

# THE MAXIMUM MODULUS OF A TRIGONOMETRIC TRINOMIAL

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**Abstract.** Let  $\Lambda$  be a set of three integers and let  $\mathcal{C}_\Lambda$  be the space of  $2\pi$ -periodic functions with spectrum in  $\Lambda$  endowed with the maximum modulus norm. We isolate the maximum modulus points  $x$  of trigonometric trinomials  $T \in \mathcal{C}_\Lambda$  and prove that  $x$  is unique unless  $|T|$  has an axis of symmetry. This enables us to compute the exposed and the extreme points of the unit ball of  $\mathcal{C}_\Lambda$ , to describe how the maximum modulus of  $T$  varies with respect to the arguments of its Fourier coefficients and to compute the norm of unimodular relative Fourier multipliers on  $\mathcal{C}_\Lambda$ . We obtain in particular the Sidon constant of  $\Lambda$ .

## 1 Introduction

Let  $\lambda_1, \lambda_2$  and  $\lambda_3$  be distinct integers and  $r_1, r_2$  and  $r_3$  positive real numbers. Given real numbers  $t_1, t_2$  and  $t_3$ , let us consider the **trigonometric trinomial**

$$(1) \quad T(x) = r_1 e^{i(t_1 + \lambda_1 x)} + r_2 e^{i(t_2 + \lambda_2 x)} + r_3 e^{i(t_3 + \lambda_3 x)}$$

for  $x \in \mathbb{R}$ . The  $\lambda$ 's are the **frequencies** of the trigonometric trinomial  $T$ ; the  $r$ 's are the **moduli** or **intensities**; and the  $t$ 's are the **arguments** or **phases** of its **Fourier coefficients**  $r_1 e^{it_1}$ ,  $r_2 e^{it_2}$  and  $r_3 e^{it_3}$ .

The maximum modulus of a trigonometric trinomial has an interpretation in plane geometry. Without loss of generality, we may assume that  $\lambda_2$  is between  $\lambda_1$  and  $\lambda_3$ . Let  $H$  be the curve with complex equation

$$(2) \quad z = r_1 e^{i(t_1 - (\lambda_2 - \lambda_1)x)} + r_3 e^{i(t_3 + (\lambda_3 - \lambda_2)x)} \quad (-\pi < x \leq \pi).$$

$H$  is a **hypotrochoid**: it is drawn by a point at distance  $r_3$  to the centre of a circle with radius  $r_1 |\lambda_2 - \lambda_1| / |\lambda_3 - \lambda_2|$  that rolls inside another circle with radius  $r_1 |\lambda_3 - \lambda_1| / |\lambda_3 - \lambda_2|$ . The maximum modulus of (1) is the maximum distance of points  $z \in H$  to a given point  $-r_2 e^{it_2}$  of the complex plane. Figure 1 illustrates the particular case  $T(x) = 4e^{-i2x} + e^{it} + e^{ix}$ .

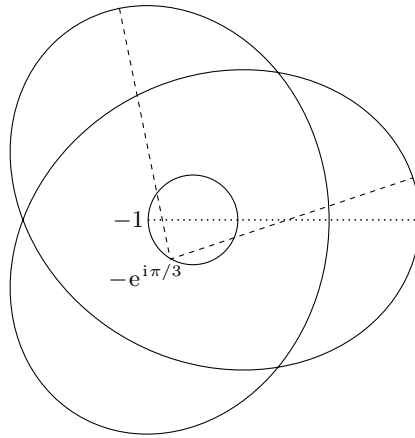


Figure 1. The unit circle, the hypotrochoid  $H$  with equation  $z = 4e^{-i2x} + e^{ix}$ , the segment from  $-1$  to the unique point on  $H$  at maximum distance and the segments from  $-e^{i\pi/3}$  to the two points on  $H$  at maximum distance.

We deduce the existence of an interval on which  $T$  attains its maximum modulus independently of the moduli of its Fourier coefficients (see Theorem 7.1 (a) for an exact statement). We prove in particular the following result.

**Theorem 1.1.** *Let  $d = \gcd(\lambda_2 - \lambda_1, \lambda_3 - \lambda_2)$  and let  $\tau$  be the distance of*

$$(3) \quad \frac{\lambda_2 - \lambda_3}{d}t_1 + \frac{\lambda_3 - \lambda_1}{d}t_2 + \frac{\lambda_1 - \lambda_2}{d}t_3$$

*to  $2\pi\mathbb{Z}$ . The trigonometric trinomial  $T$  attains its maximum modulus at a unique point modulo  $2\pi/d$ , with multiplicity 2, unless  $\tau = \pi$ .*

Theorem 1.1 shows that if there are two points of the hypotrochoid  $H$  at maximum distance to  $-r_2e^{it_2}$ , it is so only because  $-r_2e^{it_2}$  lies on an axis of symmetry of  $H$ .

We obtain a precise description of those trigonometric trinomials that attain their maximum modulus twice modulo  $2\pi/d$  (see Theorem 7.1 (c)). Their rôle becomes clear by the following result in convex geometry: they yield the exposed points of the unit ball of the ambient normed space. Let us first establish the proper functional analytic framework. Let  $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$  be the **spectrum** of the trigonometric trinomial  $T$  and write  $e_\lambda : x \mapsto e^{i\lambda x}$ . Let  $\mathcal{C}_\Lambda$  be the space of functions spanned by  $\{e_\lambda : \lambda \in \Lambda\}$ , endowed with the maximum modulus norm. Recall that a point  $P$  of a convex set  $K$  is **exposed** if there is a hyperplane that meets  $K$  only in  $P$ ;  $P$  is **extreme** if it is not the midpoint of any two other points of  $K$ .

**Theorem 1.2.** *Let  $K$  be the unit ball of the space  $\mathcal{C}_\Lambda$  and let  $P \in K$ .*

- (a) *The point  $P$  is an exposed point of  $K$  if and only if  $P$  is either a trigonometric monomial  $e^{i\alpha}e_\lambda$  with  $\alpha \in \mathbb{R}$  and  $\lambda \in \Lambda$  or a trigonometric trinomial that attains its maximum modulus, 1, at two points modulo  $2\pi/d$ . Every linear functional on  $\mathcal{C}_\Lambda$  attains its norm on an exposed point of  $K$ .*
- (b) *The point  $P$  is an extreme point of  $K$  if and only if  $P$  is either a trigonometric monomial  $e^{i\alpha}e_\lambda$  with  $\alpha \in \mathbb{R}$  and  $\lambda \in \Lambda$  or a trigonometric trinomial such that  $1 - |P|^2$  has four zeros modulo  $2\pi/d$ , counted with multiplicities.*

We describe the dependence of the maximum modulus of the trigonometric trinomial  $T$  on the arguments. The general issue has been studied for a long time; [14, 22] are two early references. In particular, the following problem has been addressed: see [6, p. 2 and Supplement].

