EXOTIC BAKER AND WANDERING DOMAINS FOR AHLFORS ISLANDS MAPS

By

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Abstract. Let X be a compact Riemann surface of genus at most 1, i.e., the Riemann sphere or a torus, and let $W \subsetneq X$ be an arbitrary domain. We construct a variety of examples of holomorphic functions $g : W \to X$ that satisfy Epstein's *Ahlfors islands property* and that have "pathological" dynamical behaviour. In particular, we show that the accumulation set of any curve tending to the boundary of W can be realized as the ω-limit set of a Baker domain of such a function. We furthermore construct Ahlfors islands maps

- with wandering domains having prescribed ω-limit sets,
- with logarithmic singularities having prescribed asymptotic curves,

and also produce examples where X is a compact hyperbolic surface. As a corollary of our method, we construct transcendental entire functions with Baker domains in which the iterates tend to infinity arbitrarily slowly.

1 Introduction

The iteration theory of holomorphic functions of one complex variable has been studied intensively over the past three decades. Originally much attention focused on the case of polynomials and rational functions. In recent years, the dynamics of transcendental entire and meromorphic functions has also received considerable attention, in part due to apparently deep connections with intriguing aspects of the rational theory. However, there are also natural and interesting dynamical examples of functions whose range is significantly larger than their domain, such as the *parabolic renormalizations* of rational functions at parabolic periodic points studied by Lavaurs [L].

In his thesis [E1], Adam Epstein introduced the concept of "finite-type maps"; this class includes all rational functions and all transcendental entire and meromorphic functions whose set of singular values is finite, but also all iterated parabolic

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renormalizations of such maps (and much more). The results in [E1] also implicitly suggest a natural generalization of arbitrary meromorphic functions: the class of *Ahlfors islands maps*, formally introduced in [E2] and further developed in [EO]. This class includes all rational, transcendental entire and transcendental meromorphic functions, and their iterates, all finite-type maps and also the class of holomorphic functions studied in [BDH].

An Ahlfors islands map is a holomorphic function $g : W \to X$, where W is an open subset of a compact Riemann surface X, that satisfies a certain transcendentality condition – the "Ahlfors islands property" – near the boundary of its domain of definition (Definition 2.1). This condition ensures that the Fatou set $F(g)$ and its complement, the Julia set $J(g) = X \setminus F(g)$, retain their usual properties. The manuscripts [E2, EO], which develop the theory of Ahlfors islands maps, are currently unpublished, but our discussion is essentially self-contained. For further background, the reader is referred to the short introduction in [R].

If g is an Ahlfors islands map, then – just as for non-linear entire and meromorphic functions – any invariant component of the Fatou set $F(g)$ is of one of finitely many types (see [M, Chapter 5] or [B1, Theorem 6]): an immediate basin of attraction for a (super-) attracting or parabolic fixed point, a rotation domain, or a **Baker domain**; the latter is a periodic component of $F(g)$ in which the iterates converge to the boundary ∂W of the domain of definition of g. If U is a Baker domain or a **wandering domain** (i.e., a component of $F(g)$ whose forward orbit is not eventually periodic), then by the **ω-limit set** $\omega(U)$ we mean the set of all points $w \in \overline{W}$ for which there is some point $z \in U$ whose orbit under g accumulates on w.

The ω-limit set of an invariant Baker domain is necessarily connected (see Lemma 2.8). Hence, if $X \setminus W$ is totally disconnected (as in the case of transcendental entire or meromorphic functions and their iterates) and g has an invariant Baker domain U, then $\omega(U)$ consists of a single point. Otherwise, it is conceivable that a Baker domain of g may have a nondegenerate continuum as its ω-limit set. Epstein asked the question whether such "exotic Baker domains" can in fact exist.

Oudkerk showed that the answer is "yes" [O]. He constructed an example where W is a simply-connected domain having a prime end with a nontrivial impression, and $g : W \to \hat{\mathbb{C}}$ has a Baker domain whose ω-limit set is exactly this prime end impression.

This example raises a number of questions, e.g., whether the boundary of the domain of an Ahlfors islands map with an exotic Baker domain can be locally connected and whether an Ahlfors islands map can have an exotic Baker domain whose ω-limit set is the whole of ∂W. More generally, one can ask, given a domain

W and a compact connected set $K \subset \partial W$, whether there is an Ahlfors islands map with a Baker domain U such that $\omega(U) = K$.

In this note, we give a complete answer to this question in the case where X is of genus 0 or 1. (That is, X is the Riemann sphere, or a torus $X = \mathbb{C}/\Gamma$, where Γ is a period lattice.)

Theorem 1.1 (Exotic Baker domains). *Let X be a compact Riemann surface of genus at most* 1. *Let $W \subsetneq X$ be nonempty, open and connected, and let $\gamma : [0, \infty) \to W$ be an injective curve with $\gamma(t) \to \partial W$ as $t \to \infty$. Let $K \subset \partial W$ be the accumulation set of γ. Then there exists an Ahlfors islands map $g : W \to X$ such that g has a Baker domain U with $\gamma \subset U$ whose ω-limit set is exactly K.*

If $X = \hat{\mathbb{C}}$, then g can furthermore be chosen to omit any given value $a \in \hat{\mathbb{C}} \setminus \gamma$.

Remark 1. If an Ahlfors islands map g has a Baker domain U, then $\omega(U)$ is the accumulation set of an injective C^∞ curve contained in U. (See Lemma 2.9.) Hence Theorem 1.1 states that any potential ω-limit set is indeed realized by a Baker domain.

Remark 2. If $X = \hat{\mathbb{C}}$ and $W \subset \mathbb{C}$, then the final statement of the theorem implies that g can be chosen to be holomorphic.

Oudkerk's example is fairly explicit, and he verifies the Ahlfors islands property directly. Instead, we use approximation theory to construct our functions, which gives us considerable flexibility.

In [RS] the *slow escaping set* of an entire (or meromorphic) function is studied, and it is shown that the Julia set always contains points whose iterates tend to ∞ arbitrarily slowly. On the other hand, there were previously no known examples of entire functions with Baker domains in which the iterates grow arbitrarily slowly. We are able to answer this question by controlling the growth of orbits inside the Baker domains we construct. Let us denote by dist_X distance with respect to the spherical metric (if $X = \hat{\mathbb{C}}$) or with respect to the unique flat metric of area 1 (if X is a torus).

Proposition 1.2 (Arbitrarily slow escape). *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers with* $\lim_{n\to\infty} \varepsilon_n = 0$. *Then the function g in Theorem 1.1 can be chosen such that the Baker domain U also has the following property: for every $z \in U$, there exists $n_0 \geq 0$ such that* $\mathrm{dist}_X(g^n(z), \partial W) \geq \varepsilon_n$ *for all $n \geq n_0$.*

Let us say a few words about the proofs of these results, beginning with Proposition 1.2 in the case where g is an entire function (i.e., $X = \hat{\mathbb{C}}$, $W = \mathbb{C}$ and $a = \infty$). It is well known from estimates of hyperbolic geometry that the iterates in a Baker domain tend to infinity very slowly provided that the domain is sufficiently "thin". So the idea is to start with an extremely thin simply-connected domain V whose boundary is a curve that tends to infinity in both directions, to let $f : V \to V$ be a holomorphic map with $f^n|_V \to \infty$ as $n \to \infty$, and to approximate f by an entire function g. The idea of using approximation theory to construct examples of entire functions with Baker domains or wandering domains was introduced by Eremenko and Lyubich in [EL].

Arakelian's Approximation Theorem allows the approximation of a function up to a prescribed euclidean error ε, but to ensure that the approximating function g still satisfies $g(V) \subset V$, this is not sufficient for thin domains V. On the other hand, it is known that functions on unbounded domains *cannot*, in general, be approximated by entire functions up to an arbitrary prescribed error function [G, Satz IV.3.4].

We deal with this difficulty by proving an approximation result (Lemma 4.2) for functions on simply-connected domains that is sufficiently precise for our purpose; this lemma may be useful also in other contexts.

