SPECTRAL ANALYSIS ON INFINITE SIERPIŃSKI FRACTAFOLDS

By

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Abstract. A fractafold, a space that is locally modeled on a specified fractal, is the fractal equivalent of a manifold. For compact fractafolds based on the Sierpiński gasket, it was shown by the first author how to compute the discrete spectrum of the Laplacian in terms of the spectrum of a finite graph Laplacian. A similar problem was solved by the second author for the case of infinite blowups of a Sierpiński gasket, where spectrum is pure point of infinite multiplicity. Both works used the method of spectral decimations to obtain explicit description of the eigenvalues and eigenfunctions. In this paper we combine the ideas from these earlier works to obtain a description of the spectral resolution of the Laplacian for noncompact fractafolds. Our main abstract results enable us to obtain a completely explicit description of the spectral resolution of the fractafold Laplacian. For some specific examples, we turn the spectral resolution into a "Plancherel formula". We also present such a formula for the graph Laplacian on the 3-regular tree, which appears to be a new result of independent interest. At the end, we discuss periodic fractafolds and fractal fields.

Contents

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1 Introduction

Analysis on fractals has been developed based on the construction of Laplacians on certain basic fractals, such as the Sierpiński gasket, the Vicsek set, the Sierpiński carpet, etc., which may be regarded as generalizations of the unit interval, in that they are both compact and have nonempty boundary. As is well known in classical analysis, it is often more interesting and sometimes simpler to deal with spaces like the circle and the line, which have no boundary, and need not be compact. The theory of analysis on manifolds is the natural context for such investigations. The notion of *fractafold*, introduced in [37], is simply the fractal equivalent: a space that is locally modeled on a specified fractal. For compact fractafolds based on the Sierpiński gasket, it was shown in [37] how to compute the spectrum of the Laplacian in terms of the spectrum of a Laplacian on a graph Γ that describes how copies of SG are glued together to make the fractafold. On the other hand, in [41] a similar problem was solved for the case of infinite blowups of SG. These are noncompact fractafolds where the graph Γ mirrors the self-similar structure of SG. Not surprisingly, the spectrum in the compact case is discrete, and the eigenvalues and eigenfunctions are described by the method of spectral decimation introduced in [11]. The surprise is that for the infinite blowups the spectrum is pure point, meaning that there is a basis of L^2 eigenfunctions (in fact compactly supported), but each eigenspace is infinite dimensional and the closure of the set of eigenvalues is a Cantor set. Again the method of spectral decimations allows an explicit description of the eigenvalues and eigenfunctions.

In this paper, we combine the ideas from these earlier works [37, 41] to obtain a description of the spectral resolution of the Laplacian for noncompact fractafolds with infinite cell graphs Γ. The graph Γ is assumed to be 3-regular, so the fractafold has no boundary. The edge graph Γ_0 is then 4-regular, and the fractafold is obtained as a limit of graphs obtained inductively from Γ_0 by filling in detail (that is, each graph triangle is eventually replaced with a copy of the Sierpiński gasket). Our first main abstract result is Theorem 2.3, which describes how to obtain the spectral resolution of the Laplacian on the fractafold from the spectral resolution of the graph Laplacian on Γ_0. This is a version of spectral decimation and uses an idea from [27] to control the L^2 norms of functions under spectral decimation.

The second main abstract result is Theorem 3.1, which shows how to obtain the spectral resolution of the graph Laplacian on Γ_0 from the spectral resolution of the graph Laplacian on Γ using ideas from [34, 40]. We note that the spectral resolution on Γ_0 may or may not contain the discrete eigenvalues equal to 6, and the explicit determination of the 6-eigenspace and its eigenprojector must be determined in a case-by-case manner. Combining the two theorems enables us to obtain a completely explicit description of the spectral resolution of the fractafold Laplacian to the extent that we are able to solve the following problems.

(a) Find the explicit spectral resolution of the graph Laplacian on Γ.
(b) Find an explicit description of the 6-eigenspace and its eigenprojector for the graph Laplacian on Γ_0.

The bulk of this paper is devoted to solving these two problems for some specific examples. However, we would like to highlight another problem that arises if we wish to turn a spectral resolution into a "Plancherel formula". Typically we write our spectral resolutions as

$$f(x) = \int_{\sigma(-\Delta)} \left(\int P(\lambda, x, y) f(y) d\mu(y) \right) dm(\lambda), \tag{1.1}$$

where $P(\lambda, x, y)$ is an explicit kernel realizing the projection onto the λ-eigenspace, i.e.,

$$-\Delta \int P(\lambda, x, y) f(y) d\mu(y) = \lambda \int P(\lambda, x, y) f(y) d\mu(y) \tag{1.2}$$

and $dm(\lambda)$ is a scalar spectral measure. (Here, neither $P(\lambda, x, y)$ nor $dm(\lambda)$ is uniquely determined, since we can clearly multiply them by reciprocal functions of λ while preserving (1.1) and (1.2).) If we write

$$P_\lambda f(x) = \int P(\lambda, x, y) f(y) d\mu(y), \tag{1.3}$$

then (1.1) resolves f into its components $P_\lambda f$ in the λ-eigenspaces. A Plancherel formula would express the squared L^2-norm $||f||_2^2$ in terms of an integral of contributions from the components $P_\lambda f$. In the case of pure point spectrum, this is straightforward, for then the λ-integral is a discrete possibly infinite sum, and we just have to take the L^2-norm of each $P_\lambda f$, so

$$||f||_2^2 = \sum_{\lambda \in \sigma(-\Delta)} ||P_\lambda f||_2^2, \tag{1.4}$$

where P_λ is the eigenprojection. The spectral measure m is the counting measure in this case.

In the case of a continuous spectrum this is decidedly not correct, and there does not appear to be a generic method to obtain the correct analog. So we pose this as a third problem.

(c) Describe explicitly a Hilbert space of λ-eigenfunctions with norm $|| \quad ||_\lambda$ such that $||P_\lambda f||_\lambda$ is finite for $m - a.e.\ \lambda$ and

$$||f||_2^2 = \int_{\sigma(-\Delta)} ||P_\lambda f||_\lambda^2 dm(\lambda). \tag{1.5}$$

This problem is interesting essentially only when the eigenspace is infinite dimensional. The resolution of this problem in some classical settings is discussed in [35] and [14]. Here we present a solution of this problem for the graph Laplacian on the 3-regular tree. This result appears to be new, and is of independent interest.

The first specific example we consider is the *tree fractafold*, discussed in Section 4, where Γ is the 3-regular tree. In this case, the solution of (a) is well known [4, 9]. We solve (b) by showing that the 6-eigenspace on Γ_0 is infinite dimensional and give an explicit tight frame for this space. We solve (c) in terms of a mean value on the tree that is in fact different from the obvious mean value. The fractafold spectrum in this example is a union of point spectrum and absolutely continuous spectrum.

In Section 5, we discuss periodic fractafolds, concentrating on a *honeycomb fractafold*, where Γ is a hexagonal lattice. In this case, the solution of (a) is also well known. Our solution of (b) gives a basis for the infinite dimensional 6-eigenspace of compactly supported functions. Finally, in Section 6, we discuss an example of a finitely ramified periodic Sierpiński fractal field (see [12]) which is not a fractafold but can be treated using our methods.

Essentially all the results of this paper can be extended to fractafolds based on the n-dimensional Sierpiński gasket, using similar methods. It seems likely that similar results could be obtained for any p.c.f. fractal for which the method of spectral decimation applies (see [1, 8, 10, 11, 13, 16, 18, 20, 26, 33, 39, 42, 43, and references therein]).

2 Set-up results for infinite Sierpiński fractafolds

2.1 Laplacian on the Sierpiński gasket. We denote by Δ_{SG} the standard Laplacian on SG, and by μ_{SG} the standard normalized Hausdorff probability measure on SG (see [17, 18, 39] for details). The Laplacian Δ_{SG} is self-adjoint on $L^2(SG, \mu_{SG})$ with appropriate boundary conditions (usually Dirichlet or Neumann). The Laplacian Δ_{SG} can be defined either probabilistically or analytically

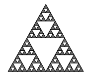


Figure 2.1. Sierpiński gasket.

using Kigami's resistance (or energy) form and the relation

$$\mathcal{E}(f, f) = -\frac{3}{2} \int_{SG} f \Delta_{SG} f d\mu_{SG}$$

for functions in the corresponding domain of the Laplacian. The energy is defined by

$$\mathcal{E}(f, f) = \lim_{n\to\infty} \left(\frac{5}{3}\right)^n \sum_{x,y\in V_n, x\sim y} (f(x) - f(y))^2 .$$

In these formulas, V_n is a finite set of $(3^{n+1} + 3)/2$ points in SG that are at the euclidean distance 2^{-n} from the neighboring points, and $\sim$ denotes the recursively defined graph structure on V_n. Note the normalization factor $3/2$; it is inserted here for convenience of computation. (See [39].)

2.2 Spectral decimation and the eigenfunction extension map. Both Dirichlet and Neumann spectra of Δ_{SG} are well known. (See [11, 39, 41]). To compute the spectrum of Δ_{SG}, one employs the so-called spectral decimation method, using inverse iterations of the polynomial $R(z) = z(5 - z)$. By convention, the eigenvalue equation is written $-\Delta_{SG} u = \lambda u$ because $-\Delta_{SG}$ is a non-negative operator. Each positive eigenvalue can be written as

$$\lambda = \lim_{m\to\infty} 5^m \lambda_m = 5^{m_0} \lim_{k\to\infty} 5^k \lambda_{k+m_0} \tag{2.1}$$

for a sequence $\{\lambda_m\}_{m=m_0}^{\infty}$ such that $\lambda_m = R(\lambda_{m+1})$ and $\lambda_{m_0} \in \{2, 5, 6\}$, which can be written as $R^{\circ k}(\lambda_{k+m_0}) \in \{2, 5, 6\}$, where the powers $R^{\circ k}$ of R are composition powers. If we let $\mathfrak{R}_k(z) = R^{\circ k}(5^{-k} z)$, then

$$R^{\circ k}(\lambda_{k+m_0}) = \mathfrak{R}_k(5^k \lambda_{k+m_0}) = \mathfrak{R}_k\left(\frac{2}{3} 5^{-m_0} 5^{m_0} 5^k \lambda_{k+m_0}\right). \tag{2.2}$$

Thus an important role is played by the function

$$\mathfrak{R}(z) = \lim_{k\to\infty} R^{\circ k}(5^{-k} z). \tag{2.3}$$

This is an analytic function, which is a classical object in complex dynamics, See [6, 7] for more details. In the context of the Laplacian on the Sierpiński gasket, this function first appeared in [28, Lemma 2.1] and [5, Remark 2.5]. (See also [15, 29] for related results). In particular, this function can be defined as the solution of the classical functional equation

$$R(\mathfrak{R}(z)) = \mathfrak{R}(5z). \tag{2.4}$$

Note that in a neighborhood of zero the inverse of the function $\mathfrak{R}$ can be defined by

$$\mathcal{R}(w) = \lim_{k\to\infty} 5^k R^{-k}(w), \tag{2.5}$$

and satisfies the functional equations

$$5\mathcal{R}(w) = \mathcal{R}(R(w)) \tag{2.6}$$

in a neighborhood of zero.

One can see from (2.2) that each nonzero eigenvalue λ satisfies

$$\lambda \in 5^{m_0}\mathfrak{R}^{-1}\{2,5,6\} \subset \bigcup_{m=0}^{\infty} 5^m\mathfrak{R}^{-1}\{2,5,6\}.$$

Some of the points in this union are so-called "forbidden eigenvalues"; the rest are so-called 2-series, 5-series and 6-series eigenvalues; see [39]. A detailed analysis shows that the spectrum of the Dirichlet Laplacian is

$$\Sigma_D = 5\left(\mathfrak{R}^{-1}\{2,5\} \cup 5\mathfrak{R}^{-1}\{5\} \bigcup_{m_0=2}^{\infty} 5^{m_0}\mathfrak{R}^{-1}\{3,5\}\right),$$

and the spectrum of the Neumann Laplacian is

$$\Sigma_N = \{0\} \cup 5\left(\mathfrak{R}^{-1}\{3\} \cup \bigcup_{m_0=1}^{\infty} 5^{m_0}\mathfrak{R}^{-1}\{3,5\}\right).$$

The multiplicities, which grow exponentially fast with k, were computed explicitly in [11] and can also be found in [1, 39, 41]. Note that because of the functional equations (2.4) and (2.6) and because $R(2) = R(3) = 6$, we have

$$5\left(\mathfrak{R}^{-1}\{2\} \cup \mathfrak{R}^{-1}\{3\}\right) = \mathfrak{R}^{-1}\{6\}.$$

We define

$$\Sigma_{ext} = 5\left(\mathfrak{R}^{-1}\{2\} \cup \bigcup_{m=0}^{\infty} 5^m\mathfrak{R}^{-1}\{5\}\right) \subset \mathfrak{R}^{-1}\{0,6\}.$$

and have the following.

Proposition 2.1. *For any $v \in \partial SG$ and any complex number $\lambda \notin \Sigma_{ext}$, there is a unique continuous function $\psi_{v,\lambda}(\cdot) : SG \to \mathbb{R}$ such that $\psi_{v,\lambda}(v) = 1$, $\psi_{v,\lambda}$ vanishes at the other two boundary points, and the pointwise eigenfunction equation $-\Delta\, \psi_{v,\lambda}(x) = \lambda\, \psi_{v,\lambda}(x)$ holds at every point $x \in SG \backslash \partial SG$.*

Naturally, $\psi_{v,\lambda}$ is called the eigenfunction extension map, which is explained in [39, Section 3.2], and the proposition is essentially the same as [39, Theorem 3.2.2].

Example 2.2. (Spectral decimation for the unit interval [0, 1]). In order to illustrate these notions, we briefly explain how they look in the more classical case of the unit interval. The operator $\Delta_{[0,1]} = d^2/dx^2$ is the standard Laplacian on [0, 1], and if $\mu_{[0,1]}$ denotes Lebesgue measure on [0, 1], $\Delta_{[0,1]}$ is self-adjoint, and

$$\mathcal{E}(f, f) = \int_0^1 (f'(x))^2 dx = -\int_{[0,1]} f\, \Delta_{[0,1]} f d\mu_{[0,1]}$$

for functions in the domain of the Dirichlet or Neumann Laplacian. The energy can also be defined by $\mathcal{E}(f, f) = \lim_{n\to\infty} 2^n \sum_{x,y \in V_n, x\sim y} (f(x) - f(y))^2$, where $V_n = \{k/2^n\}_{k=0}^{2^n}$. To compute the spectrum of $-\Delta_{[0,1]}$, one can use the spectral decimation method with inverse iterations of the polynomial $R(z) = z(4 - z)$. Each positive eigenvalue can be written as $\lambda = \lim_{m\to\infty} 4^m \lambda_m$ for a sequence $\{\lambda_m\}_{m=m_0}^{\infty}$ such that $\lambda_m = R(\lambda_{m+1})$ and $\lambda_{m_0} \in \{0, 4\}$. Then $\mathfrak{R}(z) = \lim_{k\to\infty} R^{\circ k}(4^{-k} z) = 2 - 2\cos(\sqrt{z})$ satisfies the functional equation $R(\mathfrak{R}(z)) = \mathfrak{R}(4z)$. In this case, $\sigma(-\Delta_{[0,1]}) \subset \mathfrak{R}^{-1}\{0, 4\}$, the multiplicity is one, and 0 is in the Neumann spectrum but not in the Dirichlet spectrum. The eigenfunction extension map is

$$\psi_{v,\lambda}(x) = \cos(\sqrt{\lambda}\, |x - v|) - \frac{\cos(\sqrt{\lambda})}{\sin(\sqrt{\lambda})} \sin(\sqrt{\lambda}\, |x - v|),$$

where v is 0 or 1.