**Extremal problem 1.3** (Complex Mandel'shtam problem). Find the minimum of the maximum modulus of a trigonometric polynomial with given Fourier coefficient moduli.

It appeared originally in electrical circuit theory: “L. I. Mandel'shtam communicated to me a problem on the phase choice of electric currents with different frequencies such that the capacity of the resulting current to blow (the circuit) is minimal”, tells N. G. Chebotarëv in [5, p. 396], where he discusses applications of a formula given in Section 9 that we would like to advertise.

Our main theorem solves an elementary case of the complex Mandel'shtam problem.

**Theorem 1.4.** *The maximum modulus of (1) is a strictly decreasing function of  $\tau$  as defined in Theorem 1.1. In particular,*

$$\begin{aligned} \min_{t_1, t_2, t_3} \max_x & \left| r_1 e^{i(t_1 + \lambda_1 x)} + r_2 e^{i(t_2 + \lambda_2 x)} + r_3 e^{i(t_3 + \lambda_3 x)} \right| \\ & = \max_x \left| \epsilon_1 r_1 e^{i\lambda_1 x} + \epsilon_2 r_2 e^{i\lambda_2 x} + \epsilon_3 r_3 e^{i\lambda_3 x} \right| \end{aligned}$$

*if  $\epsilon_1, \epsilon_2$  and  $\epsilon_3$  are real signs  $+1$  or  $-1$  such that  $\epsilon_i \epsilon_j = -1$ , where  $i, j, k$  is a permutation of  $1, 2, 3$  such that the power of 2 in  $\lambda_i - \lambda_j$  is greater than the power of 2 in  $\lambda_i - \lambda_k$  and in  $\lambda_k - \lambda_j$ .*

Our result shows that the maximum modulus is minimal when the phases are chosen in opposition, independently of the intensities  $r_1, r_2$  and  $r_3$ .

The decrease of the maximum modulus of (1) may be bounded as shown in the next result.

**Theorem 1.5.** *Let  $d$  and  $\tau$  be defined as in Theorem 1.1. Suppose that  $\lambda_2$  is between  $\lambda_1$  and  $\lambda_3$ . The quotient of the maximum modulus of (1) by  $|r_1 + r_2 e^{i\tau d/|\lambda_3 - \lambda_1|} + r_3|$  is a strictly increasing function of  $\tau$  unless  $r_1 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_2 - \lambda_1|$ , in which case it is constantly equal to 1.*

When  $r_1 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_2 - \lambda_1|$ , the hypocrochoid  $H$  with equation (2) is a **hypocycloid** with  $|\lambda_3 - \lambda_1|/d$  cusps: the rolling point is **on** the rolling circle. Figure 2 illustrates the particular case  $T(x) = (1/3)e^{-i2x} + e^{it} + (2/3)e^{ix}$ .

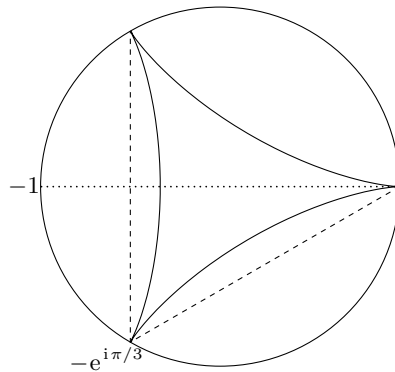


Figure 2. The unit circle, the deltoid  $H$  with equation  $z = (1/3)e^{-i2x} + (2/3)e^{ix}$ , the segment from  $-1$  to the unique point on  $H$  at maximum distance and the segments from  $-e^{i\pi/3}$  to the two points on  $H$  at maximum distance.

We may deduce from Theorem 1.5 a less precise but handier inequality.

**Theorem 1.6.** *Let  $d$  and  $\tau$  be defined as in Theorem 1.1. Let*

$$(4) \quad D = \frac{\max(|\lambda_2 - \lambda_1|, |\lambda_3 - \lambda_2|, |\lambda_3 - \lambda_1|)}{\gcd(\lambda_2 - \lambda_1, \lambda_3 - \lambda_1)}$$

*be the quotient of the diameter of  $\Lambda$  by  $d$ . Let  $t'_1, t'_2$  and  $t'_3$  be another three real numbers and  $\tau'$  defined correspondingly. If  $\tau > \tau'$ , then*

$$\begin{aligned} \max_x |r_1 e^{i(t_1 + \lambda_1 x)} + r_2 e^{i(t_2 + \lambda_2 x)} + r_3 e^{i(t_3 + \lambda_3 x)}| \\ \geq \frac{\cos(\tau/2D)}{\cos(\tau'/2D)} \max_x |r_1 e^{i(t'_1 + \lambda_1 x)} + r_2 e^{i(t'_2 + \lambda_2 x)} + r_3 e^{i(t'_3 + \lambda_3 x)}|, \end{aligned}$$

*with equality if and only if  $r_1 : r_2 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_3 - \lambda_1| : |\lambda_2 - \lambda_1|$ .*

Figure 3 illustrates the inequalities obtained in Theorems 1.5 and 1.6 for the particular case  $T(x) = 4e^{-i2x} + e^{it} + e^{ix}$ , as in Figure 1.

If we choose  $\tau' = 0$  in the last result, we get the solution to an elementary case of the following extremal problem.

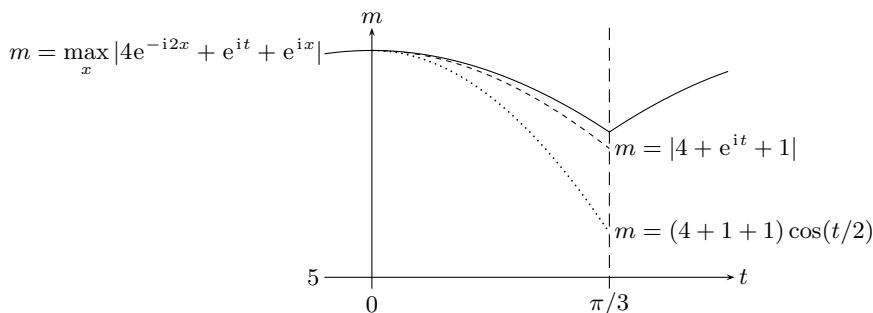


Figure 3. Let  $H$  be the hypotrochoid with equation  $z = 4e^{-i2x} + e^{ix}$ . This plot shows the maximum distance  $m$  of points  $z \in H$  to the point  $-e^{it}$  and the two estimates of this maximum distance provided by Theorems 1.5 and 1.6 for  $t \in [0, \pi/3]$ .