The same approach is used to construct Ahlfors islands maps with exotic Baker domains; the Ahlfors islands property of the approximating function is obtained by using the theory of normal families. In the case where X is a torus, we also need to replace Arakelian's theorem by its generalization for Riemann surfaces due to Scheinberg [S].

In the same general setting as Theorem 1.1, we also produce Ahlfors islands maps with wandering domains that have very precise "pathological" dynamics. We note that transcendental entire and meromorphic functions can have wandering domains of many types; see [B1] for references, and also [DS] for some non-meromorphic examples.

Theorem 1.3 (Wandering domains)**.** *Let X be a compact Riemann surface of genus at most* 1*, and let $W \subsetneq X$ be nonempty, open and connected. Also let $K \subset \partial W$ be compact. Then there exists an Ahlfors islands map $g : W \to X$ such that g has a wandering domain whose ω-limit set is exactly K.*

If $X = \hat{\mathbb{C}}$, then g can furthermore be chosen to omit any given value $a \in \hat{\mathbb{C}}$.

Finally, we can also use approximation theory to produce examples of Ahlfors islands maps with logarithmic tracts. Recall that $a \in X$ is called a **logarithmic asymptotic value** of a holomorphic function $g : U \to X$ if there is a simply-connected neighborhood D of a and a component T of $g^{-1}(D)$ such that $g : T \to D \setminus \{a\}$ is a universal covering. In this case, T is called a **logarithmic tract** of g.

It was remarked in [R] that Arakelian's theorem can be used to construct Ahlfors functions in the unit disk having a logarithmic tract that spirals out to the unit circle. We show that in fact any injective C^1 curve tending to the boundary in a domain $W \subset X$ (with X as above) can occur as an asymptotic curve for a logarithmic asymptotic value of an Ahlfors islands map $g : W \to X$. If this asymptotic value is itself in the accumulation set of the asymptotic curve, then by [R, Theorem A.1] (which generalizes a result of [BKZ]), the Hausdorff dimension of the Julia set of g is strictly greater than one.

Since the existence of logarithmic asymptotic values is a function-theoretic rather than a dynamical property, it makes sense to state our results also for *non-iterable* Ahlfors islands maps, where the domain W is a nonempty open subset of some compact Riemann surface Y, which may or may not agree with the range X. (See Remark 1 after Definition 2.1.)

Theorem 1.4 (Logarithmic tracts)**.** *Let X be a compact Riemann surface of genus at most 1 and let Y be a compact Riemann surface of arbitrary genus. Let $W \subsetneq Y$ be nonempty, open and connected. Also let $\gamma : [0, \infty) \to W$ be an injective C^1 curve with $\gamma(t) \to \partial W$ as $t \to \infty$. Then, for any $w_0 \in X$ and any Jordan neighborhood $U \subset X$ of w_0, there exists an Ahlfors islands map $g : W \to X$ such that $g(\gamma) \subset U \setminus \{w_0\}$, $g(\gamma(t)) \to w_0$ as $t \to \infty$, and such that the component of $g^{-1}(U \setminus \{w_0\})$ containing γ is mapped as a universal covering by g.*

If $X = \hat{\mathbb{C}}$, then g can furthermore be chosen to omit any given value $a \in \hat{\mathbb{C}}$ with $a \notin \overline{U} \setminus \{w_0\}$.

We note that, in particular, we obtain examples of Ahlfors islands maps with arbitrary domains whose range is a Riemann surface of genus at most 1.

Corollary 1.5 (Examples of Ahlfors islands maps)**.** *Let X and Y be compact Riemann surfaces, and let $W \subsetneq Y$ be nonempty, open and connected. If X has genus at most* 1*, then there exists an Ahlfors islands map $g : W \to X$.*

If the surface X is hyperbolic, then the question of which domains W can support Ahlfors islands maps with values in X is much more subtle. For example, if $W \subset Y$ is simply-connected, then any universal covering $g : W \to X$ is an Ahlfors islands map. On the other hand, a holomorphic map taking values in a hyperbolic surface cannot have isolated essential singularities. So if ∂W has isolated points and X is hyperbolic, then there are no Ahlfors islands maps $g : W \to X$.

Furthermore, let Y be a compact Riemann surface of genus at least two, and let $g : \mathbb{D} \to Y$ be an Ahlfors islands map (where $\mathbb{D}$ denotes the unit disk). Suppose that g has an asymptotic value $w_0 \in Y$ and that γ is an asymptotic curve for w_0.

Then, using the universal covering of Y, we can lift g to a holomorphic function $G : \mathbb{D} \to \mathbb{D}$ that also has γ as an asymptotic curve. Fatou's theorem implies that the accumulation set of γ on $\partial\mathbb{D}$ consists of a single point. (We thank Chris Bishop for this observation.) A similar argument shows that, if $D \subset Y$ is a Jordan domain and $f : D \to Y$ is an Ahlfors islands map, then f cannot have a Baker domain that spirals in to the boundary of D.

Hence there are obstructions to extending our theorems to the case where X is a hyperbolic Riemann surface. However, our methods can still be used to prove the following, slightly weaker results.

Theorem 1.6 (Ahlfors maps on hyperbolic surfaces)**.** *Theorems 1.1, 1.3 and 1.4 still hold when X is a compact hyperbolic Riemann surface, provided the phrase "there exists an Ahlfors islands map $g : W \to X$" is replaced by "there exist a domain $W' \subsetneq W$ and an Ahlfors islands map $g : W' \to X$".*

As far as we know, these are the first known examples of Ahlfors islands maps with Baker domains or wandering domains on surfaces of genus $g \geq 2$.

2 Definitions and preliminaries

Basic notation. We denote the complex plane, the Riemann sphere, the unit disk and the right half plane by $\mathbb{C}$, $\hat{\mathbb{C}}$, $\mathbb{D}$ and $\mathbb{H}$, respectively. The (euclidean) disk of radius δ around $z \in \mathbb{C}$ is denoted by $B(z, \delta)$. We also denote euclidean distance by $\text{dist}_{\mathbb{C}}$.

Closures and boundaries will usually be taken in an underlying compact Riemann surface X (which X is meant should be clear from the context). Sometimes we consider relative closures, and it is convenient to denote the relative closure of $A \subset W$ in $W \subset X$ by $\overline{A}_W := \overline{A} \cap W$.

We also denote hyperbolic distance on any hyperbolic Riemann surface W by dist_W. Then $B_W(z, \delta)$ denotes the hyperbolic disk in W of radius δ around z. (See, e.g., [M, Chapter 2] for basic definitions and results of hyperbolic geometry and in particular, *Pick's theorem*, which we shall use frequently.)

Ahlfors islands maps and normal families. Let X and Y be compact Riemann surfaces and let $W \subset Y$ be open and nonempty. It will be convenient to introduce the following terminology. If $V \subset Y$ is a connected open set that intersects the boundary of W in Y, then we call a component U of $V \cap W$ a **boundary neighborhood** of W, provided that U is hyperbolic. The latter is always the case if V is chosen sufficiently small to omit at least three points of Y.

A **Jordan domain** in X is a simply-connected domain that is bounded by a Jordan curve in X. If $V \subset X$ is a Jordan domain and $g : W \to X$ is holomorphic, then a **simple island** of g over V is a domain $I \subset W$ such that $g : I \to V$ is a conformal isomorphism.

Definition 2.1 (Ahlfors islands maps)**.** Let X, Y, and $W \subset Y$ be as above. A holomorphic function $g : W \to X$ has the **Ahlfors islands property** and g is called an **Ahlfors islands map** if there exists a number k such that the following is true.

If $V_1, \ldots, V_k$ are Jordan domains in X with pairwise disjoint closures, then for every boundary neighborhood U of W, there exists $j \in \{1, \ldots, k\}$ such that U contains a simple island of g over V_j.

Remark 1. The deinition of Ahlfors islands maps given in [R] includes the assumption that $X = Y$, in which case it is possible to consider g as a dynamical system. However, this is not required, and the above is the general definition as introduced by Epstein. For our purposes, the additional flexibility simplifies the discussion when constructing Ahlfors islands maps on the torus.

Remark 2. One of the key properties of the definition is that the composition of two Ahlfors islands maps is again an Ahlfors islands map. We do not require this fact.