For further information on this example and its relation to quantum graphs, see [30] and references therein.

2.3 Underlying graph assumptions and Sierpiński fractafolds. Let $\Gamma_0 = (V_0, E_0)$ be a finite or infinite graph. To define a Sierpiński fractafold we assume that Γ_0 is a 4-regular graph that is a union of complete graphs of 3 vertices. It can be said that Γ_0 is a regular 3-hyper-graph in which every vertex belongs to two hyper-edges. A hyper-edge in this case is a complete graph of 3 vertices; we call it a **cell** or **0-cell** of Γ_0. We denote the discrete Laplacian on Γ_0 by Δ_{Γ_0}. (In principle, these assumptions can be weakened; see Section 6 and Figure 6.1, for instance.)

Let SG be the usual compact Sierpiński gasket (see Figure 2.2). We define a Sierpiński fractafold $\mathfrak{F}$ by replacing each cell of Γ_0 with a copy of SG. We call these copies **cells or 0-cells** of the Sierpiński fractafold $\mathfrak{F}$. Naturally, the corners of the copies of the Sierpiński gasket SG are identified with the vertices of Γ_0.

A fractafold is called **infinite** if the graph Γ_0 is infinite. Otherwise, the fractafold is called **finite**. In particular, finite fractafolds are compact and infinite fractafolds are not. All the details concerning finite and infinte fractafolds can be found in [37]. In this paper, we use the notation of [37] as much as possible. (See also [40].) Since the pairwise intersections of the cells of the Sierpiński fractafold $\mathfrak{F}$ are finite, we can consider the natural measure on the Sierpiński fractafold $\mathfrak{F}$, which we also denote μ. Furthermore, since Δ_{SG} is a local operator, we can define a local Laplacian Δ on the Sierpiński fractafold $\mathfrak{F}$, as explained in [37].

2.4 Eigenfunction extension map on fractafolds. For any $v \in V_0$ and $\lambda \notin \Sigma_{ext}$, there is a unique continuous function $\psi_{v,\lambda}(\cdot) : \mathfrak{F} \to \mathbb{R}$ such that

(1) the support of $\psi_{v,\lambda}$ is contained in the union of of the cells of the Sierpiński fractafold $\mathfrak{F}$ that contain v,
(2) $\psi_{v,\lambda}(v) = 1$,
(3) the pointwise eigenfunction equation $-\Delta\psi_{v,\lambda}(x) = \lambda\psi_{v,\lambda}(x)$ holds at every point $x \in \mathfrak{F}\backslash V_0$.

For any function f_0 on Γ_0 (and any λ as above), we define the eigenfunction extension map by

$$\Psi_\lambda f_0(x) = \sum_{v \in V_0} f_0(v)\psi_{v,\lambda}(x). \tag{2.7}$$

By definition, $f = \Psi_\lambda f_0$ is a continuous extension of f_0 to the Sierpiński fractafold $\mathfrak{F}$ which is a pointwise solution to the eigenvalue equation above for all $x \in \mathfrak{F}\backslash V_0$. Moreover, it is known that if f_0 is a pointwise solution of the eigenfunction equation $-\Delta_{\Gamma_0} f_0 = \lambda_0 f_0$ on Γ_0, and $\lambda_0 \notin \{0, 6\}$, then $f = \Psi_\lambda f_0$ is a continuous extension of f_0 to the Sierpiński fractafold $\mathfrak{F}$ which is a pointwise solution of the eigenvalue equation above for all $x \in \mathfrak{F}$. Note that here $\lambda \in \mathfrak{R}^{-1}(\lambda_0)$, where $\mathfrak{R}$ is as above. The eigenfunction extension map is explained in [39, p. 69].

It is easy to see that $\Psi_\lambda : \ell^2(V_0) \to L^2(\mathfrak{F}, \mu)$ is a bounded linear operator for any $\lambda \notin \mathfrak{R}^{-1}\{2, 5, 6\}$, and its adjoint $\Psi_\lambda^* : L^2(\mathfrak{F}, \mu) \to \ell^2(V_0)$ can be computed as

$$\left(\Psi_\lambda^* g\right)(v) = \int_{\mathfrak{F}} g(x)\psi_{v,\lambda}(x)d\mu(x). \tag{2.8}$$

2.5 Spectral decomposition (resolution of the identity). Suppose that the self-adjoint discrete Laplacian Δ_{Γ_0} on Γ_0 has a spectral decomposition (resolution of the identity)

$$-\Delta_{\Gamma_0} = \int_{\sigma(-\Delta_{\Gamma_0})} \lambda dE_{\Gamma_0}(\lambda), \tag{2.9}$$

which has the form

$$-\Delta_{\Gamma_0} f_0(v) = \int_{\sigma(-\Delta_{\Gamma_0})} \lambda \sum_{u \in V_0} P_{\Gamma_0}(\lambda, u, v) f_0(u) dm_{\Gamma_0}(\lambda), \tag{2.10}$$

where $m(\cdot)$ is a spectral measure of $-\Delta$ which is a Borel measure on $\sigma(-\Delta_{\Gamma_0})$. (See Section 3 for more detail.)

We define the function $M(\lambda)$ as the infinite product

$$M(\lambda) = \prod_{m=1}^{\infty} \frac{(1-\lambda_m/5)(1-\lambda_m/2)}{(1-\lambda_m/6)(1-2\lambda_m/5)}, \tag{2.11}$$

where $\lambda = \lim_{m\to\infty} 5^m \lambda_m$ and $\lambda_m = R(\lambda_{m+1})$. The function $M(\cdot)$ is known from [27, Lemma 2.2 and Corollary 2.4]; it appears when the L^2 norm of eigenfunctions on the Sierpiński gasket is computed. This function does not depend on the fractafold, but only on the Sierpiński gasket.

We set

$$\Sigma_\infty = 5\left(\mathfrak{R}^{-1}\{2\} \cup \bigcup_{m=0}^{\infty} 5^m \mathfrak{R}^{-1}\{3,5\}\right), \quad \Sigma'_\infty = 5\left(\bigcup_{m=1}^{\infty} 5^m \mathfrak{R}^{-1}\{3,5\}\right) \subset \Sigma_\infty.$$

Note that $\Sigma_\infty \backslash \Sigma'_\infty = 5\mathfrak{R}^{-1}\{2,3,5\} \subset \mathfrak{R}^{-1}\{0,6\}$.

Theorem 2.3. *The Laplacian* Δ *is self-adjoint and*

$$\mathfrak{R}^{-1}(\sigma(-\Delta_{\Gamma_0})) \cup \Sigma'_\infty \subset \sigma(-\Delta) \subset \mathfrak{R}^{-1}(\sigma(-\Delta_{\Gamma_0})) \cup \Sigma_\infty. \tag{2.12}$$

Moreover, the spectral decomposition $-\Delta = \int_{\sigma(-\Delta)} \lambda dE(\lambda)$ *can be written as*

$$-\Delta = \int_{\mathfrak{R}^{-1}(\sigma(-\Delta_{\Gamma_0}))\backslash\Sigma_\infty} \lambda M(\lambda) \Psi^*_\lambda d\left(E_{\Gamma_0}(\mathfrak{R}(\lambda))\right)\Psi_\lambda + \sum_{\lambda\in\Sigma_\infty} \lambda E\{\lambda\}. \tag{2.13}$$

Here $E\{\lambda\}$ *denotes the eigenprojection if* λ *is an eigenvalue. (The eigenprojection is non-zero if and only if* λ *is an eigenvalue.)*

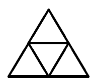
All eigenvalues and eigenfunctions of Δ *can be computed by the spectral decimation method as so-called offsprings of either localized eigenfunctions on approximating graph Laplacians or of eigenfunctions on* Γ_0*. Furthermore, the* *spectral decomposition of the Laplacian* Δ *on the Sierpiński fractafold* $\mathfrak{F}$ *has the form*

$$(2.14)\qquad -\Delta f(x) = \int_{\mathfrak{R}^{-1}(\sigma(-\Delta_{\Gamma_0}))\setminus\Sigma_\infty} \lambda \left(\int_{\mathfrak{F}} P(\lambda, x, y) f(y) d\mu(y) \right) dm(\lambda) + \sum_{\lambda\in\Sigma_\infty} \lambda E\{\lambda\} f(x),$$

where $m = m_{\Gamma_0} \circ \mathfrak{R}$ *and*

$$(2.15)\qquad P(\lambda, x, y) = M(\lambda) \sum_{u,v\in V_0} \psi_{v,\lambda}(x)\psi_{u,\lambda}(y) P_{\Gamma_0}(\mathfrak{R}(\lambda), u, v).$$

Proof. Let $\Gamma_0 = (V_0, E_0)$ be as above, and let $\Gamma_1 = (V_1, E_1)$ be a graph obtained from Γ_0 by replacing each cell of Γ_0 with the graph

 .

The three vertices of the biggest triangle replace the three vertices of each cell of Γ_0. We repeat this procedure recursively to define a sequence of discrete approximations V_n to the Sierpiński fractafold $\mathfrak{F}$. On each V_n we consider a discrete energy form; they converge as $n \to \infty$ with the same normalization as in Subsection 2.1. In the limit, we obtain a resistance form $\mathcal{E}$ of the Sierpiński fractafold $\mathfrak{F}$, and one can use the theory of resistance forms of Kigami (see [18, 19]) to define the weak Laplacian Δ on the Sierpiński fractafold $\mathfrak{F}$.

More precisely, by [19, Theorem 8.10], the resistance form is a regular Dirichlet form on $L^2(\mathfrak{F}, \mu)$, for which a self-adjoint Laplacian Δ is uniquely defined (see [19, Proposition 8.11].) One sees easily that in this case, the set of continuous compactly supported functions in $Dom\,\Delta$ such that Δf is also continuous (and compactly supported) form a core. For any such function f the Laplacian Δf can be approximated by discrete Laplacians; that is $\Delta f(x) = \lim_{n\to\infty} 5^n \Delta_n f(x)$, where Δ_n is the graph Laplacian on V_n. The limit is pointwise for each $x \in V_* = \bigcup V_n$ and is uniform on compact subsets of the Sierpiński fractafold $\mathfrak{F}$, provided Δf is continuous with compact support. The pointwise and uniform convergence of discrete Laplacians in this case is justified in the same as way in the case of the Laplacian on the Sierpiński gasket.

Using the notation of Subsections 2.1 and 2.2, we let $m_n = m_{\Gamma_0} \circ \mathfrak{R}_n$ and

$$P_n(\lambda, x, y) = M_n(\lambda) \sum_{u,v\in V_0} \psi_{v,\lambda}(x)\psi_{u,\lambda}(y) P_{\Gamma_0}(\mathfrak{R}(\lambda), u, v),$$

where $M_n(\lambda)$ is defined as the partial product in the definition of $M(\lambda)$. Set $\Sigma_n = 5\big(\mathfrak{R}_n^{-1}\{2\} \cup \bigcup_{m=0}^{n-1} 5^m \mathfrak{R}_m^{-1}\{3, 5\}\big)$ and let $E_n\lambda$ be the eigenprojection of $-\Delta_n$

corresponding to λ. Then we have the discrete version of formula (2.14) because of the computation in [1, Theorem 3.3]. (See also Sections 3 and 4 below, where $P_{\Gamma_0}(\lambda, u, v)$ is denoted by $\tilde{P}_\lambda(u, v)$.) Note that in [1, Theorem 3.3], the normalization factor is $1/(\phi R')$, where $\phi(z) = (3-2z)/\big((5-4z)(1-2z)\big)$ and $R(z) = z(5-4z)$. This produces the normalization factor

$$\frac{(5-4z)(1-2z)}{(3-2z)(5-8z)} = \frac{1}{3}\frac{(1-4z/5)(1-4z/2)}{(1-4z/6)(1-8z/5)},$$

which is the same as in (2.11). Here $4z$ replaces λ_m because of the distinction between probabilistic and graph Laplacians, and the extra factor $1/3$ appears because of the integration in (2.14).

Let u and f be continuous functions on the Sierpiński fractafold $\mathfrak{F}$ with compact support and let $v = (-\Delta + 1)^{-1} f$. The usual energy and L^2 estimates imply that $v \in Dom(\Delta)$ is continuous and square integrable and that $-\Delta v = f - v$. The discrete approximations imply that the inner product $\langle u, v\rangle_{L^2}$ equals

$$\int_{\mathfrak{R}^{-1}(\sigma(-\Delta_{\Gamma_0}))\setminus\Sigma_\infty} \frac{1}{\lambda+1} \left\langle u, \int_{\mathfrak{F}} P(\lambda, x, y) f(y) d\mu(y) \right\rangle_{L^2} dm(\lambda) \\ + \sum_{\lambda\in\Sigma_\infty} \frac{1}{\lambda+1} \langle u, E\{\lambda\} f\rangle_{L^2},$$

and so $\langle u, v\rangle_{L^2} = \int_{\sigma(-\Delta)} \langle u, dE(\lambda) f\rangle_{L^2}/(\lambda+1)$ when u, f are continuous functions with compact support. The theorem then follows by the general theory of self-adjoint operators. (See [32, Section VIII.7].) □

2.6 Infinite Sierpiński gaskets. As a collection of first examples, we consider the infinite Sierpiński gaskets, where the spectrum was analyzed in [3, 41, 31].

First, note that up to a natural isometry there is exactly one infinite Sierpiński gasket with a distinguished boundary point (and which hence is not a fractafold), and there are uncountably many non-isometric infinite Sierpiński gaskets which are fractafolds. (See [41] for more detail.)

