ON THE FURSTENBERG MEASURE AND DENSITY OF STATES FOR THE ANDERSON-BERNOULLI MODEL AT SMALL DISORDER

By

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Abstract. We establish new results on the dimension of the Furstenberg measure and the regularity of the integrated density of states for the Anderson-Bernoulli model at small disorder.

0 Summary

Let $H = \Delta + \lambda V$, where Δ is the lattice Laplacian on $\mathbb{Z}$ and $V = (V_n)_{n\in\mathbb{Z}}$ are independent random variables in $\{1, -1\}$. We assume small $|\lambda|$ and restrict the energy E to be outside of a fixed neighborhood of $\{0, 2, -2\}$. We then show that the Furstenberg measure ν_E of the corresponding $SL_2(\mathbb{R})$-cocycle

$$\begin{pmatrix} E - \lambda V_n & -1 \\ 1 & 0 \end{pmatrix}$$

has dimension at least $\gamma(\lambda)$, where $\gamma(\lambda) \overset{\lambda\to 0}{\longrightarrow} 1$. As a consequence, we derive that the integrated density of states (IDS) $\mathcal{N}(E)$ is Hölder-regular with exponent at least $s(\lambda) \overset{\lambda\to 0}{\longrightarrow} 1$.

The spectral theory of the Anderson-Bernoulli (A-B) model has been studied by various authors. It was shown by Halperin [S-T] that for fixed $\lambda > 0$, $\mathcal{N}(E)$ is not Hölder continuous of any order α larger than

$$\alpha_0 = \frac{2\log 2}{\text{Arc }\cosh(1+\lambda)}. \tag{0.1}$$

Hölder regularity for some $\alpha > 0$ has been established in several papers. In [Ca-K-M], le Page's method is used. Different approaches (including one using the super-symmetric formalism) appear in the important paper [S-V-W] that

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relies on harmonic analysis principles around the uncertainty principle. In [B1], the author proved Hölder regularity of the IDS using the Figotin-Pastur expansion of the Lyapounov exponent and martingale theory. We note that in both [S-V-W] and [B1], the Hölder exponent α remains uniform for $\lambda \to 0$. (In fact, [B1] gives an explicit exponent $\alpha(\lambda) > 1/5 + \varepsilon$ as $\lambda \to 0$.)

Thus, the result in this note just falls short of establishing the conjectured Lipschitz regularity of IDS of the A-B model for small λ. Related is the question whether the Furstenberg measure on projective space is absolutely continuous when λ is small (or even better). As pointed out at the end of the paper, a natural approach to these problems is through certain spectral gap properties which do not depend on hyperbolicity. There have been recent advances (cf. [BG1, BG2, B2]) which are based on methods from arithmetic combinatorics. But at present, this theory seems too restrictive for an application to A-B-cocycles. It does apply, however, for Schrödinger operators with single site distribution given by a measure of positive dimension.

1 Probabilistic inequalities on the Boolean cube

The following statement is a consequence of Sperner's Combinatorial Lemma[1].

Lemma 1. *Let $f = f(\varepsilon_1, \dots, \varepsilon_n)$ be a real valued function on $\{1, -1\}^n$ and let*

$$I_j = f|_{\varepsilon_j = 1} - f|_{\varepsilon_j = -1} \tag{1.1}$$

denote the j-influence, which is a function of $\varepsilon_{j'}$, $j' \neq j$.

Assume that for all $j = 1, \dots, n$,

$$I_j \geq 0 \tag{1.2}$$

(i.e., f is monotone increasing) and moreover

$$I_j \geq \kappa > 0 \text{ on } \Omega_j \cap \Omega'_j, \tag{1.3}$$

where Ω_j (respectively, Ω'_j) are subsets of $\{1, -1\}^n$ depending only on the variables $\varepsilon_1, \dots, \varepsilon_{j-1}$ (respectively, $\varepsilon_{j+1}, \dots, \varepsilon_n$). Then, for any $t \in \mathbb{R}$,

$$mes\left[|f - t| < \frac{\kappa}{2}\right] \leq \frac{1}{\sqrt{n}} + \sum_j (2 - mes\,\Omega_j - mes\,\Omega'_j). \tag{1.4}$$

[1]It was also used in [B1] and [B-K] in the context of the Anderson-Bernoulli model.

Proof. Let

$$\tilde{\Omega} = \bigcap_{1 \le j \le n} (\Omega_j \cap \Omega'_j), \tag{1.5}$$

for which

$$1 - \operatorname{mes} \tilde{\Omega} \le \sum_j (2 - \operatorname{mes} \Omega_j - \operatorname{mes} \Omega'_j). \tag{1.6}$$

We claim that the set $[|f - t| < \kappa] \cap \tilde{\Omega}$ does not contain a pair of distinct comparable elements $\varepsilon = (\varepsilon_j)_{1 \le j \le n}$ and $\varepsilon' = (\varepsilon'_j)_{1 \le j \le n}$. Assume otherwise. Let $\varepsilon < \varepsilon'$, i.e., $\varepsilon_j \le \varepsilon'_j$ for each j. Then

$$\begin{aligned} f(\varepsilon') - f(\varepsilon) &= \sum_{1 \le j \le n} \big(f(\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon'_j, \dots, \varepsilon'_n) \\ &\qquad - f(\varepsilon_1, \dots, \varepsilon_j, \varepsilon'_{j+1}, \dots, \varepsilon'_n) \big) \\ &= \sum_{\substack{1 \le j \le n \\ \varepsilon_j \ne \varepsilon'_j}} I_j(\varepsilon_1, \dots, \varepsilon_j, \varepsilon'_{j+1}, \dots, \varepsilon'_n). \end{aligned} \tag{1.7}$$

Since $\varepsilon \in \Omega_j$, $\varepsilon' \in \Omega'_j$, it follows from our assumption on $\Omega_j, \Omega_{j'}$ that

$$(\varepsilon_1, \dots, \varepsilon_j, \varepsilon'_{j+1}, \dots, \varepsilon'_n) \in \Omega_j \cap \Omega'_j$$

and hence $I_j(\varepsilon_1, \dots, \varepsilon_j, \varepsilon'_{j+1}, \dots \varepsilon'_n) \ge \kappa$ by (1.3). In particular, since $\varepsilon \ne \varepsilon'$,

$$(1.7) \ge \#\{1 \le j \le n; \varepsilon_j \ne \varepsilon'_j\}\kappa \ge \kappa,$$

which is, however, impossible if $|f(\varepsilon) - t| \le \kappa/2$ and $|f(\varepsilon') - t| \le \kappa/2$. This establishes the claim.

Therefore, by Sperner's lemma on the maximal size of subsets of $\{1, -1\}^n$ not containing any pair of distinct comparable elements, we get

$$\operatorname{mes}(\tilde{\Omega} \cap [|f - t| < \kappa]) \lesssim \frac{1}{\sqrt{n}}, \tag{1.8}$$

and (1.4) follows from (1.6) and (1.8). □

We use the following corollary to Lemma 1.

Lemma 2. *Let f and I_j be as in Lemma 1 and assume each $I_j \ge 0$. Assume further $\kappa, \delta > 0$ and for each $1 \le j < n$,*

$$f|_{\varepsilon_j = 1, \varepsilon_{j+1} = 1} - f|_{\varepsilon_j = -1, \varepsilon_{j+1} = -1} \ge \kappa \text{ for } \varepsilon \in \Omega_j, \tag{1.9}$$

where $\Omega_j \subset \{1,-1\}^n$ *is a set only depending on the variables* $\varepsilon_{j+2},\dots,\varepsilon_n$ *and such that*

$$\text{(1.10)} \qquad mes\,\Omega_j > 1-\delta.$$

Then, for all $t \in \mathbb{R}$,

$$\text{(1.11)} \qquad mes\left[|f-t| < \frac{\kappa}{2}\right] \lesssim \frac{1}{\sqrt{n}} + n\delta.$$

Proof. Assume even $n = 2m$ and write $\omega = (\varepsilon_1, \varepsilon_1', \dots, \varepsilon_m, \varepsilon_m')$ for the $\{1,-1\}^n$-variable. With this notation, let Ω_j refer to the set Ω_{2j-1}.