If $X = Y$ and $g : W \to X$ is an Ahlfors islands map, then the **Fatou set** $F(g)$ consists of all points $z \in X$ for which there exists either some $k \geq 0$ such that $g^k(z) \in X \setminus \overline{W}$ or an open neighborhood U of z in which the iterates $g^k|_U$ are all defined and form a normal family. By virtue of the Ahlfors islands property of g, the **Julia set** $J(f) = X \setminus F(f)$ is the closure of the set of repelling periodic points, and also retains its other well-known properties. In particular, if g is *non-elementary*, i.e., not a conformal isomorphism of X, then $J(g)$ is a nonempty, perfect, compact set. (We do not use these facts in this article.)

The definition of Ahlfors islands maps – and their name – is inspired by the classical Five Islands Theorem of Ahlfors, which implies that every transcendental meromorphic function is an Ahlfors islands map (with $k = 5$). We will use the following "normal families version" of this result; see e.g. [B2].

Theorem 2.2 (Ahlfors Five Islands Theorem, normal families version)**.** *Suppose $V_1, \ldots, V_5 \subset \hat{\mathbb{C}}$ are Jordan domains with pairwise disjoint closures. If $U \subset \mathbb{C}$ is a domain, and $\mathcal{F}$ is a family of meromorphic functions $f : U \to \hat{\mathbb{C}}$ that have no simple islands over any of the V_j, then $\mathcal{F}$ is a normal family.*

Remark. If the functions in $\mathcal{F}$ are holomorphic – which is the case we are interested in – the number five can be replaced by four. (An extremal example is given by the iterates of the sine function, which has no islands over any Jordan domain that contains 1, -1 or ∞.)

We use Theorem 2.2 to construct our examples by introducing a class of functions that are not normal near any point of the boundary of W. Note that such functions can exist only if X is an elliptic or parabolic manifold (i.e., X is the sphere or a torus), since any family of holomorphic functions taking values in a hyperbolic surface is normal.

Definition 2.3 (Strong non-normality)**.** Suppose that W is a hyperbolic Riemann surface.

We say that a holomorphic function $g : W \to \mathbb{C}$ is **strongly non-normal** if there exists a number $\delta > 0$ with the property that whenever $(w_n)_{n\in\mathbb{N}}$ is a sequence in W tending to the boundary of W and $\pi_n : \mathbb{D} \to W$ are universal covers with $\pi_n(0) = w_n$, the family $\{g \circ \pi_n : B_{\mathbb{D}}(0, \delta) \to \hat{\mathbb{C}}\}$ is not normal.

Suppose that $g : W \to \mathbb{C}$ is strongly non-normal, where W is a hyperbolic subdomain of a compact Riemann surface Y. Since every boundary neighborhood of W contains a sequence of hyperbolic balls of fixed diameter tending to the boundary, it follows from the normal families version of the Ahlfors Islands Theorem that $g : W \to \hat{\mathbb{C}}$ is an Ahlfors islands map.

As our definition of strong non-normality is phrased in conformally invariant terms, it is invariant under precomposition with a conformal map; this is why we have chosen it among a number of other definitions that would also be suitable for our purposes. We use the following condition to ensure that a function is strongly non-normal.

Lemma 2.4 (Sufficient condition for strong non-normality)**.** *Let W be a hyperbolic Riemann surface, and let $g : W \to \mathbb{C}$ be holomorphic. Suppose that there are relatively closed subsets $B, C \subset W$ and a positive constant Θ with the following properties:*

- *for every $w \in W$, $\operatorname{dist}_W(w, B) \le \Theta$ and $\operatorname{dist}_W(w, C) \le \Theta$;*
- *$g|_B$ is bounded;*
- *$g(c) \to \infty$ as $c \to \partial W$ within C.*

Then g is strongly non-normal.

Proof. Set $\delta := 2\Theta$; then every closed hyperbolic disk of radius $\delta/2$ intersects both B and C. Let w_n and π_n be as in the definition of strong non-normality. Then there are sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ in $\overline{B_{\mathbb{D}}(0, \delta/2)}$ such that $\pi_n(b_n) \in B$ and

$\pi_n(c_n) \in C$, and in particular $\limsup_{n\to\infty} |g(\pi_n(b_n))| < \infty$ and $g(\pi_n(c_n)) \to \infty$ as $n \to \infty$. Clearly the sequence $(g \circ \pi_n)$ cannot be normal on $B_{\mathbb{D}}(0, \delta)$, as required.□

The Arakelian-Scheinberg Approximation Theorem. We use the following terminology from [G].

Definition 2.5 (Weierstrass sets)**.** Let Y be a compact Riemann surface, and let $W \subsetneq Y$ be a nonempty open and connected set. A relatively closed set $A \subset W$ is called a **Weierstrass set** (in W) if any continuous function $f : A \to \mathbb{C}$ that is holomorphic on the interior of A can be uniformly approximated, up to an arbitrarily small error ε, by a holomorphic function $g : W \to \mathbb{C}$.

Theorem 2.6 (Arakelian-Scheinberg Approximation Theorem)**.** *A relatively closed set A in W is a Weierstrass set if and only if $\widehat{W} \setminus A$ is connected and locally connected, where $\widehat{W} = W \cup \{\infty\}$ is the one-point compactification of W.*

Remark. The case where $Y = \hat{\mathbb{C}}$ is Arakelian's theorem [G, Satz IV.2.3]; the general case is due to Scheinberg [S]. The latter more generally treats *arbitrary* non-compact Riemann surfaces W. In this setting, it turns out that there is no longer a topological characterization of Weierstrass sets, but Scheinberg gives a sufficient condition on the topology of A under which the above criterion is still necessary and sufficient. This condition includes the case where W has finite genus.

We note that it follows from Theorem 2.6 that any countable disjoint union of nonseparating compact subsets $K_n \subset W$ tending to ∂W as $n \to \infty$ is a Weierstrass set.

Our construction relies on approximating certain functions up to an error prescribed by a given function. Nersesjan's theorem [G, Satz IV.3.4] gives a necessary and sufficient condition on the set A for this to be possible with an arbitrary error function in the case where $W \subset \hat{\mathbb{C}}$. Unfortunately, this criterion excludes the case in which A has unbounded interior components.

Instead, we make use of the following trick [G, Hilfssatz IV.3.3], which also goes back to Arakelian.

Lemma 2.7 (Approximation up to an error function)**.** *Let Y be a compact Riemann surface, let $W \subsetneq Y$ be nonempty, open and connected, and let A be a Weierstrass set in W. Furthermore, let $f : A \to \mathbb{C}$ and $E : A \to \mathbb{C}$ be continuous on A and holomorphic on the interior of A.*

Then there is a holomorphic function $g : W \to \mathbb{C}$ such that

$$|g(z) - f(z)| \le |\exp(E(z))| \quad \textit{for all } z \in A.$$

Proof. Because A is a Weierstrass set, we can find a holomorphic function $H : W \to \mathbb{C}$ such that $|H(z) - (E(z) - 1)| \leq 1$ for all $z \in A$. In particular, we have $\operatorname{Re} H(z) \leq \operatorname{Re} E(z)$, and hence

$$|\exp(H(z))| = \exp(\operatorname{Re} H(z)) \leq \exp(\operatorname{Re} E(z)) = |\exp(E(z))|$$

for all $z \in A$.

Let G be a holomorphic function on W that approximates $f(z) \cdot \exp(-H(z))$ up to an error of at most 1 on A. Then the map $g(z) := G(z) \cdot \exp(H(z))$ satisfies

$$\begin{aligned} |g(z) - f(z)| &= |\exp(H(z))| \cdot |G(z) - f(z) \cdot \exp(-H(z))| \\ &\leq |\exp(H(z))| \leq |\exp(E(z))| \end{aligned}$$

for all $z \in A$. □

Remark. Note that Lemma 2.7 implies that approximation up to an arbitrary error function is possible if A is a Weierstrass set that has only compact connected components; this is a special case of the theorem of Nersesjan.

In Lemma 4.2, we use Lemma 2.7 to approximate a function given inside a simply-connected subset of W that accumulates on ∂W.