If an infinite Sierpiński gasket fractafold is build in a self-similar way, as described in [36, 41], then the spectrum on Γ_0 is pure point with two infinite series of eigenvalues of infinite multiplicity. One series of eigenvalues consists of isolated points which accumulate to the Julia set $\mathfrak{J}_R$ of the polynomial R, and the points of the other series are located on the edges of the gaps of this Julia set (the Julia set in this case is a real Cantor set of one dimensional Lebesgue measure zero). The set of eigenvalues Σ_0 on Γ_0 consists of 6 and all the pre-images of 5 and 3 under

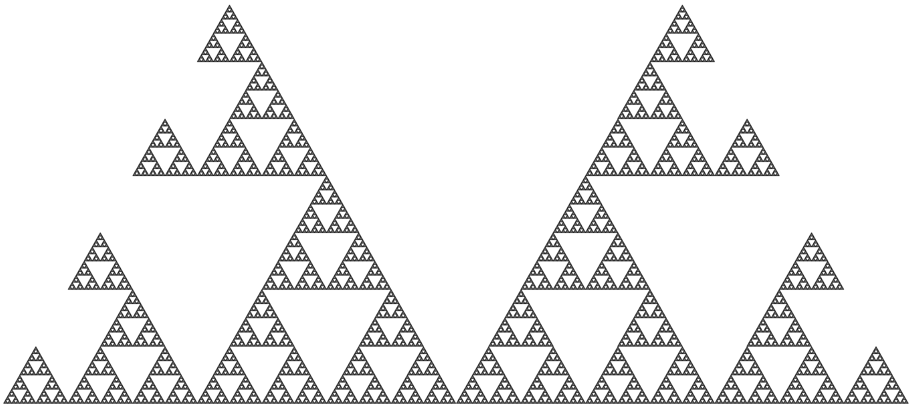


Figure 2.2. A part of an infinite Sierpiński gasket.

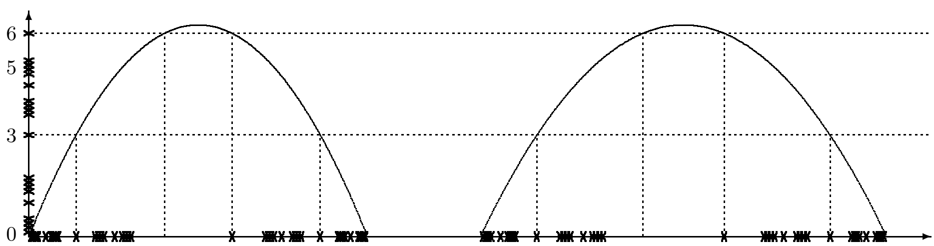


Figure 2.3. An illustration of the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$, the vertical axis contains the spectrum of $\sigma(-\Delta_{\Gamma_0})$ and the horizontal axis contains the spectrum $\sigma(-\Delta)$.

the inverse iterations of R. In this case formula (2.14) is the same as the formulas for eigenprojections in [41]. An illustration of the computation of the spectrum in Theorem 2.3 is shown in Figure 2.3, where the graph of the function $\mathfrak{R}$ is shown schematically and the locations of eigenvalues are denoted by small crosses. The spectrum $\sigma(-\Delta)$ is shown on the horizontal axis and the set of eigenvalues Σ_0 of $-\Delta_{\Gamma_0}$ is shown on the vertical axis.

A different infinite Sierpiński gasket fractafold can be constructed using two copies of an infinite Sierpiński gasket with a boundary point and joining these copies at the boundary. This fractal was first considered in [2] and has a natural axis of symmetry between left and right copies. Therefore, we can consider symmetric and anti-symmetric functions with respect to these symmetries. It was proved in [41] that the spectrum of the Laplacian restricted to the symmetric part is pure point, with a complete set of eigenfunctions with compact support. For the anti-symmetric part, the compactly supported eigenfunctions are not complete, and it was proved in [31] that the Laplacian on Γ_0 has a singular continuous component in the spectrum, supported on $\mathfrak{J}_R$, of spectral multiplicity one. As a corollary of these and our results, we have the following proposition.

Proposition 2.4. *On the Barlow-Perkins infinite Sierpiński fractafold the spectrum of the Laplacian consists of a dense set of eigenvalues $\mathfrak{R}^{-1}(\Sigma_0)$ of infinite multiplicity and of a singularly continuous component of spectral multiplicity one supported on $\mathfrak{R}^{-1}(\mathfrak{J}_R)$.*

3 General infinite fractafolds and the main results

Consider a fractafold with cell graph Γ, so Γ is an arbitrary infinite 3-regular graph. The spectrum of $-\Delta_\Gamma$ is contained in [0,6], and by the Spectral Theorem, there exist projection operators E_I corresponding to intervals $I \subseteq [0, 6]$. Because we are in a discrete setting we can say a lot more. There is a kernel function E_I on $\Gamma \times \Gamma$ such that

$$E_I f(a) = \sum_{b\in\Gamma} E_I(a, b) f(b), \tag{3.1}$$

and $I \to E_I(a, b)$ is a signed measure for each fixed a, b. Since there are countably many such measures, we can find a single positive measure μ on [0,6] such that

$$E_I(a, b) = \int_I P_\lambda(a, b) d\mu(\lambda) \tag{3.2}$$

for a function $P_\lambda(a, b)$ defined almost anywhere with respect to μ (so $P_\lambda(a, b)$ is just the Radon-Nikodym derivative of $E_I(a, b)$ with respect to μ). In fact, by a theorem of Besicovitch,

$$P_\lambda(a, b) = \lim_{\epsilon\to 0} \frac{E_{[\lambda-\epsilon,\lambda+\epsilon]}(a, b)}{\mu([\lambda - \epsilon, \lambda + \epsilon])} \tag{3.3}$$

for $\mu - a.e.\ \lambda$. (If μ is absolutely continuous, this is just the Lebesgue differential of the integral theorem.) It follows from (3.3) that

$$-\Delta_\Gamma P_\lambda(\cdot, b) = \lambda P_\lambda(\cdot, b) \tag{3.4}$$

for $\mu - a.e.\ \lambda$. Thus, if we define the pointwise projections

$$P_\lambda f(a) = \sum_{b\in\Gamma} P_\lambda(a, b) f(b), \tag{3.5}$$

the spectral resolution is

$$f = \int_{\Sigma} P_\lambda f d\mu(\lambda), \tag{3.6}$$

with

$$-\Delta_\Gamma P_\lambda f = \lambda P_\lambda f, \tag{3.7}$$

where $\Sigma \subseteq [0, 6]$ is the spectrum. In other words, (3.6) represents a general function f (we may take $f \in \ell^2(\Gamma)$, or more restrictively, a function of finite support) as an integral of λ-eigenfunctions. Note that typically $P_\lambda f$ is not in $\ell^2(\Gamma)$. Also, the measure μ and the kernel P_λ are not unique, since one may be multiplied by $g(\lambda)$ and the other divided by $g(\lambda)$ for any positive function g. We are not aware of any way to make a "canonical" choice to eliminate this ambiguity.

We also observe that the measure μ does not have a discrete atom at $\lambda = 6$. In other words, there are no $\ell^2(\Gamma)$ 6-eigenfunctions. Indeed, for a 3-regular graph, there exist 6-eigenfunctions if an only if the graph is bipartite, in which case the 6-eigenfunction alternates ± 1 on the two parts. Since we are assuming Γ is infinite, this eigenfunction is not in $\ell^2(\Gamma)$.

Let Γ_0 denote the edge graph of Γ. Then Γ_0 is 4-regular. Let Δ_{Γ_0} denote its Laplacian. Define

$$\tilde{P}_\lambda(x, y) = \frac{1}{6-\lambda} \sum_{a \in x} \sum_{b \in y} P_\lambda(a, b) \tag{3.8}$$

(there are 4 terms in the sum). Let E_6 denote the space of 6-eigenfunctions in $\ell^2(\Gamma_0)$ (this may be 0), and write $\tilde{P}_6$ for the orthogonal projection of $\ell^2(\Gamma_0)$ onto E_6.

Theorem 3.1. *The spectral resolution of* $-\Delta_{\Gamma_0}$ *is given by*

$$F = \tilde{P}_6 F + \int_{\Sigma} \tilde{P}_\lambda F d\mu(\lambda), \tag{3.9}$$

where

$$-\Delta_{\Gamma_0} \tilde{P}_\lambda F = \lambda \tilde{P}_\lambda F \tag{3.10}$$

for $\mu - a.e.\ \lambda$, *and*

$$\tilde{P}_\lambda F(x) = \sum_{y \in \Gamma_0} \tilde{P}_\lambda(x, y) F(y). \tag{3.11}$$

In particular, $spect(-\Delta_{\Gamma_0}) = \Sigma$ *(if* $E_6 = 0$*) or* $\Sigma \cup \{6\}$.

For the proof, we require some lemmas.

Following [40], we define the two sum operators $S_1 : \ell^2(\Gamma) \to \ell^2(\Gamma_0)$ and $S_2 : \ell^2(\Gamma_0) \to \ell^2(\Gamma)$ by

$$S_1 f(x) = f(a) + f(b) \qquad \text{if } x \text{ is the edge } (a, b) \tag{3.12}$$

and

$$S_2 F(a) = F(x) + F(y) + F(z) \qquad \text{if } x, y, z \text{ are the edges containing } a. \tag{3.13}$$

Lemma 3.2. *$S_2S_1 = 6I + \Delta_\Gamma$ and $S_1S_2 = 6I + \Delta_{\Gamma_0}$. In particular, S_2S_1 is invertible, S_1 is one-to-one, and S_2 is onto.*

Proof. The formulas for S_2S_1 and S_1S_2 are simple computations. Since there are no 6-eigenfunctions in $\ell^2(\Gamma)$, we obtain the invertability of S_2S_1 (see also [40]). □

It follows from Lemma 3.2 that $E_6 = (\mathrm{Im} S_1)^\perp$ and $\ell^2(\Gamma_0) = \mathrm{Im} S_1 \oplus E_6$.

Lemma 3.3. *For any $\lambda \neq 6$, $-\Delta_\Gamma f = \lambda f$ if and only if $-\Delta_{\Gamma_0} S_1 f = \lambda S_1 f$. In particular, $sp(-\Delta_{\Gamma_0}) = sp(-\Delta_\Gamma) \cup \{6\}$.*

Proof. Suppose $-\Delta_\Gamma f = \lambda f$. Since $-\Delta_\Gamma = 6I - S_2S_1$, $-S_2S_1 f = (\lambda - 6) f$. Apply S_1 to this identity and use $-\Delta_{\Gamma_0} = 6I - S_1S_2$ to obtain $-\Delta_{\Gamma_0} S_1 f = \lambda S_1 f$. Similarly, we can reverse the implications. Note that the condition $\lambda \neq 6$ implies that $S_1 f$ is not identically zero (see also [40]). □

Lemma 3.4. *Let $F \in \ell^2(\Gamma_0)$ be orthogonal to E_6 (if E_6 is nontrivial). Then $F = S_1 f$ for*

$$f = (6I + \Delta_\Gamma)^{-1} S_2 F_1. \tag{3.14}$$

Moreover,

$$\tilde{P_\lambda} = \frac{1}{6-\lambda} S_1 P_\lambda S_2. \tag{3.15}$$

Proof. For f defined by (3.4), we have $S_2S_1 f = S_2 F$ by Lemma 3.2. Since S_2 is injective on $E_6^\perp$ and $S_1 f \in E_6^\perp$, we conclude that $S_1 f = F$.

By definition, $\tilde{P_\lambda} F(x) = \sum_{y\in\Gamma_0} \big(\sum_{a\in x} \sum_{b\in y} P_\lambda(a, b) F(y)\big)/(6 - \lambda)$, and this is equivalent to (3.15) by the definition of S_1 and S_2. □

Proof of Theorem 3.1. It suffices to establish (3.9) for $F \in E_6^{\perp}$. For f defined by (3.14), we apply S_1 to (3.6), to obtain

$$F = \int S_1 P_\lambda f d\mu(\lambda) = \int S_1 P_\lambda (6I + \Delta_G)^{-1} S_2 F d\mu = \int \frac{1}{6-\lambda} S_1 P_\lambda S_2 F d\mu(\lambda),$$

since $P_\lambda(6I + \Delta_\Gamma)^{-1} = P_\lambda/(6-\lambda)$. Then (3.9) follows by (3.15). We obtain (3.10) from (3.7) and Lemma 3.3. □

In order to give an explicit form of the spectral resolution for any particular Γ, we need to solve two problems.

(a) Find an explicit formula for $P_\lambda(a, b)$.

(b) Give an explicit description of E_6 and the projection operator $\tilde{P}_6$.

In addition, there is one more problem we would like to solve in order to obtain an explicit Plancherel formula. We can always write

$$||f||^2_{\ell^2(\Gamma)} = \int_\Sigma < P_\lambda f, f > d\mu(\lambda) \tag{3.16}$$

and

$$||F||^2_{\ell^2(\Gamma_0)} = ||\tilde{P}_6 F||_2^2 + \int_\Sigma < \tilde{P}_\lambda F, F > d\mu(\lambda) \tag{3.17}$$

for a reasonable dense space of functions f and F (certainly finitely supported functions will do). What we would like is to replace $< P_\lambda f, f >$ and $< \tilde{P}_\lambda F, F >$ with expressions only involving $P_\lambda f$ and $\tilde{P}_\lambda F$ and some inner product on a space of λ-eigenfunctions. Note that from (3.2) and the fact that E_I is a projection operator, we have

$$< P_\lambda f, f > = \lim_{\epsilon \to 0} \mu([\lambda - \epsilon, \lambda + \epsilon]) \left|\left| \frac{1}{\mu([\lambda-\epsilon, \lambda+\epsilon])} E_{[\lambda-\epsilon,\lambda+\epsilon]} f \right|\right|_2^2 \tag{3.18}$$

for $\mu - a.e.\ \lambda$. This suggests the following conjecture,

Conjecture 3.5. *For μ a.e. λ there exists a Hilbert space of λ-eigenfunctions ξ_λ with inner product $<,>_\lambda$ such that $P_\lambda f \in \xi_\lambda$ for $\mu - a.e.\ \lambda$ for every $f \in \ell^2(\Gamma)$, and*

$$< P_\lambda f, f > = < P_\lambda f, P_\lambda f >_\lambda . \tag{3.19}$$

Moreover, a similar statement holds for $< \tilde{P}_\lambda F, F >$.

Our last problem is then

(c) Find an explicit description of ξ_λ and its inner product, and transfer this to $\tilde{\xi}_\lambda$ of Γ_0.

4 The tree fractafold

In this section, we study in detail the spectrum of the Laplacian on the tree fractafold TSG (Figure 4.1) whose cell graph Γ is the 3-regular tree. In a sense, this

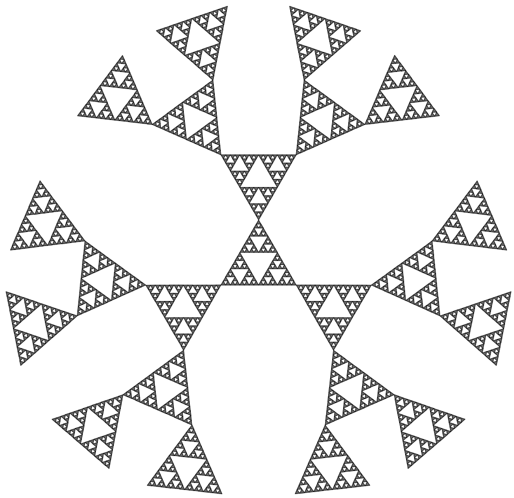


Figure 4.1. A part of the infinite Sierpiński fractafold based on the binary tree.

example becomes the "universal covering space" of all the other examples if we "fill in" all copies of SG with triangles.

