ON THE GROWTH AND RANGE OF FUNCTIONS IN MÖBIUS INVARIANT SPACES

By

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Abstract. This paper is a continuation of our earlier work and focuses on the structural and geometric properties of functions in analytic Besov spaces, primarily on univalent functions in such spaces and their image domains. We improve several earlier results.

Introduction

In this paper, we continue the study of growth properties of functions in analytic Besov spaces B^p, $1 \leq p < \infty$, and of their image domains. The spaces B^p are conformally invariant and represent a natural generalization of the classical Dirichlet space $\mathcal{D} = B^2$ of analytic functions in the unit disk whose image Riemann surface has finite area. They are also important in view of their relationship to the Bergman projection and Hankel operators on Bergman spaces. These spaces and their operators were studied extensively in the mid 80's and early 90's in [4], [21], [22], [38] and, more recently, in [36], [8], [11], [35], [7], and [9], especially from a geometric point of view.

It is a well-known fact that the only Blaschke products in B^p spaces are the finite ones. However, if the zeros of a Blaschke product lie in a Stolz angle with vertex at $z = 1$ and we multiply the product by $(1 - z)^\alpha$, then the new function that arises this way may belong to certain Besov spaces. We consider such "modified Blaschke products" in Section 2. It should be noted that this question is actually equivalent to that of membership in a weighted Bergman space of the derivative of

*The authors thankfully acknowledge partial support from the following grants. The first author: MTM2008-05561-C02-02 (MICINN) and 2009 SGR 420 (Generalitat de Catalunya). The second author: MTM2007-60854 (MICINN) and FQM-210 and P09-FQM-4468 (Junta de Andalucía). The third author: MTM2009-14694-C02-01 (MICINN). All authors were also partially supported by MTM2008-02829-E (Acciones Complementarias) and Ingenio Mathematica (i-MATH) CSD2006-00032 from MICINN, Spain, and by the Thematic Network "Harmonic and Complex Analysis and Its Applications" (European Networking Programme, ESF).

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 112 (2010)
DOI 10.1007/s11854-010-0029-9

the Blaschke product in question; a problem that has been of interest to a number of authors. We also obtain analogous results for atomic singular inner functions.

In Section 3, we discuss the growth of functions in B^p spaces. It is well known (see [21] or [38]) that a function in B^p, $1 < p < \infty$, satisfies the following growth condition:

$$|f(z)| = o\left(\left(\log\frac{1}{1-|z|}\right)^{1-1/p}\right), \quad \text{as } |z| \to 1\,. \tag{1}$$

This is known to be sharp in the following sense: in the proof of Theorem 24 of [8], a univalent function was constructed in B^p, $1 < p < \infty$, with the property that

$$|f(z_n)| \gtrsim n^{-t(1-1/p)}\left(\log\frac{1}{1-|z_n|}\right)^{1-1/p} \quad \text{as} \quad n \to \infty,$$

for $t > 1/(p-1)$ and infinitely many points z_n in the unit disk. This has recently been refined further in [9]. Here we show that when $2 < p < \infty$, estimate (1) is sharp even in a stronger sense: we prove that for a large class of functions $\alpha : [0,\infty) \to [0,\infty)$ satisfying $\alpha(t) \searrow 0$ as $t \to \infty$, there exists a univalent map f in B^p that maps the radius $[0,1)$ onto the semi-axis $[0,\infty)$ in such a fashion that

$$f(r) \gtrsim \alpha\left(\frac{1}{1-r}\right)\cdot\left(\log\frac{1}{1-r}\right)^{1-1/p}, \quad \text{as} \quad r \to 1^-\,.$$

Estimate (1) implies that for every function f in B^p with $f(0) = 0$ we have

$$\int_0^1 \frac{(1-r)^{p-2}|f(r)|^p}{r^{p-1}}dr \le C\int_0^1 \frac{(1-r)^{p-2}\left(\log\frac{1}{1-r}\right)^{p-1}}{r^{p-1}}dr < \infty,$$

while, on the other hand, our result shows that for a certain conformal map f of the unit disc $\mathbb{D}$ and a function $\alpha : [0,1) \to [0,\infty)$ that tends to zero as $r \to 1^-$ and satisfies some additional conditions, we also have

$$\int_0^1 \frac{(1-r)^{p-2}|f(r)|^p}{r^{p-1}}dr \ge \int_0^1 \frac{\alpha(r)(1-r)^{p-2}\left(\log\frac{1}{1-r}\right)^{p-1}}{r^{p-1}}dr,$$

meaning that the previous inequality is essentially sharp even for *conformal* maps that belong to B^p. However, this can still be improved. We show this in Section 4.

Consider a univalent map f in $\mathbb{D}$ which fixes the origin and denote by $d(r)$ the length of the Jordan arc onto which the truncated diameter $(-r,r)$ is mapped, $0 < r < 1$. Trivially, $|f(r)| \le d(r)$ in this case. We prove that

$$2^{2-p}\int_0^1 \frac{(1-r^2)^{p-2}d(r)^p}{r^{p-1}}dr \le \|f\|_{B^p}^p\,.$$

In particular, when $f \in \mathcal{D}$, this reduces to the simple inequality

$$\int_0^1 \frac{d(r)^2}{r} dr \le A(\Omega),$$

which resembles Ahlfors' classical length–area principle ([20], Chapter 2) and can also be considered as yet another analogue of the classical Fejér–Riesz inequality (see [14] or Chapter 3 of [13]). It clearly differs from the analogues obtained earlier by Shields [33] and Holland–Walsh [21].

In Section 5, we discuss the minimal Besov space B^1 and the image domains of certain functions in this space. We also improve further some relevant examples from [36] and [11]. In particular, we prove that a univalent function in $\mathbb{D}$ whose image is a bounded convex domain belongs to B^1. On the other hand, we also prove that there exists a univalent function in $\mathbb{D}$ that does not belong to any of the spaces B^p with $1 \le p < 2$ but whose image is a bounded starlike domain.

In Section 6, we exhibit a class of non-finitely valent self-maps of the disc in B^p that cover certain parts of the disk once, twice, three times, etc. We also make our discussion of the univalent Besov domains from the earlier paper [11] more precise in the case of the Dirichlet space. By improving a construction found in that paper, we obtain a curious geometric-topological property of domains of finite area which, to the best of our knowledge, does not seem to be well known, or at least does not seem to have an easy or well-known proof. Namely, we prove that any planar domain Ω contains a simply connected domain Ω' such that $\Omega \setminus \Omega'$ has finite length and with the property that $\int_\Omega \operatorname{dist}(w, \partial\Omega)^\alpha \, dA(w) \approx \int_{\Omega'} \operatorname{dist}(w, \partial\Omega')^\alpha \, dA(w)$, for any $\alpha \ge 0$.

1 Some important function spaces

Throughout the paper, $D(z, r)$ denotes the disk of radius r centered at z. The unit disk $D(0, 1)$ is denoted by $\mathbb{D}$.

Let H^p $(0 < p < \infty)$ denote the Hardy space of analytic functions in $\mathbb{D}$ for which the integral means

$$M_p(r, f) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

are bounded for $r \in (0, 1)$ and define the norm $\|f\|_{H^p} = \lim_{r \to 1^-} M_p(r, f)$ as usual. The space H^∞ consists of all bounded analytic functions in $\mathbb{D}$. We use the same letter to denote the values of an H^p function in the disk and its radial limits on the unit circle. All basic information on Hardy spaces needed here can be found in [13].

