

Lemma 6.2. *Let X and Y be normed spaces and X_1 and X_2 be two subspaces of X such that $X = X_1 \oplus X_2$ (topological direct sum). Then, $X \hookrightarrow Y$ if and only if $X_i \hookrightarrow Y$, $i = 1, 2$.*

The relation $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) = W_{rad} \oplus W_0$ reflects the equality $u = u_S + (u - u_S)$ with u_S the radial symmetrization of u, that is, $u_S(x) = f_u(|x|)$ with f_u given by (3.1). Then, $u_S \in W_{rad}$ and $||u_S||_{\{a,b\},(q,p)} \leq ||u||_{\{a,b\},(q,p)} \leq ||u||_{a,q} + ||\nabla u||_{b,p}$ by Lemma 3.4(ii), which proves the continuity of $u \mapsto u_S$ ($W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ and $\widetilde{W}^{1,(q,p)}_{\{a,b\}}$ contain the same radially symmetric functions and the induced norms are the same). That $u - u_S \in W_0$ and $W_{rad} \cap W_0 = \{0\}$ is trivial.

The principle outlined above is simple, but cannot always be implemented in a straightforward way, primarily because condition (i-2) in Corollary 6.1 is far from being necessary. The case $r < \min\{p, q\}$ (Section 5) is one, but not the only, example when this condition fails but the embedding holds under other assumptions. In practice, this means that Corollary 6.1 alone does not always suffice to prove that $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ under optimal conditions about c. Other arguments, most notably Theorem 5.1(but with other parameters), are needed; see the proofs of Lemma 7.1 and of Theorem 9.1.

7 Embedding theorem when $p < r \leq q$

In this section, we discuss the embedding $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ when $p < r \leq q$. Together with Theorem 5.1 (when $1 \leq r \leq \min\{p, q\}$), this settles the issue when $1 \leq r \leq q$.

Theorem 7.1. *Suppose that $a, b, c \in \mathbb{R}$ and $1 \leq p < r \leq q < \infty$. Then $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ (and hence $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow W^{1,(r,p)}_{\{c,b\}}(\mathbb{R}^N_*)$) in the following cases:*

(i) *a and $b - p$ are on the same side of $-N$ (including $-N$), $(a + N)/q \neq (b - p + N)/p$, c is in the open interval with endpoints c^0 and c^1, and $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$;*

(ii) *a and $b - p$ are strictly on opposite sides of $-N$ (hence $(a + N)/q \neq (b - p + N)/p$), c is in the open interval with endpoints c^0 and $-N$ and $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$;*

(iii) *$r = q$ and $c = a$;*

(iv) *$r \leq p^*$, $a \leq -N$ and $b - p < -N$ or $a \geq -N$ and $b - p > -N$, $c = c^1$;*

(v) *$(a + N)/q = (b - p + N)/p \neq 0$ and $c = c^1$ $(= c^0)$.*

7.1 Proof of parts (i) and (ii). In this subsection, we assume that $(a + N)/q \neq (b - p + N)/p$. Let $0 \leq \bar{\theta} \leq 1$ denote the largest value of $\theta \in [0, 1]$ such that $\theta(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$; i.e., since $r \leq q$ is assumed,

$$\bar{\theta} = \begin{cases} 1 & \text{if } r \leq p^*, \\ (r^{-1} - q^{-1})(p^{-1} - N^{-1} - q^{-1})^{-1} < 1 & \text{if } p < N \text{ and } r > p^*. \end{cases} \tag{7.1}$$

Denote the corresponding value of c by $\bar{c}$, namely,

$$\bar{c} := \bar{\theta} c^1 + (1 - \bar{\theta}) c^0, \tag{7.2}$$

so that $\bar{\theta} = \theta_{\bar{c}}$; see (1.4). Since $(a + N)/q \neq (b - p + N)/p$, the points c^0 and $\bar{c}$ coincide if and only if $\bar{\theta} = 0$, i.e., $r = q > p^*$, and then $\bar{c} = c^0 = a$ by (1.3).

Lemma 7.1. *If* $(a+N)/q \neq (b-p+N)/p$ *and* $\bar{c}$ *is given by* (7.1) *and* (7.2), *the subspace* W_0 *of* $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ *in* (6.1) *is continuously embedded into* $L^r(\mathbb{R}^N; |x|^c dx)$ *for every* c *in the interval* J *with endpoints* $\bar{c}$ *(included) and* c^0 *(not included, unless* $r = q$*).*

Proof. If $r = q$, the embedding $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for $c \in J$ follows from Corollary 6.1(i), since $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1} = 0$ by definition of J and $\theta_c r/p + (1 - \theta_c) \geq 1$ irrespective of $\theta_c \in [0, 1]$ since $r > p$ by assumption.

From now on, suppose that $r < q$ and $c^0 \notin J$. Observe that the set

$$\{c \in \mathbb{R} : W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)\}$$

is always an interval (in this statement, W_0 may be replaced with any normed space of measurable functions on $\mathbb{R}^N$). Thus, to prove that this interval contains J, it suffices to show that $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ when $c = \bar{c}$ and when $c \in J$ is arbitrarily close to c^0.

The embedding $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^{\bar{c}} dx)$ follows once again from Corollary 6.1(i), since $\bar{\theta}(p^{-1} - N^{-1} - q^{-1}) < r^{-1} - q^{-1}$ by definition of $\bar{\theta}$, and $\bar{\theta} r/p + (1 - \bar{\theta}) r/q \geq 1$ by a simple calculation (obvious if $\bar{\theta} = 1$; otherwise, use $p < N$ and $q \geq r > p^*$).

To complete the proof, assume that $c \in J$ is close to c^0, so that $\theta_c > 0$ is small. If so, condition (i-2) of Corollary 6.1 fails when $r < q$ and this corollary cannot be used. Nonetheless, using another argument, we prove that in this case, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$, a stronger result than actually needed.