We begin by solving problem (b).

Lemma 4.1. *For any fixed z in Γ_0, define $F_z(x) = (-1/2)^{d(x,z)}/\sqrt{3}$. Then $F_z \in \ell^2(\Gamma_0)$, $||F_z||_{\ell^2(\Gamma_0)} = 1$, and $F_z \in E_6$.*

Proof. Note that z has 4 neighbors $\{y_1, y_2, y_3, y_4\}$ in Γ_0 with $d(y_j, z) = 1$, so

$$-\Delta_{\Gamma_0} F_z(z) = 4F_z(z) - \sum_{j=1}^{4} F(y_j) = \frac{1}{\sqrt{3}}\left(4\left(-\frac{1}{2}\right)^0 - 4\left(-\frac{1}{2}\right)^1\right) = \frac{6}{\sqrt{3}} = 6F_z(z),$$

verifying the 6-eigenvalue equation at z.

On the other hand, if $x \neq z$, the 4 neighbors $\{y_1, y_2, y_3, y_4\}$ of x may be permuted so that $d(y_1, z) = d(x, z) - 1$, $d(y_2, z) = d(x, z)$, and $d(y_3, z) = d(y_4, z) = d(y, z) + 1$. It follows that

$$-\Delta_{\Gamma_0} F_z(x) = 4F_z(x) - \sum_{j=1}^{4} F_z(y_j) = F_z(x)\left(4 - \left(-2 + 1 - 2 \cdot \frac{1}{2}\right)\right) = 6F_z(x),$$

verifying the 6-eigenvalue equation at x. Finally,

$$||F_z||^2_{\ell^2(\Gamma_0)} = \frac{1}{3}\left(1+4\cdot\left(\frac{1}{2}\right)^2+8\cdot\left(\frac{1}{4}\right)^2+\dots\right) = \frac{1}{3}\left(1+1+\frac{1}{2}+\frac{1}{4}+\dots\right) = 1.$$

(See Figure 4.2). □

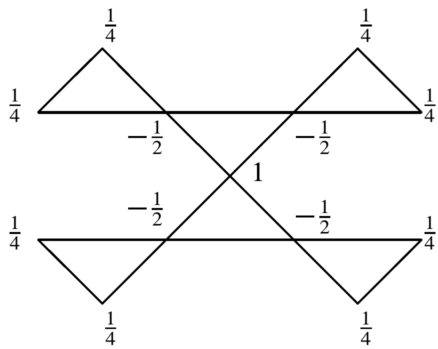


Figure 4.2. Values of $\sqrt{3}F_z$ (the center point is z).

Remark 4.2. It is easy to see from the 6-eigenvalue equation that F_z is the unique (up to a constant multiple) function in E_6 that is radial about z (a function of $d(x,z)$).

Lemma 4.3. $\sum_x F_z(x)F_y(x) = \sqrt{3}F_z(y)$.

Proof. Fix z. Then the left side is a 6-eigenfunction of y and is radial about z, so it must be a constant multiple of $F_z(y)$. To compute the constant, set $y = z$; the left side is 1, while $F_z(z) = 1/\sqrt{3}$. □

Definition 4.4. Let $\tilde{P}_6(x,y) = F_x(y)/\sqrt{3} = (-1/2)^{d(x,y)}/3$ and define the operator

$$\tilde{P}_6 F(x) = \sum_y \tilde{P}_6(x,y)F(y). \tag{4.1}$$

Theorem 4.5. *$\tilde{P}_6$ is the orthogonal projection $\ell^2(\Gamma_0) \to E_6$.*

Proof. Lemma 4.3 shows that $\tilde{P}_6 F_z = F_z$. Now we claim that the functions F_z span E_6. Indeed, if F is in E_6 and is orthogonal to F_z, then we can radialize F about z to obtain a function $\tilde{F}$ that is still in E_6 and orthogonal to F_z. Since $\tilde{F}$ must be a multiple of F_z, it follows that it is identically zero. Since $\tilde{F}(z) = F(z)$, it follows that $F(z) = 0$. Since this holds for every z, the orthogonal complement of the span of F_z is zero. This shows that $\tilde{P_6}$ is the identity on E_6. Also, $\tilde{P_6}E_6^{\perp} = 0$, by the orthogonality of different parts of the spectrum. □

Note that $\{F_z\}$ is not an orthonormal basis of E_6, since $< F_z, F_y > = \sqrt{3}F_z(y)$ by Lemma 4.3. The next result shows that it is a tight frame.

Theorem 4.6. *For any $F \in E_6$,*

$$\sum_z | < F, F_z > |^2 = 3||F||^2_{\ell^2(\Gamma_0)} \tag{4.2}$$

Proof. We may write $F = \sum_y a(y)F_y$. Then

$$||F||^2_{\ell^2(\Gamma_0)} = \sum_y \sum_z a(y)\bar{a(z)}\sqrt{3}F_z(y).$$

But $< F, F_z > = \sum_y a(y)\sqrt{3}F_z(y)$, and so

$$\begin{aligned} \sum_z | < F, F_z > |^2 &= 3\sum_z \sum_y \sum_{y'} a(y)\overline{a(y')}F_y(z)F_{y'}(z) \\ &= 3\sum_y \sum_{y'} a(y)\overline{a(y')}F_{y'}(y) = 3||F||^2_{\ell^2(\Gamma_0)}. \end{aligned} \tag{4.3}$$

□

It follows from polarizing (4.3) that we may also write

$$\tilde{P_6}F = \frac{1}{3}\sum_z < F, F_z > F_z.$$

The solution of problem (a) is due to Cartier [4]. We outline the solution following [9].

Definition 4.7. Let $z \in \mathbb{C}$ with $2^{2z-1} \neq 1$. Let

$$c(z) = \frac{1}{3}\frac{2^{1-z} - 2^{z-1}}{2^{-z} - 2^{z-1}}, \quad c(1-z) = \frac{1}{3}\frac{2^{-z} - 2^z}{2^{-z} - 2^{z-1}}$$

and $\varphi_z(n) = c(z)2^{-nz} + c(1-z)2^{-n(1-z)}$.

Remark 4.8. Note that $c(z)$ and $c(1-z)$ are characterized by the identities $c(z) + c(1-z) = 1$ and $c(z)2^{-z} + c(1-z)2^{z-1} = c(z)2^z + c(1-z)2^{1-z}$, which imply $\varphi_z(0) = 1$ and $\varphi_z(1) = \varphi_z(-1)$.

Theorem 4.9. *For any fixed $y \in \Gamma$, let $f_y(x) = \varphi_z(d(x,y))$. Then*

$$-\Delta_\Gamma f_y = (3 - 2^z - 2^{1-z})f_y, \tag{4.4}$$

and f_y may be characterized as the unique $(3 - 2^z - 2^{1-z})$-eigenfunction which is radial about y and satisfies $f_y(y) = 1$.

Proof. Uniqueness follows from the eigenvalue equation. To verify the eigenvalue equation, we do the computation separately for $x \neq y$ and $x = y$. For $x \neq y$, note that x has two neighbors, x_1 and x_2, with $d(x_1, y) = d(x_2, y) = d(x, y) + 1$ and one neighbor x_3, with $d(x_3, y) = d(x, y) - 1$; so the eigenvalue equation is immediate. On the other hand, y has three neighbors, x_1, x_2, x_3, with $d(x_j, y) = 1$, and the eigenvalue equation follows from $\varphi_z(1) = \varphi_z(-1)$. □

Note that there is no choice of z that makes f_y belong to $\ell^2(\Gamma)$. However, the choice $z = 1/2+it$ gets close. Indeed, $|\varphi_{1/2+it}(d(x, y))|^2 \approx \sum_n 2^n \cdot 2^{-n}$ just diverges. So it is natural to conjecture that these eigenfunctions give the spectral resolution of $-\Delta_\Gamma$ on $\ell^2(\Gamma)$. In fact, the following proposition is the content of [9, p. 61].

Proposition 4.10. *Let* $0 \le t \le \pi/\log 2$, $\lambda(t) = 3 - 2\sqrt{2}\cos(t \log 2)$, *and* $\sum = [3 - 2\sqrt{2}, 3 + 2\sqrt{2}]$. *Define*

$$P_t f(x) = \sum_y \varphi_{1/2+it}(d(x, y)) f(y). \tag{4.5}$$

Then

$$f(x) = \int_0^{\pi/\log 2} P_t f(x) dm(t) \tag{4.6}$$

for the measure

$$dm(t) = \frac{\log 2}{3\pi} \left| c\left(\frac{1}{2} + it\right)\right|^{-2} dt = \frac{(3 \log 2) \sin^2(t \log 2)}{\pi(1 + 2\sin^2(t \log 2))} dt. \tag{4.7}$$

Note that $P_t f$ of (4.5) satisfies $-\Delta_\Gamma P_t f = \lambda(t) P_t(f)$ and that, because of periodicity, choosing $t \in [0, \pi/\log 2]$ imposes no restriction.

It is convenient to change notation so that the eigenvalue λ rather than t is the parameter. We easily compute that for λ as in Proposition 4.10,

$$t = \frac{1}{\log 2} \cos^{-1}\left(\frac{3 - \lambda}{2\sqrt{2}}\right).$$

Note that $d\lambda = 2\sqrt{2} \log 2 \sin(t \log 2) dt$, $\sin^2(t \log 2) = (-\lambda^2 + 6\lambda - 1)/8$, and $1 + 2\sin^2(t \log 2) = (-\lambda^2 + 6\lambda + 3)/4$. If we write $P_\lambda = P_t$, the spectral resolution becomes

$$f(x) = \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} P_\lambda f(x) dm(\lambda) \quad \text{for } dm(\lambda) = \frac{3\sqrt{-\lambda^2 + 6\lambda - 1}}{\sqrt{2}\pi(-\lambda^2 + 6\lambda + 3)} d\lambda.$$

Now suppose $F \in \ell^2(\Gamma_0) \cap E_6^\perp$. Then we may write

$$F = S_1 f \quad \text{for } f = (6I + \Delta_\Gamma)^{-1} S_2 F \in \ell^2(\Gamma).$$

Indeed,we know that 6 is in the resolvent of $-\Delta_\Gamma$; so f is well defined, and then $S_2 S_1 f = S_2 F$ by Lemma 3.2. Since S_2 is injective on $E_6^\perp$ and $S_1 f \in E_6^\perp$, we conclude that $S_1 f = F$.

By Proposition 4.10, we have

$$S_1 f = \int_\Sigma S_1 P_\lambda f dm(\lambda), \tag{4.8}$$

and of course $-\Delta_{\Gamma_0} S_1 P_\lambda f = \lambda S_1 P_\lambda f$ by Lemma 3.3; so we define $\tilde{P_\lambda} F = S_1 P_\lambda f$ and obtain the spectral resolution

$$F = \int_\Sigma \tilde{P_\lambda} F dm(\lambda). \tag{4.9}$$

Note that $P_\lambda(6I + \Delta_\Gamma)^{-1} = P_\lambda/(6-\lambda)$, so $\tilde{P_\lambda} F = S_1 P_\lambda S_2 F/(6-\lambda)$.

We may write this quite explicitly, as follows.

Lemma 4.11. *Define*

$$\psi_z(n) = \tilde{c}(z)2^{-nz} + \tilde{c}(1-z)2^{-n(1-z)} \tag{4.10}$$

for $\tilde{c}(z) = (2 + 2^{-z} + 2^z)c(z)$. *Then*

$$S_1 P_\lambda S_2 F(x) = \frac{1}{3}\sum_y \psi_{1/2+it}(d(x,y))F(y). \tag{4.11}$$

Note that $\psi_z(n) = 2\varphi_z(n) + \varphi_z(n+1) + \varphi_z(n-1)$.

Proof. $S_2 F(b) = \sum_{y\sim b} F(y)$. There are three terms in the sum, and $y \sim b$ means the edge y joins b and one of its neighbors in Γ. We compute

$$P_\lambda S_2 F(a) = \sum_{b\in\Gamma}\sum_{y\sim b} \varphi_{\frac{1}{2}+it}(d(a,b))F(y) \tag{4.12}$$

and

$$S_1 P_\lambda S_2 F(x) = \sum_{a\sim x}\sum_{b\in\Gamma}\sum_{y\sim b} \varphi_{\frac{1}{2}+it}(d(a,b))F(y), \tag{4.13}$$

where $a \sim x$ means that a is one of the vertices in the edge x.

Suppose $x \neq y$, and let $n = d(x,y)$ with $n \geq 1$, (Figure 4.3 shows the Γ_0 graph for $n = 2$). Then $x \sim a_1$ and $x \sim a_2$, while $y \sim b_1$ and $y \sim b_2$, with $d(a_1,b_2) = d(a_2,b_1) = n$, $d(a_1,b_1) = n-1$, and $d(a_2,b_2) = n+1$. The result follows in this case.

If $x = y$, then $d(x,y) = 0$ and $a_1 = b_1, a_2 = b_2$; so $d(a_1,b_2) = d(a_2,b_1) = 1$ and $d(a_1,b_1) = d(a_2,b_2) = 0$. The result follows because $\varphi_{1/2+it}(-1) = \varphi_{1/2+it}(1)$. □

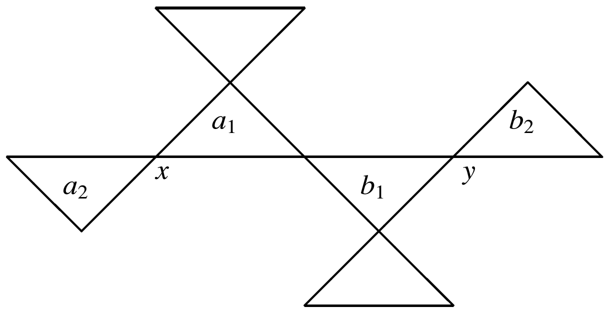


Figure 4.3. Graph Γ_0

Theorem 4.12. *Any $F \in \ell^2(\Gamma_0)$ has the explicit spectral resolution*

$$F = \tilde{P}_6 F + \int_\Sigma \tilde{P}_\lambda F dm(\lambda) \tag{4.14}$$

for

$$\tilde{P}_\lambda F(x) = \frac{1}{3(6-\lambda)} \sum_y \psi_{1/2+it}(d(x,y))F(y). \tag{4.15}$$

The Theorem follows by combining Lemma 4.11 and Proposition 4.10. We note that the proof of Proposition 4.10 involves an explicit computation of the resolvent $(\lambda I + \Delta_\Gamma)^{-1}$ for λ outside the spectrum of $-\Delta_\Gamma$, followed by a contour integral to obtain the spectral resolution from the resolvent. We sketch some of these ideas and then show how to carry out a similar proof for Theorem 4.12.

On $\ell^2(\Gamma)$ we define

$$H_z f(a) = \sum_b 2^{-zd(a,b)} f(b). \tag{4.16}$$

A direct computation shows that

$$(\lambda I + \Delta_\Gamma) H_z f = (2^{-z} - 2^z) f \tag{4.17}$$

for $\lambda = 3 - 2^z - 2 \cdot 2^{-z}$.