Let $dA(z) = r dr d\theta$ be Lebesgue area measure on $\mathbb{C}$. For $0 < p < \infty$, $-1 < \alpha < \infty$, denote by A^p_α the weighted Bergman space of all analytic functions in the disk with the finite weighted L^p area norm

$$\|f\|^p_{A^p_\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty .$$

The standard (unweighted) Bergman space is $A^p = A^p_0$. As an important example of a class of basic functions in Bergman spaces, we mention that (for a real parameter α)

$$(1-z)^{-\alpha} \in A^p \text{ if and only if } \alpha p < 2. \tag{2}$$

This is easy to check by integrating in polar coordinates centered at $z = 1$, rather than at the origin.

For $a \in \mathbb{D}$, define the Möbius map $\varphi_a : \mathbb{D} \to \mathbb{D}$ by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in \mathbb{D}.$$

Then φ_a is an involutive conformal mapping from $\mathbb{D}$ onto itself. Let $\mathrm{Aut}(\mathbb{D})$ denote the group of all conformal mappings from $\mathbb{D}$ onto itself. It is well known that $\mathrm{Aut}(\mathbb{D})$ coincides with the set of all Möbius transformations from $\mathbb{D}$ onto itself:

$$\mathrm{Aut}(\mathbb{D}) = \text{Möb}(\mathbb{D}) = \{\lambda\varphi_a : a \in \mathbb{D},\ |\lambda| = 1\}.$$

A space X of analytic functions in $\mathbb{D}$, equipped with a semi-norm ρ, is said to be **conformally invariant** or **Möbius invariant** if whenever $f \in X$, then also $f \circ \varphi \in X$ for any $\varphi \in \mathrm{Aut}(\mathbb{D})$ and, moreover, $\rho(f \circ \varphi) \le C\rho(f)$ for some positive constant C and all $f \in X$.

The **Bloch space** $\mathcal{B}$ consists of all analytic functions f in $\mathbb{D}$ with bounded invariant derivative:

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)\,|f'(z)| < \infty.$$

The **little Bloch space** $\mathcal{B}_0$ is the closure of the polynomials in the topology given by the above norm of $\mathcal{B}$ and consists of all functions f analytic in $\mathbb{D}$ for which

$$\lim_{|z|\to 1} (1-|z|^2)|f'(z)| = 0.$$

A classical source for Bloch functions is [3]. The Bloch space is conformally invariant. In fact, Rubel and Timoney [32] proved that it is the largest one in the sense that any reasonable Möbius invariant space of analytic functions in $\mathbb{D}$ can be continuously injected into the Bloch space.

The space **BMOA** consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on the unit circle $\partial\mathbb{D}$ as defined by F. John and L. Nirenberg. Equivalently, an analytic function $f \in \mathbb{D}$ belongs to *BMOA* if and only if $\sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^p} < \infty$, for some (or, equivalently, for all) $p \in (0, \infty)$. It is well known that $H^\infty \subset BMOA \subset \mathcal{B}$.

The space **VMOA** was defined by Sarason and is the set of all analytic functions $f \in \mathbb{D}$ that satisfy $\lim_{|a|\to 1} \|f \circ \varphi_a - f(a)\|_{H^p} = 0$ for some (or, equivalently, for all) finite positive p. Alternatively, *VMOA* is the closure of the polynomials in the *BMOA* norm topology. The space *BMOA* is also conformally invariant and we have $H^\infty \subset BMOA \subset \mathcal{B}$ and $VMOA \subset \mathcal{B}_0$. A lot of information about the spaces *BMOA* and *VMOA* can be found in [5] and [18].

The **analytic Besov space** B^p is defined for $1 < p < \infty$ as the set of all analytic functions in the disk such that

$$\|f\|_{B^p} = \left((p-1)\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}dA(z)\right)^{1/p} < \infty.$$

The B^p spaces are conformally invariant; i.e, $\|f \circ \varphi\|_{B^p} = \|f\|_{B^p}$ for all disk automorphisms φ. A very important member of this scale of spaces is the Dirichlet space $B^2 = \mathcal{D}$. The **minimal Besov space** B^1 is defined as the space of all functions f analytic in $\mathbb{D}$ for which $f'' \in A^1$. It is well known [4] that

$$B^1 \subset \Lambda(1,1) \subset \bigcap_{1<p<\infty} B^p,$$

where $\Lambda(1,1)$ denotes the Lipschitz space of all analytic functions in $\mathbb{D}$ such that $f' \in H^1$.

2 Modified Blaschke products

It is well known that no inner function other than a finite Blaschke product can belong to any B^p space (see, for example, [24], [11]). For an infinite Blaschke product B, the function $|B'|^p$ may or may not be integrable; this depends to a great extent on the behavior of B near its zeros. See [1], [2], [24], or [19] for more details and further references. However, it turns out that if the zeros of a Blaschke product B are all located in a Stolz angle with vertex at, say, $z = 1$, then multiplying B by the simple outer "correction factor" $(1-z)^\alpha$ may place it in the space B^p. Such results are natural in view of the well-known fact that a Blaschke product of this kind can be extended analytically across the entire unit circle minus the point $z = 1$ [16, Theorem 6.1] and also because a statement of this type is equivalent to the problem of the membership of B' in the weighted

Bergman space A^p_{p-2}, which again falls within the circle of problems mentioned above.

Theorem 1. *Let B be an infinite Blaschke product whose zeros belong to a Stolz angle with vertex at* 1.

(i) *If $\alpha > 2$, then the modified function $(1-z)^\alpha B(z)$ belongs to B^1.*

(ii) *If $0 < \alpha \le 2$, then $(1-z)^\alpha B(z) \in \bigcap_{2/\alpha < p < \infty} B^p$.*

The following result is used in the proof of Theorem 1.

Lemma 1. *If the zeros $\{a_n\}_{n=1}^\infty$ of a Blaschke product B all lie in a Stolz angle with vertex at* 1*, then there exists a positive constant C such that*

$$|1-z|^2|B'(z)| \le C \quad \text{and} \quad |1-z|^4|B''(z)| \le C, \quad z \in \mathbb{D}.$$

Proof. The first inequality is explicit in the course of the proof of Theorem 2.3 in [19]. In fact, it was proved there that the constant C can be taken to be of the form

$$C = K \sum_{n=1}^\infty (1-|a_n|), \tag{3}$$

where the constant K depends only on the aperture of the Stolz angle.