Define $\tilde{c} := ((b-p)(q-r) + a(r-p))/(q-p)$ and note that by (1.6), $\theta_{\tilde{c}} = p(q-r)/(r(q-p)) \in (0, 1)$ (recall $p < r < q$), so that $\tilde{c} \neq c^0$. Both the open intervals with endpoints c^0 and either $\tilde{c} \neq c^0$ or $\bar{c} \neq c^0$ consist of convex combinations of c^0 and c^1. Thus, they intersect along a nonempty open interval having c^0 as an endpoint. As a result, it suffices to show that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c close enough to c^0 in the open interval $\widetilde{J}$ with endpoints c^0 and $\tilde{c}$.

Given any such c, set $\sigma := c - a(r-p)/(q-p)$ and $\gamma := (q-r)/(q-p) \in (0, 1)$. If $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$, write $|x|^c |u|^r = |x|^\sigma |u|^{p\gamma} |x|^{c-\sigma} |u|^{r-p\gamma}$ and use Hölder's inequality to obtain

$$\int_{\mathbb{R}^N} |x|^c |u|^r dx \leq \left(\int_{\mathbb{R}^N} |x|^{\sigma/\gamma} |u|^p dx\right)^\gamma \left(\int_{\mathbb{R}^N} |x|^a |u|^q dx\right)^{1-\gamma}. \tag{7.3}$$

By Theorem 5.1(i) and (ii) with c replaced with d and r replaced with p (since $p = \min\{p, q\}$), there is a nonempty open interval I with one endpoint $d^0 := p(a+N)/q - N$ and another endpoint between d^0 and $d^1 := b - p$ (specifically, $b - p$ or $-N$) such that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^p(\mathbb{R}^N; |x|^d dx)$ when $d \in I$.

When c is moved from c^0 to $\tilde{c}$, $d := \sigma/\gamma = (c(q-p) - a(r-p))/(q-r)$ moves from d^0 to $b-p$. Therefore, $d \in I$ for c in some nonempty open subinterval $\widetilde{I}$ of $\widetilde{J}$ having c^0 as an endpoint. From the above, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^p(\mathbb{R}^N; |x|^d dx)$ when $c \in \widetilde{I}$. By Corollary 2.1, this embedding is accounted for by a multiplicative inequality of the type (2.3) (with c replaced with d and r replaced with p), namely, $||u||_{d,p} \leq C||\nabla u||^{\theta_d}_{b,p}||u||^{1-\theta_d}_{a,q}$ with $\theta_d := (d-d^0)/(d^1-d^0)$. Since $d = \sigma/\gamma$, the substitution into (7.3) yields, when $c \in \widetilde{I}$, the inequality $||u||_{c,r} \leq C||\nabla u||^{\nu}_{b,p}||u||^{1-\nu}_{a,q}$ for $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$, where $\nu = (p\gamma\theta_d)/r \in (0,1)$. In turn, this implies a corresponding additive (i.e., embedding) inequality. □

Proof of part (i). If $\bar{\theta} = 0$ in (7.1) (so that $r = q > p^*$), no c in the open interval with endpoints $c^0 = a$ and $c^1 = (q(b-p+N))/p - N$ satisfies $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$, since $\theta_c > 0$ and $\theta_c \leq \bar{\theta} = 0$ are contradictory. Thus, there is nothing to prove.

Lemma 7.1 ensures that if $0 < \bar{\theta} \leq 1$, so that $\bar{c} \neq c^0$, then $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the half-open interval J with endpoints $\bar{c}$ (included) and c^0 (not included, unless $r = q$). Meanwhile, by Theorem 4.1(i), $W_{rad} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the open interval with endpoints c^0 and c^1 (since $\breve{\theta} \leq 0$ when $r \leq q$). Thus, by Lemma 6.2, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the intersection of these two intervals. By definition of $\bar{\theta}$, this intersection is the set of those c in the open interval with endpoints c^0 and c^1 such that $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$. □

Proof of part (ii). Again, it is not restrictive to assume that $0 < \bar{\theta} \leq 1$. By Theorem 4.1(ii), $W_{rad} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the open interval with endpoints c^0 and $-N$ (since $\breve{\theta} \leq 0$) and, by Lemma 7.1, $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the half-open interval with endpoints c^0 and $\bar{c}$ ($\neq c^0$ since $\bar{\theta} > 0$), including $\bar{c}$ but not c^0. Hence, by Lemma 6.2, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the intersection of these two intervals, which is the set of those c in the open interval with endpoints c^0 and $-N$ such that $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$. □

7.2 Proof of parts (iii), (iv) and (v). Since part (iii) is obvious, it remains only to prove (iv) and (v). The proof of (iv) in the case $(a+N)/q \neq (b-p+N)/p$ (so that $c^1 \neq c^0$) follows from Lemma 6.2, Theorem 4.1(iii), and Corollary 6.1(i) (recall $\theta_{c^1} = 1$ and $p < r \leq p^*$). The use of Corollary 6.1(ii) instead of Corollary 6.1(i) yields (v), which in turn implies (iv) when $(a+N)/q = (b-p+N)/p$.