Limit sets. For completeness, we note two simple facts about ω-limit sets.

Lemma 2.8 (Limit sets). *Let X be a compact Riemann surface, let $W \subset X$ be open and nonempty, and let $g : W \to X$ be an Ahlfors islands map. Also let U be a Baker domain or a wandering domain of g.*

If $z \in U$ and (n_k) is a sequence such that $g^{n_k}(z) \to w_0 \in \omega(U)$, then $g^{n_k}|_U \to w_0$ locally uniformly. In particular, $\omega(U)$ coincides with the set of limit points of the sequence $(g^n(z))$ for every $z \in U$.

If U is an invariant Baker domain, then there is a curve $\alpha \subset U$ whose accumulation set is $\omega(U)$. In particular, $\omega(U)$ is connected.

Proof. Set $\widetilde{U} := \bigcup_{j \geq 0} g^n(U)$. Then, by assumption on U, every limit function of the normal family $\{g^n|_U\}$ takes values in the nowhere dense set $\partial\widetilde{U}$, and hence is constant. In particular, with notation as in the lemma, every limit function of the sequence $(g^{n_k}|_U)$ is the constant function w_0. This proves the first claim.

If U is an invariant Baker domain and $z \in U$ is arbitrary, then we can pick some curve $\alpha_0 \subset U$ connecting z and $g(z)$. Then $\alpha := \bigcup_{j \geq 0} g^j(\alpha_0)$ is the desired curve for the final claim. □

We can make the final part of the preceding lemma more precise as follows.

Lemma 2.9 (Curves in Baker domains). *Let X be a compact Riemann surface, and let $U \subset X$ be a hyperbolic domain. Let $f : U \to U$ be holomorphic, and suppose that* $\mathrm{dist}_X(f^n(z), \partial U) \to 0$ *as $n \to \infty$. Let $z \in U$ and let K be the set of limit points of the sequence $(f^n(z))$. Then there exists an injective C^∞ curve $\alpha : [0, \infty) \to U$ whose accumulation set is K.*

Remark. We state this lemma only for completeness; it is not used in the rest of the article.

Proof. Let C be the hyperbolic distance between z and $f(z)$. By Pick's theorem, C is an upper bound for the hyperbolic distance between $f^n(z)$ and $f^{n+1}(z)$, for all n. So $V := \bigcup_{n\geq 0} B_U(f^n(z), 2C)$ is a connected subdomain of U. Since the diameter of $B_U(f^n(z), 2C)$ in X tends to zero as $n \to \infty$, we have $\partial V \cap \partial U = K$.

Now pick an injective C^∞ curve $\alpha : [0, \infty) \to V$ that tends to ∂U as $t \to \infty$ and passes through all $f^n(z)$. Then the accumulation set of α is precisely K (it contains K since it passes through all $f^n(z)$ and is contained in K by choice of V).

(That the existence of such a curve α follows from standard methods of topology should be plausible to the reader, but we sketch a proof for completeness. First note that it is sufficient to prove this statement without the requirement that α be C^∞, as we can approximate any injective curve by a smooth and injective one. As in the previous lemma, there is a curve β with the desired properties, but β is not necessarily injective. This curve has an injective subcurve β_0 that tends to ∂U, but might not contain all the points $f^n(z)$. This is easily corrected by connecting the missing points, one by one, to β_0 using a piece of the curve β, modifying the result so as to obtain an injective curve.) $\square$

3 Wandering domains

Proof of Theorem 1.3. There are three different cases to consider:

(a) $X = \hat{\mathbb{C}}$ and W is a plane or punctured plane, in which case we may assume without loss of generality that $W = \mathbb{C}$ or $W = \mathbb{C} \setminus \{0\}$;

(b) $X = \hat{\mathbb{C}}$ and W is a hyperbolic domain;

(c) $X = \mathbb{C}/\Gamma$ is a torus and W is a hyperbolic domain.

The proof is similar for all three cases, so we treat them in parallel, and remark on differences in the appropriate places. (We note that Theorem 1.3 is well known in case (a), when W is parabolic. We nonetheless include the proof in this case, since many of the arguments reoccur later.)

If $X = \hat{\mathbb{C}}$, let $\pi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation with $\pi(\infty) = a$. (Here a is the value that our map g will omit; recall the statement of the theorem). Otherwise,

$X = \mathbb{C}/\Gamma$ is a torus, and we let $\pi : \mathbb{C} \to X$ be the natural projection.

The basic structure of the proof is as follows. We pick a sequence $(z_k)_{k\in\mathbb{N}}$ in W, $k \in \mathbb{N}$, that accumulates exactly on the given compact set $K \subset \partial W$ (and tends to the boundary "sufficiently quickly"). We also pick a sequence of small Jordan domains $(\Delta_k)_{k\in\mathbb{N}}$ such that $z_k \in \Delta_k$, $\overline{\Delta_k} \subset W$ and the $\overline{\Delta_k}$ are pairwise disjoint, and let $\Delta'_k \subset \mathbb{C}$ be a component of $\pi^{-1}(\Delta_k)$. If $X = \hat{\mathbb{C}}$, we assume that z_k and Δ_k are chosen so that $a \notin \overline{\Delta_k}$ for all k. Thus every Δ'_k is a bounded Jordan domain in $\mathbb{C}$.

Then we choose a function $f : \bigcup_{k=1}^{\infty} \Delta_k \to \mathbb{C}$ with $f(\overline{\Delta_k}) \subset \Delta'_{k+1}$ for all k, and approximate this map, using Lemma 2.7, by a holomorphic function $h : W \to \mathbb{C}$ that still takes $\overline{\Delta_k}$ into Δ'_{k+1}. This construction is done in such a way that we ensure that h (considered as a function $W \to \hat{\mathbb{C}}$) is an Ahlfors islands map.

Because π is a covering map onto X, the function $g := \pi \circ h : W \to X$ is clearly also an Ahlfors islands map. We have $g(\overline{\Delta_k}) \subset \Delta_{k+1}$, and if z_k and Δ_k are suitably chosen, we can ensure that each Δ_k is contained in a wandering domain of g, completing the proof. Note that, if $X = \hat{\mathbb{C}}$, then g omits the value a.

We now provide the details.

Choice of z_k and Δ_k in the hyperbolic case. First suppose that W is a hyperbolic domain. Then we may suppose that the sequence (z_k) mentioned above is chosen such that

$$\operatorname{dist}_W(z_{k+1}, \{z_1, \dots, z_k\}) \to \infty \quad \text{as } k \to \infty. \tag{3.1}$$

The only additional requirement on our sequence (Δ_k) of Jordan domains in this case is that the hyperbolic diameter of Δ_k should be bounded by some constant M, independently of k.

Choice of z_k and Δ_k in the parabolic case. If $W = \mathbb{C}$ or $W = \mathbb{C} \setminus \{0\}$, we require that $\left|\log|z|\right| > k \cdot \left|\log|w|\right|$ whenever $z \in \Delta_{k+1}$ and $w \in \Delta_k$. This can be ensured by letting $\left|\log|z_k|\right|$ grow sufficiently quickly and picking Δ_k sufficiently small.

Choice of B and C. In addition, we choose two discrete countable subsets sets $B, C \subset W$, disjoint from each other and all $\overline{\Delta_k}$, such that the accumulation sets of B and of C in X coincide exactly with ∂W. (These sets are used to ensure that the function h is an Ahlfors islands map.)

If W is hyperbolic, we also require that these sets are chosen such that points of W have uniformly bounded hyperbolic distance from both B and C (as in Lemma 2.4). Note that this is possible because we required the Δ_k to have uniformly bounded hyperbolic diameters.

Definition of f and approximation. We set $A := B \cup C \cup \bigcup_{k=1}^{\infty} \overline{\Delta_k}$. Observe that A is a Weierstrass set by Theorem 2.6.