Note that $(3-\lambda)/(2\sqrt{2}) = \cosh((z - 1/2)\log 2)$, and that in order to have H_z bounded on $\ell^2(\Gamma)$ we need $\Re z > 1/2$. This shows that $spect(-\Delta_\Gamma) = \Sigma$ and $(\lambda I + \Delta_\Gamma)^{-1} = H_z/(2^{-z} - 2^z)$ for $z \notin \Sigma$.

On $\ell^2(\Gamma_0)$ we define

$$\tilde{H}_z F(x) = \sum_y 2^{-zd(x,y)} F(y). \tag{4.18}$$

Lemma 4.13. *$spect(-\Delta_{\Gamma_0})^{-1} = \Sigma \cup \{6\}$ and $(\lambda I+\Delta)^{-1} = \tilde{H}_z/(2\cdot 2^{-z}-2^z-1)$ for $z \notin spect(-\Delta_{\Gamma_0})$.*

Proof. Note that $\tilde{H}_z$ is bounded on $\ell^2(\Gamma_0)$ for $\Re z > 1/2$. Also $\lambda = 6$ corresponds to $z = 1+\pi i/\log 2$, for which $2\cdot 2^{-z} - 2^z - 1 = 2(-1/2)-(2)-1 = 0$. Now fix x and consider its four neighbors, x_1, x_2, x_3, x_4 (so $d(x, x_j) = 1$). For any fixed $y \neq x$, we may order them so that $d(x_1, y) = d(x_2, y) = d(x, y)+1$, $d(x_3, y) = d(x, y)$, $d(x_4, y) = d(x, y)-1$. It follows that

$$
\begin{aligned}
(\lambda I + \Delta_{\Gamma_0})\tilde{H}_z F(x) &= (\lambda-4)\tilde{H}_z F(x) + \sum_j \tilde{H}_z F(x_j) \\
&= (\lambda-4)F(x) + \sum_j 2^{-z}F(x) \\
&\quad + (\lambda-4)\sum_{y\neq x} 2^{-zd(x,y)}F(y) + \sum_j \sum_{y\neq x} 2^{-zd(x,y)}F(y) \\
&= (2\cdot 2^{-z} - 2^z - 1)F(x),
\end{aligned}
\tag{4.19}
$$

and the result follows. □

For $f \in \ell^2(\Gamma)$, we have

$$
f = \frac{1}{2\pi i}\int_\gamma (\lambda I + \Delta_\Gamma)^{-1} f d\lambda \tag{4.20}
$$

for any contour γ that circles Σ once in the counterclockwise direction. We choose γ as shown in Figure 4.4 and take the limit as $\delta \to 0^+$. The contribution from the vertical segments goes to zero, so

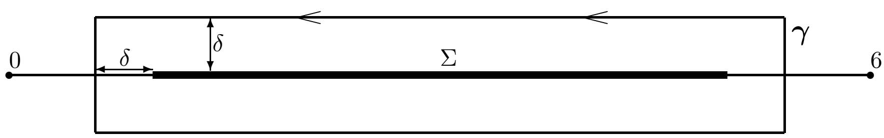


Figure 4.4. The contour γ for integration in (4.20).

$$
f = \lim_{\delta\to 0^+} \frac{1}{2\pi i}\int_\Sigma \left((\lambda - i\delta + \Delta_\Gamma)^{-1} - (\lambda + i\delta + \Delta_\Gamma)^{-1}\right) f d\lambda. \tag{4.21}
$$

If $z = 1/2+\epsilon+it$ for $\epsilon > 0$ then $(3-\lambda)/(2\sqrt{2}) = \cos(t\log 2 - i\epsilon\log 2)$ and

$$
\lambda = 3 - 2\sqrt{2}\cos(t\log 2)\cosh(\epsilon\log 2) - i2\sqrt{2}\sinh(\epsilon\log 2)\sin(t\log 2). \tag{4.22}
$$

For $t > 0$, we have $\lambda \approx 3 - 2\sqrt{2}\cos(t\log 2) - i\delta$, while for $t < 0$, we have $\lambda \approx 3 - 2\sqrt{2}\cos(t\log 2) + i\delta$ with $\delta > 0$. Thus

$$\lim_{\delta\to 0^+}\left((\lambda - i\delta + \Delta_\Gamma)^{-1} - (\lambda + i\delta + \Delta_\Gamma)^{-1}\right) f = \frac{1}{2^{-1/2-it} - 2^{1/2+it}} H_{1/2+it} f - \frac{1}{2^{-1/2+it} - 2^{1/2-it}} H_{1/2-it} f, \tag{4.23}$$

so we obtain

$$f = \frac{1}{2\pi i}\int_0^{\pi/\log 2}\left(\frac{1}{2^{-1/2-it} - 2^{1/2+it}} H_{1/2+it} f - \frac{1}{2^{-1/2+it} - 2^{1/2-it}} H_{1/2-it} f\right) 2\sqrt{2}\log 2\sin(t\log 2)dt. \tag{4.24}$$

This is the same as $f = \int_0^{\pi/\log 2} P_t f dm(t)$.

For $F \in \ell^2(\Gamma_0)$,

$$F = \frac{1}{2\pi i}\int_\gamma (\lambda I + \Delta_{\Gamma_0})^{-1} F d\lambda + \frac{1}{2\pi i}\int_{\gamma'} (\lambda I + \Delta_{\Gamma_0})^{-1} F d\lambda, \tag{4.25}$$

where γ is as before and γ' is a small circle about 6. Taking the limit, we obtain

$$F = \lim_{\delta\to 0^+}\frac{1}{2\pi i}\int_\Sigma\left((\lambda - i\delta + \Delta_{\Gamma_0})^{-1}F - (\lambda + i\delta + \Delta_{\Gamma_0})^{-1}F\right)d\lambda + \lim_{\delta\to 0^+}\frac{1}{2\pi i}\int_0^{2\pi}(6 + \delta e^{i\theta} + \Delta_{\Gamma_0})^{-1} F i\delta e^{i\theta} d\theta. \tag{4.26}$$

As before, we can write the first term as

$$\frac{\sqrt{2}\log 2}{\pi i}\int_\Sigma\left(\frac{1}{2^{1/2-it} - 2^{1/2+it} - 1}\tilde{H}_{1/2+it}F - \frac{1}{2^{1/2+it} - 2^{1/2-it} - 1}\tilde{H}_{1/2-it}F\right)\sin(t\log 2)dt, \tag{4.27}$$

which we identify with $\int_\Sigma \tilde{P}_\lambda F dm(\lambda)$, while the second term is $\tilde{P}_6 F$.

Next we discuss an explicit Plancherel formula on Γ, given in terms of the modified mean inner product

$$< f, g >_M = \lim_{N\to\infty}\frac{1}{N}\sum_{d(x,x_0)\le N} f(x)\overline{g(x)}. \tag{4.28}$$

We deal with eigenspaces for which the limit exists and is independent of the point x_0. Note that this is not the usual mean on Γ, since the cardinality of the ball $\{x : d(x, x_0) \leq N\}$ is $O(2^n)$; but it is tailor made for functions of growth rate $O(2^{-d(x,x_0)/2})$, which is exactly the growth rate of our eigenfunctions.

We expect that analogous results are valid for k-regular trees for all k; but to keep the discussion simple we deal only with the case $k = 3$, which we need for our applications.

Lemma 4.14. *For all n and t,*

$$\varphi_{1/2+it}(n) = \frac{1}{3}\left(3\cos(nt\log 2) + \frac{\sin(nt\log 2)}{\tan(t\log 2)}\right) 2^{-n/2}. \tag{4.29}$$

Proof. From the definition,

$$\varphi_{1/2+it}(n) = \left(2\Re\big(c(\tfrac{1}{2} + it)\big)2^{-itn}\right)2^{-n/2}.$$

The result follows from the explicit formula for $c(1/2+it)$ and some trigonometric identities. □

In what follows, we write φ for $\varphi_{1/2+it}$ to simplify the notation.

Lemma 4.15. *Let*

$$b(\lambda) = 8 + \frac{1}{\sin^2(t\log 2)} = 8\left(\frac{-\lambda^2 + 6\lambda}{-\lambda^2 + 6\lambda - 1}\right). \tag{4.30}$$

Then for any integers k and j,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} 2^{n+\frac{k}{2}}\varphi(n)\varphi(n+k) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} 2^{n+j+\frac{k}{2}}\varphi(n+j+k)\varphi(n+j)$$

$$= \frac{1}{18} b(\lambda)\cos(kt\log 2). \tag{4.31}$$

Proof. It is easy to see that (4.31) is independent of j, so we take $j = 0$. Then by (4.29),

$$2^{n+\frac{k}{2}}\varphi(n)\varphi(n+k) = \frac{1}{9}\left(3\cos(nt\log 2) + \frac{\sin(nt\log 2)}{\tan(t\log 2)}\right)\bigg(3\cos(nt\log 2)\cos(kt\log 2)$$
$$-3\sin(nt\log 2)\sin(kt\log 2) + \frac{\sin(nt\log 2)\cos(kt\log 2)}{\tan(t\log 2)} + \frac{\cos(nt\log 2)\sin(kt\log 2)}{\tan(t\log 2)}\bigg).$$

Now use the identities $\lim_{N\to\infty}\big(\sum_{n=1}^{N}\cos^2 n\alpha\big)/N = \lim_{N\to\infty}\big(\sum_{n=1}^{N}\sin^2 n\alpha\big)/N = 1/2$ and $\lim_{N\to\infty}\big(\sum_{n=1}^{N}\cos n\alpha \sin n\alpha\big)/N = 0$ to see that the limit in (4.31) equals

$$\frac{1}{18}\left(9\cos(kt\log 2) + \frac{3\sin(kt\log 2)}{\tan(t\log 2)} - \frac{3\sin(kt\log 2)}{\tan(t\log 2)} + \frac{\cos(kt\log 2)}{\tan^2(t\log 2)}\right)$$
$$= \frac{1}{18} b(\lambda)\cos(kt\log 2). \quad \square$$

Lemma 4.16. *For any λ in the interior of Σ and $x_1 \in \Gamma$, $< P_\lambda \delta_{x_1}, P_\lambda \delta_{x_1} >_M$ exists and is independent of the base point x_0, and*

$$< P_\lambda \delta_{x_1}, P_\lambda \delta_{x_1} >_M = \frac{1}{12} b(\lambda). \tag{4.32}$$

Proof. $P_\lambda \delta_{x_1}(x) = \varphi(d(x, x_1))$ and $\varphi(n) = O(2^{-n/2})$ by (4.29). It follows easily that the limit, if it exists, is independent of the choice of x_0. Indeed, if $d(x_0, x_0') = k$, then $B_{n-k}(x_0') \subseteq B_n(x_0) \subseteq B_{n+k}(x_0')$, and the division by N in (4.28) makes the difference go to zero as N goes to infinity. We prove the existence of the limit by computing (4.32) with $x_0 = x_1$.

Note that there are exactly $3 \cdot 2^{n-1}$ points x with $d(x, x_0) = n$ for $n \geq 1$, and we can ignore the point $x = x_1$ in computing the limit. Thus

$$< P_\lambda \delta_{x_1}, P_\lambda \delta_{x_1} >_M = \lim_{N \to \infty} \frac{3}{2N} \sum_{n=1}^{N} 2^n \varphi(n)^2 = \frac{1}{12} b(\lambda)$$

by Lemma 4.15. □

Lemma 4.17. *Suppose $d(x_1, x_2) = k$ and λ is in the interior of Σ. Then $< P_\lambda \delta_{x_1}, P_\lambda \delta_{x_2} >_M$ exists and is independent of the base point x_0, and*

$$< P_\lambda \delta_{x_1}, P_\lambda \delta_{x_2} >_M = \frac{1}{12} b(\lambda) \varphi(k). \tag{4.33}$$

Proof. The proof of independence of the base point is the same as in Lemma 4.16, so we compute the limit for $x_0 = x_1$. Except for a few points when n is small that don't enter into the limit, we may partition the points with $d(x, x_1) = n$ as follows:

2^n points with $d(x, x_2) = n + k$,
2^{n-j-1} points with $d(x, x_2) = n + k - 2j$ for $1 \leq j \leq k - 1$,
2^{n-k} points with $d(x, x_k) = n - k$.

This implies

$$\begin{aligned} &< P_\lambda \delta_{x_1}, P_\lambda \delta_{x_2} >_M \\ &= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \Bigg(2^n \varphi(n) \varphi(n + k) + \frac{1}{2} \sum_{j=1}^{k-1} 2^{n-j} \varphi(n) \varphi(n + k - 2j) \\ &\qquad + 2^{n-k} \varphi(n) \varphi(n - k) \Bigg) \\ &= \frac{1}{18} b(\lambda) 2^{-k/2} \Bigg(\cos(kt \log 2) + \frac{1}{2} \sum_{j=1}^{k-1} \cos(k - 2j)t \log 2 + \cos(kt \log 2) \Bigg), \end{aligned}$$

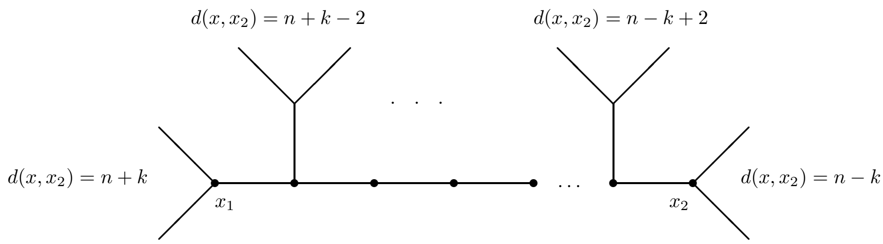


Figure 4.5. Partition of points x with $d(x, x_1) = n$.

by Lemma 4.15.

However, the trigonometric identity $\sin(a)\sum_{j=0}^{k-1}\cos(k-2j)a = \sin(ka)\cos(a)$ implies

$$\begin{aligned} &2\cos(kt\log 2) + \frac{1}{2}\sum_{j=1}^{k-1}\cos(k-2j)t\log 2 \\ &= \frac{3}{2}\cos(kt\log 2) + \frac{1}{2}\sum_{j=0}^{k-1}\cos(k-2j)t\log 2 \\ &= \frac{3}{2}\left(\cos(kt\log 2) + \frac{1}{3}\frac{\sin(kt\log 2)}{\tan(t\log 2)}\right) = \frac{3}{2}\varphi(k)2^{k/2} \end{aligned}$$

by Lemma 4.14, which implies (4.33). □

Theorem 4.18. *Suppose f has finite support. Then*

$$< P_\lambda f, f > = 12b(\lambda)^{-1} < P_\lambda f, P_\lambda f >_M . \tag{4.34}$$

Proof. Since $< P_\lambda \delta_{x_1}, \delta_{x_2} > = \varphi(d(x_1, x_2))$, we can rewrite (4.33) as

$$< P_\lambda \delta_{x_1}, \delta_{x_2} > = 12b(\lambda)^{-1} < P_\lambda \delta_{x_1}, P_\lambda \delta_{x_1} >_M ,$$

and (4.34) follows by linearity. □

Corollary 4.19. *Let $f \in \ell^2(\Gamma)$. Then for μa.e. λ, $< P_\lambda f, P_\lambda f >_M$ exists, and*

$$||f||^2_{\ell^2(\Gamma)} = \int_\Sigma < P_\lambda f, P_\lambda f >_M \; 12b(\lambda)^{-1} d\mu(\lambda). \tag{4.35}$$

Proof. For f of finite support, (4.35) follows from (4.34) and (3.16). It then follows for $f \in \ell^2(\Gamma)$ by routine limiting arguments. □

To complete the solution of problem (c) for this example, we need to transfer the result from Γ to Γ_0. Define the modified mean inner product on Γ_0 by (4.28) again, where f and g are functions on Γ_0 and x and x_0 vary in Γ_0.