8 Embedding theorem when $r > q \geq 1$ and $r \geq p$

Throughout this section, we assume that $r > q \geq 1$ and $r \geq p$. It follows from Theorem 2.2 and Theorem 2.1(i) that if also ($p < N$ and) $r > p^*$, then $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ is not continuously embedded into any $L^r(\mathbb{R}^N; |x|^c dx)$. Thus, it is not restrictive to confine attention to the case $r \leq p^*$. Since $q < r$, it follows that $q < p^*$. If $(a+N)/q \neq (b-p+N)/p$, the combination $r > q$ and $q < p^*$ (i.e., the fact that $q^{-1} + N^{-1} - p^{-1} > 0$) shows that the necessary condition for the embedding $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ given in Theorem 2.2(i) is $\theta_c \geq \bar{\theta} > 0$, where

$$\bar{\theta} = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right)^{-1}. \tag{8.1}$$

This formula is the same as in (7.1), but now $\bar{\theta}$ is the *smallest* value of $\theta \in [0, 1]$ such that $\theta(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$. Note that indeed $\bar{\theta} \leq 1$, because $r \leq p^*$ (and $\bar{\theta} = 1$ if and only if $r = p^*$). Equivalently, c must belong to the closed interval with endpoints $\bar{c} := \bar{\theta}c^1 + (1-\bar{\theta})c^0$ (as in (7.2)) and c^1.

In addition, $p \leq r < \infty$ ensures that the subspace W_0 in (6.1) is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$ for c in the closed interval with endpoints $\bar{c}$ and c^1. This follows from part (i) of Corollary 6.1 since $r\theta_c/p + r(1-\theta_c)/q \geq 1$ irrespective of $\theta_c \in [0, 1]$. We record this result for future reference.

Lemma 8.1. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p, q, r < \infty$ be such that $r > q$, $r \geq p$, and $(a+N)/q \neq (b-p+N)/p$. If $\bar{c}$ is given by* (8.1) *and* (7.2)*, then $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the closed interval with endpoints $\bar{c}$ and c^1.*

Lemma 8.2. *Let $a, b \in \mathbb{R}$ and $1 \leq p, r < \infty$, $1 \leq q < r \leq p^*$ be such that $(a+N)/q \neq (b-p+N)/p$. If $\breve{\theta} := (1-q/r)(q/p'+1)^{-1}$ and $\bar{\theta}$ is given by* (8.1)*, then $0 < \breve{\theta} \leq \bar{\theta}$.*

Proof. The result follows from an explicit calculation using the fact that $q < p^*$. □

Theorem 8.1. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p, q, r < \infty$, $q < r \leq p^*$, and $r \geq p$. Then $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ (and hence $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow W^{1,(r,p)}_{\{c,b\}}(\mathbb{R}^N_*)$) in the following cases:*

(i) *a and $b-p$ are on the same side of $-N$ (including $-N$), $(a+N)/q \neq (b-p+N)/p$, c is in the open interval with endpoints c^0 and c^1, and $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$;*

(ii) *a and $b-p$ are strictly on opposite sides of $-N$ (hence $(a+N)/q \neq (b-p+N)/p$), c is in the open interval with endpoints c^0 and $-N$, and $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$;*

(iii) *either* $a \leq -N$ *and* $b-p < -N$ *or* $a \geq -N$ *and* $b-p > -N$*, and* $c = c^1$*;*
(iv) $a = -N$, $b = p - N$ *and* $c = c^0$ $(= c^1 = -N)$*.*

Proof. (i) Since $\breve{\theta} \leq \bar{\theta}$ by Lemma 8.2, it follows from Theorem 4.1(i) that $W_{rad} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for every c in the half-open interval J with endpoints $\bar{c}$ (included) and c^1 (not included). Therefore, by Lemmas 8.1 and 6.2, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for $c \in J$. By definition of $\bar{\theta}$, it is plain that J consists of those c in the open interval with endpoints c^0 and c^1 such that $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$.

(ii) Once again by Lemma 8.2, $\breve{\theta} \leq \bar{\theta}$ while $\theta_c \geq \bar{\theta}$ for every c satisfying the specified conditions. Thus, the result follows from Theorem 4.1(ii), Lemma 8.1, and Lemma 6.2.

(iii) The result follows from Corollary 6.1(1) and (ii), Theorem 4.1(iii), and Lemma 6.2.

(iv) The result follows from Corollary 6.1(ii), Theorem 4.1(v) and Lemma 6.2. □

9 Embedding theorem when $1 \leq q < r < p$

If $q < r < p$, then $r < p^*$ and $q < p^*$. Thus, as in the previous section, $\bar{\theta}$ in (8.1) is the smallest $\theta \in [0, 1]$ such that $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$. Clearly, $\bar{\theta} \in (0, 1)$.

Theorem 9.1. *Suppose that* $a, b, c \in \mathbb{R}$ *and* $1 \leq q < r < p < \infty$*. Then* $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ *(and hence* $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow W^{1,(r,p)}_{\{c,b\}}(\mathbb{R}^N_*)$*) in the following cases:*

(i) a *and* $b - p$ *are on the same side of* $-N$ *(including* $-N$*),* $(a+N)/q \neq (b-p+N)/p$*,* c *is in the open interval with endpoints* c^0 *and* c^1*, and* $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$*;*
(ii) a *and* $b - p$ *are strictly on opposite sides of* $-N$ *(hence* $(a+N)/q \neq (b-p+N)/p$*),* c *is in the open interval with endpoints* c^0 *and* $-N$*, and* $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$*;*
(iii) $(a+N)/q = (b-p+N)/p \neq 0$ *and* $c = c^0 (= c^1)$*;*
(iv) $a = -N, b = p - N$ *and* $c = c^0$ $(= c^1 = -N)$*.*

Proof. (i) Let $\tilde{c} := ((b-p)(q-r)+a(r-p))/(q-p)$ as in the proof of Lemma 7.1, so that by (1.4), $\tilde{\theta} := \theta_{\tilde{c}} = p(q-r)/(r(q-p)) \in (0,1)$. If c is in the half-open interval with endpoints c^0 (not included) and $\tilde{c}$ (included), then $0 < \theta_c \leq \tilde{\theta}$. A routine verification shows that condition (i-2) of Corollary 6.1 holds. Another simple verification reveals that condition (i-1) holds if and only if $\theta_c \geq \bar{\theta}$ and $\tilde{\theta} > \bar{\theta}$. Thus, by Corollary 6.1, $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if c is in the closed interval K with endpoints $\bar{c}$ in (7.2) and $\tilde{c}$.