Consider the partition $\{1,-1\}^{2m} = \bigcup_{S\subset\{1,\dots,m\}} V_S$ with

$$\text{(1.12)} \qquad V_S = \{\omega; \varepsilon_j = \varepsilon_j' \text{ if } j\in S \text{ and } \varepsilon_j \neq \varepsilon_j' \text{ if } j \notin S\}.$$

Thus

$$\text{(1.13)} \qquad \text{mes}\left[|f-t| < \frac{\kappa}{2}\right] = \sum_{S\subset\{1,\dots,m\}} \text{mes}\left[V_S \cap |f-t| < \frac{\kappa}{2}\right].$$

Fix $S \subset \{1,\dots,m\}$.

We consider f on V_S as a function of $(\varepsilon_j)_{j\in S}$ with the other variables $(\varepsilon_j, \varepsilon_j')_{j\notin S}$ fixed. Denoting this function on $\{1,-1\}^{|S|}$ by $g = g(\varepsilon_j; j\in S)\}$, we have by our assumption (1.9), for $j \in S$,

$$\begin{aligned} I_j(g)(\varepsilon_j, j\in S) &= f(\varepsilon_1,\varepsilon_1',\dots,\varepsilon_{j-1},\varepsilon_{j-1}',1,1,\varepsilon_{j+1},\varepsilon_{j+1}',\dots,\varepsilon_n,\varepsilon_n') \\ &\quad - f(\varepsilon_1,\varepsilon_1',\dots,\varepsilon_{j-1},\varepsilon_{j-1}',-1,-1,\varepsilon_{j+1},\dots,\varepsilon_n') \\ &\geq \kappa \end{aligned}$$

provided

$$\begin{aligned} (\varepsilon_k)_{k\in S} \in \Omega_j' &= \{(\varepsilon_k)_{k\in S}; \big((\varepsilon_k,\varepsilon_k)_{k\in S}, (\varepsilon_k,\varepsilon_k')_{k\notin S}\big) \in \Omega_j\} \\ &= (\Omega_j \cap V_S)\,(\varepsilon_k,\varepsilon_k'; k\notin S) \subset \{1,-1\}^{|S|}, \end{aligned}$$

which depends only on $(\varepsilon_k)_{k\in S, k>j}$. (Recall that we have fixed the variables outside S).

Applying Lemma 1 to g (with $\Omega_j = \{1,-1\}^{|S|}$ for all $j\in S$), we obtain

$$\begin{aligned} \#\left[\omega\in V_S; |f(\omega)-t| < \frac{\kappa}{2}\right] &\leq \frac{\#V_S}{|S|^{1/2}} + \sum_{j\in S}\sum_{\varepsilon_k\neq\varepsilon_k', k\notin S}\big(2^{|S|} - \#(\Omega_j\cap V_S)(\varepsilon_k,\varepsilon_k'; k\notin S)\big) \\ \text{(1.14)} \qquad &= \frac{\#V_S}{|S|^{\frac{1}{2}}} + \sum_{j\in S}\#(V_S\backslash\Omega_j). \end{aligned}$$

Summing over $S \subset \{1, \dots, m\}$ gives

$$(1.15) \qquad \begin{aligned} (1.13) &\leq 2^{-m} \sum_{S \subset \{1,\dots,m\}} \frac{1}{|S|^{1/2}} + \sum_{j=1}^{n} \operatorname{mes}(\Omega \backslash \Omega_j) \\ &\lesssim \left(\frac{m}{2}\right)^{-1/2} + n\delta \end{aligned}$$

and hence (1.11). □

2 Application to the Anderson-Bernoulli model

Consider the projective action of $SL_2(\mathbb{R})$ on $P_1(\mathbb{R}) \simeq \mathbb{T} = \mathbb{R}/\mathbb{Z}$, defined for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

by

$$(2.1) \qquad e^{i\tau_g(\theta)} = \frac{(a\cos\theta + b\sin\theta) + i(c\cos\theta + d\sin\theta)}{[(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2]^{1/2}}.$$

Then

$$(2.2) \qquad \begin{aligned} (\tau_g)'(\theta) &= \frac{\sin^2 \tau_g(\theta)}{(c\cos\theta + d\sin\theta)^2} \\ &= \frac{1}{[(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2]^{1/2}} \end{aligned}$$

and

$$(2.3) \qquad \|g\|^2 \geq (\tau_g)' \geq \frac{1}{\|g\|^2}.$$

Consider the Anderson-Bernoulli model (A-B model)

$$(2.4) \qquad H_\lambda(\varepsilon) = \lambda\varepsilon_n\delta_{nn'} + \Delta$$

with $\varepsilon = (\varepsilon_n)_{n\in\mathbb{Z}} \in \{1, -1\}^{\mathbb{Z}}$ at small disorder $\lambda > 0$ (Here, Δ stands for the usual lattice Laplacian).

The corresponding transfer operators $M_N(E) \in SL_2(\mathbb{R})$ are given by

$$(2.5) \qquad \begin{aligned} M_N &= M_N(E;\varepsilon) \\ &= \begin{pmatrix} E - \lambda\varepsilon_N & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - \lambda\varepsilon_{N-1} & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - \lambda\varepsilon_1 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \prod_N^1 g_E(\varepsilon_j). \end{aligned}$$

Considering ε_j $(1 \leq j \leq N)$ as a continuous variable on $[-1, 1]$ and the corresponding partial derivative ∂_j, we have for the projective action

$$\tau_{M_N} = \tau_{g_E(\varepsilon_N)\cdots g_E(\varepsilon_{j+1})^o}\ \tau_{g_E(\varepsilon_j)^o}\tau_{g_E(\varepsilon_{j-1})\cdots g_E(\varepsilon_1)}$$

and

$$(\partial_j \tau_{M_N})(\theta) = \tau'_{g_E(\varepsilon_N)\cdots g_E(\varepsilon_{j+1})}(\tau_{g_E(\varepsilon_j)\cdots g_E(\varepsilon_1)}(\theta)).(\partial_j \tau_{g_E})(\tau_{g_E(\varepsilon_{j-1})\cdots g_E(\varepsilon_1)}). \tag{2.6}$$

Since

$$\operatorname{cotg} \tau_{g_E(\varepsilon)}(\theta) = (E - \lambda\varepsilon) - \frac{\sin\theta}{\cos\theta}, \tag{2.7}$$

we have

$$\begin{aligned}(\partial_\varepsilon \tau_{g_E})(\theta) &= \lambda . \sin^2 \tau_{g_E(\varepsilon)}(\theta) \\ &= \lambda \frac{\cos^2\theta}{\cos^2\theta + ((E-\lambda\varepsilon)\cos\theta - \sin\theta)^2} \sim \lambda \cos^2\theta.\end{aligned} \tag{2.8}$$

From (2.3), (2.6), and (2.8), we have

$$\begin{aligned}(\partial_j \tau_{M_N})(\theta) &\gtrsim \frac{\lambda}{\|g_E(\varepsilon_N)\cdots g_E(\varepsilon_{j+1})\|^2} \cos^2 \tau_{g_E(\varepsilon_{j-1})\cdots g_E(\varepsilon_1)}(\theta) \\ &= \frac{\lambda}{\|M_{N-j}(E; \varepsilon_{j+1}, \ldots, \varepsilon_N)\|^2} \cos^2 \tau_{M_{j-1}(E,\varepsilon)}(\theta).\end{aligned} \tag{2.9}$$

In order to deal with the issue of $\cos \tau_{M_{j-1}(E,\varepsilon)}(\theta)$ being small, note that by (2.7), for all θ, ε,

$$|\cos\theta| + |\cos \tau_{g_E(\varepsilon)}(\theta)| > c. \tag{2.10}$$

Hence (2.9) implies

$$(\partial_j \tau_{M_N})(\theta) + (\partial_{j+1} \tau_{M_N})(\theta) \gtrsim \frac{\lambda}{\|M_{N-j}(E; \varepsilon_{j+1}, \ldots, \varepsilon_N)\|^2} \tag{2.11}$$

(for all θ).