Let us now prove the second inequality. We argue as on pp. 676–677 of [19]. Write

$$b_n(z) = \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z}, \quad B(z) = \prod_{n=1}^\infty b_n(z), \quad B_n(z) = \frac{B(z)}{b_n(z)}.$$

The zero set of the Blaschke product B_n is $\{a_k\}_{k \ne n}$, trivially a subset of the zero set of B. Thus, by applying condition (3) in the observation above to both B_n and B, it follows that

$$|1-z|^2|B_n'(z)| \le K \sum_{k=1}^\infty (1-|a_k|) = C_1, \tag{4}$$

for all $z \in \mathbb{D}$ and all positive integers n. Since $B'(z) = \sum_{n=1}^\infty b_n'(z) \cdot B_n(z)$ and

$$B''(z) = \sum_{n=1}^\infty b_n''(z) \cdot B_n(z) + \sum_{n=1}^\infty b_n'(z) \cdot B_n'(z),$$

we have

$$|B''(z)| \le 2 \sum_{n=1}^\infty \frac{1-|a_n|^2}{|1-\overline{a}_n z|^3} |B_n(z)| + \sum_{n=1}^\infty \frac{1-|a_n|^2}{|1-\overline{a}_n z|^2} |B_n'(z)|. \tag{5}$$

Using (5), Lemma 2.1 of [19] and (4), we easily obtain $|1-z|^4|B''(z)| \le C$, as claimed. □

Proof of Theorem 1. Let us begin by observing that

$$[(1-z)^\alpha B(z)]' = -\alpha(1-z)^{\alpha-1}B(z) + (1-z)^\alpha B'(z)$$

and

$$(6)\quad [(1-z)^\alpha B(z)]'' = \alpha(\alpha-1)(1-z)^{\alpha-2}B(z) - 2\alpha(1-z)^{\alpha-1}B'(z) + (1-z)^\alpha B''(z).$$

Suppose first that $\alpha > 2$. Using Lemma 1, we see that

$$|(1-z)^{\alpha-2}B(z)| \le C,\ |(1-z)^{\alpha-1}B'(z)| \le \frac{C}{|1-z|^{3-\alpha}},\ |(1-z)^\alpha B''(z)| \le \frac{C}{|1-z|^{4-\alpha}}$$

for all $z \in D$. In view of $4-\alpha < 2$ and (2), this implies that $[(1-z)^\alpha B(z)]'' \in A^1$, which is the same as saying that $(1-z)^\alpha B(z) \in B^1$.

Suppose now that $0 < \alpha \le 2$ and $p > 2/\alpha$. Since $\alpha > 0$, again using (2), we see that $\int_{\mathbb{D}} |(1-z)^{\alpha-2}|dA(z) < \infty$. Hence $(1-z)^\alpha \in B^1 \subset B^p$. This and the fact that B is bounded imply that $\int_{\mathbb{D}} |(1-z)^{\alpha-1}B(z)|^p(1-|z|^2)^{p-2}\, dA(z) < \infty$. Therefore, it follows that $(1-z)^\alpha B(z) \in B^p$ if and only if

$$\int_{\mathbb{D}} |1-z|^{\alpha p}|B'(z)|^p(1-|z|^2)^{p-2}dA(z) < \infty.$$

Using Lemma 1 and the Schwarz–Pick lemma, we get

$$\begin{aligned}
&|1-z|^{\alpha p}|B'(z)|^p(1-|z|^2)^{p-2}\\
&\quad = \left(|1-z|^2|B'(z)|\right)^{\frac{\alpha p}{2}} [(1-|z|^2)|B'(z)|]^{p-\frac{\alpha p}{2}} \cdot (1-|z|^2)^{\frac{\alpha p}{2}-2}\\
&\quad \le C^{\frac{\alpha p}{2}}[1-|B(z)|^2]^{p-\frac{\alpha p}{2}}(1-|z|^2)^{\frac{\alpha p}{2}-2}\\
&\quad \le C^{\frac{\alpha p}{2}}(1-|z|^2)^{\frac{\alpha p}{2}-2},
\end{aligned}$$

and the latter function is integrable since $\alpha p/2 - 2 > -1$ by assumption. □

Recall that an **interpolating Blaschke product** is a Blaschke product whose zero sequence is uniformly separated (equivalently, interpolating for H^∞; see [13, Chapter 9]). It follows from a well-known paper by Newman [27] that for those Blaschke products whose zeros belong to a Stolz angle, the following conditions on their zeros are all equivalent:

- the zeros form an interpolating sequence,
- the zeros form a uniformly discrete sequence,
- the zeros are a finite union of exponential sequences.

Assuming any one of these conditions improves many integrability properties of B' and allows us to place the function $(1-z)^\alpha B(z)$ into a significantly smaller space. We prove the following result.

Theorem 2. *If the zeros of an interpolating Blaschke product B belong to a Stolz angle with vertex at* 1 *and* $\alpha > 0$, *then* $(1-z)^{\alpha}B(z) \in B^1$.

As noted above, if B is a Blaschke product with zeros in a Stolz angle with vertex at 1, then $(1-z)^2 B'(z)$ and $(1-z)^4 B''(z)$ belong to H^∞. In the following lemma, we obtain an improved result of this kind by assuming in addition that the sequence is interpolating. This result is used in the proof of Theorem 2.

Lemma 2. *Suppose that* $\varepsilon > 0$ *and* B *is an interpolating Blaschke product whose sequence of zeros is contained in a Stolz angle with vertex at* 1. *Then* $(1-z)^{1+\varepsilon}B'(z) \in H^\infty$ *and* $(1-z)^{2+\varepsilon}B''(z) \in H^\infty$.

Proof. Let $\{a_n\}$ be the sequence of zeros of B. Using the above-mentioned result of Newman, we know that it is a finite union of exponential sequences. Let us set as before

$$b_n(z) = \frac{|a_n|}{a_n}\frac{a_n - z}{1-\overline{a}_n z}, \quad B(z) = \prod_{n=1}^{\infty} b_n(z), \quad B_n(z) = \frac{B(z)}{b_n(z)}. \tag{7}$$

Then

$$B'(z) = \sum_{n=1}^{\infty} b_n'(z)\cdot B_n(z),$$

whence

$$|B'(z)| \le \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\overline{a}_n z|^2}|B_n(z)| \le \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\overline{a}_n z|^2}.$$

Applying Lemma 2.1 of [19], we obtain

$$|(1-z)^{1+\varepsilon}B'(z)| \le \sum_{n=1}^{\infty} \left|\frac{1-z}{1-\overline{a}_n z}\right|^{1+\varepsilon} \left|\frac{1-|a_n|^2}{1-\overline{a}_n z}\right|^{1-\varepsilon} (1-|a_n|^2)^{\varepsilon} \le C\sum_{n=1}^{\infty}(1-|a_n|)^{\varepsilon}. \tag{8}$$

The last sum is finite because $\varepsilon > 0$ and $\{a_n\}$ is a finite union of exponential sequences. Hence, we have proved that $(1-z)^{1+\varepsilon}B'(z) \in H^\infty$.

On the other hand, just as in the proof of Lemma 1, we have

$$|B''(z)| \le 2\sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\overline{a}_n z|^3}|B_n(z)| + \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\overline{a}_n z|^2}|B_n'(z)|. \tag{9}$$

Applying (8) to B_n instead of B and with $\varepsilon/2$ replacing ε, we see that there exists a positive constant A such that

$$|(1-z)^{1+\frac{\varepsilon}{2}}B_n'(z)| \le A, \quad z \in \mathbb{D}, \quad n = 1, 2, \ldots.$$

Using this and Lemma 2.1 of [19] in (9) yields

$$
\begin{aligned}
|(1-z)^{2+\varepsilon}B''(z)| \le & 2\sum_{n=1}^{\infty}\left|\frac{1-z}{1-\overline{a}_n z}\right|^{2+\varepsilon}\left|\frac{1-|a_n|^2}{1-\overline{a}_n z}\right|^{1-\varepsilon}(1-|a_n|^2)^{\varepsilon} \\
& + A\sum_{n=1}^{\infty}\left|\frac{1-z}{1-\overline{a}_n z}\right|^{1+\frac{\varepsilon}{2}}\left|\frac{1-|a_n|^2}{1-\overline{a}_n z}\right|^{1-\frac{\varepsilon}{2}}(1-|a_n|^2)^{\varepsilon/2} \\
\le & C_1\sum_{n=1}^{\infty}(1-|a_n|^2)^{\varepsilon} + C_2\sum_{n=1}^{\infty}(1-|a_n|^2)^{\varepsilon/2} < \infty.
\end{aligned}
$$