**Extremal problem 1.7.** Find the minimum of the maximum modulus of a trigonometric polynomial with given spectrum, Fourier coefficient arguments and moduli sum.

**Theorem 1.8.** Let  $\tau$  be defined as in Theorem 1.1 and  $D$  be given by (4). Then

$$\frac{\max_x |r_1 e^{i(t_1 + \lambda_1 x)} + r_2 e^{i(t_2 + \lambda_2 x)} + r_3 e^{i(t_3 + \lambda_3 x)}|}{r_1 + r_2 + r_3} \geq \cos(\tau/2D),$$

with equality if and only if  $\tau = 0$  or  $r_1 : r_2 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_3 - \lambda_1| : |\lambda_2 - \lambda_1|$ .

The dependence of the maximum modulus of (1) on the arguments may also be expressed as properties of relative multipliers. Given three real numbers  $t_1, t_2$  and  $t_3$ , the linear operator on  $\mathcal{C}_\Lambda$  defined by  $e_{\lambda_j} \mapsto e^{it_j} e_{\lambda_j}$  is a **unimodular relative Fourier multiplier**: it multiplies each Fourier coefficient of elements of  $\mathcal{C}_\Lambda$  by a fixed unimodular number; let us denote it by  $(t_1, t_2, t_3)$ . See [9] for general background on relative multipliers.

**Theorem 1.9.** The unimodular relative Fourier multiplier  $(t_1, t_2, t_3)$  has norm

$$\cos((\pi - \tau)/2D) / \cos(\pi/2D),$$

where  $\tau$  is defined as in Theorem 1.1 and  $D$  is given by (4), and attains its norm exactly at functions of the form

$$r_1 e^{i(u_1 + \lambda_1 x)} + r_2 e^{i(u_2 + \lambda_2 x)} + r_3 e^{i(u_3 + \lambda_3 x)}$$

with  $r_1 : r_2 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_3 - \lambda_1| : |\lambda_2 - \lambda_1|$  and

$$\frac{\lambda_2 - \lambda_3}{d} u_1 + \frac{\lambda_3 - \lambda_1}{d} u_2 + \frac{\lambda_1 - \lambda_2}{d} u_3 = \pi \pmod{2\pi}.$$

The maximum of the norm of unimodular relative Fourier multipliers is the **complex unconditional constant** of the canonical basis  $(e_{\lambda_1}, e_{\lambda_2}, e_{\lambda_3})$  of  $\mathcal{C}_\Lambda$ . As

$$r_1 + r_2 + r_3 = \max_x |r_1 e^{i\lambda_1 x} + r_2 e^{i\lambda_2 x} + r_3 e^{i\lambda_3 x}|,$$

this constant is the minimal constant  $C$  such that

$$r_1 + r_2 + r_3 \leq C \max_x |r_1 e^{i(u_1 + \lambda_1 x)} + r_2 e^{i(u_2 + \lambda_2 x)} + r_3 e^{i(u_3 + \lambda_3 x)}|$$

and is therefore the **Sidon constant** of  $\Lambda$ . It is also the solution to the following extremal problem.

**Extremal problem 1.10** (Sidon constant problem). Find the minimum of the maximum modulus of a trigonometric polynomial with given spectrum and Fourier coefficient moduli sum.

Setting  $\tau = \pi$  in Theorem 1.9, we obtain the following result.

**Corollary 1.11.** *The Sidon constant of  $\Lambda$  is  $\sec(\pi/2D)$ , where  $D$  is given by (4). It is attained exactly at functions of the form given in Theorem 1.9.*

Finally, we would like to stress that each of the above results gives rise to open questions if the set  $\Lambda$  is replaced by *any* set of four integers.

Let us now give a brief description of this article. In Sections 2 and 3, we use carefully the invariance of the maximum modulus under rotation, translation and conjugation to reduce the arguments  $t_1, t_2$  and  $t_3$  of the Fourier coefficients of the trigonometric trinomial  $T$  to the variable  $\tau$ . Section 4 shows how further to reduce this study to the trigonometric trinomial

$$(5) \quad r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}$$

with  $k$  and  $l$  positive coprime integers and  $t \in [0, \pi/(k + l)]$ . In Section 5, we prove that (5) attains its maximum modulus for  $x \in [-t/k, t/l]$ . Section 6 studies the variations of the modulus of (5) for  $x \in [-t/k, t/l]$ . It turns out that it attains its absolute maximum only once on that interval; this yields Theorem 1.1. Section 7 restates the results of the two previous sections for a general trigonometric trinomial  $T$ . Section 8 is dedicated to the proof of Theorem 1.2. In Section 9, we compute the directional derivative of the maximum modulus of (5) with respect to the argument  $t$  and prove Theorems 1.4, 1.5, 1.6 and 1.8. In Section 10, we prove Theorem 1.9 and show how to lift unimodular relative Fourier multipliers to operators of convolution with a linear combination of two Dirac measures. Section 11 replaces our computation of the Sidon constant in a general context and describes the initial motivation for this research.

Some of these results appeared previously, with a different proof, in [19, II.10] and in [17].

**Notation.** Throughout this article,  $\lambda_1, \lambda_2$  and  $\lambda_3$  are three distinct integers,  $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$  and  $d = \gcd(\lambda_2 - \lambda_1, \lambda_3 - \lambda_2)$ . If  $\lambda$  is an integer,  $e_\lambda$  is the function  $x \mapsto e^{i\lambda x}$  of the real variable  $x$ . A *trigonometric polynomial* is a linear combination of functions  $e_\lambda$ ; it is a *monomial*, *binomial* or *trinomial* if this linear combination has one, two or three nonzero coefficients, respectively. The normed space  $\mathcal{C}_\Lambda$  is the three-dimensional space of complex functions spanned by  $e_\lambda$  with  $\lambda \in \Lambda$ , endowed with the maximum modulus norm. The Dirac measure  $\delta_x$  is the linear functional  $T \mapsto T(x)$  of evaluation at  $x$  on the space of continuous functions. Given three real numbers  $t_1, t_2$  and  $t_3$ , the linear operator on  $\mathcal{C}_\Lambda$  defined by  $e_{\lambda_j} \mapsto e^{it_j} e_{\lambda_j}$  is a *unimodular relative Fourier multiplier* denoted by  $(t_1, t_2, t_3)$ .