In order to satisfy (1.9), we need an upper bound on $\|M_n(E; \varepsilon_1, \ldots, \varepsilon_n)\|$. This function can be analyzed using the Figotin-Pastur expansion.

Let

$$E = 2\cos\kappa, \quad 0 \leq \kappa \leq \pi, \tag{2.12}$$

$$V_n = -\frac{\varepsilon_n}{\sin\kappa}, \tag{2.13}$$

where we assume $\delta_0 < |E| < 2-\delta_0$ and hence κ stays away from $0, \pi/2, \pi$. (Here, δ_0 is a fixed constant independent of λ.)

The Figotin-Pastur formula gives

$$\frac{1}{N}\log\|M_N(E,\varepsilon)\| = \frac{1}{2N}\sum_1^N \log\left(1+\lambda V_n \sin 2(\varphi_n+\kappa)+\lambda^2 V_n^2\sin^2(\varphi_n+\kappa)\right) \tag{2.14}$$

with

$$\zeta_n = e^{2i\varphi_n} \tag{2.15}$$

recursively given by

$$\zeta_{n+1} = \mu\zeta_n + i\frac{\lambda}{2}V_n\frac{(\mu\zeta_n-1)^2}{1-i\lambda V_n(\mu\zeta_n-1)/2} \tag{2.16}$$

and

$$\mu = e^{2i\kappa}. \tag{2.17}$$

Note that by (2.13) and (2.16), ζ_n depends only on $\varepsilon_{n'}$ for $n' \leq n-1$.

Expanding (2.14), we obtain

$$(2.14) = \frac{\lambda^2}{8N}\sum_1^N V_n^2 \tag{2.18}$$

$$+\frac{\lambda}{2N}\sum_1^N V_n\sin(\varphi_n+\kappa) \tag{2.19}$$

$$-\frac{\lambda^2}{4N}\sum_1^N V_n^2\cos 2(\varphi_n+\kappa) \tag{2.20}$$

$$+\frac{\lambda^2}{8N}\sum_1^N V_n^2\cos 4(\varphi_n+\kappa) \tag{2.21}$$

$$+O(\lambda^3)$$

and

$$(2.18) = \frac{\lambda^2}{8\sin^2\kappa} = \frac{\lambda^2}{2(4-E^2)}. \tag{2.22}$$

By (2.16) and (2.17),

$$|1-\mu|\left|\sum_1^N \zeta_n\right| < 1+O(\lambda N)$$

$$\left|\sum_1^N \xi_n\right| < \frac{O(\lambda N)}{\sin^2\kappa} \tag{2.23}$$

and similarly,

$$\Big|\sum_1^N \zeta_n^2\Big| < \frac{O(\lambda N)}{\sin^2 2\kappa} < O(\lambda N). \tag{2.24}$$

Since $2\cos 2(\varphi_n+\kappa) = \mu\zeta_n + \bar\mu\bar\zeta_n$ and $2\cos 4(\varphi_n+\kappa) = \mu^2\zeta_n^2 + \bar\mu^2\bar\zeta_n^2$, (2.23) and (2.24) imply

$$(2.20), (2.21) = O(\lambda^3). \tag{2.25}$$

Thus,

$$\begin{aligned}(2.19) &= \frac{-\lambda}{2N\sin\kappa}\sum_1^N \varepsilon_n \sin(\varphi_n+\kappa)\\ &= -\frac{\lambda}{2N\sin\kappa}\sum_1^N \varepsilon_n d_n(\varepsilon_{n'}; n' < n),\end{aligned} \tag{2.26}$$

which is a martingale difference sequence, with

$$\begin{aligned}\sum_1^N |d_n|^2 = \sum_1^N \sin^2 2(\varphi_n+\kappa) &< \frac{N}{2} + \frac{1}{2}\Big|\sum_1^N \mu^2\zeta_n^2 + \bar\mu^2\bar\zeta_n^2\Big|\\ &< \Big(\frac{1}{2} + O(\lambda)\Big)N.\end{aligned} \tag{2.27}$$

In conclusion,

$$\frac{1}{N}\log\|M_N(E;\varepsilon)\| = \frac{\lambda^2}{8\sin^2\kappa} - \frac{\lambda}{2N\sin\kappa}\sum_1^N \varepsilon_n d_n + O(\lambda^3), \tag{2.28}$$

and the Lyapounov exponent satisfies

$$L(E) = \frac{\lambda^2}{8\sin^2\kappa} + O(\lambda^3). \tag{2.29}$$

From martingale theory and (2.28), we get for $a > 0$ the large deviation inequality

$$\begin{aligned}\operatorname{mes}\Big[\varepsilon \ \Big|\frac{1}{N}\log\|M_N(E;\varepsilon)\| - L(E)\Big| > aL(E)\Big] &< e^{-\left(a^2\lambda^2/(16\sin^2\kappa)+O(\lambda^3)\right)N}\\ &< e^{-\left(a^2L(E)/2+O(\lambda^3)\right)N}.\end{aligned} \tag{2.30}$$

In particular, taking $a > 2$ (and λ small), we have

$$\operatorname{mes}[\varepsilon | \log\|M_N(E;\varepsilon)\| > aN\lambda^2] < e^{-ca^2\lambda^2 N}. \tag{2.31}$$

Returning to (2.11), we take

$$N \sim \lambda^{-2}. \tag{2.32}$$

For $1 \leq j \leq N$, it follows from (2.31) that

$$\begin{aligned} &\operatorname{mes}\,[(\varepsilon_{j+1},\dots,\varepsilon_N);\|M_{N-j}(E;\varepsilon_{j+1},\dots,\varepsilon_N)\| > e^{C_1(\log N)^{1/2}}] \\ &\qquad \leq \exp\Big\{\Big[-cC_1^2\lambda^2\frac{\log N}{(\lambda^2(N-j))^2} + O(\lambda^3)\Big](N-j)\Big\} \\ &\qquad < e[-cC_1^2\log N + O(\lambda)] \\ &\qquad < N^{-C_1}. \end{aligned} \tag{2.33}$$

Recalling (2.11), we see that Lemma 2 may be applied to the function of $\varepsilon \in \{1,-1\}^N$, $f = \tau_{M_N(E;\varepsilon)}(\theta)$, with $\kappa \sim \lambda e^{-2C_1(\log N)^{1/2}}$ and $\delta < N^{-C_1} < N^{-10}$, for a suitable choice of constant C_1.

Hence, we have proved the following result.

Lemma 3. *For small λ, $N \sim \lambda^{-2}$, $\delta_0 < |E| < 2-\delta_0$, and $\theta \in \mathbb{T}$, the distributional inequality*

$$mes\,[\varepsilon;|\tau_{M_N(\varepsilon)}(\theta) - t| < \lambda e^{-C|\log\lambda|^{1/2}}] \leq C\lambda \tag{2.34}$$

holds for all t, where C is some constant.

3 Dimension of the Furstenberg measure

Fixing E as above, denote by $\nu_E = \nu$ the Furstenberg measure on $\mathbb{T}$ for the random walk associated with the probability measure on $SL_2(\mathbb{R})$,

$$\mu = \frac{1}{2}\delta_{\Lambda^-} + \frac{1}{2}\delta_{\Lambda^+}, \tag{3.1}$$

where

$$\Lambda^- = \begin{pmatrix} E-\Lambda & -1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda^+ = \begin{pmatrix} E+\Lambda & -1 \\ 1 & 0 \end{pmatrix}$$

Thus for all N,

$$\int_{SL_2(\mathbb{R})} \varphi(g)\mu^{(N)}(dg) = \int_{\{1,-1\}^N} \varphi\big(M_N(E,\varepsilon)\big)d\varepsilon. \tag{3.2}$$

The measure ν is μ-stationary, i.e.,

$$\nu = \int (\tau_g)_*[\nu]\mu(dg) \tag{3.3}$$

and

$$\langle \nu, f \rangle = \lim_{N\to\infty} \int f\big(\tau_{M_N(\varepsilon)}(\theta)\big) d\varepsilon \tag{3.4}$$

for all $f \in C(\mathbb{T})$ and $\theta \in \mathbb{T}$.