Lemma 4.20. *For any integers k and j,*

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 2^{n+\frac{k}{2}} \psi(n)\psi(n+k) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 2^{n+j+\frac{k}{2}} \psi(n+j)\psi(n+j+k) \\ &= \frac{(6-\lambda)^2}{36} b(\lambda) \cos(kt \log 2). \end{aligned} \tag{4.36}$$

Proof. As in the proof of Lemma 4.15, it is clear that (4.36) is independent of j, so we may take $j = 0$. Since $\psi(k) = 2\varphi(k) + \varphi(n-1) + \varphi(n+1)$, we may reduce (4.36) to (4.31) as follows:

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 2^{n+k/2} \psi(n)\psi(n+k) \\ &\quad= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 2^{n+k/2} \big(2\varphi(n) + \varphi(n-1) + \varphi(n+1)\big) \\ &\qquad \times \big(2\varphi(n+k) + \varphi(n+k-1) + \varphi(n+k+1)\big) \\ &\quad= \frac{b(\lambda)}{18} \bigg[\left(4 + 2 + \frac{1}{2}\right) \cos(kt \log 2) + 2\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) (\log(k+1)t \log 2 \\ &\qquad + \log(k-1)t \log 2) + \cos(k+2)t \log 2 + \cos(k-2)t \log 2 \bigg] \\ &\quad= \frac{b(\lambda)}{18} \cos kt \log 2 \left[\left(4 + 2 + \frac{1}{2}\right) + \psi\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) \cos t \log 2 + 2 \cos 2t \log 2 \right] \\ &\quad= \frac{b(\lambda)}{18} \cos kt \log 2 \left(\frac{3}{\sqrt{2}} + 2 \cos t \log 2 \right)^2. \end{aligned}$$

Now (4.36) follows, since $3/\sqrt{2} + 2 \cos t \log 2 = (6-\lambda)/\sqrt{2}$. □

Lemma 4.21. *For any λ in the interior of Σ and $x_1 \in \Gamma_0$, $< \tilde{P}_\lambda \delta_{x_1}, \tilde{P}_\lambda \delta_{x_1} >_M$ exists and is independent of the base point x_0, and*

$$< \tilde{P}_\lambda \delta_{x_1}, \tilde{P}_\lambda \delta_{x_1} >_M = \frac{b(\lambda)}{162}. \tag{4.37}$$

Proof. The proof that the limit is independent of the base point is the same as in Lemma 4.16, so we compute (4.36) with $x_0 = x_1$. Note that for $n \geq 1$, there are exactly $4 \cdot 2^{n-1}$ points x in V_0 with $d(x, x_1) = n$. For such points, $\tilde{P}_\lambda \delta_{x_1}(x) = \psi(n)/\big(3(6-\lambda)\big) = \big(2\varphi(n) + \varphi(n-1) + \varphi(n+1)\big)/\big(3(6-\lambda)\big)$. Thus

$$< \tilde{P}_\lambda \delta_{x_1}, \tilde{P}_\lambda \delta_{x_1} >_M = \frac{1}{(6-\lambda)^2} \cdot \frac{2}{9} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{\infty} 2^n \left(2\varphi(n) + \varphi(n-1) + \varphi(n+1)\right)^2,$$

and (4.37) follows from (4.36). □

Lemma 4.22. *Suppose $d(x_1, x_2) = k$ and λ is in the interior of Σ. Then $< \tilde{P}_\lambda \delta_{x_1}, \tilde{P}_\lambda \delta_{x_2} >_M$ exists and is independent of the base point, and*

$$< \tilde{P}_\lambda \delta_{x_1}, \tilde{P}_\lambda \delta_{x_2} >_M = \frac{b(\lambda)}{36} \cdot \frac{1}{3(6-\lambda)} \psi(k). \tag{4.38}$$

Proof. As before we can take the base point $x_0 = x_1$. For $n > k$, we can sort the 2^{n+1} points x with $d(x, x_1) = n$ as follows:
2^n points with $d(x, x_2) = n + k$,
2^{n-j} points with $d(x, x_2) = n + k - 2j + 1$ for $1 \le j \le k$, and
2^{n-k} points with $d(x, x_2) = n - k$.

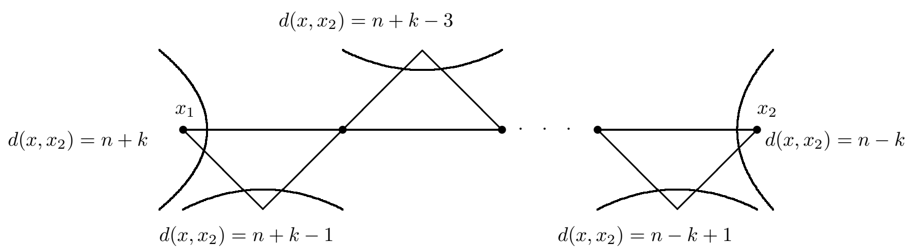


Figure 4.6. Partition of points x with $d(x, x_1) = n$.

Thus we have

$$\begin{aligned} < \tilde{P}_\lambda \delta_{x_1}, \tilde{P}_\lambda \delta_{x_2} >_M &= \frac{1}{(6-\lambda)^2} \cdot \frac{1}{9} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \psi(n) \\ &\quad \left(2^n \psi(n+k) + \sum_{j=1}^{k} 2^{n-j} \psi(n+k-2j+1) + 2^{n-k}\psi(n-k)\right) \\ &= \frac{b(\lambda)}{9 \cdot 36} 2^{-k/2} \Bigg[\cos(kt \log 2) + \frac{1}{\sqrt{2}} \sum_{j=1}^{k} \cos(k-2j+1)t \log 2 \\ &\quad + \cos(kt \log 2) \Bigg] \end{aligned}$$

by (4.36).

To complete the proof, we need to show

$$\frac{2^{-k/2}}{9}\left[2\cos(kt\log 2)+\frac{1}{\sqrt{2}}\sum_{j=1}^{k}\cos(k-2j+1)t\log 2\right]$$
$$=\frac{1}{3(6-\lambda)}(2\varphi(k)+\varphi(k-1)+\varphi(k+1)).$$

As we saw in the proof of Lemma 4.17,

$$\varphi(k)=\frac{2}{3}2^{-k/2}(2\cos(kt\log 2)+\frac{1}{2}\sum_{j=1}^{k-1}\cos(k-2j)t\log 2),$$

so

$$\begin{aligned}2\varphi(k)+\varphi(k-1)+\varphi(k+1)=\frac{2}{3}2^{-k/2}\Big(&4\cos(kt\log 2)+\sum_{j=1}^{k-1}\cos(k-2j)\log 2\\ &+2\sqrt{2}\cos(k-1)t\log 2\\ &+\frac{\sqrt{2}}{2}\sum_{j=1}^{k-2}\cos(k-2j-1)t\log 2\\ &+\sqrt{2}\cos(k+1)t\log 2\\ &+\frac{1}{2\sqrt{2}}\sum_{j=1}^{k}\cos(k-2j+1)t\log 2\Big),\end{aligned}$$

and the result follows by standard trigonometric identities. □

Theorem 4.23. *Suppose F has finite support on Γ_0. Then*

$$< \tilde{P_\lambda}F, F > = 36b(\lambda)^{-1} < \tilde{P_\lambda}F, \tilde{P_\lambda}F >_M . \tag{4.39}$$

Proof. Since $< \tilde{P_\lambda}\delta_{x_1}, \delta_{x_2} > = \psi(d(x_1,x_2))/\big(3(6-\lambda)\big)$ we can rewrite (4.37) as $< \tilde{P_\lambda}\delta_{x_1}, \delta_{x_2} > = 36b(\lambda)^{-1} < \tilde{P_\lambda}\delta_{x_1}, \tilde{P_\lambda}\delta_{x_1} >_M$, and (4.39) follows by linearity.□

Corollary 4.24. *Let $F \in \ell^2(\Gamma_0)$. Then for $\mu - a.e.$ λ in Σ, $< \tilde{P_\lambda}F, \tilde{P_\lambda}F >_M$ exists, and*

$$||F||^2_{\ell^2(\Gamma_0)} = ||\tilde{P_6}F||_2^2 + \int_\Sigma < \tilde{P_\lambda}F, \tilde{P_\lambda}F >_M 36b(\lambda)^{-1}d\mu(\lambda).$$

Proof. The proof is the same as for Corollary 4.19. □

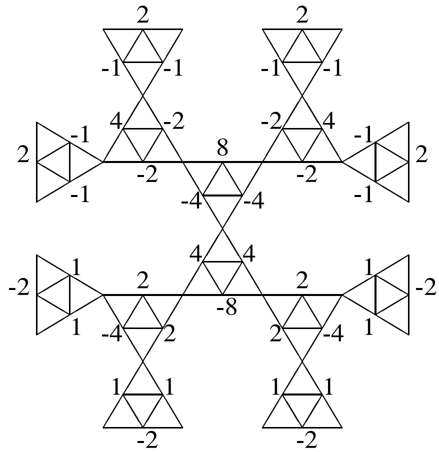


Figure 4.7. A part of Γ_1 with a 5-eigenfunction (values not shown are equal to zero).

We end this section with a description of 5-series eigenfunctions on the graph Γ_1. (Note there are no 5-eigenfunctions on the graph Γ_0). One sees easily that on Γ_1 there are no finitely supported 5-eigenfunctions, there are no radially symmetric 5-eigenfunctions, and that 5-eigenfunctions do not correspond to cycles. By by an argument similar to that used in Theorem 4.5, one can show that eigenfunctions in Figure 4.7 (with their translations, rotations and reflections) are complete in the eigenspace E_5 on Γ_1. We do not give an explicit formula for the 5-eigenprojections on Γ_n. One can see that for each $n > 1$, there are eigenfunctions on Γ_n that resemble those in Figure 4.7, and also finitely supported 5-eigenfunctions. (See Remark 5.1).

5 Periodic fractafolds

Remark 5.1. Note that on a periodic graph, linear combinations of compactly supported eigenfunctions are dense in an eigenspace (see [23, Theorem 8], [22] and [24, Lemma 3.5]).

The computation of compactly supported 5- and 6- series eigenfunctions is discussed in detail in [37, 41]; the eigenfunctions with compact support are complete in the corresponding eigenspaces. In particular, [37, 41] show that any 6-series finitely supported eigenfunction on Γ_{n+1} is the continuation of any finitely supported function on Γ_n, and the corresponding continuous eigenfunction on the Sierpiński fractafold $\mathfrak{F}$ can be computed using the eigenfunction extension map on fractafolds; see Subsection 2.4. Similarly, any 5-series finitely supported eigenfunction on Γ_{n+1} can be described by a cycle of triangles (homology) in Γ_n, and the corresponding continuous eigenfunction on the Sierpiński fractafold $\mathfrak{F}$ is computed using the eigenfunction extension map on fractafolds.

Example 5.2. (The ladder fractafold). Here Γ is the ladder graph consisting of two copies of $\mathbb{Z}$, $\{a_k\}$ and $\{b_k\}$, with $a_k \sim b_k$ and Γ_0 consisting of three copies

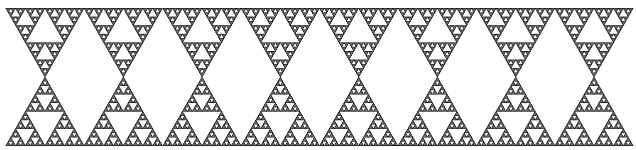


Figure 5.1. A part of the infinite Ladder Sierpiński fractafold.

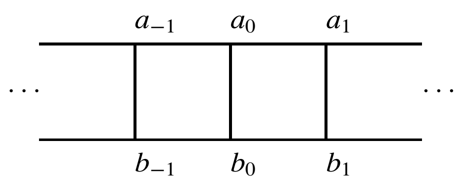


Figure 5.2. Γ graph for the ladder fractafold

of $\mathbb{Z}$, $\{x_{k+1/2}\}$, $\{w_k\}$, $\{y_{k+1/2}\}$ with w_k joined to $x_{k-1/2}, x_{k+1/2}, y_{k-1/2}$, and $y_{k+1/2}$, where $x_{k+1/2}$ is the edge $[a_k, a_{k+1}]$, $x_{y+1/2}$ is the edge $[b_k, b_{k+1}]$ and w_k is the edge

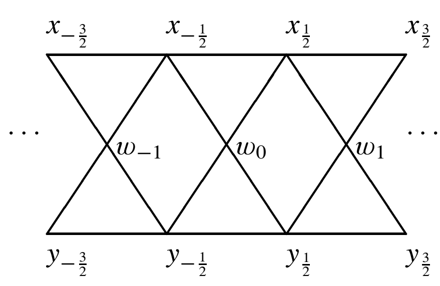


Figure 5.3. Γ_0 graph for the Ladder Fractafold

$[a_k, b_k]$.

It is easy to see that the spectrum of $-\Delta_\Gamma$ is $[0, 6]$, with the even functions $\varphi_\theta(a_k) = \varphi_\theta(b_k) = \cos k\theta$ or $\sin k\theta$, $0 \le \theta \le \pi$ corresponding to $\lambda = 2 - 2\cos\theta$

in $[0,4]$ and the odd functions $\psi_\theta(a_k) = -\psi_\theta(b_k) = \cos k\theta$ or $\sin k\theta$, $0 \le \theta \le \pi$ corresponding to $\lambda = 4 - 2\cos\theta$ in $[2,6]$.

These transfer to eigenfunctions of $-\Delta_{\Gamma_0}$

$$\begin{aligned}
\tilde\varphi_\theta(x_{k+1/2}) &= \tilde\varphi_\theta(y_{k+1/2}) = \cos(k+1/2)\theta\cos\theta/2 \quad \text{or} \ \sin(k+1/2)\theta\cos\theta/2,\\
\tilde\varphi_\theta(w_k) &= \cos k\theta \quad \text{or} \ \sin k\theta,\\
\tilde\psi_\theta(x_{k+1/2}) &= -\tilde\psi_\theta(y_{k+1/2}) = \cos(k+1/2)\theta \quad \text{or} \ \sin(k+1/2)\theta,\\
\tilde\psi_\theta(w_k) &= 0,
\end{aligned}$$

with the same eigenvalues. It is also easy to see that there are no $\ell^2(\Gamma_0)$ eigenfunctions corresponding to $\lambda = 6$ (or for any λ value whatsoever). Thus $-\Delta_{\Gamma_0}$ has absolutely continuous spectrum $[0,6]$ with multiplicity 2 in $[0,2]$ and $[4,6]$ and multiplicity 4 in $[2,4]$.