By Theorem 4.1(i), $W_{rad} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if c is in the half-open interval with endpoints $\breve{c} := \breve{\theta}c^1 + (1-\breve{\theta})c^0$ (included) and c^1 (not included) and, by Lemma 8.2, this interval contains K. Thus, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ when $c \in K$, by Lemma 6.2.

This is not yet the desired result, but $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^{\tilde{c}} dx)$, since $\tilde{c} \in K$, so that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow W^{1,(r,p)}_{\{\tilde{c},b\}}(\mathbb{R}^N_*)$. Now $(\tilde{c}+N)/r \neq (b-p+N)/p$ (because $\tilde{\theta} < 1$), and $\tilde{c}$ and $b-p$ are on the same side of $-N$ (because the same is true of a and $b-p$). Therefore, Theorem 5.1(i) with a and q replaced with $\tilde{c}$ and r, respectively (use $r = \min\{p, r\}$), gives $W^{1,(r,p)}_{\{\tilde{c},b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the open interval with endpoints $\tilde{c}$ and c^1.

Altogether, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ for c in the union of K with the open interval with endpoints $\tilde{c}$ and c^1, that is, the half-open interval with endpoints $\bar{c}$ and c^1. By definition of $\bar{\theta}$, this interval is the set of those c (included) in the open interval with endpoints c^0 and c^1 (not included) such that $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$.

(ii) If $\theta_{-N} \leq \bar{\theta}$, there is nothing to prove, since no c satisfies the required conditions. Suppose then that $\theta_{-N} > \bar{\theta}$. On the one hand, as in the proof of (i) above, $W_0 \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if c is in the (nonempty) closed interval K with endpoints $\bar{c}$ and $\tilde{c}$. On the other hand, since $\breve{\theta} \leq \bar{\theta}$ by Lemma 8.2 and $\bar{\theta} < \theta_{-N}$, it follows from part (ii) of Theorem 4.1 that $W_{rad} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if c is in the half-open interval $\breve{J}$ with endpoints $\breve{c} := \breve{\theta}c^1 + (1-\breve{\theta})c^0$ (included) and $-N$ (not included). Therefore, by Lemma 6.2, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ when $c \in K \cap \breve{J}$.

Since $\breve{\theta} \leq \bar{\theta} < \theta_{-N}$, it follows that $\bar{c} \in K \cap \breve{J}$ is an endpoint of $K \cap \breve{J}$. Since also $\bar{\theta} < \tilde{\theta}$, the second endpoint can only be $-N$ or $\tilde{c}$. If $\theta_{-N} \leq \tilde{\theta}$, then $K \cap \breve{J}$ is the half-open interval with endpoints $\bar{c}$ (included) and $-N$ (not included). If $\theta_{-N} > \tilde{\theta}$, then $K \cap \breve{J}$ is the closed interval with endpoints $\bar{c}$ and $\tilde{c}$. Yet, once again, $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ when c is in the half-open interval with endpoints $\bar{c}$ (included) and $-N$ (not included), as shown below. This proves the desired result since, by definition of $\bar{\theta}$, this interval consists of those c in the open interval with endpoints c^0 and $-N$ such that $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$.

To complete the proof, note that by (1.6), $\theta_{-N} > \tilde{\theta}$ implies that $(\tilde{c}+N)/r$ and $(a+N)/q$ and hence also $\tilde{c}+N$ and $a+N$ have the same (nonzero) sign, so that $\tilde{c}$ and $b-p$ are strictly on opposite sides of $-N$. As in the proof of (i) above, but now by Theorem 5.1(ii) with a and q replaced with $\tilde{c}$ and r, respectively, it follows that $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ when c is in the union of the closed interval with endpoints $\bar{c}$ and $\tilde{c}$ with the open interval with endpoints $\tilde{c}$ and $-N$, that is, the half-open interval with endpoints $\bar{c}$ (included) and $-N$ (not included), as claimed.

(iii) This follows Theorem 4.1(iv), Corollary 6.1(ii), and Lemma 6.2.

(iv) The argument is the same as in the proof of Theorem 8.1(iv). □

10 Generalized CKN inequalities

If $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$, then $r \leq \max\{p^*, q\}$ by Theorem 2.2, and c is in the closed interval with endpoints c^0 and c^1 by Theorem 2.1(i). It was shown in Corollary 2.1 that if, in addition, $(a+N)/q \neq (b-p+N)/p$, then the embedding is accounted for by the multiplicative inequality

$$(10.1) \qquad ||u||_{c,r} \leq C||\nabla u||^{\theta_c}_{b,p}||u||^{1-\theta_c}_{a,q},$$

with θ_c given by (1.4). When $a, b, c > -N$ and $u \in C^\infty_0(\mathbb{R}^N)$, such inequalities coincide with some of the CKN inequalities proved in [6].

Using variational methods, Catrina and Costa [7] (see also [8]) recently obtained (10.1) with best constant C in the case $p = q = r = 2$, $c = (a+b)/2 - 1$ (so that $\theta_c = 1/2$) and $u \in C^\infty_0(\mathbb{R}^N_*)$ under no a priori limitation on a and b. Their result does not imply (10.1) for $u \in W^{1,(2,2)}_{\{a,b\}}(\mathbb{R}^N_*)$ or that the best constant is the same; see Subsection 11.3.