## 2 Isometric relative Fourier multipliers

The rôle of (3) is explained by the following lemma.

**Lemma 2.1.** *Let  $t_1, t_2$  and  $t_3$  be real numbers. The unimodular relative Fourier multiplier  $M = (t_1, t_2, t_3)$  is an isometry on  $\mathcal{C}_\Lambda$  if and only if*

$$(6) \quad \frac{\lambda_2 - \lambda_3}{d}t_1 + \frac{\lambda_3 - \lambda_1}{d}t_2 + \frac{\lambda_1 - \lambda_2}{d}t_3 \in 2\pi\mathbb{Z},$$

*i.e., it is a unimodular multiple of a translation: there are real numbers  $\alpha$  and  $v$  such that  $Mf(x) = e^{i\alpha} f(x - v)$  for all  $f \in \mathcal{C}_\Lambda$  and all  $x \in \mathbb{R}$ .*

**Proof.** If  $M$  is a unimodular multiple of a translation by a real number  $v$ , then

$$\left| r_1 e^{i(t_1 + \lambda_1 v)} + r_2 e^{i(t_2 + \lambda_2 v)} + r_3 e^{i(t_3 + \lambda_3 v)} \right| = r_1 + r_2 + r_3,$$

which holds if and only if

$$(7) \quad t_1 + \lambda_1 v = t_2 + \lambda_2 v = t_3 + \lambda_3 v \quad \text{modulo } 2\pi.$$

There exists  $v$  satisfying (7) if and only if equation (6) holds, as (7) means that there exist integers  $a_1$  and  $a_3$  such that

$$v = \frac{t_2 - t_1 + 2\pi a_1}{\lambda_1 - \lambda_2} = \frac{t_2 - t_3 + 2\pi a_3}{\lambda_3 - \lambda_2}.$$

If  $t_1, t_2$  and  $t_3$  are three real numbers satisfying (6), let  $v$  be such that (7) holds. Then

$$\begin{aligned} & r_1 e^{i(t_1 + u_1 + \lambda_1 x)} + r_2 e^{i(t_2 + u_2 + \lambda_2 x)} + r_3 e^{i(t_3 + u_3 + \lambda_3 x)} \\ &= e^{i(t_2 + \lambda_2 v)} \left( r_1 e^{i(u_1 + \lambda_1(x-v))} + r_2 e^{i(u_2 + \lambda_2(x-v))} + r_3 e^{i(u_3 + \lambda_3(x-v))} \right) \end{aligned}$$

for all real numbers  $u_1, u_2, u_3$  and  $x$ .

### 3 The arguments of the Fourier coefficients of a trigonometric trinomial

We have used a translation and a rotation to reduce the three arguments of the Fourier coefficients of a trigonometric trinomial to a single variable. Use of the involution  $\overline{f(-x)}$  of  $\mathcal{C}_\Lambda$  allows us to restrict even further the domain of that variable.

**Lemma 3.1.** *Let  $t_1, t_2$  and  $t_3$  be real numbers and let  $\tilde{t}_2$  be the representative of*

$$(8) \quad \frac{\lambda_2 - \lambda_3}{\lambda_3 - \lambda_1}t_1 + t_2 + \frac{\lambda_1 - \lambda_2}{\lambda_3 - \lambda_1}t_3$$

*modulo  $2\pi/|\lambda_3 - \lambda_1|$  in  $[-\pi d/|\lambda_3 - \lambda_1|, \pi d/|\lambda_3 - \lambda_1|]$ .*

(a) *There are real numbers  $\alpha$  and  $v$  such that*

$$(9) \quad r_1 e^{i(t_1 + \lambda_1 x)} + r_2 e^{i(t_2 + \lambda_2 x)} + r_3 e^{i(t_3 + \lambda_3 x)} \\ = e^{i\alpha} (r_1 e^{i\lambda_1(x-v)} + r_2 e^{i(\tilde{t}_2 + \lambda_2(x-v))} + r_3 e^{i\lambda_3(x-v)})$$

*for all  $x$ .*

(b) *Let  $t = |\tilde{t}_2|$  be the distance of (8) to  $(2\pi d/|\lambda_3 - \lambda_1|)\mathbb{Z}$ . There is a sign  $\epsilon \in \{+1, -1\}$  such that*

$$|r_1 e^{i(t_1 + \lambda_1 x)} + r_2 e^{i(t_2 + \lambda_2 x)} + r_3 e^{i(t_3 + \lambda_3 x)}| \\ = |r_1 e^{i\lambda_1 \epsilon(x-v)} + r_2 e^{i(t + \lambda_2 \epsilon(x-v))} + r_3 e^{i\lambda_3 \epsilon(x-v)}|$$

*for all  $x$ .*

**Proof.** (a). The argument  $\tilde{t}_2$  is chosen so that the relative multiplier  $(t_1, t_2 - \tilde{t}_2, t_3)$  is an isometry.

(b). If  $\tilde{t}_2$  is negative, take the conjugate under the modulus of the right hand side in (9).

**Remark 3.2.** This proves the following periodicity formula:

$$|r_1 e^{i\lambda_1 x} + r_2 e^{i(t + 2\pi d/(\lambda_3 - \lambda_1) + \lambda_2 x)} + r_3 e^{i\lambda_3 x}| \\ = |r_1 e^{i\lambda_1(x-v)} + r_2 e^{i(t + \lambda_2(x-v))} + r_3 e^{i\lambda_3(x-v)}|$$

for all  $x$  and  $t$ , where  $v$  satisfies  $\lambda_1 v = 2\pi d/(\lambda_3 - \lambda_1) + \lambda_2 v = \lambda_3 v$  modulo  $2\pi$ , that is,

$$v = \frac{2m\pi}{\lambda_3 - \lambda_1}, \quad \text{with } m \text{ an inverse of } \frac{\lambda_3 - \lambda_2}{d} \text{ modulo } \frac{\lambda_3 - \lambda_1}{d}.$$



### 4 The frequencies of a trigonometric trinomial

We may suppose without loss of generality that  $\lambda_1 < \lambda_2 < \lambda_3$ . Let  $k = (\lambda_2 - \lambda_1)/d$  and  $l = (\lambda_3 - \lambda_2)/d$ . Then

$$|r_1 e^{i\lambda_1 x} + r_2 e^{i(t+\lambda_2 x)} + r_3 e^{i\lambda_3 x}| = |r_1 e^{-ik(dx)} + r_2 e^{it} + r_3 e^{il(dx)}|.$$