Our goal is to show that for small λ, the dimension of ν_E is close to 1.

The main inequality is the following.

Lemma 4. *Let $h \in SL_2(\mathbb{R})$ be such that $\|h\| \sim \lambda^{-1/10}$. Let $N \sim \lambda^{-1}$ and $I \subset \mathbb{T}$ an interval of size $|I| < \lambda$. Then*

$$\int (\tau_{M_N(\varepsilon)h})_*[\nu](I) d\varepsilon \le e^{C|\log\lambda|^{1/2}} \Big\{ \max_{|J|<\lambda^{1/10}|I|} \nu(J) + \lambda^{1/30} \max_{|J|\le|I|} \nu(J) + \max_{\lambda^{-1/10}<D<\lambda^{-1/5}} \frac{1}{D} \max_{|J|<D.|I|} \nu(J) \Big\}, \tag{3.5}$$

where J is an interval.

Proof. Write

$$\int \big(M_N(\varepsilon)h\big)_*[\nu](I) d\varepsilon = \sum_{0\le k\lesssim N} \int_{[\|M_N(\varepsilon)\|\sim 2^k]} \nu\big(\tau_{h^{-1}}\tau_{M(\varepsilon)^{-1}}(I)\big) d\varepsilon. \tag{3.6}$$

From (2.31),

$$\text{mes}\,[\|M_N(\varepsilon)\| \sim 2^k] < e^{-ck^2} \tag{3.7}$$

and, if $\|M_N(\varepsilon)\| \sim 2^k$, then $\tau_{h^{-1}}\tau_{M_N(\varepsilon)^{-1}}(I)$ is contained in an interval $J \in \mathbb{T}$ of size at most $\|h\|^2 4^k |I|$. Thus the k^{th} summands in (3.6) are certainly bounded by

$$e^{-ck^2} \max_{|J|<4^k\|h\|^2|I|} \nu(J). \tag{3.8}$$

Next, restrict $k \lesssim (\log N)^{1/2}$ and ε to $[\|M_N(\varepsilon)\| \sim 2^k]$.

Let $R_1, \dots, R_M$ be a partition of $\mathbb{T}$ into intervals of size $1/M \sim \lambda$. We have the estimates

(3.9)

$$\begin{aligned} \int_{[\|M_N(\varepsilon)\|\sim 2^k]} \nu\big(\tau_{h^{-1}}\tau_{M_N(\varepsilon)^{-1}}(I)\big) d\varepsilon &\le \sum_{m=1}^{M} \int_{[\|M_N(\varepsilon)\|\sim 2^k]} \nu\big(\tau_{h^{-1}}(\tau_{M_N(\varepsilon)^{-1}}(I) \cap R_M)\big) d\varepsilon \\ &\le \sum_{m=1}^{M} \text{mes}\,\big[\varepsilon; \|M_N(\varepsilon)\| \sim 2^k \text{ and } \tau_{M_N(\varepsilon)}(R_m) \cap I \neq \varnothing\big] \\ &\quad \cdot \max\{\nu(J), |J| \le 4^k D_m |I|\}, \end{aligned}$$

where

$$D_m = \max_{\theta\in R_m} |\tau'_{h^{-1}}(\theta)|. \tag{3.10}$$

Fix $\theta_m \in R_m$ and $\psi \in I$. Then $\tau_{M_N(\varepsilon)}(R_m)$ is contained in an $4^k/M$-neighborhood of $\tau_{M_N(\varepsilon)}(\theta_m)$, and hence

$$|\tau_{M_N(\varepsilon)}(\theta_m) - \psi| \lesssim \frac{4^k}{M} + |I| \lesssim \frac{4^k}{M} \tag{3.11}$$

since $\tau_{M_N(\varepsilon)}(R_m) \cap I \neq \phi$. In view of Lemma 3, mes $[\varepsilon; (3.11)] < 4^k e^{C(\log N)^{1/2}}/M$ by (2.34) and a suitable partition of the interval $[\psi - 4^k/M, \psi + 4^k/M]$. Hence, for k as above,

$$\text{mes}\,[\varepsilon; \|M_N(\varepsilon)\| \sim 2^k \text{ and } \tau_{M_N(\varepsilon)}(R_m) \cap I \neq \varnothing] < e^{C(\log N)^{12}}\lambda. \tag{3.12}$$

Let

$$h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By (2.2), $\tau'_{h^{-1}}(\theta) = \big((a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2\big)^{-1}$ and

$$\frac{1}{\|h\|^2} \lesssim \tau'_{h^{-1}}(\theta) \lesssim \min\Big(\frac{1}{\|h\|^2\|\theta-\theta_h\|^2}, \|h\|^2\Big) \tag{3.13}$$

for some $\theta_h \in \mathbb{T}$. Thus

$$\frac{1}{\|h\|^2} \lesssim D_m \lesssim \min\Big[\frac{1}{\|h\|^2\|\theta_m-\theta_h\|^2}, \|h\|^2\Big]. \tag{3.14}$$

Hence, given $D > 0$,

$$\#\{1 \le m \le M; D_m \sim D\} \lesssim 1 + \frac{M}{\|h\| D^{1/2}}. \tag{3.15}$$

From (3.12) and (3.15), we obtain the estimate

$$\begin{aligned}(3.9) &< e^{C(\log N)^{1/2}}\lambda(\log N)\Big\{\max_{\|h\|^{-2}<D<\|h\|^2} \frac{M}{\|h\|D^{1/2}}\big(\max_{|J|<4^kD|I|}\nu(J)\big)\Big\}\\ &< e^{C'(\log N)^{1/2}}\big(\max_{\|h\|^{-2}<D<\|h\|^2}\frac{1}{D^{1/2}\|h\|}\max_{|J|<D|I|}\nu(J)\big),\end{aligned} \tag{3.16}$$

since $k \lesssim (\log N)^{1/2}$ and J is a union of 4^k intervals of size at most $D.|I|$.

We distinguish several contributions.

(i) For $D < \|h\|^{-1}$, estimate (3.16) by

$$e^{C'(\log N)^{1/2}} \max_{|J|<|I|/\|h\|} \nu(J) < e^{C'|\log\lambda|^{1/2}} \max_{|J|<\lambda^{1/10}|I|} \nu(J). \tag{3.17}$$

(ii) For $1 > D > \|h\|^{-1}$, we have $D^{1/2}\|h\| > \|h\|^{1/2} \gtrsim \lambda^{-1/20}$, and we may bound (3.16) by

$$\lambda^{1/30} \max_{|J| \le |I|} \nu(J). \tag{3.18}$$

(iii) For $1 \le D \le \|h\|$, bound (3.16) by

$$e^{C'|\log\lambda|^{1/2}} \frac{D^{1/2}}{\|h\|} \max_{|J|\le|I|} \nu(J) \le \lambda^{1/30} \max_{|J|\le|I|} \nu(J). \tag{3.19}$$

(iv) For $\|h\| < D < \|h\|^2$, estimate by

$$\frac{e^{C|\log\lambda|^{1/2}}}{D} \max_{|J|<D|I|} \nu(J). \tag{3.20}$$

Collecting the contributions (3.17) - (3.20) gives (3.5). □

Next, we return to (3.3). Writing $\mu = \delta_{g_1}/2 + \delta_{g_2}/2$, we make the following construction. Assume

$$\nu = \int (\tau_g)_*[\nu]\mu_1(dg), \tag{3.21}$$

where μ_1 is some discrete probability measure on $SL_2(\mathbb{R})$ such that

$$\|g\| < 2\lambda^{-1/10} \text{ for } g \in \operatorname{supp} \mu_1. \tag{3.22}$$

If $g \in \operatorname{supp} \mu_1$ and $\|g\| < \lambda^{-1/10}$, by (3.3), $(\tau_g)_*[\nu] = (\tau_{gg_1})_*[\nu]/2 + (\tau_{gg_2})_*[\nu]/2$. Define $\mu_2 = \sum_{\|g\|\ge\lambda^{-1/10}} \mu_1(g)\delta_g + (1/2)\sum_{\|g\|<\lambda^{-1/10}} \mu_1(g)(\delta_{gg_1} + \delta_{gg_2})$, which still satisfies (3.21).