The CKN inequalities also incorporate the limiting case $(a+N)/q = (b-p+N)/p$ (when θ_c in (1.4) is not defined). It is therefore natural to ask whether the embedding $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ can be characterized by similar multiplicative inequalities when $(a+N)/q = (b-p+N)/p$, so that $c = c^0$ $(= c^1)$ is the only possible value.

The next lemma is, roughly speaking, a "multiplicative" analog of Lemma 6.2. Recall the definition (6.1) of the subspace W_0, as well as the shorthand W_{rad} for the subspace of radially symmetric functions of $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$.

Lemma 10.1. *Let $a, b, c \in \mathbb{R}$ and $1 \leq p, q, r < \infty$. If for some $\theta \in [0,1]$, $||u||_{c,r} \leq C||\nabla u||^{\theta}_{b,p}||u||^{1-\theta}_{a,q}$ for every $u \in W_{rad} \cup W_0$, then $||u||_{c,r} \leq C||\nabla u||^{\theta}_{b,p}||u||^{1-\theta}_{a,q}$ (with a possibly different C) for every $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$.*

Proof. Let $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$. Then $u = u_S + (u - u_S)$, where $u_S \in W_{rad}$ and $u - u_S \in W_0$. By Lemma 3.4(ii), $||u_S||_{a,q} \leq ||u||_{a,q}$ and $||\partial_\rho u_S||_{b,p} \leq ||\partial_\rho u||_{b,p} \leq ||\nabla u||_{b,p}$. Becasue of the fact that $||\partial_\rho u_S||_{b,p} = ||\nabla u_S||_{b,p}$, the inequality $||u_S||_{c,r} \leq C||\nabla u_S||^{\theta}_{b,p}||u_S||^{1-\theta}_{a,q}$ yields $||u_S||_{c,r} \leq C||\nabla u||^{\theta}_{b,p}||u||^{1-\theta}_{a,q}$.

Also, $||u - u_S||_{a,q} \leq ||u||_{a,q} + ||u_S||_{a,q} \leq 2||u||_{a,q}$ and $||\nabla(u - u_S)||_{b,p} \leq ||\nabla u||_{b,p} + ||\nabla u_S||_{b,p} \leq M||\nabla u||_{b,p}$ for some $M > 0$ independent of u. Thus, $||u - u_S||_{c,r} \leq C||\nabla u||^{\theta}_{b,p}||u||^{1-\theta}_{a,q}$. As a result, $||u||_{c,r} \leq ||u_S||_{c,r} + ||u - u_S||_{c,r} \leq 2C||\nabla u||^{\theta}_{b,p}||u||^{1-\theta}_{a,q}$. □

Theorem 10.1. *Let $a, b \in \mathbb{R}$ and $1 \leq p, q, r < \infty$ be such that $(a+N)/q = (b-p+N)/p \neq 0$ and let $c = c^0 = c^1$.*

(i) *If $p \leq r \leq p^*$, there is a constant $C > 0$ such that*

$$||u||_{c^1,r} \leq C||\nabla u||_{b,p}, \quad \text{for all } u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*). \tag{10.2}$$

(ii) *If $r = p = q$ or if $p \neq q$ and $\min\{p, q\} \leq r \leq \max\{p, q\}$, there is a constant $C > 0$ such that*

$$||u||_{c^1,r} \leq C||\nabla u||^{\theta}_{b,p}||u||^{1-\theta}_{a,q}, \quad \text{for all } u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*), \tag{10.3}$$

where $\theta = 0$ if $r = p = q$ and $\theta = p(r-q)/(r(p-q))$ if $p \neq q$.

Proof. (i) This follows from Lemma 10.1 together with (4.6) and (6.6).
(ii) This follows from Lemma 10.1 together with (4.7) and (6.7). □

Remark 6. It follows from the estimates (10.2) and (10.3) that if $(a+N)/q = (b-p+N)/p \neq 0$ and $p \leq r \leq \min\{p^*, \max\{p, q\}\}$, then

$$||u||_{c^1,r} \leq C||\nabla u||^{\theta}_{b,p}||u||^{1-\theta}_{a,q} \tag{10.4}$$

for $u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ with $\theta = 1$ and $\theta = \underline{\theta}$, where

$$\underline{\theta} = \begin{cases} p(r-q)/r(p-q) & \text{if } p \neq q, \\ 0 & \text{if } p = r = q. \end{cases}$$

Hence, (10.4) holds with $\theta \in [\underline{\theta}, 1]$, and so θ is not unique if $r > p \neq q$ or if $r = p = q$. This is actually trivial if $r = q \geq p$ (because (10.3) is trivial), but not in the other cases, namely, $p < r < q \leq p^*$ or $p < N$ and $p < r \leq p^* < q$.

Clearly, (10.2) is an N-dimensional weighted Hardy-type inequality, apparently new when $q \neq p$. When $q = p$, so that $a = b - p \neq -N$, it is proved in [22, p. 309] and was obtained earlier for $u \in C^{\infty}_0(\mathbb{R}^N_*)$ by Gatto, Gutiérrez and Wheeden [9], who showed that $p \leq r \leq p^*$ is already necessary in that setting. A number of special cases of (10.2) for various classes of smooth functions with compact support can be found in both the older and the recent literature ([10], [13], [28], among others). Inequality (10.3), which is meaningless when $q = p$, seems to be known only if $a, b, c > -N$ and $u \in C_0^\infty(\mathbb{R}^N)$, when it is one of the CKN inequalities.