A homothety by  $d^{-1}$  allows us to restrict our attention to the function

$$f(t, x) = |r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}|^2$$

for  $x \in \mathbb{R}$ , with  $k$  and  $l$  two positive coprime numbers and  $t \in [0, \pi/(k + l)]$ . We have

$$(10) \quad f(-t, x) = f(t, -x)$$

and Remark 3.2 shows that

$$(11) \quad f(t + 2\pi/(k + l), x) = f(t, x - 2m\pi/(k + l))$$

for all  $x$  and  $t$ , where  $m$  is the inverse of  $l$  modulo  $k + l$ . In particular, if  $t = \pi/(k + l)$ , we have the symmetry relation

$$(12) \quad f(\pi/(k + l), x) = f(\pi/(k + l), 2m\pi/(k + l) - x).$$

### 5 Location of the maximum point

The purpose of our first proposition is to deduce the existence of a small interval on which a trigonometric trinomial attains its maximum modulus. Note that a trigonometric binomial attains its maximum modulus at a point that depends only on the phase of its coefficients:

- $|r_1 e^{-ikx} + r_2 e^{it}|$  attains its maximum at  $-t/k$  independently of  $r_1$  and  $r_2$ ,
- $|r_1 e^{-ikx} + r_3 e^{ilx}|$  attains its maximum at 0 independently of  $r_1$  and  $r_3$ ,
- $|r_2 e^{it} + r_3 e^{ilx}|$  attains its maximum at  $t/l$  independently of  $r_2$  and  $r_3$ .

The next proposition shows that if the point at which a trigonometric trinomial attains its maximum modulus changes with the intensities (i.e., moduli) of its coefficients, it changes very little; we get bounds for this point which are independent of the intensities.

**Proposition 5.1.** *Let  $k$  and  $l$  be positive coprime integers; let  $r_1, r_2$  and  $r_3$  be three positive real numbers; and let  $t \in [0, \pi/(k + l)]$ . Set*

$$f(x) = |r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}|^2$$

for  $x \in \mathbb{R}$ .

- (a) *The function  $f$  attains its absolute maximum in the interval  $[-t/k, t/l]$ .*  
 (b) *If  $f$  attains its absolute maximum at a point  $y$  outside of  $[-t/k, t/l]$  modulo  $2\pi$ , then  $t = \pi/(k+l)$  and  $2m\pi/(k+l) - y$  lies in  $[-t/k, t/l]$  modulo  $2\pi$ , where  $m$  is the inverse of  $l$  modulo  $k+l$ .*

**Proof.** (a). We have

$$(13) \quad f(x) = r_1^2 + r_2^2 + r_3^2 + 2 \cdot (r_1 r_2 \cos(t + kx) + r_1 r_3 \cos((k+l)x) + r_2 r_3 \cos(t - lx)).$$

Let us prove that  $f$  attains its absolute maximum on  $[-t/k, t/l]$ . Let  $y$  be outside of  $[-t/k, t/l]$  modulo  $2\pi$ . Let  $I$  be the set of all  $x \in [-t/k, t/l]$  such that

$$\begin{cases} \cos(t + kx) \geq \cos(t + ky) \\ \cos((k+l)x) \geq \cos((k+l)y) \\ \cos(t - lx) \geq \cos(t - ly). \end{cases}$$

Note that if  $x \in [-t/k, t/l]$ , then

$$\begin{cases} t + kx \in [0, (k+l)t/l] \\ (k+l)x \in [-(k+l)t/k, (k+l)t/l] \\ t - lx \in [0, (k+l)t/k], \end{cases}$$

and that  $(k+l)t/k, (k+l)t/l \in [0, \pi]$ . Let

- $\alpha$  be the distance of  $t/k + y$  to  $(2\pi/k)\mathbb{Z}$ ,
- $\beta$  be the distance of  $y$  to  $(2\pi/(k+l))\mathbb{Z}$ ,
- $\gamma$  be the distance of  $t/l - y$  to  $(2\pi/l)\mathbb{Z}$ .

Then

$$(14) \quad I = [-t/k, t/l] \cap [-t/k - \alpha, -t/k + \alpha] \cap [-\beta, \beta] \cap [t/l - \gamma, t/l + \gamma].$$

Let us check that  $I$  is the nonempty interval

$$(15) \quad I = [\max(-t/k, -\beta, t/l - \gamma), \min(t/l, -t/k + \alpha, \beta)].$$

In fact, we have the following triangular inequalities:

- $-\beta \leq -t/k + \alpha$ , because  $t/k$  is the distance of  $(t/k + y) - y$  to  $(2\pi/k(k+l))\mathbb{Z}$ ;
- $t/l - \gamma \leq -t/k + \alpha$ , because  $t/l + t/k$  is the distance of  $(t/k + y) + (t/l - y)$  to  $(2\pi/k)\mathbb{Z}$ ;
- $t/l - \gamma \leq \beta$ , because  $t/l$  is the distance of  $(t/l - y) + y$  to  $(2\pi/l(k+l))\mathbb{Z}$ .

The other six inequalities that are necessary to deduce (15) from (14) are obvious.

(b). We have proved in (a) that there exists  $x \in [-t/k, t/l]$  such that  $\cos(t+kx) \geq \cos(t+ky)$ ,  $\cos((k+l)x) \geq \cos((k+l)y)$  and  $\cos(t-lx) \geq \cos(t-ly)$ . In fact, at least one of these inequalities is strict unless there are signs  $\delta, \epsilon, \eta \in \{-1, 1\}$  such that  $t+kx = \delta(t+ky)$ ,  $t-lx = \epsilon(t-ly)$  and  $(k+l)x = \eta(k+l)y$  modulo  $2\pi$ . Two of these three signs are equal, and the corresponding two equations imply the third one with the same sign. This system is therefore equivalent to

$$\begin{cases} k(x-y) = 0 \\ l(x-y) = 0 \end{cases} \quad \text{or} \quad \begin{cases} k(x+y) = -2t \\ l(x+y) = 2t \end{cases}$$

modulo  $2\pi$ . The first pair of equations yields  $x = y$  modulo  $2\pi$  because  $k$  and  $l$  are coprime. Let  $m$  be an inverse of  $l$  modulo  $k+l$ ; then the second pair of equations is equivalent to

$$\begin{cases} 2(k+l)t = 0 \\ x+y = 2mt \end{cases}$$

modulo  $2\pi$ . Therefore,  $g$  does not attain its absolute maximum at  $y$  unless  $t = \pi/(k+l)$  and  $2m\pi/(k+l) - y \in [-t/k, t/l]$ .