Thus $(1-z)^{2+\varepsilon}B''(z) \in H^{\infty}$ as claimed. □

Proof of Theorem 2. Take $\varepsilon \in (0,\alpha)$. Bearing in mind (6) and using Lemma 2, we deduce that

$$
\begin{aligned}
|[(1-z)^{\alpha}B(z)]''| & \lesssim \left(\frac{|B(z)|}{|1-z|^{2-\alpha}} + \frac{|B'(z)|}{|1-z|^{1-\alpha+\varepsilon}} + \frac{|B''(z)|}{|1-z|^{-\alpha}}\right) \\
& \lesssim \left(\frac{1}{|1-z|^{2-\alpha}} + \frac{1}{|1-z|^{2-\alpha+\varepsilon}} + \frac{1}{|1-z|^{2-\alpha+\epsilon}}\right).
\end{aligned}
$$

Since $2-\alpha+\varepsilon < 2$, the result follows readily. □

It is natural to look for analogues of our theorems 1 and 2 for an atomic singular inner function S instead of a Blaschke product. For simplicity, we consider only the case $\alpha = 1$. Relying on a theorem of Mateljević and Pavlović [25], we prove the following result.

Theorem 3. *Let S denote the atomic singular inner function with mass concentrated at* 1

$$
S(z) = \exp\left(\frac{z+1}{z-1}\right).
$$

Then the modified function $(1-z)S(z)$ belongs to $\bigcap_{1<p<\infty} B^p$ but not to $\Lambda(1,1)$.

Proof. Set $F(z) = (1-z)S'(z)$ $(z \in \mathbb{D})$. It is clear that $(1-z)S(z) \in \Lambda(1,1)$ if and only if $F \in H^1$ and that $(1-z)S(z) \in B^p$ if and only if $F \in A^p_{p-2}$ $(p > 1)$. The technique used in the proof of the main theorem from [25] yields

$$
M_1(r,F) \asymp \log\frac{1}{1-r} \text{ and} \tag{10}
$$

$$
M_p^p(r,F) \asymp (1-r)^{\frac{1}{2}(1-p)}, \quad 1 < p < \infty. \tag{11}
$$

Trivially, (10) implies that $F \notin H^1$ and (11) implies that $F' \in B^p$ whenever $1 < p < \infty$. □

3 On the rate of growth of Besov functions

A complex-valued function defined in $\mathbb{D}$ is said to be **univalent** if it is analytic and one-to-one. We refer to [12], [28] and [30] for the theory of such functions. Throughout the paper, $\mathcal{U}$ stands for the class of all univalent functions in $\mathbb{D}$. It is often useful to consider certain normalized subclasses of $\mathcal{U}$ such as

$$S = \{f \in \mathcal{U} : f(0) = 0, f'(0) = 1\}.$$

The following "big-Oh" estimate for the growth of B^p functions $(1 < p < \infty)$ is well known [38, Theorem 9]:

$$|f(z)| = \mathrm{O}\left(\left(\log \frac{1+|z|}{1-|z|}\right)^{1-1/p}\right), \quad \text{as } |z| \to 1^-. \tag{12}$$

A similar "little-oh" statement is also well known.

Proposition 1. *Whenever $f \in B^p$, $1 < p < \infty$, we have*

$$|f(z)| = \mathrm{o}\left(\left(\log \frac{1}{1-|z|}\right)^{1-1/p}\right), \quad \textit{as } |z| \to 1^-.$$

The following question comes to mind immediately: can we find a function in B^p that grows at the power-logarithmic rate times a prescribed function that tends to zero? Our first result in this section shows that the answer is affirmative if $2 < p < \infty$. In order to produce such an example, we use a construction similar to the those used in the papers [8], [11], or [9], with one additional element. In addition to controlling simultaneously the geometry of the domain and the growth of the derivative of the conformal map of the disk onto it, we also control the growth of the proper conformal map along a whole radius.

In what follows, for two positive functions of $r \in (0,1)$ we write $\phi(r) \gtrsim \psi(r)$, as $r \to 1$, to indicate that $\phi(r) \geq m\,\psi(r)$ for some fixed positive m and all r sufficiently close to 1. Likewise, $\phi(r) \asymp \psi(r)$, as $r \to 1$, means that $\phi(r) \gtrsim \psi(r)$ and $\psi(r) \gtrsim \phi(r)$, as $r \to 1$.

Theorem 4. *Let $2 \leq p < \infty$ and let $\alpha : [0,\infty) \to (0,\infty)$ be a decreasing function that satisfies the following conditions.*

(i) $\int_e^\infty \alpha(t)^{p-1}/t \log t \, dt < \infty$.

(ii) *There exists a positive constant C such that $\alpha(et) \geq C\alpha(t)$, for all $t > 0$.*

Then there exists a univalent map f in the disk such that $f \in B^p$ and $f(0) = 0$. Moreover, $f(r)$ is real and positive for all positive r, and

$$f(r) \gtrsim \alpha\left(\frac{1}{1-r}\right) \cdot \left(\log \frac{1}{1-r}\right)^{1-1/p}, \quad \textit{as} \quad r \to 1^-.$$

Proof. We may assume without loss of generality that $\alpha(0) < 1$. Define

$$\beta:\ [0,\infty)\to(0,\infty)\quad\text{by}\quad \beta(x)=\alpha\left(\frac{1}{2}e^x\right).$$

Clearly, β is a decreasing function in $[0,\infty)$ and $\lim_{x\to\infty}\beta(x)=0$. Also, β satisfies the following two conditions:
(a) $\beta(x+1)\geq C\beta(x)$, for all $x>0$;
(b) $\int_1^\infty \beta(x)^{p-1}/x\,dx<\infty$.
Observe that

$$\beta\left(\log\frac{2}{1-r}\right)=\alpha\left(\frac{1}{1-r}\right),\quad 0<r<1. \tag{13}$$

Note also that as product of two decreasing functions, $\beta(x)^{p-1}/x$ is decreasing in $[0,\infty)$.

Now let us consider the simply connected domain $\Omega = Q\cup\Omega_1$, where

$$Q=\{x+iy:-1<x\leq 1,\ |y|<\beta(1)\},\quad \Omega_1=\left\{x+iy:x>1,\ |y|<\frac{\beta(x)}{x^{1/(p-1)}}\right\}.$$

Let f be the conformal mapping from the unit disk $\mathbb{D}$ onto Ω that satisfies $f(0)=0$ and $f'(0)>0$. The uniqueness of f and the fact that the domain Ω is symmetric with respect to the real axis readily imply that f takes on real positive values on the radius $(0,1)$.