By Corollary 2.1, the inequality (sharper than (10.1))

$$||u||_{c,r} \leq C||\partial_\rho u||_{b,p}^{\theta_c}||u||_{a,q}^{1-\theta_c}, \quad \text{for all } u \in \widetilde{W}_{\{a,b\}}^{(q,p)},$$

holds if $(a+N)/q \neq (b-p+N)/p$, c is in the closed interval with endpoints c^0 and c^1, and $\widetilde{W}_{\{a,b\}}^{(q,p)} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$. Necessary and sufficient conditions for this embedding were given in Theorem 5.1 for the case $r \leq \min\{p, q\}$, where it is also shown that $\widetilde{W}_{\{a,b\}}^{(q,p)} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if $(a+N)/q = (b-p+N)/p \neq 0$, $r = p\ (\leq q)$ and $c = c^0 = c^1$. It follows from Theorem 4.1(iii) and Lemma 5.1 that in this case,

$$||u||_{b-p,p} \leq C||\partial_\rho u||_{b,p}, \quad \text{for all } u \in \widetilde{W}_{\{a,b\}}^{(q,p)}.$$

The only case in which the embedding $W_{\{a,b\}}^{1,(q,p)}(\mathbb{R}_*^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ is true but *not* equivalent to a multiplicative inequality arises in Theorem 1.1(vi), when $N \geq 2$ (if u is radially symmetric, or $N = 1$; see (4.8)).

Theorem 10.2. *If $q < r \leq p^*$, then $W_{\{-N,p-N\}}^{1,(q,p)}(\mathbb{R}_*^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^{-N} dx)$ for all N but if $N \geq 2$, then $||u||_{-N,r} \leq C||\nabla u||_{p-N,p}^{\theta}||u||_{-N,q}^{1-\theta}$ fails to hold for any $C > 0$ and $\theta \in [0, 1]$.*

Proof. The embedding statement is Theorem 1.1(vi). The inequality can hold only if $\theta = \breve{\theta} := (1 - q/r)(q/p' + 1)^{-1}$. This follows by choosing $u(x) = g(\ln |x|)$ with $g \in C_0^\infty(\mathbb{R})$ and reversing the steps of the proof of Theorem 4.1(v) (by [6], (4.10) cannot hold with $\theta \neq \breve{\theta}$ when $g \in C_0^\infty(\mathbb{R})$ is arbitrary).

Next, the method of proof of Theorem 2.2 with $a = c = -N$ and $b = p - N$ shows that if $||u||_{-N,r} \leq C||\nabla u||_{p-N,p}^{\theta}||u||_{-N,q}^{1-\theta}$, then $\theta(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$. Upon substituting the only possible value $\theta = \breve{\theta}$, a short calculation yields $q(N-1) \geq r(N-1)$. If $N \geq 2$, this implies that $q \geq r$, which contradicts the fact that $q < r \leq p^*$. □

It is easily verified that, consistent with Theorem 10.2 and its proof, when $a = b - p = c = -N$, $\theta = \breve{\theta}$, and $N \geq 2$, Lemma 10.1 is not applicable because condition (i) of Lemma 6.1 fails, so that (6.2) cannot be used.

11 Examples

11.1 Embedding of unweighted spaces into $L^r(\mathbb{R}^N; |x|^c dx)$. We spell out the special case of Theorem 1.1 when $a = b = 0$. It is noteworthy that $W^{1,(q,p)}(\mathbb{R}_*^N) = W^{1,(q,p)}(\mathbb{R}^N) = \{u \in L^q(\mathbb{R}^N) : \nabla u \in (L^p(\mathbb{R}^N))^N\}$ if $N \geq 2$, with the same norm; see Remark 7 below. At any rate, if $a = b = 0$, then θ_c in (1.4) is defined if and only if $p^{-1} - N^{-1} - q^{-1} \neq 0$, i.e., $q \neq p^*$, and then $\theta_c = ((c+N)/rN - 1/q)(p^{-1} - N^{-1} - q^{-1})^{-1}$. Therefore, the condition $\theta_c(p^{-1} - N^{-1} - q^{-1}) \leq r^{-1} - q^{-1}$ in Theorem 1.1(i) and (ii) is just $c \leq 0$. It follows that $W^{1,(q,p)}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$ if and only if $r \leq \max\{p^*, q\}$ and one of the following conditions holds:

(i) $p \leq N, q \neq p^*$, and $c \leq 0$ is in the open interval with endpoints $rN/q - N$ and $r(N-p)/p - N$ (a nonempty set if $r < \max\{p^*, q\}$);

(ii) $p > N$ and either $r \leq q$ and $-N < c < rN/q - N$ (≤ 0) or $r > q$ and $-N < c \leq 0$;

(iii) $r = q$ and $c = 0$;

(iv) $p < N, p \leq r \leq p^*$, and $c = r(N-p)/p - N$ (≤ 0, since $r \leq p^*$).

Since $N/q = N/p - 1$ implies $p < N$ and $q > p$, Theorem 1.1(v) coincides with (iv) above. Theorem 1.1(vi) is not applicable.

If $r \leq \min\{p, q\}$, conditions (i)-(iv) are necessary and sufficient conditions for $\widetilde{W}^{1,(q,p)} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)$, where $\widetilde{W}^{1,(q,p)} := \widetilde{W}^{1,(q,p)}_{\{0,0\}}$ is unweighted (Theorem 5.1). They take the simpler form

(i') $p \leq N, q \neq p^*$, and c is in the open interval with endpoints $rN/q - N$ and $r(N-p)/p - N$ (hence $c < 0$);

(ii') $p > N$ and $-N < c < rN/q - N$ (≤ 0);

(iii') $q \leq p, r = q$, and $c = 0$;

(iv') $p \leq q, p < N, r = p$, and $c = -p$.

When $c = 0$, conditions (i) to (iv) become

(i) $p < N$ and r is in the closed interval with endpoints p^* and q or

(ii) $p \geq N$ and $r \geq q$.

This is, of course, well known, especially when $p = q$.