Example 5.3. (The honeycomb fractafold). Here Γ is the hexagonal graph consisting of the triangular lattice $\mathcal{L}$ generated by $(1,0)$ and $(1/2, \sqrt{3}/2)$ and the displaced lattice $\mathcal{L} + (1/2, \sqrt{3}/6)$. We denote by $a(j,k)$ the points $j(1,0) + k(1/2, \sqrt{3}/2)$ of $\mathcal{L}$ and by $b(j,k)$ the points $a(j,k) + (1/2, \sqrt{3}/6)$ of the displaced lattice, with edges $a(j,k) \sim b(j,k)$, $a(j,k) \sim b(j-1,k)$ and $a(j,k) \sim b(j,k-1)$. The eigenfunctions of $-\Delta_\Gamma$ have the form

$$\begin{aligned}
\varphi_{u,v}(a(j,k)) &= e^{2\pi i(ju+kv)}\\
\varphi_{u,v}(b(j,k)) &= \gamma e^{2\pi i(ju+kv)},
\end{aligned}$$

where $(u,v) \in [0,1]\times[0,1]$ and γ depends on u, v. Let $1 + e^{2\pi iu} + e^{2\pi iv} = re^{i\theta}$ in polar coordinates (so r and θ are functions of u, v). Note that $0 \le r \le 3$. Then the eigenvalue equation requires $\gamma^2 = e^{2i\theta}$ or $\gamma = \pm e^{i\theta}$ with corresponding eigenvalues $\lambda = 3 \mp r$ (so the choice $\pm$ yields the intervals $[0,3]$ and $[3,6]$ in $spect(-\Delta_\Gamma)$).

We can write the explicit spectral resolution as follows. For $f \in \ell^2(\Gamma)$, define

$$\hat f_a(u,v) = \sum_j \sum_k e^{-2\pi i(ju+kv)} f(a(j,u))$$

and

$$\hat f_b(u,v) = \sum_j \sum_k e^{-2\pi i(ju+kv)} f(b(j,u)).$$

We can invert these, so that

$$\begin{aligned}
\begin{Bmatrix} f(a(j,k)) \\ f(b(j,k)) \end{Bmatrix} &= \int_0^1\int_0^1 \begin{Bmatrix} 1 \\ e^{i\theta} \end{Bmatrix} e^{2\pi i(ju+kv)} \frac{1}{2}(\hat f_a(u,v) + e^{-i\theta}\hat f_b(u,v))\,du\,dv\\
&\quad + \int_0^1\int_0^1 \begin{Bmatrix} 1 \\ -e^{i\theta} \end{Bmatrix} e^{2\pi i(ju+kv)} \frac{1}{2}(\hat f_a(u,v) - e^{-i\theta}\hat f_b(u,v))\,du\,dv.
\end{aligned}$$

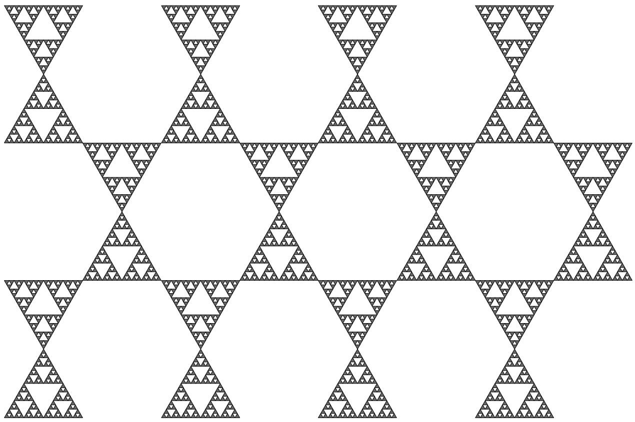


Figure 5.4. A part of the infinite periodic Sierpiński fractafold based on the hexagonal (honeycomb) lattice.

Define $\lambda_{\pm}(u, v)$ by

$$\lambda_{\pm}(u, v) = 3 \mp \sqrt{3 + 2\cos 2\pi u + 2\cos 2\pi v + 2\cos 2\pi(u - v)}.$$

For $0 \le \lambda \le 3$, we define u_θ and v_θ by solving $\lambda_+(u, v) = \lambda$; similarly for $3 \le \lambda \le 6$, we solve $\lambda_-(u, v) = \lambda$. We then define

$$\begin{Bmatrix} P_\lambda f(a(j,k)) \\ P_\lambda f(b(j,k)) \end{Bmatrix} = \int_0^{2\pi} \begin{Bmatrix} 1 \\ \pm e^{i\theta} \end{Bmatrix} e^{2\pi i(ju_\theta + kv_\theta)} \frac{1}{2}(\hat{f}_a(u_\theta, v_\theta) \pm e^{-i\theta}\hat{f}_b(u_\theta, v_\theta)) \left| \frac{\partial(u_\theta, v_\theta)}{\partial(\lambda, \theta)} \right| d\theta, \tag{5.1}$$

to obtain $f = \int_0^6 P_\lambda f d\lambda$ with $-\Delta_\Gamma P_\lambda f = \lambda P_\lambda f$. This solves problem (a).

To solve problem (b), we identify the space E_6 in $\ell^2(\Gamma_0)$. We may regard Γ_0 as an infinite union of hexagons, each vertex belonging to exactly two hexagons. For any fixed hexagon H, define ψ_H to take alternate values ± 1 around the vertices of H, and to be zero elsewhere. It is easy to see that ψ_H in is E_6. If $\{H_j\}$ is an enumeration of all the hexagons in Γ_0, then $\sum c_j \psi_{H_j}$ (finite sum) is in E_6.

Lemma 5.4. *Suppose $u \in E_6$ has compact support. Then $u = \sum c_j \psi_{H_j}$ (finite sum).*

Proof. Suppose $supp(u) \subseteq \bigcup_{j \in A} H_j$ We show that there exists $j_0 \in A$ and c_{j_0} such that $supp(u - c_{j_0}\psi_{H_{j_0}}) \subseteq \bigcup_{j \in A \setminus \{j_0\}} H_j$. The proof is then completed by induction.

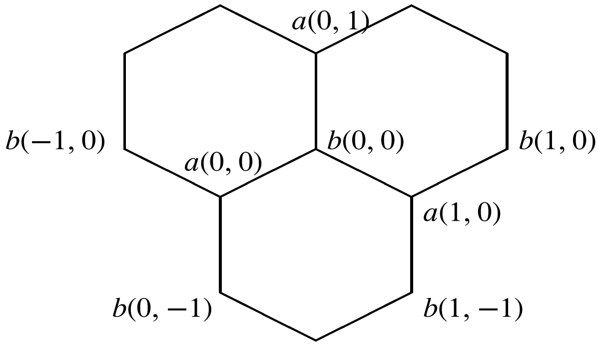


Figure 5.5. A part of the Hexagonal graph

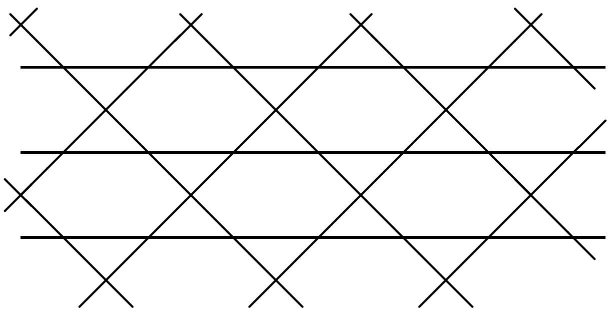


Figure 5.6. A part of the graph Γ_0 for the honeycomb fractafold

We choose j so that H_j lies in the top row and right-most down-right slanting diagonal of $\bigcup_{j \in A} H_j$. In Figure 5.7 above, $j' = 0$, and u vanishes on H_1, H_2, and H_3. So $u(x_1) = 0$, $u(x_2) = 0$, $u(x_3) = 0$. But $u(x_3) + u(x_4) + u(y_{34}) = 0$ because $E_6 = ker(S_2)$, and $u(y_{34}) = 0$ since $y_{34} \in H_3$. So $u(x_4) = 0$. A similar argument shows that $u(x_6) = 0$. The only vertex left in H_0 is x_5. By subtracting off $u(x_5)\psi_{H_5}$, we can make u vanish on H_0.

We can systematically go across the top row in $supp(u)$ from right to left and remove each hexagon, only changing u on the row below it. Eventually, u is supported on just one row, and $u(x) = 0$ unless x is one of the dotted points in Figure 5.8.

Let H_0 be the right most hexagon, where u is not identically zero; $u\,|_{H_1} = 0$ implies $u(x_1) = u(x_2) = u(x_6) = 0$. Considering the triangle below the row, we get $u(x_5) = 0$, and similarly, $u(x_3) = 0$. Considering the triangle above x_4, we get $u(x_4) = 0$. So $u\,|_{H_0} = 0$. □

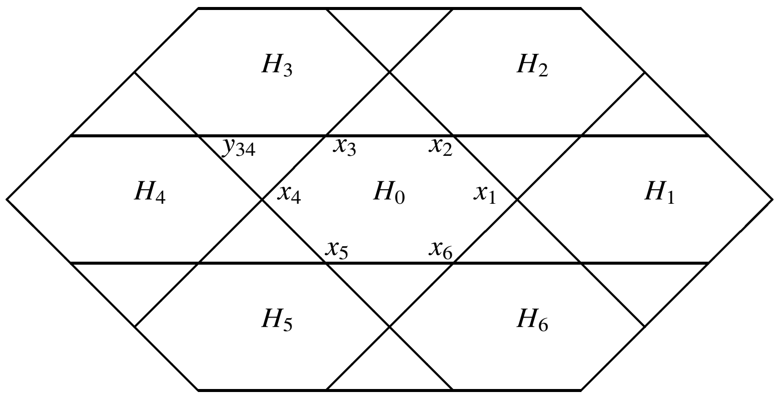


Figure 5.7. Labels of hexagons and points

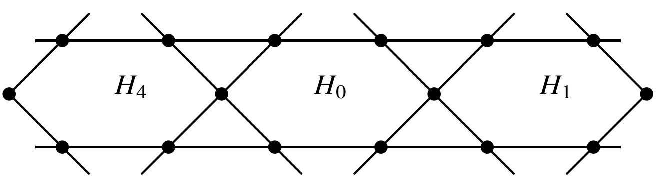


Figure 5.8. A row of hexagons

Corollary 5.5. *A function of compact support is in E_6 if and only if*

$$u(x_1) + u(x_2) + u(x_3) = 0$$

for every triangle $\{x_1, x_2, x_3\}$ in Γ_0.

Proof. The identity clearly holds for each ψ_H, hence for all compactly supported functions in E_6. Conversely, every point x in Γ_0 lies in exactly two triangles. Summing the identity for those two triangles yields the 6-eigenvalue equation at the point x. □

The functions $\{\psi_{H_j}\}$ do not form a tight frame, and it seems unlikely that they even form a frame (the lower frame bound is doubtful), so they do not seem well suited for describing $\tilde{P}_6$. We can, however, find an orthonormal basis of E_6 that consists of translates of a single function, but we pay the price that the function is not compactly supported.

We change notation to index the hexagons in Figure 5.6 by the lattice

$$[j, k] = j\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} + k\begin{Bmatrix} 1/2 \\ \sqrt{3}/2 \end{Bmatrix}.$$

Note that hexagon $H_{[j,k]}$ has six neighbors $H_{[j',k']}$ for

$$[j', k'] = [j, k] + \{[1, 0], [-1, 0], [0, 1], [0, -1], [1, -1], [-1, 1]\}.$$

To describe a function

$$F = \sum_{\mathbb{Z}^2} f([j, k])\psi_{H_{[j,k]}}, \tag{5.2}$$

it suffices to give the discrete Fourier transform $\hat{f}(a, b)$ for $(a, b) \in [0, 1] \times [0, 1]$ given by

$$\hat{f}(a, b) = \sum_{\mathbb{Z}^2} f([j, k])e^{-2\pi i(aj+bk)}, \tag{5.3}$$

for then

$$f([j, k]) = \int_0^1 \int_0^1 e^{2\pi i(aj+bk)}\hat{f}(a, b)dadb. \tag{5.4}$$

In fact, we construct $\hat{f}(a, b)$ directly; then we substitute this into (5.4) and finally into (5.2) to obtain our function in E_6.

The basic observation is that each point in Γ_0 lies in exactly two neighboring hexagons, and the values of ψ_H for those two hexagons are ± 1. Thus

$$< F, F >_{\ell^2(\Gamma_0)} = \sum |f([j, k]) - f([j', k'])|^2$$

for f of the form (5.2), where the sum is over all neighboring pairs, and by polarization, we have

$$< F, G >_{\ell^2(\Gamma_0)} = \sum \left(f([j, k]) - f([j', k'])(\overline{g([j, k]) - g([j', k'])})\right) \tag{5.5}$$

if F and G are of the form (5.2). Now we substitute (5.4) into (5.5), to obtain

$$\begin{aligned} < F, G >_{\ell^2(\Gamma_0)} = \int_0^1 \int_0^1 \sum_{\mathbb{Z}^2} e^{2\pi i(aj+bk)}\hat{f}(a, b)[6 - e^{2\pi ia} - e^{-2\pi ia} - e^{2\pi ib} \\ - e^{-2\pi ib} - e^{2\pi i(a-b)} - e^{2\pi i(b-a)}]\overline{g([j, k])}dadb \end{aligned} \tag{5.6}$$

because of the form of the neighboring relation between $[j, k]$ and $[j', k']$. But then we can evaluate the sum in (5.6) using (5.3), to obtain

$$\begin{aligned} < F, G >_{\ell^2(\Gamma_0)} = \int_0^1 \int_0^1 2(3 - \cos(2\pi a) - \cos(2\pi b) - \cos(2\pi(a - b))) \\ \hat{f}(a, b)\overline{\hat{f}(a, b)}dadb. \end{aligned} \tag{5.7}$$

Lemma 5.6. *The functions* $\tau_{p,q}F = \sum_{\mathbb{Z}^2} f([j,k]+[p,q])\psi_{H_{[j,k]}}$ *form an orthonormal basis of* E_6 *for* $[p,q] \in \mathbb{Z}^2$ *if and only if*

$$|\hat{f}(a,b)| = \frac{1}{\sqrt{2(3-\cos(2\pi a)-\cos(2\pi b)-\cos(2\pi(a-b)))}}. \tag{5.8}$$

Proof. We note that for $\tau_{p,q}f([j,k]) = f([j,k]+[p,q])$, we have

$$(\tau_{p,q}f)\hat{}(a,b) = e^{2\pi i(ap+bq)}\hat{f}(a,b) \tag{5.9}$$

from (5.3), so

$$< F, \tau_{p,q}F >_{\ell^2(\Gamma_0)} = \int_0^1\int_0^1 e^{-2\pi i(ap+bq)} 2(3-\cos(2\pi a)-\cos(2\pi b) - \cos(2\pi(a-b)))|\hat{f}(a,b)|^2 da db \tag{5.10}$$

by (5.9) and (5.7). But the right side of (5.10) is $\delta(p,q)$ if and only if

$$2\,(3-\cos(2\pi a)-\cos(2\pi b)-\cos(2\pi(a-b)))\,|\hat{f}(a,b)|^2$$

is identically one, and this is equivalent to (5.8). □

We are free to choose any phase in (5.8); since it is not clear what is to be gained, we simply choose $\hat{f}(a,b)$ to be positive. Note that the only singularity of $\hat{f}$ is near $(0,0)$, where it behaves like $(a^2+b^2)^{-1/2}$, so the singularity is integrable, but not square integrable. Thus (5.4) is everywhere finite and decays as $O\left((j^2+k^2)^{-1/2}\right)$. Although f is not in $\ell^2(\mathbb{Z}^2)$, we do have $F \in \ell^2(\Gamma_0)$.