The criterion for membership of conformal maps of the disk in B^p obtained in [36] and [8] implies that $f\in B^p$ if and only if

$$\int_\Omega d_\Omega^{p-2}(w)dA(w)<\infty, \tag{14}$$

where d_Ω denotes the (Euclidean) distance from w to the boundary of Ω. It is clear that, for a point $w=x+iy$ in Ω,

$$d_\Omega(w)\leq\frac{\beta(x)}{x^{1/(p-1)}}. \tag{15}$$

Using the definition of Ω, Fubini's theorem, and (15), we obtain

$$\begin{aligned}\int_\Omega d_\Omega^{p-2}(w)dA(w) &= \int_Q d_\Omega^{p-2}(w)dA(w)+2\int_1^\infty\int_0^{\frac{\beta(x)}{x^{1/(p-1)}}}\left(\frac{\beta(x)}{x^{1/(p-1)}}\right)^{p-2}dydx\\ &\leq \int_Q d_\Omega^{p-2}(w)dA(w)+2\int_1^\infty\left(\frac{\beta(x)}{x^{1/(p-1)}}\right)^{p-1}dx\\ &= \int_Q d_\Omega^{p-2}(w)dA(w)+2\int_1^\infty\frac{\beta(x)^{p-1}}{x}\,dx.\end{aligned}$$

Taking into account that $p \geq 2$ and condition (b), we see that the last two integrals are finite. This shows that $f \in B^p$.

Since $\alpha(0) < 1$, it follows that $\beta(x)/x^{1/(p-1)} < 1$ for all $x > 1$. Then, using the fact that $\beta(x)/x^{1/(p-1)}$ is a decreasing function, we see that $d_\Omega(x) \geq \beta(x+1)/(x+1)^{1/(p-1)}$ for all $x > 1$. By condition (a), this implies that

$$d_\Omega(x) \geq C\,\frac{\beta(x)}{x^{1/(p-1)}}, \quad x > 1. \tag{16}$$

Since f is a univalent map, we have $d_\Omega(f(z)) \asymp (1-|z|^2)|f'(z)|$ (see, e.g, [31, Corollary 1.4]). As mentioned above, f maps the radius $[0, 1)$ to the positive part of the real axis and it is obvious that f is an increasing positive function when restricted to this radius. Using (15) and (16), we have

$$d_\Omega\left(f(r)\right) \asymp \frac{\beta\left(f(r)\right)}{f(r)^{1/(p-1)}}, \quad \text{as } r \to 1.$$

Using the properties mentioned above, the fact that f fixes the origin, the crude estimate (12), and integration by parts, we get

$$\begin{aligned} f(r) &= \int_0^r f'(s)ds \\ &\asymp \int_0^r \frac{d_\Omega(f(s))}{1-s^2}ds \\ &\asymp \int_0^r \frac{\beta(f(s))}{p(1-s)f(s)^{1/(p-1)}}ds \\ &\geq \int_0^r \frac{\beta(f(s))}{p(1-s)(\log\frac{1}{1-s})^{1/p}}ds \\ &= \beta(f(r))\left(\log\frac{1}{1-r}\right)^{1-1/p} - \int_0^r \left(\log\frac{1}{1-s}\right)^{1/p} d(\beta\circ f)(s) \\ &\geq \beta(f(r))\left(\log\frac{1}{1-r}\right)^{1-1/p}, \end{aligned} \tag{17}$$

taking into account that both β and $\beta \circ f$ are decreasing.

Now, because $\beta(1) < \pi/2$, it follows that Ω is contained in the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < \frac{\pi}{2}\}$. This means that f is subordinate to the function $F(z) = \log((1+z)/(1-z))$ (see [13, Chapter 1]); and it then follows (see, e.g, [30, Chapter 2]) that

$$f(r) \leq \log\frac{1+r}{1-r} \leq \log\frac{2}{1-r}, \quad 0 < r < 1.$$

Using this observation, (17), and the fact that β is decreasing, and keeping in mind (13), we deduce that

$$f(r) \gtrsim \alpha\left(\frac{1}{1-r}\right)\left(\log\frac{1}{1-r}\right)^{1/p}, \quad \text{as } r \to 1.$$

□

In particular, for each $\varepsilon > 0$, one can take $\alpha(x) = (\log x)^{-\varepsilon}$ $(x > 2)$ in Theorem 4 to obtain the following.

Proposition 2. *Suppose that* $2 \leq p < \infty$ *and* $0 < \lambda < 1 - 1/p$. *Then there exists a univalent map* f *in the disk such that* $f \in B^p$. *Moreover,* $f(r)$ *is real and positive for all positive* r *and*

$$f(r) \gtrsim \left(\log \frac{1}{1-r}\right)^{\lambda}, \quad \text{as } r \to 1^-.$$

In a short note on the Dirichlet space [37], Yamashita improved an earlier observation by Cowling [10] by showing that for each constant $\lambda \in (0, 1/2)$, there exists a function $f \in \mathcal{D}$ such that

$$\liminf_{r \to 1^-} \left(\log \frac{1}{1-r}\right)^{-\lambda} M(r, f) \geq 1, \tag{18}$$

where $M(r, f) = \max\{|f(z)| : |z| = r\}$. We note that this is implied by our Proposition 2 in the case $p = 2$.

Note that if we argue as in the proof of Theorem 4, but take $\alpha(t) = 1$ for all t, we obtain the following result.

Theorem 5. *There exists a univalent map* f *defined in the disk such that* $f \in \bigcap_{2<p<\infty} B^p$, $f(0) = 0$, $f(r)$ *is real and positive for all positive* r, *and*

$$f(r) \gtrsim \left(\log \frac{1}{1-r}\right)^{1/2}, \quad \text{as } r \to 1^-.$$

Next, we extend Proposition 2 to the spaces B^p with $1 < p < 2$.

Proposition 3. *Suppose that* $1 < p \leq 2$ *and* $0 < \lambda < 1 - 1/p$. *Then, there exists a univalent function* f *in* B^p *such that* $f(r)$ *is real and positive for all positive* r *and*

$$f(r) \gtrsim \left(\log \frac{1}{1-r}\right)^{\lambda}, \quad \text{as } r \to 1^-.$$

Proof. Set $F(z) = \log(e/(1-z))$, $(z \in \mathbb{D})$. Then F is univalent in $\mathbb{D}$ and $\operatorname{Re} F(z) > 0$ for all $z \in \mathbb{D}$. Take λ with $0 < \lambda < 1 - 1/p$. It then follows that the function

$$f(z) = \left(\log \frac{e}{1-z}\right)^{\lambda}, \quad z \in \mathbb{D},$$

is also univalent. It is also easily seen that $f \in B^p$. □

The natural question of obtaining an analogue of Theorem 4 for $1 < p \leq 2$ remains open.

4 A length–area analogue of the Fejér–Riesz inequality

4.1 The area function. Given a conformal map f of $\mathbb{D}$ onto a simply connected planar domain Ω, denote by $L(r)$ the length of the image by f of the circle $\{z : |z| = r\}$. Consider also the **area function**

$$a(r) = \int_{D(0,r)} |f'|^2 dA = \int_0^r \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 d\theta \rho d\rho \tag{19}$$

associated with a conformal map f of $\mathbb{D}$ onto a simply connected domain Ω. The following estimate is known as the **length–area principle**:

$$\frac{1}{2\pi} \int_0^r \frac{L(\rho)^2}{\rho} d\rho \leq a(r), \tag{20}$$

with equality when f is the identity map and $\Omega = \mathbb{D}$ (see [20], Theorem 2.1, Chapter 1 of [17], or the proof of Proposition 2.2 of [31]). The length-area principle was first proved by Ahlfors in 1930 and later exploited in the works of M. Cartwright.