We now define a continuous function $f : A \to \mathbb{C}$ as follows:

(a) $f|_{\Delta_k}$ is a holomorphic function that extends continuously to $\partial\Delta_k$ such that $f(\overline{\Delta_k}) \subset \Delta'_{k+1}$ (recall that Δ'_{k+1} is a component of $\pi^{-1}(\Delta_{k+1})$);

(b) $f|_B$ is bounded;

(c) $f(c) \to \infty$ as $c \to \partial W$ within C.

We also define a locally constant function $e(z) : A \to (0, \infty)$ by setting $e \equiv 1$ on B and C, and $e(z) := \mathrm{dist}_{\mathbb{C}}(f(\Delta_k), \partial\Delta'_{k+1})$ on $\overline{\Delta_k}$.

Now we can apply Lemma 2.7 to f and $E = \log e$ to obtain a holomorphic function $h : W \to \mathbb{C}$ such that $|h(z) - f(z)| \leq e(z) \quad$ for all $z \in A$.

Set $g := \pi \circ h$. We have $g(\Delta_k) \subset \pi(\Delta'_{k+1}) = \Delta_{k+1}$, so Δ_k is contained in the Fatou set of g for all k. Furthermore, the iterates of g, restricted to Δ_k, tend to ∂W, so the Fatou component containing Δ_k is either a Baker domain (or pre-image component of a Baker domain) or a wandering domain.

Proof that h is an Ahlfors islands map. If W is hyperbolic, then it follows from Lemma 2.4 and our definitions that h is strongly non-normal, and hence an Ahlfors islands map.

If W is parabolic, then it follows from the fact that h is bounded on B and tends to infinity on C that each point of ∂W is an essential singularity of h. Hence h is an Ahlfors islands map by the classical Five Islands Theorem.

As noted above, g is then also an Ahlfors islands map.

Each Δ_k is contained in a wandering domain. First suppose that W is hyperbolic. If Δ_1 were contained in an eventually periodic domain, then there would be some $k \in \mathbb{N}$ and $l > k$ such that z_k and z_l can be connected by a curve in the Fatou set $F(g)$. Recall that the diameter of Δ_k in the hyperbolic distance of W is at most M. Hence we would have, by Pick's theorem, that

$$\begin{aligned} \mathrm{dist}_{F(f)}(z_k, z_l) &\geq \mathrm{dist}_{F(f)}(g^m(z_k), g^m(z_l)) \geq \mathrm{dist}_W(g^m(z_k), g^m(z_l)) \\ &\geq \mathrm{dist}_W(z_{k+m}, z_{l+m}) - 2M \to \infty \end{aligned}$$

as $m \to \infty$ by (3.1). This is a contradiction.

If W is parabolic, then we can likewise show that Δ_k must be contained in a wandering domain. Indeed, since $g(\Delta_k) \subset \Delta_{k+1}$, we have

$$\left|\log|g^{k+1}(z)|\right| \geq k \cdot \left|\log|g^k(z)|\right| \tag{3.2}$$

for all $z \in \Delta_1$.

It is well known that there is a bound on the speed of escape in Baker domains [B1, Lemma 7]. More precisely, if z is eventually mapped to a Baker domain of period l, then for sufficiently large k and a suitable constant C, we have

$$\left|\log|g^{k+l}(z)|\right| \leq C \cdot \left|\log|g^k(z)|\right|.$$

But this would contradict (3.2). Thus Δ_1 is contained in a wandering domain, and hence so is each Δ_k.

By construction and Lemma 2.8, the ω-limit set of these wandering domains is exactly K, as claimed. □

4 Baker domains

To prove Theorem 1.1, we begin by forming a simply-connected domain around a given curve γ. The following lemma shows that this is always possible. This result is surely known, but we do not know of a reference, and therefore include a proof in Section 7.

Lemma 4.1 (Simply-connected domain around a curve)**.** *Let W be a non-compact Riemann surface, with a metric d on W (compatible with the topology). Furthermore, let $\gamma : [0, \infty) \to W$ be an injective curve such that $\gamma(t) \to \infty$ as $t \to \infty$ (in the one-point compactification $\widehat{W}$). Also let $\delta : [0, \infty) \to (0, \infty)$ be continuous. Then there exists a simply-connected domain $V \subset W$ with $\gamma \subset V$ and a conformal isomorphism $\phi : \mathbb{H} \to V$ such that*

(a) $V \subset \bigcup_{t\geq 0}\{z \in W : d(z, \gamma(t)) < \delta(t)\}$*;*

(b) *ϕ extends continuously to a homeomorphism between the closures $\overline{\mathbb{H}}_{\mathbb{C}}$ and $\overline{V}_W$ of $\mathbb{H}$ and V in $\mathbb{C}$ respectively, W;*

(c) *$\phi(z) \to \infty$ in $\widehat{W}$ as $z \to \infty$ in $\mathbb{H}$;*

(d) *if $\alpha \subset V$ is any curve that tends to ∞ in $\widehat{W}$, then points of α have uniformly bounded distance to γ, and vice versa.*

If γ is C^1, then furthermore the domain V can be chosen such that

(e) $\operatorname{Re}\phi^{-1}(\gamma(t)) \to \infty$ *as* $t \to \infty$.

Remark 1. We do not use (e) in the proof of Theorem 1.1, but require it in the next section to prove Theorem 1.4.

Remark 2. Suppose (under the hypotheses of Lemma 4.1) that W is a hyperbolic subdomain of some compact Riemann surface Y and d is the hyperbolic metric on W. Then (d) implies that any curve $\alpha \subset V$ tending to ∂W has the same accumulation set as γ.

Given a domain V as in the lemma, we aim to construct a holomorphic function $g : W \to X$ that maps V into itself. To do so, we require an approximation theorem that allows us to control the error in the (potentially very thin) strip V. The following result provides the kind of control that we are looking for.

Lemma 4.2 (Control of approximation on simply-connected sets). *Let X be a compact Riemann surface, let $W \subsetneq X$ be a domain, and let $A \subset W$ be a Weierstrass set. Suppose that $V \subset \mathbb{C}$ is a simply-connected domain, let $\phi : \mathbb{H} \to V$ be a Riemann map, and set $V' := \phi(\{\zeta \in \mathbb{H} : \operatorname{Re}\zeta \geq 1\})$. Now let $f : A \to \mathbb{C}$ be continuous on A and holomorphic in the interior of A. Suppose furthermore that A can be written as the disjoint union of two relatively closed subsets, A_1 and A_2, and that $f(A_1) \subset V'$. Then for every $\varepsilon > 0$, there exists a holomorphic function $g : W \to \mathbb{C}$ such that*

(a) *$|f(z) - g(z)| \leq \varepsilon$ for $z \in A_2$, and*

(b) *for all $z \in A_1$, $g(z) \in V$ and $\left|\phi^{-1}(f(z)) - \phi^{-1}(g(z))\right| \leq \varepsilon$.*

Proof. Consider the function

$$\Psi : V \to \mathbb{C} \setminus \{0\};\ z \mapsto \lambda \cdot \left(z - \phi\left(\phi^{-1}(z) + \frac{1}{2}\right)\right),$$

where $\lambda > 0$. We claim that λ can be chosen such that $B(z, |\Psi(z)|) \subset V$ for all $z \in V'$ and furthermore

$$\phi^{-1}(B(z, |\Psi(z)|)) \subset B(\phi^{-1}(z), \varepsilon). \tag{4.1}$$

Indeed, let $z \in V'$ and set $\zeta := \phi^{-1}(z)$. By Koebe's theorem, there is a constant C such that $|\phi(\zeta) - \phi(\zeta + 1/2)| \leq C|\phi'(\zeta)|$ and

$$\phi(B(\zeta, \varepsilon)) \supset B\big(\phi(\zeta), \min(1, \varepsilon) \cdot |\phi'(\zeta)|/4\big).$$

Hence, if $\lambda < \min(1, \varepsilon)/4C$, the claim follows.

Since V is simply-connected and $\Psi(z) \neq 0$, there exists a holomorphic function $\psi : V \to \mathbb{C}$ such that $\exp(\psi(z)) = \Psi(z)$ for $z \in V$.