Theorem 5.7. *Let*

$$\tilde{f}([j,k]) = \int_0^1\int_0^1 \frac{e^{2\pi i(aj+bk)}}{\sqrt{2\,(3-\cos(2\pi a)-\cos(2\pi b)-\cos(2\pi(a-b)))}} da db. \tag{5.11}$$

Then $\left\{\sum_{\mathbb{Z}^2} \tau_{p,q}\tilde{f}([j,k])\psi_{H_{[j,k]}}\right\}$ *is an orthonormal basis of* E_6, *and*

$$\tilde{P_6}F(x) = \sum_{[p,q]\in\mathbb{Z}^2}\left(\sum_{y\in\Gamma_0}\sum_{[j,k]\in\mathbb{Z}^2} \tau_{p,q}\tilde{f}([j,k])\psi_{H_{[j,k]}}(y)F(y)\right) \sum_{[j',k']\in\mathbb{Z}^2} \tau_{p,q}\tilde{f}([j',k'])\psi_{H_{[j',k']}}(x). \tag{5.12}$$

Proof. This is an immediate consequence of Lemma 5.6. □

6 Non-fractafold examples

Theorem 2.3 can be applied to examples which are not fractafolds. We assume that $\Gamma_0 = (V_0, E)$ is a finite or infinite graph which is a union of complete graphs of 3 vertices. (It can be said that Γ_0 is a 3-hyper-graph.) In principle, we can allow Γ_0 to have unbounded degrees as well as loops and multiple edges; but in this section, we keep everything simple and assume that Γ_0 is a regular graph. As before, we call each of these complete 3-graphs a cell, or 0-cell, of Γ_0. We denote the discrete Laplacian on Γ_0 by Δ_{Γ_0}. We define a **finitely ramified Sierpiński fractal field** $\mathfrak{F}$ by replacing each cell of Γ_0 with a copy of SG. We call these copies cells, or 0-cells, of $\mathfrak{F}$. Naturally, the corners of the copies of the Sierpiński gasket SG are identified with the vertices of Γ_0. See [12] for fractal fields, not necessarily finitely ramified. Since the pairwise intersections of the cells of $\mathfrak{F}$ are finite, we can consider the natural measure on $\mathfrak{F}$, which we also denote by μ. Furthermore, since Δ_{SG} is a local operator, we can define a local Laplacian Δ on $\mathfrak{F}$ in the

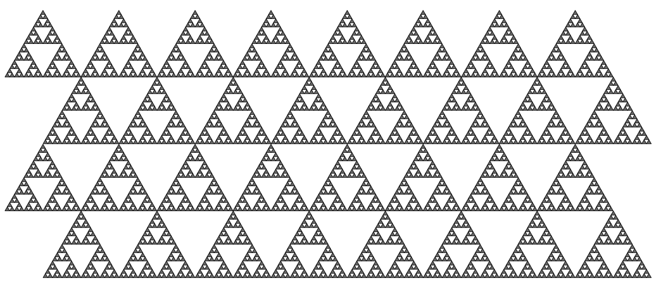


Figure 6.1. A part of the periodic triangular lattice finitely ramified Sierpiński fractal field. This fractal field is not a fractafold.

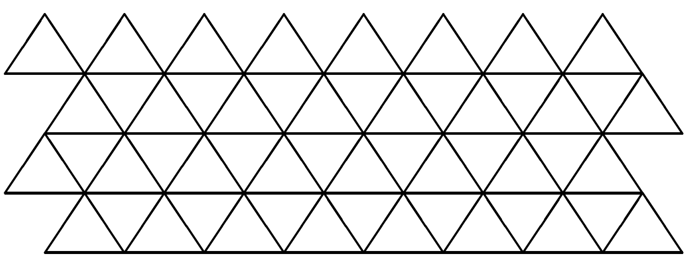


Figure 6.2. A part of the infinite triangular lattice, the Γ_0 graph for the fractal field in Figure 6.1.

same way as explained in [37]. (This means that the sum of normal derivatives is zero at every junction point.) Most of our results can be easily generalized for the finitely ramified Sierpiński fractal fields. For instance, Theorem 2.3 is essentially still valid. One change needed is that on the graph Γ, we have to consider the probabilistic Laplacian (which is explained in [26, 33]) and multiply it by 4 to align with the normalization of the Laplacian on the Sierpiński gasket.

In the example shown in Figure 6.2, the spectrum on Γ_0 is $[0, 8]$ for the adjacency matrix Laplacian, and the spectrum is $[0, 4/3]$ for the probabilistic Laplacian. Thus $\Sigma_0 = [0, 16/3]$. In this particular case, the spectrum is absolutely continuous by the classical theory. (See [21, 22, 23, 24, 25] and the references within for a sample of relevant recent results on periodic Laplacians.) Combining the methods described in this paper, we obtain the following proposition (see also Figure 6.3).

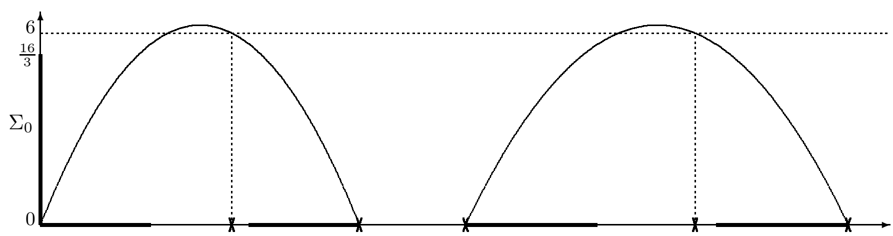


Figure 6.3. Computation of the spectrum on the triangular lattice finitely ramified Sierpiński fractal field.

Proposition 6.1. *The Laplacian on the periodic triangular lattice finitely ramified Sierpiński fractal field consists of absolutely continuous spectrum and pure point spectrum. The absolutely continuous spectrum is* $\mathfrak{R}^{-1}[0, 16/3]$. *The pure point spectrum consists of two infinite series of eigenvalues of infinite multiplicity. The series* $5\mathfrak{R}^{-1}\{3\} \subsetneq \mathfrak{R}^{-1}\{6\}$ *consists of isolated eigenvalues, and the series* $5\mathfrak{R}^{-1}\{5\} = \mathfrak{R}^{-1}\{0\}\backslash\{0\}$ *is at the gap edges of the a.c. spectrum. The eigenfunction with compact support are complete in the p.p. spectrum. The spectral resolution is given by* (2.14).

It is straightforward to generalize such a result for other finitely ramified Sierpiński fractal fields (see, in particular, Remark 5.1).

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References

[1] N. Bajorin, T. Chen, A. Dagan, C. Emmons, M. Hussein, M. Khalil, P. Mody, B. Steinhurst, and A. Teplyaev, *Vibration modes of* $3n$*-gaskets and other fractals*, J. Phys. A: Math Theor. **41** (2008) 015101; *Vibration spectra of finitely ramified, symmetric fractals*, Fractals **16** (2008), 243–258.

[2] M. T. Barlow and E. A. Perkins, *Brownian motion on the Sierpiński gasket*, Probab. Theory Related Fields **79** (1988), 543–623.

[3] J. Béllissard, *Renormalization group analysis and quasicrystals*, Ideas and Methods in Quantum and Statistical Physics (Oslo, 1988), Cambridge Univ. Press, Cambridge, 1992, pp. 118–148.

[4] P. Cartier, *Harmonic analysis on trees*, Harmonic Analysis on Homogeneous Spaces, Amer. Math. Soc., Providence, RI, 1973, pp. 419-424.

[5] J. DeGrado, L. G. Rogers and R. S. Strichartz, *Gradients of Laplacian eigenfunctions on the Sierpinski gasket*, Proc. Amer. Math. Soc. **137** (2009), 531-540.

[6] G. Derfel, P. Grabner, and F. Vogl, *The zeta function of the Laplacian on certain fractals*, Trans. Amer. Math. Soc. **360** (2008), 881–897.

[7] G. Derfel, P. Grabner, and F. Vogl, *Complex asymptotics of Poincaré functions and properties of Julia sets*, Math. Proc. Cambridge Philos. Soc. **145** (2008), 699–718.

[8] S. Drenning and R. Strichartz, *Spectral decimation on Hambly's homogeneous hierarchical gaskets*, Illinois J. Math. **53** (2009), 915–937.

[9] A. Figa-Talamanca and C. Nebbia, *Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees*, Cambridge University Press, Cambridge, 1991.

[10] D. Ford and B. Steinhurst, *Vibration spectra of the m-tree fractal*, Fractals **18** (2010), 157–169.

[11] M. Fukushima and T. Shima, *On a spectral analysis for the Sierpiński gasket*, Potential Anal. **1** (1992), 1–35.

[12] B. M. Hambly and T. Kumagai, *Diffusion processes on fractal fields: heat kernel estimates and large deviations*, Probab. Theory Related Fields **127** (2003), 305–352.

[13] K. Hare, B. Steinhurst, A. Teplyaev, and D. Zhou, *Disconnected Julia sets and gaps in the spectrum of Laplacians on symmetric finitely ramified regular fractals*, Math. Res. Let. (2012), to appear.

[14] A. Ionescu, *On the Poisson transform on symmetric spaces of real rank one*, J. Funct. Anal. **174** (2000), 513–523.

[15] M. Ionescu, Erin P. J. Pearse, L. G. Rogers, Huo-Jun Ruan, and R. S. Strichartz, *The resolvent kernel for PCF self-similar fractals*, Trans. Amer. Math. Soc. **362** (2010), 4451-4479.

[16] J. Jordan, *Comb graphs and spectral decimation*, Glasg. Math. J. **51** (2009), 71–81.

[17] J. Kigami, *A harmonic calculus on the Sierpiński spaces*, Japan J. Appl. Math. **6** (1989), 259–290.

[18] J. Kigami, *Analysis on Fractals*, Cambridge University Press, Cambridge, 2001.

[19] J. Kigami, *Harmonic analysis for resistance forms*, J. Funct. Anal. **204** (2003), 399–444.

[20] B. Krön and E. Teufl, *Asymptotics of the transition probabilities of the simple random walk on self-similar graph*, Trans. Amer. Math. Soc. **356** (2003), 393–414.

[21] P. Kuchment, *On the Floquet theory of periodic difference equations*, Geometrical and Algebraical Aspects in Several Complex Variables (Cetraro, 1989), EditEl, Rende, 1991, pp. 201–209.

[22] P. Kuchment, *Floquet Theory for Partial Differential Equations*, Birkhäuser Verlag, Basel, 1993.

[23] P. Kuchment, *Quantum graphs II. Some spectral properties of quantum and combinatorial graphs*, J. Phys. A. **38** (2005), 4887–4900.

[24] P. Kuchment and O. Post, *On the spectra of carbon nano-structures*, Comm. Math. Phys. **275** (2007), 805-826.

[25] P. Kuchment and B. Vainberg, *On the structure of eigenfunctions corresponding to embedded eigenvalues of locally perturbed periodic graph operators*, Comm. Math. Phys. **268** (2006), 673–686.

[26] L. Malozemov and A. Teplyaev, *Self-similarity, operators and dynamics*, Math. Phys. Anal. Geom. **6** (2003), 201–218.

[27] R. Oberlin, B. Street, and R. S. Strichartz, *Sampling on the Sierpinski gasket*, Experiment. Math. **12** (2003), 403–418.

[28] K. Okoudjou, L. G. Rogers, and R. S. Strichartz, *Generalized eigenfunctions and a Borel theorem on the Sierpinski gasket.* Canad. Math. Bull. **52** (2009), 105-116.

[29] K. Okoudjou, L. G. Rogers, and R. S. Strichartz, *Szegö limit theorems on the Sierpiński gasket*, J. Fourier Anal. Appl. **16** (2010), 434-447.

[30] O. Post, *Equilateral quantum graphs and boundary triples*, Analysis on Graphs and its Applications, Amer. Math. Soc., Providence, RI, 2008, pp. 469–490.

[31] J.-F. Quint, *Harmonic analysis on the Pascal graph*, J. Funct. Anal. **256** (2009), 3409–3460.

[32] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I. Functional Analysis*, Academic Press, 1980.

[33] T. Shima, *On eigenvalue problems for Laplacians on p.c.f. self-similar sets*, Japan J. Indust. Appl. Math. **13** (1996), 1–23.

[34] T. Shirai, *The spectrum of infinite regular line graphs*, Trans. Amer. Math. Soc. **352** (2000), 115–132.

[35] R. S. Strichartz, *Harmonic analysis as spectral theory of Laplacians*. J. Funct. Anal. **87** (1989) 51–148, Corrigendum, J. Funct. Anal. **109** (1992), 457–460.

[36] R. S. Strichartz, *Fractals in the large*, Canad. J. Math. **50** (1998), 638–657.

[37] R. S. Strichartz, *Fractafolds based on the Sierpiński gasket and their spectra*, Trans. Amer. Math. Soc. **355** (2003), 4019–4043.

[38] R. S. Strichartz, *Laplacians on fractals with spectral gaps have nicer Fourier series*. Math. Res. Lett. **12** (2005), 269–274.

[39] R. S. Strichartz, *Differential Equations on Fractals: a Tutorial*, Princeton University Press, Princeton, NJ, 2006.

[40] R. S. Strichartz, *Transformation of spectra of graph Laplacians*, Rocky Mountain J. Math. **40** (2010), 2037–2062.

[41] A. Teplyaev, *Spectral analysis on infinite Sierpiński gaskets* J. Funct. Anal. **159** (1998), 537–567.

[42] D. Zhou, *Spectral analysis of Laplacians on the Vicsek set*, Pacific J. Math. **241** (2009), 369–398.

[43] D. Zhou, *Criteria for spectral gaps of Laplacians on fractals*, J. Fourier Anal. Appl. **16** (2010), 76–96.

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