4.2 The Féjer–Riesz inequality. The classical (sharp) inequality of Fejér and Riesz

$$\int_{-1}^1 |f(x)|^p dx \leq \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta = \pi \|f\|_{H^p}^p$$

states that the injection map $H^p \subset L^p((-1, 1), dx)$ is bounded and its norm is $\pi^{1/p}$. The inequality is of particular interest in the following special case. Let F map the disk onto a Jordan domain. If the boundary of the domain is rectifiable (equivalently, if $F' \in H^1$), then the length of the image of the diameter $(-1, 1)$ of the disk is at most half the length of the image of the boundary (cf. Chapter 3 of [13] or the original source [14]).

It is of some interest is to obtain analogous statements for the Dirichlet space or, more generally, for analytic Besov spaces. Since, for a function $f \in \mathcal{D}$, the integral $\int_{-1}^1 |f'(x)|^2 dx$ need not be convergent, any correct analogue of Fejér–Riesz must have a different form. The following was obtained by A. L. Shields [33]:

If $f \in \mathcal{D}$, then

$$\int_0^1 |f(r)|^2 \frac{r^2}{1-r} \left(\log \frac{1}{1-r}\right)^{-2} dr \leq c \int_{\mathbb{D}} |f'|^2 dA,$$

for some universal constant $c > 0$.

Holland and Walsh ([21], Theorem 4) improved this estimate by showing that if $f \in \mathcal{D}$ and $f(0) = 0$, then

$$\int_0^1 \left(\int_0^r M_\infty(\rho, f')d\rho \right)^2 \frac{1}{1-r} \left(\log \frac{1}{1-r} \right)^{-2} dr \le c \int_{\mathbb{D}} |f'|^2 dA,$$

where c is a concrete value and $M_\infty(\rho, f')$ stands for the maximum modulus of f' over the circle of radius r centered at the origin. They actually obtained a more general B^p version, $1 < p < \infty$.

While such results are of interest in themselves and for $p = 2$ also have a relationship with the multipliers of the Dirichlet space, it still seems reasonable to look for a statement more suited for the context of the length of the image of a segment under a conformal map, more along the lines of the original Fejér–Riesz theorem. We now prove such a statement, which at the same time resembles the length–area principle (20) and gives us a concrete value of the constant.

4.3 Some new inequalities for B^p functions. Denote by $d(r)$ the length of the image under F of the truncated diameter $(-r, r)$. By applying first the Fejér–Riesz inequality and then the length–area principle, we obtain

$$\int_0^R \frac{d(r)^2}{r} dr \le \frac{1}{4} \int_0^R \frac{L(r)^2}{r} dr \le \frac{\pi}{2} a(R).$$

However, this bound can be improved and also generalized to the case of arbitrary analytic Besov space B^p $(1 < p < \infty)$, as follows.

Theorem 6. *Let f be a conformal map in $\mathbb{D}$ and let $d(r)$ denote the length of the Jordan arc $f((-r, r))$. If $f \in B^p$, then*

$$2^{2-p}(p-1) \int_0^1 \frac{(1-r^2)^{p-2} d(r)^p}{r^{p-1}} dr \le \|f\|_{B^p}^p.$$

In particular, for a conformal map f of $\mathbb{D}$ onto a domain Ω of finite area,

$$\int_0^1 \frac{d(r)^2}{r} dr \le A(\Omega).$$

Proof. Start off with a function g which is simply analytic in $\mathbb{D}$. Apply the standard Fejér–Riesz inequality to the dilations $g_r(z) = g(rz)$, $0 < r < 1$ to obtain

$$\int_{-r}^r |g(x)|^p dx \le \frac{r}{2} \int_0^{2\pi} |g(re^{i\theta})|^p d\theta. \tag{21}$$

Also, the Hölder inequality yields

$$\int_{-r}^r |g(x)|^p dx \ge (2r)^{1-p} \left(\int_{-r}^r |g(x)| dx \right)^p. \tag{22}$$

By putting together (21) and (22) and integrating over (0, 1) with respect to $(1-r^2)^{p-2}dr$, we get

$$2^{2-p}\int_0^1 \frac{(1-r^2)^{p-2}}{r^{p-1}}\left(\int_{-r}^{r}|g(x)|dx\right)^p dr \le \int_{\mathbb{D}}|g(z)|^p(1-|z|^2)^{p-2}dA(z).$$

Now multiply both sides by $p-1$ and then replace g by f' where f is a conformal map to obtain the desired statements. □

It would be interesting to determine the best possible bound in the inequality

$$\int_0^1 \frac{d(r)^2}{r}\,dr \le cA(\Omega).$$

Theorem 6 tells us that $c \le 1$, while the function $F(z) \equiv z$ shows that $c \ge 2\pi^{-1}$. The best constant issue here seems more difficult than for the Fejér–Riesz inequality, as we have to control $d(r)$ for *each* value of $r \in (0,1)$.

5 Some special univalent functions in Besov spaces

5.1 Univalent lacunary series in Besov spaces. As usual, by a **lacunary series** (also called **power series with Hadamard gaps**) we mean a power series of the form $\sum_{k=0}^{\infty} a_k z^{n_k}$, where $n_{k+1}/n_k \ge q > 1$ for all k. A characterization of the lacunary series in the space B^p $(1 \le p < \infty)$ is well known (see [21] or [11], for example). Namely, Theorem D on p. 55 of [11] asserts that if $1 \le p < \infty$ and f is an analytic function in $\mathbb{D}$ given by a lacunary power series $\sum_{k=0}^{\infty} a_k z^{n_k}$, then $f \in B^p$ if and only if

$$\sum_{k=0}^{\infty} n_k |a_k|^p < \infty.$$

A result of Pommerenke [29] and Fuchs [15] (see also Theorem 5.13 on p. 152 of [30]) asserts that if f is a univalent function in the disk, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and f has Hadamard gaps, then $a_n = \mathrm{O}(n^{-1})$ as $n \to \infty$. We improve this result by showing that lacunary series which are univalent functions actually belong to the smallest conformally invariant space B^1. In the proof, we use a deep theorem of Murai [26], which asserts that if f is given by a lacunary power series, $f(z) = \sum_k a_k z^{n_k}$ and also $\sum_k |a_k| = \infty$, then f takes on every complex value infinitely often in $\mathbb{D}$.

Theorem 7. *If f is a univalent function in $\mathbb{D}$ given by a power series with Hadamard gaps, $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$, then $f \in B^1$ and $\sum_{k=0}^{\infty} n_k|a_k| < \infty$.*

Proof. Observe that $f'(z) = \sum_k n_k a_k z^{n_k - 1}$ is also given by a power series with Hadamard gaps. Since f is univalent in $\mathbb{D}$, f' does not vanish there. Hence Murai's theorem implies that $\sum_k n_k |a_k| < \infty$. By [11, Theorem D, p. 55], it follows that $f \in B^1$. □

5.2 Univalent maps, geometric properties, and Besov spaces. We now consider some very special univalent maps (those that map the disk onto a bounded convex or a bounded starlike domain) and discuss their membership in Besov spaces. Recall that a domain Ω is called **starlike** if there exists a point $p \in \Omega$ such that, for every other point $w \in \Omega$, the segment $[p, w]$ is contained in Ω.