We define

$$E : A \to \mathbb{C}; \quad z \mapsto \begin{cases} \psi(f(z)) & z \in A_1 \\ \log \varepsilon & z \in A_2. \end{cases}$$

Since A_1 and A_2 are relatively closed, the function E is continuous on A and holomorphic in the interior of A. Applying Lemma 2.7 to f and E, we obtain a holomorphic function $g : W \to \mathbb{C}$ such that $|g(z) - f(z)| \leq |\exp(E(z))|$ for $z \in A$.

So we have $|g(z) - f(z)| \leq \varepsilon$ on A_2. Furthermore, suppose that $z \in A_1$, so $f(z) \in V'$ by hypothesis. Then, by the above, $g(z) \in B(f(z), |\Psi(f(z))|) \subset V$ and

$$\phi^{-1}(g(z)) \in \phi^{-1}(B(f(z), |\Psi(f(z))|)) \subset B(\phi^{-1}(f(z)), \varepsilon)$$

by (4.1). This completes the proof. □

We are now ready to prove Theorem 1.1 and Proposition 1.2.

Proof of Theorem 1.1. Let π be defined as in the proof of Theorem 1.3. Thus $\pi : \mathbb{C} \to X$ is a projection when X is a torus; otherwise, $\pi : \hat{\mathbb{C}} \to X$ is a Möbius transformation taking ∞ to a given point $a \in \hat{\mathbb{C}} \setminus \gamma$. Also pick discrete sets B and C, disjoint from γ, as in the proof of Theorem 1.3.

We now apply Lemma 4.1, taking d to be a flat metric (if W is the plane or the punctured plane) or the hyperbolic metric in W (otherwise). The function δ is any continuous function into the positive reals with the property that $d(\gamma(t), B \cup C) > \delta(t)$ for all $t \in [0, \infty)$.

Let V be the simply-connected domain given by Lemma 4.1, and let $\phi : \mathbb{H} \to V$ be the corresponding conformal isomorphism, whose continuous extension to $\overline{\mathbb{H}}_{\mathbb{C}}$ we also denote by ϕ. Let $\tilde{V}$ be a component of $\pi^{-1}(V)$; then $\tilde{V} \subset \mathbb{C}$ is a simply-connected domain and $\pi : \tilde{V} \to V$ is a conformal isomorphism. Let $\tilde{\phi} : \mathbb{H} \to \tilde{V}$ be the conformal isomorphism satisfying $\pi \circ \tilde{\phi} = \phi$.

We use Lemma 4.2 to construct a strongly non-normal function $h : W \to \mathbb{C}$ that maps $\overline{V}_W$ into $\tilde{V}$.

To do so, set $A := \overline{V}_W \cup B \cup C$. Note that A is a Weierstrass set by Theorem 2.6. Define a function $f : A \to \mathbb{C}$, continuous on A and holomorphic in the interior of A, such that

(a) $f(z) := \tilde{\phi}(\phi^{-1}(z) + 2)$ for $z \in \overline{V}_W$;
(b) f is bounded on B;
(c) $f(c) \to \infty$ as $c \to \partial W$ within C.

Set $A_1 := \overline{V}_W$ and $A_2 := B \cup C$. Then we can apply Lemma 4.2 (with some $\varepsilon < 1$), to obtain a holomorphic function $h : W \to \mathbb{C}$. As in the proof of Theorem 1.3, h is an Ahlfors islands map, as is $g := \pi \circ h$.

We have $h(\overline{V}_W) \subset \tilde{V}$ by Lemma 4.2 (b). Let us define $G : \overline{\mathbb{H}}_{\mathbb{C}} \to \mathbb{H}$ by

$$G(\zeta) := \phi^{-1}(g(\phi(\zeta))) = \tilde{\phi}^{-1}(h(\phi(\zeta))).$$

Then, for any $\zeta \in \overline{\mathbb{H}}_{\mathbb{C}}$, we have (setting $z = \phi(\zeta)$)

$$|G(\zeta) - (\zeta + 2)| = |\tilde{\phi}^{-1}(h(z)) - \tilde{\phi}^{-1}(f(z))| \leq \varepsilon < 1, \tag{4.2}$$

again using Lemma 4.2 (b).

In particular, $\operatorname{Re} G^n(\zeta) \to \infty$ as $n \to \infty$; hence if $z \in V$, then the sequence $(g^n(z))$ lies in V and has no accumulation point in W. By Lemma 2.8, there is a curve $\alpha \subset V$ whose accumulation set is exactly the accumulation set of the orbit of z. By the choice of V, the accumulation set of α is exactly the accumulation set K of γ (recall Remark 2 after Lemma 4.1).

Hence V is contained in a Baker domain U with $\omega(U) = K$, as required. $\square$

Proof of Proposition 1.2. We now show how to modify the proof of Theorem 1.1 in order to obtain the extra claim in Proposition 1.2. We use the same notation as in the preceding proof.

First we show that the claim holds for all w in the Baker domain U such that $g^k(z) \in V$ for some $k \geq 0$, provided the function $\delta(t)$ was chosen sufficiently small, depending on the sequence (ε_n). Then we indicate how to modify the construction to ensure that V is an *absorbing set* for U, i.e., that every point of U is mapped to V under iteration.

First note that we may assume without loss of generality that (ε_n) is a strictly decreasing sequence (otherwise, we consider the sequence $\left(1/n + \max_{k\geq n} \varepsilon_k\right)$ instead).

Fix $w_0 := \gamma(0)$ as a base point. If $w \in V$, then by (4.2), the point $G^n(\phi^{-1}(w))$ is contained in the disk of radius $n\varepsilon$ around $\phi^{-1}(w) + 2n$. It follows that

$$\operatorname{dist}_V(w_0, g^n(w)) = \operatorname{dist}_{\mathbb{H}}(\phi^{-1}(w_0), G^n(\phi^{-1}(w))) = O(\log n) \quad \text{as } n \to \infty. \tag{4.3}$$

On the other hand, we can clearly let $\delta(t)$ tend to zero sufficiently rapidly to ensure that, for all sufficiently large n and all $v \in V$ with $d(v, \partial W) \leq \varepsilon_n$,

$$\operatorname{dist}_V(w_0, v) > n.$$

Now let w be a point such that $g^k(w) \in V$ for some $k \in \mathbb{N}$. By (4.3), we have $\operatorname{dist}_V(w_0, g^{n+k}(w)) < n$ for sufficiently large n. So $d(g^{n+k}(w), \partial W) > \varepsilon_n > \varepsilon_{n+k}$, as desired.

It remains to show that we can ensure that V is an absorbing set for U. To do so, we modify the construction of f by also requiring that there are many points near the boundary of V that are not in the Baker domain U. This ensures that points of $U \setminus V$ have very large hyperbolic distance from the "central" part of V, from which it follows that the orbits of all points of U must enter V eventually.

More precisely, after picking V, B and C, we also pick a discrete closed set $Z \subset W \setminus (\overline{V} \cup B \cup C)$ such that the hyperbolic metric of the set $W' := W \setminus Z$ satisfies

$$\operatorname{dist}_{W'}(\phi(2n), W \setminus V) \to \infty \quad \text{as } n \to \infty. \tag{4.4}$$

Also let $\Delta \subset W$ be a disk whose closure is disjoint from the sets $\overline{V}$, B, C and Z, and let $\tilde{\Delta} \subset \mathbb{C}$ be another disk with $\pi(\tilde{\Delta}) \subset \Delta$.

Instead of letting the set A from the construction of h consist only of $\overline{V}_W$, B and C, we now set $A := \overline{V}_W \cup B \cup C \cup \overline{\Delta} \cup Z$. We define f as before on the first three sets, and such that $\overline{f(\Delta)} \subset \tilde{\Delta}$ and $\overline{f(Z)} \subset \tilde{\Delta}$.

If we choose ε sufficiently small in our application of Lemma 4.2, the approximating map h also has the above properties. Thus $g = \pi \circ h$ satisfies $\overline{g(\Delta)} \subset \Delta$, so Δ is contained in the basin of an attracting fixed point of g. Since $g(Z) \subset \Delta$, all points of Z are also attracted to this fixed point, and in particular $U \subset W'$.