From the positivity of the Lyapounov exponent, an iteration of this process clearly produces a discrete probability measure $\tilde{\mu}$ on $SL_2(\mathbb{R})$ such that

$$\nu = \int (\tau_g)_*[\nu]\tilde{\mu}(dg) \tag{3.23}$$

and

$$\lambda^{-1/10} < \|g\| < 2\lambda^{-1/10} \text{ for } g \in \operatorname{supp} \tilde{\mu}. \tag{3.24}$$

Also, by (3.2), $\nu = \int (\tau_{M_N(\varepsilon)})_*[\nu] d\varepsilon$. Thus, taking $N \sim \lambda^{-2}$ in (3.23) gives

$$\nu = \int \Big[\int (\tau_{M_N(\varepsilon)h})_*[\nu] d\varepsilon \Big] \tilde{\mu}(dh). \tag{3.25}$$

From (3.25) and Lemma 4, we obtain the following inequality.

Lemma 5. *For an interval $I \subset \mathbb{T}$ of size at most λ,*

$$\nu(I) \le e^{C|\log\lambda|^{1/2}} \Big[\max_{|J|<\lambda^{1/10}|I|} \nu(J) + \max_{\lambda^{-1/10}<D<\lambda^{-1/5}} \frac{1}{D} \max_{|J|<D|I|} \nu(J) \Big]. \tag{3.26}$$

If we iterate (3.26) r times, assuming $\lambda^{-r/5}|I| < \lambda$, we obtain

$$\nu(I) \le 2^r e^{C|\log\lambda|^{1/2} r} \frac{1}{D_1} \nu(J) \tag{3.27}$$

for some interval J of size $|J| < D_1\delta_1|I|$, where $D_1 > 1$, $0 < \delta_1 < 1$, and $D_1\delta_1^{-1} > \lambda^{-r/10}$.

Theorem 1. *For $\delta_0 < |E| < 2-\delta_0$, the dimension of the Furstenberg measure $\nu_E^{(\lambda)}$ for the A-B model is at least $\alpha(\lambda) \overset{\lambda\to 0}{\longrightarrow} 1$.*

It is known from random matrix product theory that the Furstenberg measure ν has positive dimension $\alpha > 0$. Hence the right side of (3.27) is at most

$$\begin{aligned} &\lesssim C^{r|\log\lambda|^{1/2}} \frac{1}{D_1}|J|^\alpha < C^{r|\log\lambda|^{1/2}} \frac{\delta_1^\alpha}{D_1^{1-\alpha}}|I|^\alpha \\ &\qquad < C^{r|\log\lambda|^{1/2}} \lambda^{(r/20)\ \min(\alpha,1-\alpha)}|I|^\alpha. \end{aligned} \tag{3.28}$$

With a constant $\gamma > 0$ (independent of $\lambda = o(1)$) satisfying

$$\gamma < \alpha < 1-\gamma, \tag{3.29}$$

(3.28) and the restriction on r would imply for $\lambda < \lambda(\gamma)$,

$$\nu(I) < (C^{|\log\lambda|^{1/2}}\lambda^{\gamma/20})^r|I|^\alpha < \lambda^{\gamma r/30}|I|^\alpha$$

and

$$\nu(I) < \Big(\frac{|I|}{\lambda}\Big)^{\gamma/6}|I|^\alpha \lesssim |I|^{\alpha+\gamma/6}. \tag{3.30}$$

But (3.30) would imply that ν has dimension at least $\alpha + \gamma/6$, a contradiction. Thus, in order to prove Theorem 1, it suffices to have a uniform lower bound in λ for $\dim \nu_E^{(\lambda)}$. This is what we establish next.

Lemma 6. *Under the assumption of Theorem 1,* $\dim \nu_E^{(\lambda)} > \gamma > 0$ *for some γ independent of λ.*

Proof. Write $M = M_N(E;\varepsilon)$ as

$$M = \frac{(v_-^\perp \otimes v_+)\lambda_+ + (v_+^\perp \otimes v_-)\lambda_+^{-1}}{\langle v_+, v_-^\perp\rangle} \tag{3.31}$$

with v_+ (respectively, v_-) the expanding (respectively, contracting) direction. Then

$$\|M\| \sim \frac{|\lambda_+|}{|v_+ \wedge v_-|}. \tag{3.32}$$

For unit vectors $u, w \in \mathbb{R}^2$, we deduce from (3.31) that

$$\begin{aligned} \frac{\|Mu\|}{\|M\|} &= \|\langle v_-^\perp, u\rangle v_+ + \lambda_+^{-2}\langle v_+^\perp, u\rangle v_-\| \\ &= (1+\lambda_+^{-2})|\langle v_-^\perp, u\rangle| + O\Big(\frac{|v_+ \wedge v_-|}{\lambda_+^2}\Big) \\ &\overset{(3.32)}{\leq} (1+\lambda_+^{-2})|\langle v_-^\perp, u\rangle| + O\Big(\frac{1}{\|M\|}\Big) \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} \frac{|\langle Mu, w\rangle|}{\|M\|} &= |\langle v_-^\perp, u\rangle\langle v_+, w\rangle + \lambda^{-2}\langle v_+^\perp, u\rangle\langle v_-, w\rangle| \\ &\geq (1+\lambda^{-2})|\langle v_-^\perp, u\rangle|\ |\langle v_+, w\rangle| - 2\lambda_+^{-2}|v_+ \wedge v_-| \\ &\geq |\langle v_-^\perp, u\rangle|\ |\langle v_+, w\rangle| + O\Big(\frac{1}{\|M\|}\Big). \end{aligned} \tag{3.34}$$

Hence, given an arc I of size η centered at v, we have

$$\begin{aligned} &\mathbb{P}\Big[\varepsilon; v_- \in I \text{ where } v_- \text{ is contracting direction of } M_N(\varepsilon)\Big] \\ &\quad\overset{(3.33)}{\leq} \mathbb{P}\Big[\varepsilon; \frac{\|M_N(\varepsilon)u\|}{\|M_N(\varepsilon)\|} < 2\eta + O\Big(\frac{1}{\|M_N(\varepsilon)\|}\Big)\Big] \\ &\quad\leq \mathbb{P}\Big[\varepsilon; \|M_N(\varepsilon)\| < e^{\lambda^2 N/20}\Big] \qquad (3.35) \\ &\qquad + \mathbb{P}\Big[\varepsilon; \frac{\|M_N(\varepsilon)u\|}{\|M_N(\varepsilon)\|} < 3\eta\Big], \qquad (3.36) \end{aligned}$$

provided

$$\eta > e^{-\lambda^2 N/20}. \tag{3.37}$$

Recalling (2.29) and (2.30), we have

$$\operatorname{mes}\Big[\varepsilon \ \Big|\frac{1}{N}\frac{\log\|M_N(\varepsilon)\|}{L(E)} - 1\Big| > a\Big] < e^{\big(-(a^2/2)L(E)+O(\lambda^3)\big)N} \tag{3.38}$$

with $\lambda^2/8 < L(E) < O(\lambda^2)$.

Hence,

$$(3.35) < e^{-(\lambda^2/50+O(\lambda^3))N} < e^{-\lambda^2 N/60} \overset{(3.37)}{\longrightarrow} < \eta^{1/3} \tag{3.39}$$

for λ small enough.