Remark 7. That $W^{1,(q,p)}(\mathbb{R}^N_*) = W^{1,(q,p)}(\mathbb{R}^N)$ with the same norm if $N > 1$ can be seen as follows. First, it suffices to show that if $u \in W^{1,(q,p)}(\mathbb{R}^N_*)$ has bounded support, then $u \in W^{1,(q,p)}(\mathbb{R}^N)$ and has the same norm. Now, if $u \in W^{1,(q,p)}(\mathbb{R}^N_*)$ has bounded support, then $u \in W^{1,\min\{p,q\}}(\mathbb{R}^N_*) = W^{1,\min\{p,q\}}(\mathbb{R}^N)$, for example by [11, p. 52]. Thus, as a distribution on $\mathbb{R}^N$, ∇u is a function, so that its restriction to $\mathbb{R}^N_*$ coincides with ∇u as a distribution on $\mathbb{R}^N_*$. Since the latter is in $(L^q(\mathbb{R}^N))^N$, the same is true of the former, which proves the claim.

11.2 Embedding of weighted spaces into $L^r(\mathbb{R}^N)$. The necessary and sufficient conditions for $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N)$ are given by Theorem 1.1 with $c = 0$. If so, $\theta_0 = (N/r - (a+N)/q)((b-p+N)/p - (a+N)/q)^{-1}$ in (1.4) when $(a+N)/q \neq (b-p+N)/p$ and these conditions become (after some work) $r \leq \max\{p^*, q\}$ plus one of the following:

(i) either $-N \leq a < N(q/r-1)$, $b > p+N(p/r-1)$, and $a(p/r-1+p/N) \leq b(q/r-1)$, or

(ii) $a > N(q/r-1)$, $b < p+N(p/r-1)$ and $a(p/r-1+p/N) \geq b(q/r-1)$;

(iii) $r = q$ and $a = 0$;

(iv) $p \leq r \leq p^*$, $b = p+N(p/r-1)$ $(\leq p)$ and $a \geq -N$;

(v) $r \geq \min\{p, q\}$, $a = N(q/r-1)$ and $b = p+N(p/r-1)$.

In (i)-(ii) above, the condition $\theta_0(p^{-1}-N^{-1}-q^{-1}) \leq r^{-1}-q^{-1}$ is accounted for by $a(p/r-1+p/N) \leq b(q/r-1)$ or its reverse, as the case may be. By Remark 1, this condition holds if $r \leq \min\{p^*, q\}$, which of course is corroborated by a direct verification.

11.3 Embedding when $p = q$. If $p = q$, then $r \leq \max\{p^*, q\}$ is simply $r \leq p^*$ and $(a+N)/q \neq (b-p+N)/p$ if and only if $a \neq b-p$. The condition $\theta_c(p^{-1}-N^{-1}-q^{-1}) < r^{-1}-q^{-1}$ in Theorem 1.1(i) and (ii) transforms into $\theta_c \geq N/p - N/r$, which is not a restriction when $r \leq p$. Also, part (v) is now a special case of part (iv).

If, in addition, $p = q = r$, Theorem 5.1 is applicable to the larger space $\widetilde{W}^{1,(p,p)}_{\{a,b\}}$. Furthermore, $c^0 = a$ and $c^1 = b-p$, and so $\widetilde{W}^{1,(p,p)}_{\{a,b\}} \hookrightarrow L^p(\mathbb{R}^N; |x|^c dx)$ if and only if either

(i) a and $b-p$ are on the same side of $-N$, not both equal to $-N$, and c is in the closed interval with endpoints a and $b-p$, or

(ii) a and $b-p$ are strictly on opposite sides of $-N$, and c is in the half-open interval with endpoints a (included) and $-N$ (not included).

These are also necessary and sufficient conditions for

$$W^{1,(p,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^p(\mathbb{R}^N; |x|^c dx).$$

When $p = q = r = 2$ and $c = (a+b)/2 - 1$, it follows from [7] that

$$C^\infty_0(\mathbb{R}^N_*) \hookrightarrow L^2(\mathbb{R}^N; |x|^c dx),$$

unless $a = b-2 = -N$. If (for example) $a < -N$ and $b > -a+2-2N$, then $b-2 > -N$, so that a and $b-2$ are on opposite sides of $-N$; but, since $c = (a+b)/2-1 > -N$, condition (ii) above does not hold when $p = 2$, and so $W^{1,(2,2)}_{\{a,b\}}(\mathbb{R}^N_*)$ is not continuously embedded into $L^2(\mathbb{R}^N; |x|^c dx)$. This shows that $C^\infty_0(\mathbb{R}^N_*)$ cannot be dense in $W^{1,(2,2)}_{\{a,b\}}(\mathbb{R}^N_*)$. Hence, in general, the embedding or other inequalities for $W^{1,(p,q)}_{\{a,b\}}(\mathbb{R}^N_*)$ are not implied by those for $C^\infty_0(\mathbb{R}^N_*)$.

11.4 A generalization. Let $B \subset \mathbb{R}^N$ be an open ball centered at the origin. If the space $\{u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) : \operatorname{Supp} u \subset \overline{B}\}$ is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$, it is also continuously embedded into $L^r(\mathbb{R}^N; |x|^d dx)$ when $d \geq c$. Likewise, if $\{u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) : \operatorname{Supp} u \subset \mathbb{R}^N \backslash B\}$ is continuously embedded into $L^r(\mathbb{R}^N; |x|^c dx)$, it is also continuously embedded into $L^r(\mathbb{R}^N; |x|^d dx)$ when $d \leq c$.