Recall that, by (4.2), the orbit of the point $w_1 := \phi(0)$ satisfies

$$|\phi^{-1}(g^n(w_1)) - 2n| \leq n$$

for all $n \geq 0$. Hence, for $n \geq 1$, the hyperbolic distance $\mathrm{dist}_V(g^n(w_1), \phi(2n))$ is bounded by some constant C. If $w \in U$, then $\mathrm{dist}_U(g^n(w), g^n(w_1)) \leq \mathrm{dist}_U(w, w_1)$ by Pick's theorem. So if $w \in U$, then $\mathrm{dist}_U(\phi(2n), g^n(w)) \not\to \infty$ as $n \to \infty$. Hence, by (4.4), we have $g^n(w) \in V$ for sufficiently large n, as claimed. $\square$

5 Logarithmic tracts

Proof of Theorem 1.4. If $X = \hat{\mathbb{C}}$, let us assume that $a \neq w_0$. (The case $a = w_0$ is similar, but actually slightly simpler. We comment on this at the end of the proof.)

Once again we pick $\pi : \mathbb{C} \to X$ to be a projection when X is a torus, and otherwise a Möbius transformation with $\pi(\infty) = a$; we also require that π be chosen such that $\pi(0) = w_0$. Again we additionally pick discrete sets B and C as in the proof of Theorem 1.3.

Use Lemma 4.1 to pick a simply-connected domain V around γ whose closure is disjoint from B and C, and let $\phi : \mathbb{H} \to V$ be the corresponding Riemann map. Additionally, let U' be the component of $\pi^{-1}(U)$ containing 0; then U' is a Jordan domain in $\mathbb{C}$. Also let $O' \subset \mathbb{C}$ be a Jordan domain containing $\overline{U'}$, but sufficiently close to U' to ensure that π is still injective on $\overline{O'}$. (That this is possible follows from the "plane separation theorem"; see [W, Chapter VI, Theorem (3.1)].) Finally, set $O := \pi(O')$. Set $K := \sup_{z \in O'} |z|$.

We now let $f : V \to O' \setminus \{0\}$ be a universal covering with $f(\partial V \cap W) = \partial O'$. Since $\phi^{-1}(\gamma)$ is bounded away from $\partial\mathbb{H}$ (see Lemma 4.1(e)), we may also assume that f is chosen such that $f(\gamma) \subset U'$. As usual, we denote the extension to $\overline{V}_W$ by f also. Define f on B and C as in the proof of Theorems 1.3 and 1.1.

Using Lemma 2.7, we approximate f by a holomorphic function $h : W \to \mathbb{C}$ such that $|h - f|$ is uniformly bounded on $B \cup C$, and such that

$$|h(z) - f(z)| < \varepsilon \cdot |f(z)| \tag{5.1}$$

for $z \in \overline{V}_W$, where $\varepsilon < 1/2$ is chosen so small that $\varepsilon < \mathrm{dist}_{\mathbb{C}}(\partial U', f(\gamma))/K$ and $\varepsilon < \mathrm{dist}_{\mathbb{C}}(\partial O', U')/K$.

Then we have $h(\gamma) \subset U'$ and $h(z) \neq 0$ for all $z \in V$. Furthermore, the component T of $h^{-1}(U')$ containing γ does not intersect ∂V, and hence is completely contained in V. By the maximum principle, T is simply-connected.

We can thus form a logarithm $H := \log h : T \to \mathbb{C}$. Likewise, we can take a logarithm $F := \log f : V \to \mathbb{C}$. Note that $F : V \to F(V)$ is a conformal isomorphism by definition, and that $H(T) \subset F(V)$.

The approximation condition (5.1) implies that there is some $C > 0$ such that

$$|H(z) - F(z)| < C \tag{5.2}$$

for all $z \in T$. Consequently, it follows that $H(z) \to \infty$ as $z \to \partial T \cap \partial W$ in T. We also have $H(z) \to \partial U'$ as $z \to \partial T \cap W \subset V$. Hence H is a proper map, and therefore has a degree $d \geq 1$.

We claim that $d = 1$. It suffices to show that H has no critical points in T. So let $c \in T$, and pick some simple closed curve $\alpha \subset H(T)$ that does not pass through any of the – finitely many – critical values of H and such that α surrounds both $H(c)$ and a disk $D(z_0, C)$ of radius C around some point $z_0 \in H(T)$. This is possible because $H(T)$ contains a left half plane.

The component β of $H^{-1}(\alpha)$ that surrounds c is a simple closed curve. By choice of α, and by (5.2), the curves $H \circ \beta$ and $F \circ \beta$ (where we fix some parametrization of β) have the same winding number $N \neq 0$ around z_0. Since F is a conformal isomorphism, we must have $N = 1$, so c is not a critical point of H, as required.

So we have seen that H is a conformal isomorphism, and hence h maps T as a universal covering map over U', as claimed. Because $\operatorname{Re} \phi^{-1}(\gamma(t)) \to \infty$ as $t \to \infty$, we also have $h(\gamma(t)) \to 0$ as $t \to \infty$. Hence the map $g := \pi \circ h$ has the stated properties.

In the case where $X = \hat{\mathbb{C}}$ and $a = w_0$, the proof proceeds essentially as above except that now $\pi(\infty) = a = w_0$. In this case, U' and O' are neighborhoods of ∞. Also, we require the approximation to satisfy $|h(z) - f(z)| < \varepsilon$ instead of (5.1); thus, we can actually apply Theorem 2.6 directly instead of using Lemma 2.7. □

6 Hyperbolic surfaces

Proof of Theorem 1.6. The proofs of the three statements in the case where X is hyperbolic are entirely analogous to those in the case of genus ≤ 1, except that here we choose the map π to be a universal covering map from a subset of the complex plane to X, e.g. $\pi : \mathbb{D} \to X$.

The constructions then proceed as before. The composition $g := \pi \circ h$ is again an Ahlfors islands map and has the desired properties, but the domain of definition $\widetilde{W} \subset W$ of g is almost certainly not connected.

To obtain also the claim in Theorem 1.6 that g can be chosen with W' connected, we can simply restrict g to the component of $\widetilde{W}$ containing γ in the case of Baker domains or logarithmic tracts (Theorems 1.1 and 1.4). However, in the case of wandering domains (Theorem 1.3), it is possible that different components of the orbit of the wandering domain belong to different components of $\widetilde{W}$. To prevent this from happening, we need to modify the construction from the proof of Theorem 1.3 slightly.

This is easy to do: Indeed, choosing the sets B and C as well as the domains Δ_k as before, we connect the disks Δ_k by arcs to form a "tree" $T \subset W \setminus (B \cup C)$, which is a Weierstrass set whose interior coincides precisely with $\bigcup_{k=1}^{\infty} \Delta_k$. We can then carry out the approximation in a way that ensures that $h(T) \subset \mathbb{D}$ (recall that $\mathbb{D}$ is the domain of definition of π), and hence such that all domains Δ_k are contained in the same component of the domain of definition of g. $\square$

7 Proof of Lemma 4.1

Proof of Lemma 4.1. We begin by noting that we will construct V with the additional property that

(d') if $\alpha \subset V$ is any curve that tends to ∞ in $\widehat{W}$, then $d(\alpha, \gamma(t)) < \delta(t)$ for all sufficiently large t.

Clearly we may assume without loss of generality that δ is bounded from above, say $\delta(t) \leq 1$, so (d'), together with (a), does indeed imply part (d) of the lemma. Furthermore, we may assume (modifying the function δ as necessary) that d is a natural conformal metric on W (i.e., the hyperbolic metric if W is hyperbolic, or the euclidean metric if $W = \mathbb{C}$ or $W = \mathbb{C}/\mathbb{Z}$). Let us denote balls with respect to this metric by $B_d(z, r)$.

We may furthermore assume for simplicity that δ is non-increasing, and that the closed ball $\overline{B_d(\gamma(t), \delta(t))}$ is homeomorphic to the closed unit disk (i.e., the injectivity radius of W at $\gamma(t)$ is larger than $\delta(t)$).