Next, we point out that in the analysis from (2.14) to (2.28), the formula (2.28) is equally valid for $(1/N)\log\|M_N(E;\varepsilon)(u)\|$ with $u \in S^1$ arbitrary (as a consequence of the argument). Thus, we can write

$$\frac{1}{N}\log\|M_N(\varepsilon)\| = L(E) - \frac{\lambda}{2N\sin\kappa}\sum_1^N \varepsilon_n d_n + O(\lambda^3) \tag{3.40}$$

and

$$\frac{1}{N}\log\|M_N(\varepsilon)(u)\| = L(E) - \frac{\lambda}{2N\sin\kappa}\sum_1^N \varepsilon_n d_n' + O(\lambda^3), \tag{3.41}$$

so that

$$\log\frac{\|M_N(\varepsilon)\|}{\|M_N(\varepsilon)(u)\|} = \frac{\lambda}{2\sin\kappa}\sum_1^N \varepsilon_n(d_n' - d_n) + O(N\lambda^3), \tag{3.42}$$

where d_n, d_n' depend on $\varepsilon_1, \ldots, \varepsilon_{n-1}$.

Letting $1 > t > 0$ be a parameter, write

$$\begin{aligned}(3.36) &< (3\eta)^t \int \Big(\frac{\|M_N(\varepsilon)\|}{\|M_N(\varepsilon)(u)\|}\Big)^t d\varepsilon \\ &< (3\eta)^t e^{O(N\lambda^3 t)} \int e^{\lambda t \sum_1^N \varepsilon_n(d_n'-d_n)/(2\sin\kappa)} d\varepsilon \\ &< (3\eta)^t e^{O(N\lambda^3 t)} e^{C\lambda^2 t^2 N},\end{aligned} \tag{3.43}$$

where the constant C only depends on E.

Choosing N such that

$$\eta \sim e^{-\lambda^2 N/10^3}, \tag{3.44}$$

we satisfy (3.37), and it follows from (3.43) and appropriate choice of t that

$$(3.36) < (3\eta)^{t-C(\lambda t+t^2)} < \eta^{c_1}$$

(again for λ small enough) with $c_1 > 0$ independent of λ.

Hence, we have shown that with N satisfying (3.44),

$$\text{mes}\,[\varepsilon; \upsilon_- \in I \text{ where } \upsilon_- \text{ is contracting vector of } M_N(\varepsilon)] < \eta^{c_1}. \tag{3.45}$$

Since υ_+ is the contracting vector of $M_N(\varepsilon)^{-1}$, we obtain a similar statement for the expanding vector. Therefore, we have proved that for any pair of η-intervals I_+, I_- in S^1,

$$\begin{aligned}&\text{mes}\,[\varepsilon; \upsilon_+ \in I_+ \text{ or } \upsilon_- \in I_- \text{ with } \upsilon_+ \text{ (respectively, } \upsilon_-) \text{ expanding} \\ &\qquad \text{(respectively, contracting) direction of } M_N(\varepsilon)] < 2\eta^{c_1}\end{aligned} \tag{3.46}$$

for N satisfying (3.44).

Returning to (3.34), we have

$$\begin{aligned}\mathbb{P}[\varepsilon; \frac{|\langle M_N(\varepsilon)u, w\rangle|}{\|M_N(\varepsilon)\|} < \eta_1] \\ \le \mathbb{P}[\varepsilon; \|M_N(\varepsilon)\| < 1/\eta_1] + \mathbb{P}[\varepsilon; |\langle v_-^\perp, u\rangle| \lesssim \sqrt{\eta_1}] \\ + \mathbb{P}[\varepsilon; |\langle v_+, w\rangle| \lesssim \sqrt{\eta_1}].\end{aligned} \tag{3.47}$$

Taking $\eta = \eta_1^{1/2}$ and N as in (3.44), we find by (3.46) that the last two terms in (3.47) are at most $O(\eta_1^{c_1/2})$, and by (3.38) that the first term is bounded by mes $[\varepsilon; \|M_N(\varepsilon)\| < e^{\lambda^2 N/500} < e^{-\lambda^2 N/60} < \eta_1$.

Hence

$$(3.47) \lesssim \eta_1^{c_1/2} \text{ with } \eta_1 \sim e^{-\lambda^2 N/500}. \tag{3.48}$$

Returning to the Furstenberg measure $\nu = \nu_E^{(\lambda)}$, we have for $I \subset \mathbb{T}$ a small arc of size η_1, by (3.4),

$$\nu(I) = \lim_{N'\to\infty} \mathbb{P}\left[\varepsilon \middle| \frac{M_{N'}(\varepsilon)e_1}{\|M_{N'}(\varepsilon)e_1\|} \in I\right].$$

Take N as in (3.48) and $N' > N$. If w denotes the center of I, then

$$\frac{|\langle M_{N'}e_1, w^\perp\rangle|}{\|M_{N'}e_1\|} < \eta_1. \tag{3.49}$$

Fix $\varepsilon_1, \dots, \varepsilon_{N'-N}$ and let

$$u = \frac{M_{N'-N}(\varepsilon_1, \dots, \varepsilon_{N'-N})(e_1)}{\|M_{N'-N}(\varepsilon_1 \dots, \varepsilon_{N'-N})(e_1)\|}.$$

We have

$$\frac{M_{N'}e_1}{\|M_{N'}e_1\|} = \frac{M_N(\varepsilon_{N'-N+1}, \dots, \varepsilon_{N'})(u)}{\|M_N(\varepsilon_{N'-N+1}, \dots, \varepsilon_{N'})u\|}.$$

Thus (3.49) implies

$$\frac{|\langle M_N(\cdots)u, w^\perp\rangle|}{\|M_N(\cdots)\|} < \eta_1 \tag{3.50}$$

for which, by (3.48), the measure in $\varepsilon_{N'-N+1}, \dots, \varepsilon_{N'}$ is at most $\eta_1^{c_1/2}$. Therefore

$$\nu(I) \lesssim |I|^{c_1/2}. \tag{3.51}$$

This proves that $\dim \nu \ge c_1/2$, uniformly in λ. □

This also completes the proof of Theorem 1.

4 Density of states

Let $u, w \in S^1$, $\eta > 0$ small. It follows from (3.34) that

$$\begin{aligned}\lim_{N\to\infty} \text{mes}\Big[\varepsilon; \frac{|\langle M_N(\varepsilon)u, w\rangle|}{\|M_N(\varepsilon)\|} < \eta\Big] \\ &\leq \lim_{N\to\infty} \text{mes}\Big[\varepsilon; |\langle v_+, w\rangle|.|\langle v_-^\perp, u\rangle| < \eta \text{ with } v_+, v_- \text{ the} \\ &\qquad \text{eigenvectors of } M_N(\varepsilon)\Big] \\ &= \lim_{N\to\infty} \text{mes}\Big[(\varepsilon, \varepsilon'); |\langle v_+, w\rangle|.|\langle v_+', u^\perp\rangle| < \eta \text{ with } v_+ \text{ (respectively, } v_+') \\ &\qquad \text{expanding direction of } M_N(\varepsilon),\ \text{(respectively, } M_N(\varepsilon'))\Big] \\ &\lesssim \log\frac{1}{\eta}. \max_{\eta_1.\eta_2=\eta} \nu_E(I_{\eta_1}(w^\perp)).\nu_E(I_{\eta_2}(u)) \ll \eta^\gamma,\end{aligned} \tag{4.1}$$

where we have used (3.4) and the independence of v_+, v_- as functions of ε as $N \to \infty$. Here, $\gamma < \dim \nu_E^{(\lambda)}$ and $\gamma = \gamma(\lambda) \to 1$ as $\lambda \to 0$.