Theorem 8. *If f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a bounded convex domain, then $f \in B^1$.*

Proof. Without loss of generality, we may suppose that $f \in S$; otherwise, subtract $f(0)$ and divide by $f'(0)$, which does not change any of the properties of $f(\mathbb{D})$. Since the boundary of a bounded convex domain is rectifiable [31, p. 65], it follows by the classical theorem of Riesz that $f' \in H^1$. By [30, Corollary 2.4, p. 46], we know that $zf'(z)$ is a starlike function and hence univalent. It is clearly also an H^1 function. Hence $(zf'(z))' \in A^1$, which implies that $f \in B^1$. Here we have used a result due to Pommerenke [28] (see also [6]) which states that if $g \in \mathcal{U}$ and $g \in H^1$, then $g' \in A^1$. □

The natural question arises whether Theorem 8 remains true for starlike domains. In [11, Theorem 2.3], by improving an example due to Walsh and Campbell from [36], we constructed a Jordan domain such that a univalent map of the disk onto it is not in $\bigcup_{1<p<2} B^p$. Here we prove that this can actually be improved even further. We may ask in addition that the domain be starlike. In particular, this shows that the answer to our question is negative in a very strong sense.

Theorem 9. *There exists a univalent function f that maps $\mathbb{D}$ onto a starlike Jordan domain but such that $f \notin \bigcup_{1<p<2} B^p$.*

Proof. Our function f is a Riemann map of $\mathbb{D}$ onto a Jordan domain $\Omega \subset \mathbb{D}$ which is starlike with respect to the origin. The domain Ω is of the form

$$\Omega = \mathbb{D} \setminus \Big(\bigcup_{n=2}^{\infty} S_n \Big),$$

where each S_n is a typical Carleson box S_n defined as follows.

$$h_n = \frac{1}{(n+1)\log^2[e(n+1)]}, \quad n = 1, 2, \dots;$$

$$\delta_n = \frac{1}{\log^2[e(n+1)]}, \quad n = 1, 2, \ldots;$$

$$\theta_1 = 0, \quad \theta_n = 2\sum_{j=1}^{n-1} h_j \text{ if } (n \geq 2);$$

$$S_n = \{z = re^{i\theta} : 1 - \delta_n < r < 1,\ \theta_n < \theta < \theta_n + h_n\}.$$

We note that $\theta_n < \theta_n + h_n < \theta_n + 2h_n = \theta_{n+1}$ and

$$\sum_{n=1}^{\infty} h_n \leq \int_1^{\infty} \frac{dx}{x\log^2(ex)} = 1.$$

It then follows easily that Ω is in fact a Jordan domain contained in $\mathbb{D}$ and starlike with respect to 0. Set also

$$T_n = \{z = re^{i\theta} : 1 - \delta_{n+1} < r < 1,\ \theta_n + h_n < \theta < \theta_{n+1}\}.$$

It is clear that the area $|T_n|$ of T_n satisfies $|T_n| \asymp \delta_{n+1}h_n$. Also, there exists $C > 0$ such that

$$w \in T_n \implies d_\Omega(w) \leq Ch_n, \quad n = 1, 2, \ldots.$$

Then it follows that, whenever $p < 2$,

$$\int_{\mathbb{D}} d_\Omega(w)^{p-2}dA(w) \geq C\sum_{n=1}^{\infty}\int_{T_n} d_\Omega(w)^{p-2}dA(w) \geq C\sum_{n=1}^{\infty} h_n^{p-1}\delta_{n+1} = \infty.$$

Using Walsh's criterion, we deduce that $f \notin B^p$ if $1 \leq p < 2$. □

6 Domains related to the Dirichlet space

6.1 Some non-finitely valent self-maps of the disk onto itself. It is not difficult to see that many surjective self-maps of the disk onto itself exist. Clearly, every finite Blaschke product has this property in view of Rouché's theorem, while no infinite Blaschke product can have it. We now give an example of a class of maps that cover certain parts of the disk once, twice, three times, etc., and are, thus, neither multivalent nor infinitely valent. Similar arguments have certainly been known for some time. For the lack of an earlier reference, we mention [23]. The added value here is that we can easily tell that such a map belongs to an arbitrary Besov space B^p, $1 < p \leq 2$, if the image domain satisfies a certain simple geometric condition.

If f is an analytic function in $\mathbb{D}$ and $z \in \mathbb{D}$, we let $d_f(z)$ denote the radius of the largest disk centered at $f(z)$ which is the one-to-one image under f of a domain

G_z with $z \in G_z \subset \mathbb{D}$. Schwarz's Lemma applied to the branch of f^{-1} defined in the disk of radius d(z) about the point f(z) yields

$$d_f(z) \le (1 - |z|^2)|f'(z)|. \tag{23}$$

Now we can state our result.

Theorem 10. *Suppose that* $1 < p \le 2$ *and that we are given an increasing sequence of real numbers* $0 = r_0 < r_1 < r_2 < \cdots < 1$ *and a sequence of positive integers* $\{k_j\}_{j=1}^{\infty}$ *satisfying*

$$\sum_{j=1}^{\infty} k_j (r_j - r_{j-1})^{p-1} < \infty. \tag{24}$$

Then there exists a function $f \in B^p$ *which maps* $\mathbb{D}$ *into itself and which takes on each value in the annulus* $A_j = \{z : r_{j-1} < |z| < r_j\}$ *exactly* k_j *times.*

Proof. Suppose that $1 < p \le 2$. Set $x_j = \log r_j$ $(j \ge 1)$ and

$$R_1 = \{z = x + iy : x \le x_1, 0 \le y \le 2\pi k_1\},$$

$$R_j = \{z = x + iy : x_{j-1} \le x \le x_j, 0 \le y \le 2\pi k_j\}, \quad j \ge 2.$$

Let Ω be the interior of the union of the sets R_j, $j = 1, 2, \ldots$. Clearly, Ω is a simply connected domain contained in the quadrant $\{x + iy : x < 0, y > 0\}$. Let F be a conformal mapping from $\mathbb{D}$ onto Ω and $f(z) = \exp F(z)$, $z \in \mathbb{D}$. Set

$$A_j = \{z \in \mathbb{D} : r_{j-1} < |z| < r_j\}.$$

We have $F(\mathbb{D}) = \mathbb{D} \setminus \{0\}$. The exponential maps takes R_j onto the annulus A_j, covering it k_j times. Furthermore, it is easy to see that, for $j \ge 2$,

- if $w \in A_j$, $r_{j-1} < |w| < (r_{j-1} + r_j)/2$ and $f(z) = w$, then $|w| - r_{j-1} \le d_f(z) \le (1 - |z|^2)|f'(z)|$.
- if $w \in A_j$, $(r_{j-1} + r_j)/2 < |w| < r_j$ and $f(z) = w$, then $r_j - |w| \le d_f(z) \le (1 - |z|^2)|f'(z)|$.

Using this and the fact that $1 < p \le 2$, we deduce that

$$\begin{aligned} &\int_{\mathbb{D}} (1 - |z|^2)^{p-2} |f'(z)|^p \, dA(z) \\ &\le C_1 + C_2 \sum_{j=2}^{\infty} k_j \left(\int_{r_{j-1}}^{(r_{j-1}+r_j)/2} (\rho - r_{j-1})^{p-2} d\rho + \int_{(r_{j-1}+r_j)/2}^{r_j} (r_j - |\rho|)^{p-2} d\rho \right) \\ &\le C_1 + C_2 \sum_{j=2}^{\infty} k_j (r_j - r_{j-1})^{p-1}. \end{aligned} \tag{25}$$

Thus, (24) implies that $f \in B^p$. ∎

6.2 A topological-metric theorem for domains of finite area. In [11], we described the so-called univalent B^p domains. In order to do that, we used a special construction of a simply connected subdomain of a given domain with equally good integrability properties. There are, however, similar constructions that improve the properties obtained there. The purpose of this section is to give such a construction and prove the following result.