We now define a function $\hat{\delta}(t)$ as follows. For $t_0 \in [0, \infty)$, let

$$t_- := \min\big\{t \in [0, t_0] : \gamma\big((t, t_0]\big) \subset B_d(\gamma(t_0), \delta(t_0)/2)\big\} \quad \text{and}$$
$$t_+ := \max\big\{t \in [t_0, \infty) : \gamma\big([t_0, t)\big) \subset B_d(\gamma(t_0), \delta(t_0)/2)\big\}.$$

We define

$$\hat{\delta}(t_0) := \frac{1}{2} \cdot \inf_{t \notin [t_-, t_+]} d(\gamma(t), \gamma(t_0)) > 0.$$

Note that $\hat{\delta}(t_0) \le \delta(t_0)/4$ because $d(\gamma(t_+), \gamma(t_0)) = \delta(t_0)/2$.

Let us set $V_1 := \bigcup_{t\ge 0} B_d(\gamma(t), \hat{\delta}(t))$ and study the set $\widetilde{V_1}$, where $\widetilde{U}$ denotes the union of a set U and its compact complementary components. The construction ensures that

(1) $\widetilde{V_1}$ is simply-connected;
(2) $\widetilde{V_1} \subset \bigcup_{t\ge 0}\{z \in W : d(z, \gamma(t)) < \delta(t)\}$;
(3) if $t_3 \gg t_2 \gg t_1$, then $\gamma(t_1)$ and $\gamma(t_3)$ belong to different connected components of $\widetilde{V_1} \setminus B_d(\gamma(t_2), \delta(t_2))$.

Property (3) implies, in particular, that $\widetilde{V_1}$ – and hence any subset V of $\widetilde{V_1}$ – satisfies (d′). Now let $\phi : \mathbb{H} \to \widetilde{V_1}$ be a conformal isomorphism, which we can assume normalized so that $\phi^{-1}(\gamma)$ accumulates at ∞. Then (3) ensures – e.g., by using the theory of prime ends – that $\phi(z) \to \infty$ in $\widehat{W}$ iff $z \to \infty$ in $\mathbb{H}$, and in particular that $\phi^{-1}(\gamma)$ accumulates only at ∞. Thus the domain $\widetilde{V_1}$ has all the properties required in the statement of the theorem, apart possibly from (b). However, we can choose a domain $H \subset \mathbb{H}$ such that $\partial H \cap \mathbb{C}$ is a simple curve contained in $\mathbb{H}$ and such that $\phi^{-1}(\gamma) \subset H$. Then $V := \phi(H)$ has the desired properties.

Now let us suppose that the curve γ is C^1, and use a different construction to obtain a domain also satisfying (e). First we extend γ to be a C^1 curve on $(-1, \infty)$ and reparametrise if necessary to ensure that γ is ‘unit speed’. For $t > -1$, let $I(t)$ denote the line segment with centre at $\gamma(t)$, length $2\varepsilon(t)$ and normal to the curve γ at $\gamma(t)$. Here $\varepsilon : (-1, \infty) \to (0, \infty)$ is a sufficiently rapidly decreasing convex function. The fact that γ is locally uniformly C^1 implies that, if ε is chosen sufficiently small, then $V = \bigcup_{-1<t<\infty} I(t)$ forms a simply-connected tubular neighbourhood of $\gamma|_{[0,\infty)}$ in W. It is easy to check that V satisfies parts (a), (b), (c) and (d) if we choose the function ε small enough.

To prove part (e), let $z^+(t)$ and $z^-(t)$ denote the endpoints of $I(t)$, and let

$$\partial^+ V = \bigcup_{-1<t<\infty} z^+(t) \quad \text{and} \quad \partial^- V = \bigcup_{-1<t<\infty} z^-(t).$$

We claim that, provided the function ε was chosen sufficiently small, there exists an absolute constant c, $0 < c < 1$, such that, for all $t \ge 0$,

$$\omega(\gamma(t), \partial^+ V, V) \le c, \quad \text{and} \quad \omega(\gamma(t), \partial^- V, V) \le c, \tag{7.1}$$

where $\omega(z, E, V)$ denotes the harmonic measure in V of the set $E \subset \partial V$ at the point $z \in V$. The inequalities (7.1) imply that for large t, the point $\phi^{-1}(\gamma(t))$ lies in a set of the form $\{\zeta : \operatorname{Re} \zeta \ge C|\zeta|\}$, where $C > 0$, which proves part (e).

To prove (7.1), note that for each $t > 0$, the line segment $I(t)$ lies on the symmetric axis of a right-angled isosceles triangle, $\Delta(t)$ say, with vertex at $z^+(t)$, height $4\varepsilon(t)$ and base outside the domain V. Hence, by the maximum principle, $\omega(\gamma(t), \partial^+ V, V)$ is dominated by the harmonic measure in $\Delta(t)$ of the union of the two equal-length sides of $\Delta(t)$ at $\gamma(t)$, and this value is an absolute constant in $(0, 1)$. This proves (7.1). □

Remark. Note that the proof shows that, provided γ is C^1, the domain V can be chosen such that $\phi^{-1}(\gamma(t)) \to \infty$ *nontangentially*. It is not difficult to see that, without a regularity assumption on γ, this cannot always be achieved.

For the conclusion of the lemma, we only require that $\phi^{-1}(\gamma)$ eventually enters every right half plane; i.e. that it tends to ∞ *horocyclically*. This can be achieved under much weaker conditions; e.g. it is sufficient to assume that there is a sequence (t_j) with $t_j \to \infty$ such that γ is differentiable at t_j for all j.

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References

[BDH] I. Noel Baker, Patricia Domínguez, and Matthew E. Herring, *Functions meromorphic outside a small set: completely invariant domains*, Complex Var. Theory Appl. **49** (2004), 95–100.

[BKZ] Krzysztof Barański, Bogusława Karpińska, and Anna Zdunik, *Hyperbolic dimension of Julia sets of meromorphic maps with logarithmic tracts*, Int. Math. Res. Not. IMRN **2009**, 615–624.

[B1] Walter Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), 151–188,

[B2] Walter Bergweiler, *A new proof of the Ahlfors five islands theorem*, J. Anal. Math. **76** (1998), 337–347.

[DS] Patricia Domínguez and Guillermo Sienra, *Doubly connected wandering domains for meromorphic functions*, Complex Var. Elliptic Equ. **55** (2010), 407–416.

[E1] Adam L. Epstein, *Towers of finite type complex analytic maps*, Ph.D. thesis, City Univ. of New York, 1995, available at http://pcwww.liv.ac.uk/ lrempe/adam/thesis.pdf.

[EL] Alexandre E. Eremenko and Mikhail Ju. Ljubich, *Examples of entire functions with pathological dynamics*, J. London Math. Soc. (2) **36** (1987), 458-468.

[E2] Alexandre E. Eremenko and Mikhail Ju. Ljubich, *Dynamics of finite type complex analytic maps I: Global structure theory*, manuscript.

[EO] Adam Epstein and Richard Oudkerk, *Iteration of Ahlfors and Picard functions which overflow their domains*, manuscript.

[G] Dieter Gaier, *Vorlesungen über Approximation im Komplexen*, Birkhäuser Verlag, Basel, 1980.

[L] Pierre Lavaurs, *Systèmes dynamiques holomorphes: explosion de points periodiques paraboliques*, doctoral thesis, Université de Paris-Sud, Orsay, France, 1989.

[M] John Milnor, *Dynamics in One Complex Variable*, third ed., Princeton University Press, Princeton, NJ, 2006.

[O] Richard Oudkerk, *Ahlfors functions with exotic Baker domains*, manuscript, 2004.

[R] Lasse Rempe, *Hyperbolic dimension and radial Julia sets of transcendental functions*, Proc. Amer. Math. Soc. **137** (2009), 1411–1420.

[RS] Philip J. Rippon and Gwyneth M. Stallard, *Slow escaping points of meromorphic functions*, Trans. Amer. Math. Soc. **363** (2011), 4171–4201.

[S] Stephen Scheinberg, *Uniform approximation by functions analytic on a Riemann surface*, Ann. of Math (2) **108** (1978), 257–298.

[W] Gordon T. Whyburn, *Analytic Topology*, American Mathematical Society, New York, 1942.

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