It is easily seen that (4.1) implies that for given $K > 1$ and large enough N (depending on K),

$$\max_{u,w\in S^1} \mathbb{E}\Big[\frac{\|M_N\|}{|\langle M_N u, w\rangle|} \wedge K\Big] < K^{1-\gamma}. \tag{4.2}$$

Here, $M_N = M_N(E)$ and (4.2) clearly remains valid for unit vectors $u, w \in \mathbb{C}^2$ and E replaced by $z = E + iy$ with $0 < y < y_N$ small enough (depending on N). Next, take $N' > N$ and consider

$$\frac{\|M_{[0,N']}(z;\varepsilon)\|.\|M_{]N',2N']}(z;\varepsilon)\|}{\|M_{[0,2N']}(z,\varepsilon)\|}. \tag{4.3}$$

Fixing $\varepsilon_{N'+1}, \dots, \varepsilon_{2N'}$, we obtain a unit vector $\zeta \in \mathbb{C}^2$ (depending on these variables) such that

$$(4.3) = \frac{\|M_{[0,N']}(z,\varepsilon)\|}{\|M_{[0,N']}(z,\varepsilon)(\zeta)\|} \tag{4.4}$$

and

$$(4.4) \lesssim \sum_{i,j=1,2} \frac{|\langle M_{[0,N']}(z,\varepsilon)e_i, e_j\rangle|)}{|\langle M_{[0,N'']}(z,\varepsilon)\zeta, e_j\rangle|}.$$

Fix also $\varepsilon_1, \dots, \varepsilon_{N'-N}$ and let ζ_1 be a unit vector in $\mathbb{C}^2$ parallel to $M^*_{[0,N'-N]}(z,\varepsilon)e_j$. Then

$$\frac{|\langle M_{[0,N']}(z,\varepsilon)e_i, e_j\rangle|}{|\langle M_{[0,N']}(z,\varepsilon)\zeta, e_j\rangle|} \leq \frac{\|M_{[0,N[}(z,\varepsilon)\|}{|\langle M_{[0,N[}(z,\varepsilon)\zeta, \zeta_1\rangle|}, \tag{4.5}$$

where ζ, ζ_1 do not depend on $\varepsilon_0, \dots, \varepsilon_{N-1}$. Thus

$$\min\big((4.3), K\big) \lesssim \min\big((4.5), K\big). \tag{4.6}$$

The expectation of (4.6) in $\varepsilon_0, \dots, \varepsilon_{N-1}$ (with other variables fixed), (4.2), and the subsequent remark give the estimate

$$\mathbb{E}_{\varepsilon_0,\dots,\varepsilon_{N-1}}[(4.6)] \lesssim K^{1-\gamma}. \tag{4.7}$$

Hence, also

$$\mathbb{E}[\min\big((4.3), K\big)] \lesssim K^{1-\gamma} \tag{4.8}$$

holds for $z = E + iy$ with $y > 0$ small enough (depending on K) and $N' > N'(K)$.

Denoting by $\mathcal{N}$ the IDS, recall that $\bar{\partial}\mathcal{N}(z) = \mathbb{E}[G(0, 0, z)]$, $z = E + iy$, where $G(z) = (H - z)^{-1}$ is the Green's function and $\mathcal{N}(z)$ is the harmonic extension of $\mathcal{N}$ to $\operatorname{Im} z > 0$.

Fix z with $\operatorname{Im} z > 0$. Then from the resolvent identity and the positivity of the Lyapounov exponent, we obtain

$$G(0, 0, z) = \lim_{\substack{\Lambda=[-a,b] \\ a,b\to\infty}} G_\Lambda(0, 0, z) \text{ a.s.}$$

and, by Cramer's rule,

$$|G(0, 0, z)| \leq \underline{\lim}_{N'\to\infty} \frac{\|M_{[-N',0]}(z, \varepsilon)\| \; \|M_{[0,N']}(z, \varepsilon)\|}{\|M_{[-N',N']}(z; \varepsilon)\|}. \tag{4.9}$$

Hence by (4.8),

$$\mathbb{E}[|G(0, 0, z)| \wedge K] \leq \underline{\lim}_{N'\to\infty}\mathbb{E}[\cdots \wedge K] \lesssim K^{1-\gamma} \tag{4.10}$$

if $y > 0$ is small enough (depending on K). Letting $y \to 0$, we get

$$\mathbb{E}[|G(0, 0, E + io)| \wedge K] < K^{1-\gamma}. \tag{4.11}$$

It follows from (4.11) that for $0 < \gamma_1 < \gamma$,

$$\mathbb{E}[|G(0, 0, E + io)|^{\gamma_1}] < C. \tag{4.12}$$

Recall that we have assumed $\delta_0 < |E| < 2 - \delta_0$. Using the subharmonicity of $|G(0, 0, z)|^{\gamma_1}$ on $\operatorname{Im} z > 0$, we deduce from (4.12) that for fixed $z = E + iy$, $y > 0$,

$$|\bar{\partial}\mathcal{N}(z)| \leq \mathbb{E}[|G(0, 0, z)|] < \frac{1}{y^{1-\gamma_1}}\mathbb{E}[|G(0, 0, z)|^{\gamma_1}] < \frac{C}{y^{1-\gamma_1}}. \tag{4.13}$$

Hence $\mathcal{N}$ is γ_1-Hölder for all $\gamma_1 < \gamma$.

This proves the following result.

Theorem 2. *For $\delta_0 < |E| < 2 - \delta_0$, the IDS of the A-B model with λ-disorder is s-Hölder regular, with $s \to 1$ as $\lambda \to 0$.*

5 Further comments

If one aims at going further and proving the Lipschitz regularity of the IDS, it seems reasonable to prove that the Furstenberg measures on the projective line $P_1(\mathbb{R}) \simeq \mathbb{T}$ are at least absolutely continuous. This is far from an obvious issue. In fact, it was conjectured in [K-L] that if ν is a finitely supported probability measure on $SL_2(\mathbb{R})$, then its Furstenberg measure on $P_1(\mathbb{R})$ is always singular. This conjecture was disproved in [B-P-S] using a probabilistic construction reminiscent of random Bernoulli-convolutions. An explicit example was given recently in [B2, B-Y], based on a construction from [B3] (which relies on an extension of the spectral gap theory for $SU(2)$ from [BG1] to $SL_2(\mathbb{R})$). A rough description is as follows. One produces a finite subset $\mathcal{G} \subset SL_2(\mathbb{R}) \cap \mathrm{Mat}_{2\times 2}(q)$, q a fixed large integer, such that $\log(\#\mathcal{G}) \sim \log q$, $\mathcal{G}$ generates freely the free group on $\#\mathcal{G}$ generators, and moreover $\mathcal{G}$ is contained in a small neighborhood of the identity (depending on q). It is shown that there is a spectral gap for the projective representation ρ, in the following sense. Denote the probability measure on $SL_2(\mathbb{R})$ by

$$\nu = \frac{1}{(\#\mathcal{G})} \sum_{g\in\mathcal{G}} \delta_g, \tag{5.1}$$

Let $f \in L^2(\mathbb{T})$, $\|f\|_2 = 1$ and assume $\hat{f}(n) = 0$ for $|n| > K$, where $K = K(q)$ is a sufficiently large constant. Then

$$\frac{1}{(\#\mathcal{G})} \Big\| \sum_{g\in\mathcal{G}} \rho_g f \Big\|_2 < \frac{1}{2}, \tag{5.2}$$

where $\rho_g f = (\tau_g')^{-1/2}(f \circ \tau_g)$ and τ_g the action on $\mathbb{T}$ defined for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

by

$$e^{i\tau_g(\theta)} = \frac{(a\cos\theta + b\sin\theta) + i(c\cos\theta + d\sin\theta)}{[(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2]^{\frac{1}{2}}}. \tag{5.3}$$

Since $g \in \mathcal{G}$ are close to identity, (5.2) clearly implies that for f as above

$$\frac{1}{(\#\mathcal{G})} \Big\| \sum_{g\in\mathcal{G}} (f \circ \tau_g) \Big\|_2 < \frac{3}{4}. \tag{5.4}$$

From (5.4), one may then derive easily that ν has an a.c. Furstenberg measure with C^k-density, where k can be made arbitrarily large.