Theorem 11. *Let Ω be a planar domain. Then there exists a simply connected domain $\Omega' \subset \Omega$ such that $\Omega \setminus \Omega'$ has σ-finite length.*

As an immediate consequence of this, we have

Corollary 1. *A planar domain contains a simply connected domain with the same area.*

Remarks.

- The domain constructed in the proof proposed here has the property that, for every finite $\alpha \geq 0$,

$$\int_{\Omega} d(w, \partial\Omega)^{\alpha}\, dA(w) \approx \int_{\Omega'} d(w, \partial\Omega')^{\alpha}\, dA(w).$$

  Consequently, this provides an alternative construction for the proof of Theorem 2.1 in [11].
- The ideas in the proof of our final result are analogous to those provided in the aforementioned result. Nevertheless, since the new statement has a different flavor, we present a self-contained proof, making clear exactly where the improvements are made with respect to [11].

Proof of Theorem 11. As in [11], we use the family of dyadic squares provided by the Whitney's decomposition of Ω [34, p. 16]. That is, we start our argument with a countable collection of closed dyadic squares $\{Q_n\}$ with pairwise disjoint interiors and such that $\bigcup Q_n = \Omega$, and such that, for any two squares with non-empty intersection, the ratio of their sides is bounded from both above and below. We recall that such squares can be chosen such that, for each w in some Q_n, $d(w, \partial\Omega)$ is comparable to diam (Q_n).

We say that two squares of our family are neighbors if they have more than a single common boundary point, that is, if one side of one of these two squares is a subset of some side of the other. Since neighboring squares have sides of comparable length, it is clear that the number of neighbors of any square of the decomposition is bounded by an absolute constant.

Now, choose an arbitrary square Q^* and refer to it as the pivoting square. Given another square Q, as in [11], let us consider a Jordan arc γ in Ω connecting some point of Q^* to some point in Q. The selection of our squares together with a simple compactness argument assures us that there are finitely many squares in our decomposition which intersect γ.

On the other hand, it is clear that for each Q, we can choose a finite sequence of our squares in such a way that the first one is Q^*, the last one is Q, and each square is a neighbor of the preceding one. Keeping this in mind, we can assign to each square Q the minimum of the cardinal numbers of such sequences. We refer to this positive integer as the generation of Q, understanding that Q^* belongs to generation zero. With this definition, it is clear that the squares of the first generation are precisely the neighbors of Q^* and that if some square belongs to generation N, then it has a neighbor of generation $N-1$. Moreover, since the number of neighbors of a square is bounded, an inductive argument allows us to deduce that there are only finitely many squares in each generation.

Following the scheme proposed in [11], we reorder our sequence of squares (Q_n), starting with Q^*, and continue with all the squares from the first generation, then with all the squares of the second generation, and so on. Observe that this new sequence has the property that for a given square Q_n, the square Q_{n+1} belongs either to the same generation or to the next one.

Now, recall that if Q_n and Q_m are neighbors, they share a segment on their boundaries. Let us denote by $C_{n,m}$ a small open subsegment properly included in $Q_n \cap Q_m$, that is, $C_{n,m}$ is a segment without its endpoints, contained in both Q_n and Q_m. Observe that with this selection, $\mathrm{Int}(Q_n) \cup \mathrm{Int}(Q_m) \cup C_{n,m}$ is a domain.

We construct the domain Ω' inductively. Let Ω_1 be the interior of Q^*. In order to construct the domain Ω_2, consider the interior of the square Q_2. Since Q_2 is a neighbor of Q_1, we can consider the segment $C_{1,2}$ and define $\Omega_2 = \mathrm{Int}(Q_1) \cup \mathrm{Int}(Q_2) \cup C_{1,2}$. In this way, our domains Ω_n consists of the union of the interiors of the squares $Q_1, \ldots, Q_n$ and a certain union of some segments $C_{n,m}$ that connect every square Q_k with one of the previous ones which must be its neighbor and must also belong to the preceding generation. That is, if we suppose that Ω_n has been constructed and satisfies the properties just mentioned, in order to construct Ω_{n+1}, proceed as follows. Take Q_{n+1} and suppose it belongs to generation N. Then choose k with $1 \le k \le n$ so that Q_k belongs to generation $N-1$ and Q_k and Q_{n+1} are neighbors and define $\Omega_{n+1} = \Omega_n \cup \mathrm{Int}(Q_{n+1}) \cup C_{k,n+1}$.

Finally, set $\Omega' = \bigcup \Omega_n$. Observe that our domain Ω' has the following properties.

- Ω' contains the interiors of all squares Q_n.

- Since segments $C_{n,m}$ have been chosen to be completely contained in the intersection of Q_n and Q_m, every square of generation N is connected to one square of the previous generation and to at most one of the generation $N+1$. Consequently, for each n, Ω_n is simply connected.
- As a trivial consequence, it follows that Ω' is also simply connected.

It is now clear that $\Omega \setminus \Omega'$ is a subset of the union of the boundaries of all squares Q_n, that is, $\Omega \setminus \Omega'$ is contained in a countable union of segments. □

As previously mentioned, the domain Ω' just constructed has the property that for any $\alpha \geq 0$, the integrals

$$\int_\Omega d(w, \partial\Omega)^\alpha \, dA(w) \quad \text{and} \quad \int_{\Omega'} d(w, \partial\Omega')^\alpha \, dA(w)$$

are comparable, with constants depending only on α (but not on the geometry of Ω!). Here is a simple proof.

Since Ω and Ω' differ only by a set of Lebesgue measure zero and $\Omega' \subset \Omega$, we have trivially

$$\int_{\Omega'} d(w, \partial\Omega')^\alpha \, dA(w) \leq \int_\Omega d(w, \partial\Omega)^\alpha \, dA(w).$$

On the other hand, using the fact that for any square Q,

$$\int_Q d(w, \partial Q)^\alpha \, dA(w) = C_\alpha \, (\operatorname{diam} Q)^{\alpha+2},$$

and recalling the properties of the Whitney decomposition, we get

$$\begin{aligned}
\int_{\Omega'} d(w, \partial\Omega')^\alpha \, dA(w) &= \sum_{j=1}^\infty \int_{Q_j} d(w, \partial\Omega')^\alpha \, dA(w) \\
&\geq \sum_{j=1}^\infty \int_{Q_j} d(w, \partial Q_j)^\alpha \, dA(w) = C_\alpha \sum_{j=1}^\infty (\operatorname{diam} Q_j)^{\alpha+2} \\
&\geq 4C_\alpha \sum_{j=1}^\infty \int_{Q_j} (\operatorname{diam} Q_j)^\alpha \, dA(w) \\
&\geq \tilde{C}_\alpha \sum_{j=1}^\infty \int_{Q_j} d(w, \partial\Omega)^\alpha \, dA(w) \\
&= \tilde{C}_\alpha \int_\Omega d(w, \partial\Omega)^\alpha \, dA(w).
\end{aligned}$$

□

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(Received April 20, 2009)