**Remark 5.2.** This proposition is a complex counterpart to Lemma 2.1. *i*) in [21], where cosine trinomials are investigated.

## 6 Uniqueness of the maximum point

Note that

$$\begin{aligned} r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx} &= r_3 e^{-il(-x)} + r_2 e^{it} + r_1 e^{ik(-x)} \\ &= r'_1 e^{-ik'x'} + r_2 e^{it} + r'_3 e^{il'x'} \end{aligned}$$

with  $r'_1 = r_3$ ,  $r'_3 = r_1$ ,  $k' = l$ ,  $l' = k$  and  $x' = -x$ . We may therefore suppose without loss of generality that  $kr_1 \leq lr_3$ .

Our second proposition studies the points at which a trigonometric trinomial attains its maximum modulus. Note that if  $k = l = 1$ , the derivative of  $|f|^2$  has at most 4 zeros, so that the modulus of  $f$  has at most two maxima and attains its absolute maximum in at most two points. Proposition 6.1 shows that this is true in general and that if it attains its absolute maximum in two points, it is so only because of the symmetry given by (12).

**Proposition 6.1.** *Let  $k$  and  $l$  be positive coprime integers; let  $r_1, r_2$  and  $r_3$  be three positive real numbers such that  $kr_1 \leq lr_3$ ; and let  $t \in ]0, \pi/(k+l)[$ . Set*

$$f(x) = |r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}|^2$$

for  $x \in [-t/k, t/l]$ .

- (a) *There is a point  $x^* \in [0, t/l]$  such that  $df/dx > 0$  on  $] -t/k, x^*$  [ and  $df/dx < 0$  on  $] x^*, t/l$  [.*
- (b) *There are three cases:*
  1.  *$f$  attains its absolute maximum at 0 if and only if  $kr_1 = lr_3$ ;*
  2.  *$f$  attains its absolute maximum at  $t/l$  if and only if  $l = 1, t = \pi/(k + 1)$  and  $k^2r_1r_2 + (k + 1)^2r_1r_3 - r_2r_3 \leq 0$ ;*
  3. *otherwise,  $f$  attains its absolute maximum in  $]0, t/l$  [.*
- (c) *The function  $f$  attains its absolute maximum with multiplicity 2 unless  $l = 1, t = \pi/(k + 1)$  and  $k^2r_1r_2 + (k + 1)^2r_1r_3 - r_2r_3 = 0$ , in which case it attains its absolute maximum at  $\pi/(k + 1)$  with multiplicity 4.*

**Proof.** (a). By Proposition 5.1, the derivative of  $f$  has a zero in  $[-t/k, t/l]$ . Let us study the sign of this derivative. Equation (13) yields

$$(16) \quad \frac{1}{2} \frac{df}{dx}(x) = -kr_1r_2 \sin(t + kx) - (k + l)r_1r_3 \sin((k + l)x) + lr_2r_3 \sin(t - lx).$$

We wish to compare  $\sin(t + kx)$  with  $\sin(t - lx)$ . Note that

$$\sin(t + kx) - \sin(t - lx) = 2 \sin((k + l)x/2) \cos(t + (k - l)x/2),$$

and that if  $x \in [-t/k, t/l]$ , then

$$\begin{aligned} -\pi/2 &\leq -\pi/2k \leq -(k + l)t/2k \leq (k + l)x/2 \leq (k + l)t/2l \leq \pi/2l \leq \pi/2 \\ 0 &\leq t + (k - l)x/2 \leq \begin{cases} t + (l - k)t/2k = (k + l)t/2k & \text{if } k \leq l \\ t + (k - l)t/2l = (k + l)t/2l & \text{if } l \leq k \end{cases} \leq \pi/2. \end{aligned}$$

Suppose that  $x \in [-t/k, 0[$ . Then it follows that  $\sin(t + kx) \leq \sin(t - lx)$  and also  $\sin((k + l)x) \leq 0$ , with equality if and only if  $k = 1$  and  $-x = t = \pi/(1 + l)$ . Since  $kr_1 \leq lr_3$ , this yields

$$(17) \quad \frac{1}{2} \frac{df}{dx}(x) \geq -(k + l)r_1r_3 \sin((k + l)x) \geq 0,$$

with equality if and only if  $k = 1$  and  $-x = t = \pi/(1 + l)$ .

Suppose that  $x \in [0, t/l]$ . If  $l \geq 2$ , then

$$\begin{cases} t + kx \in [t, (k + l)t/l] \subset [t, \pi/2] \\ (k + l)x \in [0, (k + l)t/l] \subset [t, \pi/2] \\ t - lx \in [0, t] \subset [0, \pi/3], \end{cases}$$

so that the second derivative of  $f$  is strictly negative on  $[0, t/l]$ ; its derivative is strictly decreasing on this interval, and (a) is proved. If  $l = 1$ , let  $g(x) = f(t - x)$

for  $x \in [0, t]$ . We have to prove that there is a point  $x^*$  such that  $dg/dx > 0$  on  $]0, x^*[$  and  $dg/dx < 0$  on  $]x^*, t[$ . Put  $\alpha = (k + 1)t$ ; then

$$\frac{1}{2} \frac{dg}{dx}(x) = kr_1r_2 \sin(\alpha - kx) + (k + 1)r_1r_3 \sin(\alpha - (k + 1)x) - r_2r_3 \sin x,$$

and it suffices to prove that

$$(18) \quad \frac{1}{2 \sin x} \frac{dg}{dx}(x) = kr_1r_2 \frac{\sin(\alpha - kx)}{\sin x} + (k + 1)r_1r_3 \frac{\sin(\alpha - (k + 1)x)}{\sin x} - r_2r_3$$

is a strictly decreasing function of  $x$  on  $]0, \alpha/(k + 1)[$ . Let us study the sign of

$$\frac{d}{dx} \frac{\sin(\alpha - kx)}{\sin x} = \frac{-k \cos(\alpha - kx) \sin x - \sin(\alpha - kx) \cos x}{\sin^2 x}$$

for  $\alpha \in ]0, \pi]$  and  $x \in ]0, \alpha/k[$ . If  $k = 1$ , then

$$-k \cos(\alpha - kx) \sin x - \sin(\alpha - kx) \cos x = -\sin \alpha \leq 0,$$