It should be pointed out that the contractive properties (5.2) and (5.4) do not exploit hyperbolicity (at least in the usual sense), as the Lyapounov exponent of the random matrix product corresponding to ν is small.

Returning to the A-B-model with small λ, let

$$\mu = \frac{1}{2}\delta_{\Lambda^-} + \frac{1}{2}\delta_{\Lambda^+}, \tag{5.5}$$

where

$$\Lambda^- = \begin{pmatrix} E-\Lambda & -1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda^+ = \begin{pmatrix} E+\Lambda & -1 \\ 1 & 0 \end{pmatrix}.$$

and let $\nu^{(\ell)}_{\lambda,E}$ be its ℓ-fold convolution. It seems reasonable to believe that

$$\Big\| \sum_g \nu^{(\ell)}_{\lambda,E}(g)(f\circ\tau_g)\Big\|_2 \le \frac{1}{2}\|f\|_2 \tag{5.6}$$

for $f\in L^2(\mathbb{T})$, $\hat{f}(n)=0$, $|n|>K(\lambda)$, where ℓ is some positive integer independent of λ, or at least $\ell = o(\lambda^{-2})$. Such a property would then again imply a.c. and a certain smoothness of the Furstenberg measure. Unfortunately, available technology to establish spectral gaps (as developed in [BG1]) so far requires algebraic matrix elements of bounded height, and hence does not apply to (5.5).

One may however combine the methods from [BG1] with those of [S-T] to prove the following result, which seems new. (Compare also with the results from [K-S].)

Theorem 3. *Let $H=\Delta+V$ be a random Schrödinger operator on $\mathbb{Z}$, where $V=(V_n)_{n\in\mathbb{Z}}$ are i.i.d.'s with distribution given by a compactly supported measure β on $\mathbb{R}$ of positive dimension; thus there exists $\kappa>0$ such that*

$$\beta(I)\lesssim |I|^\kappa \text{ for intervals } I\subset\mathbb{R}. \tag{5.7}$$

Then H has C^∞ density of states.

We sketch the proof.

For fixed E, let μ_E be the probability measure on $SL_2(\mathbb{R})$ obtained as image measure of β under the map

$$v\mapsto \begin{pmatrix} E-v & -1 \\ 1 & 0. \end{pmatrix} \tag{5.8}$$

In light of [S-T], it suffices to show that for some fixed convolution power ℓ, the measure $\mu_1=\mu_E^{(\ell)}$ on $SL_2(\mathbb{R})$ gives a smoothing convolution operator on $P_1(\mathbb{R})$. Thus there exists $\alpha>0$ such that for $f\in H^s(\mathbb{T})$ and $s\ge 0$,

$$\Big\| \int (f\circ\tau_g)\mu_1(dg)\Big\|_{H^{s+\alpha}} \lesssim \|f\|_{H^s}, \tag{5.9}$$

where H^S denotes the usual Sobolev space with norm

$$\|f\|_{H^s} = \left(\sum(1+|n|)^{2s}|\hat{f}(n)|^2\right)^{1/2}.$$

Letting $x = E - v$, one has

$$\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} xyz - x - z & 1 - xy \\ yz - 1 & -y \end{pmatrix}, \tag{5.10}$$

and recalling (5.7), one sees that $\mu_E^{(3)}$ certainly has the property that

$$\mu_E^{(3)}(\mathfrak{S}_\delta) \lesssim \delta^{\kappa'} \text{ for all } \delta > 0 \tag{5.11}$$

if $\mathfrak{S}$ is a proper algebraic subvariety of $SL_2(\mathbb{R})$ of bounded degree and $\mathfrak{S}_\delta$ is a δ-neighborhood of $\mathfrak{S}$. Here, $\kappa' > 0$ depends on the degree bound.

Let P_δ, $\delta > 0$, denote an approximate identity on $SL_2(\mathbb{R})$. Using (5.11), an extension of the 'Flattening Lemma' from [BG1] to $SL_2(\mathbb{R})$ (note that, up to complexification, $SU(2)$ and $SL_2(\mathbb{R})$ have the same Lie-algebra, and our analysis is local), permits us to conclude the following.

Lemma 7. *Fix $0 < \varepsilon < 1$. There exits $\ell = \ell(\varepsilon) \in \mathbb{Z}_+$ such that for all $\delta > 0$,*

$$\|\mu_E^{(3\ell)} * P_\delta\|_\infty < \delta^{-\varepsilon}. \tag{5.12}$$

In particular, $\mu_E^{(3\ell)}$ has dimension at least $3 - \varepsilon$.

Lemma 7 is the crucial step in the proof and depends on "arithmetic combinatorics" in groups. (See [BG1] and related references for more details.)

Taking $\varepsilon = 10^{-3}$ and $\ell = \ell(\varepsilon)$ given by Lemma 7, we can now prove that $\mu_1 = \mu_E^{(3\ell)}$ satisfies (5.9). This is clearly a consequence of the following statement.

Lemma 8. *Let $f \in L^2(\mathbb{T})$, $\|f\|_2 = 1$ and supp $\hat{f} \subset [2^k, 2^{k+1}] \cup [-2^{k+1}, -2^k]$ with k sufficiently large. Then*

$$\left\| \int (f \circ \tau_g)\mu_1(dg) \right\|_2 < 2^{-k\kappa} \tag{5.13}$$

for some $\kappa > 0$.

Proof. We summarize the argument from [B2].

Let $G = SL_2(\mathbb{R})$ and take $\delta = 4^{-k}$, so that, by assumption on f, the left side of (5.13) can be replaced by

$$\left\| \int (f \circ \tau_g)(\mu_1 * P_\delta)(dg) \right\|_2. \tag{5.14}$$

Using (5.12), one obtains

$$(5.14)^2 \lesssim \delta^{-2\varepsilon} \iint_{G\times G} |\langle f \circ \tau_{g_1}, f \circ \tau_{g_2}\rangle| \Omega(g_1)\Omega(g_2) dg_1 dg_2,$$

where $0 \le \Omega \le 1$ is a suitable compactly supported function on G (depending on the support of β). Next, the Cauchy-Schwarz inequality gives

$$(5.15) \quad (5.14)^4 \lesssim \delta^{-4\varepsilon} \iiiint_{G\times G\times \mathbb{T}\times\mathbb{T}} f(\tau_{g_1}x)\bar{f}(\tau_{g_2}x)\bar{f}(\tau_{g_1}y)f(\tau_{g_2}y) \Omega(g_1)\Omega(g_2)dg_1dg_2dxdy.$$

To estimate (5.15), we proceed as follows. Fix $x, y \in \mathbb{T}$ and $g_1 \in G$, and consider the integral in g_2

$$(5.16) \qquad \int \bar{f}(\tau_g x) f(\tau_g y)\Omega(g)dg.$$

The point here is that if one specifies $\tau_g x \in \mathbb{T}$, there remains an average in $\tau_g y$ to be exploited, when integrating in g (unless x and y are very close). More precisely, if $\|x-y\| < 2^{-h/10}$, then $|(5.16)| < 2^{-k}\|f\|_1^2$ and the contribution in (5.15) is at most $\delta^{-4\varepsilon}2^{-k}\|f\|_1^4 < 2^{-k/2}$. The contribution of $\|x-y\| < 2^{-k/10}$ in (5.15) is easily estimated by

$$\iint_{\|x-y\|<2^{-k/10}} \Big[\int_G |f(\tau_g x)|\ |f(\tau_g y)|\Omega(g)dg\Big]^2 dxdy$$
$$\le \iint_{\|x-y\|<2^{-k/10}} \Big[\int_G |f(\tau_g x)|^2\Omega(g)dg\Big]\Big[\int_G |f(\tau_g y)|^2\Omega(g)dg\Big]dxdy$$
$$\lesssim 2^{-k/10}\|f\|_2^4,$$

and (5.13) follows. □

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