With this remark and a cut-off argument, Theorem 1.1 can be extended to more general weighted spaces. Let $x_1, \ldots, x_k \in \mathbb{R}^N$ be distinct points, $a_1, \ldots, a_k, a_\infty, b_1, \ldots, b_k, b_\infty \in \mathbb{R}$, and $1 \leq r \leq p, q < \infty$. For $a, b \in \mathbb{R}$, call $J(a, b) := \{c \in \mathbb{R} : W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)\}$ the interval of admissible c characterized in Theorem 1.1, with endpoints $c_-(a, b) \leq c_+(a, b)$ and let $c_1, \ldots, c_k, c_\infty$ be such that $c_i > c_-(a_i, b_i)$, $1 \leq i \leq k$, and $c_\infty < c_+(a_\infty, b_\infty)$ (the endpoints may be included if they are in the admissible interval). If w_a, w_b and[6] w_c are positive weights on $\mathbb{R}^N \backslash \{x_1, \ldots, x_k\}$ such that $w_a(x) = |x - x_i|^{a_i}$, $w_b(x) = |x - x_i|^{b_i}$, $w_c(x) = |x - x_i|^{c_i}$ for x near x_i, $i = 1, \ldots, k$ and $w_a(x) = |x|^{a_\infty}$, $w_b(x) = |x|^{b_\infty}$, $w_c(x) = |x|^{c_\infty}$ for large $|x|$, then the space $W^{1,(q,p)}_{\{w_a,w_b\}}(\mathbb{R}^N \backslash \{x_1, \ldots, x_k\}) := \{u \in L^1_{loc}(\mathbb{R}^N \backslash \{x_1, \ldots, x_k\}) : u \in L^q(\mathbb{R}^N; w_a(x)dx), \nabla u \in (L^q(\mathbb{R}^N; w_b(x)dx))^N\}$ is continuously embedded into $L^r(\mathbb{R}^N; w_c(x)dx)$.

A somewhat heuristic yet compelling reason that such conditions should be optimal is simple. As pointed out above, membership in $L^r(\mathbb{R}^N; |x|^c dx)$ of functions with support in a closed ball $\overline{B}$ about the origin is unaffected by increasing c. Thus, the value of the upper end $c_+(a, b)$ can only be dictated by the behavior of functions with support bounded away from 0. The optimality of the lower end $c_-(a, b)$ is justified by a similar argument. However, this rationale is meaningless when $J(a, b) = \varnothing$. The simplest way around the difficulty in this case is to rely on the related fact that for functions with support in $\overline{B}$, membership in $W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N_*)$ is unaffected by increasing a or b. Thus $c_-(a, b)$ can be defined by inceasing a or b until $J(a, b)$ becomes nonempty. Likewise, a or b can be decreased to define $c_+(a, b)$. This may or may not produce the best possible conditions. Because of space limitations, a more detailed investigation of the optimality issue by more sophisticated procedures (elaboration on Remark 5) is not attempted here.

Naturally, the weights need only to "look like" (not coincide with) power weights in the vicinity of the points x_i (or infinity). This remark clarifies two things. First, w_a, w_b and w_c need not actually have power-like singularities at the same points: This case is reduced to the previous one by adding points as needed and setting the corresponding a_i, b_i or c_i equal to 0. Next, the cut-off argument is technically simplified, and nothing is changed, if it is assumed that $w_a(x) = |x - x_1|^{a_\infty}$, $w_b(x) = |x - x_1|^{b_\infty}$, $w_c(x) = |x - x_1|^{c_\infty}$ for large $|x|$ (otherwise,

[6] Here, a, b and c are just indices.

the origin plays a technical role even when it is not one of the points x_i). Theorem 1.1 is recovered when $k = 1, x_1 = 0$, and $a_1 = a_\infty, b_1 = b_\infty, c_1 = c_\infty$.

If only $k = 1$ and $x_1 = 0$, Theorem 4.1 too can be generalized to obtain the embedding of the subspace W_{rad} of radially symmetric functions in $W^{1,(q,p)}_{\{w_a,w_b\}}(\mathbb{R}^N_*)$ into $L^r(\mathbb{R}^N; w_c(x)dx)$ under the conditions $c_1 > c^{rad}_-(a_1, b_1)$ and $c_\infty < c^{rad}_+(a_\infty, b_\infty)$, where $c^{rad}_\pm(a, b)$ denote the endpoints of the admissible interval in Theorem 4.1. Once again, $c^{rad}_-(a_1, b_1)$ and $c^{rad}_+(a_\infty, b_\infty)$ may be included if they are in the admissible interval and can also be defined when the admissible interval is empty.

When $1 < p = q < N$ and $w_b = 1$ (so that $b_1 = b_\infty = 0$), the embedding into $L^r(\mathbb{R}^N; w_c(x)dx)$ of the closure C_{rad} of the space of radially symmetric functions in $C^\infty_0(\mathbb{R}^N) \cap L^p(\mathbb{R}^N; w_a(x)dx)$ equipped with the $W^{1,(p,p)}_{\{w_a,1\}}(\mathbb{R}^N_*)$ norm, has recently been investigated by Su et al. [28, Theorems 1 and 2]. They assume that $a_1, c_1, a_\infty, c_\infty$ are given and find the admissible values of r under the implicit assumption $r \geq p$. The reformulation in terms of lower (upper) bounds about c_1 (c_∞) given a_1, a_∞ and r is conceptually trivial, but quite messy and technical in practice. Accordingly, we do not elaborate beyond the remark that, because C_{rad} is usually *smaller* than W_{rad}, the embedding may be true under conditions more general than $c_1 \geq c^{rad}_-(a_1, 1)$ and $c_\infty \leq c^{rad}_+(a_\infty, 1)$. On the other hand, the case $0 < r < p$ and all others ($p = 1, p \geq N, q \neq p, b_1 \neq 0, b_\infty \neq 0$) can be handled by the method outlined above.

Acknowledgements. I am grateful to Professor V. Maz'ya for valuable information about the early historical development of the subject and other connections with the existing literature.

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(Received May 28, 2011 and in revised form August 23, 2011)