and the inequality is strict unless  $\alpha = \pi$ . We conclude the proof of (a) by showing by induction on  $k$  that

$$k \cos(\alpha - kx) \sin x + \sin(\alpha - kx) \cos x > 0$$

for all  $k \geq 2$ ,  $\alpha \in ]0, \pi]$  and  $x \in ]0, \alpha/k[$ . Let  $k \geq 1$  and  $x \in ]0, \alpha/(k + 1)[$ . Then

$$\begin{aligned} & (k + 1) \cos(\alpha - (k + 1)x) \sin x + \sin(\alpha - (k + 1)x) \cos x \\ &= (k + 1) \cos(\alpha - kx) \cos x \sin x + (k + 1) \sin(\alpha - kx) \sin^2 x \\ & \quad + \sin(\alpha - kx) \cos^2 x - \cos(\alpha - kx) \sin x \cos x \\ &= (k \cos(\alpha - kx) \sin x + \sin(\alpha - kx) \cos x) \cos x + (k + 1) \sin(\alpha - kx) \sin^2 x \\ & \geq (k + 1) \sin(\alpha - kx) \sin^2 x > 0 \end{aligned}$$

(b). 1. By Proposition 5.1 and (a),  $f$  attains its absolute maximum at 0 if and only if 0 is a critical point for  $f$ . We have

$$\frac{1}{2} \frac{df}{dx}(0) = (lr_3 - kr_1)r_2 \sin t \geq 0,$$

and equality holds if and only if  $kr_1 = lr_3$ .

2. We have

$$\frac{1}{2} \frac{df}{dx}(t/l) = (-kr_1r_2 - (k + l)r_1r_3) \sin((k + l)t/l) \leq 0,$$

and equality holds if and only if  $l = 1$  and  $t = \pi/(k + 1)$ . Let  $l = 1$  and  $t = \pi/(k + 1)$ . Using the notation introduced in the last part of the proof of (a), we need to characterise the case in which  $g$  has a maximum at 0. As  $\alpha = \pi$  and

$$(19) \quad \frac{1}{2 \sin x} \frac{dg}{dx}(x) = k^2r_1r_2 + (k + 1)^2r_1r_3 - r_2r_3 + o(x)$$

is a strictly decreasing function of  $x$  on  $]0, \pi/(k + 1)[$ ,  $g$  has a maximum at 0 if and only if  $k^2r_1r_2 + (k + 1)^2r_1r_3 - r_2r_3 \leq 0$ .

(c). If  $l \geq 2$ , the second derivative of  $f$  is strictly negative on  $[0, t/l]$ . If  $l = 1$ , the derivative of (18) is strictly negative on  $]0, \alpha/(k + 1)[$ ; it follows that the second derivative of  $g$  can vanish only at 0. By (b) 2,  $g$  has a maximum at 0 only if  $t = \pi/(k + 1)$ ; then

$$(20) \quad \frac{1}{2} \frac{d^2g}{dx^2}(0) = k^2r_1r_2 + (k + 1)^2r_1r_3 - r_2r_3$$

$$(21) \quad \frac{1}{2} \frac{d^4g}{dx^4}(0) = -k^4r_1r_2 - (k + 1)^4r_1r_3 + r_2r_3.$$

If (20) vanishes, then the sum of (21) with (20) yields

$$\frac{1}{2} \frac{d^4g}{dx^4}(0) = -k(k + 1)r_1((k - 1)kr_2 + (k + 1)(k + 2)r_3) < 0.$$

**Remark 6.2.** We were able to prove directly that the system

$$\begin{cases} f(x) = f(y) \\ \frac{df}{dx}(x) = \frac{df}{dx}(y) = 0 \\ \frac{d^2f}{dx^2}(x), \frac{d^2f}{dx^2}(y) \leq 0 \end{cases}$$

implies  $x = y$  modulo  $2\pi$  or  $t = \pi/(k + l)$  and  $x + y = 2m\pi/(k + l)$ , but our computations are very involved and opaque.

**Remark 6.3.** This proposition is a complex counterpart to Lemma 2.1. ii) in [21].

**Remark 6.4.** Suppose that  $l = k = 1$ . If  $t \in ]0, \pi/2[$ , it is necessary to solve a generally irreducible quartic equation in order to compute the maximum of  $f$ . If  $t = \pi/2$ , it suffices to solve a linear equation and one gets the following expression for  $\max_x |r_1e^{-ix} + ir_2 + r_3e^{ix}|$ :

$$\begin{cases} (r_1 + r_3)\sqrt{1 + r_2^2/4r_1r_3} & \text{if } |r_1^{-1} - r_3^{-1}| < 4r_2^{-1} \\ r_2 + |r_3 - r_1| & \text{otherwise.} \end{cases}$$

This formula appears in [1, (3.1)]. In the first case, the maximum is attained at the two points  $x^*$  such that  $\sin x^* = r_2(r_3 - r_1)/4r_1r_3$ .

**Remark 6.5.** Suppose that  $l = 1$  and  $k = 2$ . If  $t \in ]0, \pi/3[$ , it is necessary to solve a generally irreducible sextic equation in order to compute the maximum of  $f$ .

If  $t = \pi/3$ , it suffices to solve a quadratic equation and one gets the following expression for  $\max_x |r_1 e^{-i2x} + r_2 e^{i\pi/3} + r_3 e^{ix}|$  : if  $r_1^{-1} - 4r_3^{-1} < 9r_2^{-1}$ , then its square is equal to

$$r_1^2 + \frac{2}{3}r_2^2 + r_3^2 + r_1 r_2 + 2r_1 r_3 \left[ \left( \left( \frac{r_2}{3r_3} \right)^2 + \frac{r_2}{3r_1} + 1 \right)^{3/2} - \left( \frac{r_2}{3r_3} \right)^3 \right]$$

and the maximum is attained at the two points  $x^*$  such that

$$2 \cos(\pi/3 - x^*) = \left( \left( \frac{r_2}{3r_3} \right)^2 + \frac{r_2}{3r_1} + 1 \right)^{1/2} - \frac{r_2}{3r_3};$$

otherwise, it is equal to  $-r_1 + r_2 + r_3$ .

### 7 The maximum modulus points of a trigonometric trinomial

If we undo all the reductions made in Sections 3 and 4 and at the beginning of Section 6, we get the following theorem.

**Theorem 7.1.** *Let  $\lambda_1, \lambda_2$  and  $\lambda_3$  be distinct integers such that  $\lambda_2$  is between  $\lambda_1$  and  $\lambda_3$ . Let  $r_1, r_2$  and  $r_3$  be positive real numbers. Given real numbers  $t_1, t_2$  and  $t_3$ , consider the trigonometric trinomial*

$$(22) \quad T(x) = r_1 e^{i(t_1 + \lambda_1 x)} + r_2 e^{i(t_2 + \lambda_2 x)} + r_3 e^{i(t_3 + \lambda_3 x)}$$

for  $x \in \mathbb{R}$ . Let  $d = \gcd(\lambda_2 - \lambda_1, \lambda_3 - \lambda_2)$  and choose integers  $a_1$  and  $a_3$  such that

$$\tau = \frac{\lambda_2 - \lambda_3}{d} (t_1 - 2\pi a_1) + \frac{\lambda_3 - \lambda_1}{d} t_2 + \frac{\lambda_1 - \lambda_2}{d} (t_3 - 2\pi a_3)$$

satisfies  $|\tau| \leq \pi$ . Let  $\tilde{t}_1 = t_1 - 2\pi a_1$  and  $\tilde{t}_3 = t_3 - 2\pi a_3$ .

(a) *The trigonometric trinomial  $T$  attains its maximum modulus at a unique point of the interval bounded by  $(\tilde{t}_1 - t_2)/(\lambda_2 - \lambda_1)$  and  $(t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$ . More precisely,*

- *if  $r_1|\lambda_2 - \lambda_1| \leq r_3|\lambda_3 - \lambda_2|$ , then this point lies between  $(\tilde{t}_1 - \tilde{t}_3)/(\lambda_3 - \lambda_1)$  and  $(t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$ ;*
- *if  $r_1|\lambda_2 - \lambda_1| \geq r_3|\lambda_3 - \lambda_2|$ , then this point lies between  $(\tilde{t}_1 - \tilde{t}_3)/(\lambda_3 - \lambda_1)$  and  $(\tilde{t}_1 - t_2)/(\lambda_2 - \lambda_1)$ ;*
- *$T$  attains its maximum modulus at  $(\tilde{t}_1 - \tilde{t}_3)/(\lambda_3 - \lambda_1)$  if and only if  $r_1|\lambda_2 - \lambda_1| = r_3|\lambda_3 - \lambda_2|$  or  $\tau = 0$ .*

(b) *The function  $T$  attains its maximum modulus at a unique point modulo  $2\pi/d$ , and with multiplicity 2, unless  $|\tau| = \pi$ .*

(c) Suppose that  $|\tau| = \pi$ , i. e.,

$$(23) \quad \frac{\lambda_2 - \lambda_3}{d}t_1 + \frac{\lambda_3 - \lambda_1}{d}t_2 + \frac{\lambda_1 - \lambda_2}{d}t_3 = \pi \pmod{2\pi}.$$

Let  $s$  be a solution to  $2t_1 + \lambda_1 s = 2t_2 + \lambda_2 s = 2t_3 + \lambda_3 s$  modulo  $2\pi$ ;  $s$  is unique modulo  $2\pi/d$ . Then  $T(s - x) = e^{i(2t_2 + \lambda_2 s)} \overline{T(x)}$  for all  $x$ . Suppose that  $|\lambda_3 - \lambda_2| \leq |\lambda_2 - \lambda_1|$ . There are three cases:

1. if  $\lambda_2 - \lambda_1 = k(\lambda_3 - \lambda_2)$  with  $k \geq 2$  integral and

$$r_1^{-1} - k^2 r_3^{-1} \geq (k + 1)^2 r_2^{-1},$$

then  $T$  attains its maximum modulus  $-r_1 + r_2 + r_3$  only at  $x = (t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$  modulo  $2\pi/d$ , with multiplicity 2 if the inequality is strict and with multiplicity 4 if there is equality;

2. if  $\lambda_2 - \lambda_1 = \lambda_3 - \lambda_2$  and

$$|r_1^{-1} - r_3^{-1}| \geq 4r_2^{-1},$$

then  $T$  attains its maximum modulus  $r_2 + |r_3 - r_1|$  at a unique point  $x$  modulo  $2\pi/d$ , with multiplicity 2 if the inequality is strict and with multiplicity 4 if there is equality. This point is  $(t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$  if  $r_1 < r_3$ , and  $(\tilde{t}_1 - t_2)/(\lambda_2 - \lambda_1)$  if  $r_3 < r_1$ ;

3. otherwise,  $T$  attains its maximum modulus at exactly two points  $x$  and  $y$  modulo  $2\pi/d$ , with multiplicity 2, where  $x$  lies strictly between  $(\tilde{t}_1 - t_2)/(\lambda_2 - \lambda_1)$  and  $(t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$ , and  $x + y = s$  modulo  $2\pi/d$ .

Note that  $s - x = x$  modulo  $2\pi/d$  in Cases 1 and 2.

## 8 Exposed and extreme points of the unit ball of $\mathcal{C}_\Lambda$

The characterisation of the maximum modulus points of a trigonometric trinomial allows us to compute the exposed and the extreme points of the unit ball of  $\mathcal{C}_\Lambda$ .

**Lemma 8.1.** (a) A trigonometric trinomial with a given spectrum that attains its maximum modulus at two given points modulo  $2\pi/d$  is determined by its value at these points.

(b) The trigonometric trinomials with a given spectrum that attain their maximum modulus with multiplicity 4 at a given point and have a given value at this point lie on a parabola.

**Proof.** We use the notation of Theorem 7.1. Without loss of generality, we may suppose that  $\lambda_1 = -k$ ,  $\lambda_2 = 0$  and  $\lambda_3 = l$  with  $k$  and  $l$  positive coprime



integers. Let  $x$  and  $y$  be real numbers that are different modulo  $2\pi/d$ , let  $\vartheta$  and  $\zeta$  be real numbers and let  $\varrho$  be a positive real number.

(a) Let us prove that at most one trigonometric trinomial  $T$  attains its maximum modulus at  $x$  and  $y$  and satisfies  $T(x) = \varrho e^{i\vartheta}$  and  $T(y) = \varrho e^{i\zeta}$ . Translating  $T$  by  $(x + y)/2$ , we may suppose that  $x + y = 0$ . Dividing  $T$  by  $e^{i(\vartheta+\zeta)/2}$ , we may suppose that  $\vartheta + \zeta = 0$ . As  $T$  attains its maximum modulus at the two points  $x$  and  $y$ , we have  $s = x + y = 0$  and  $2t_1 - ks = 2t_2 = 2t_3 + ls = \vartheta + \zeta = 0$  modulo  $2\pi$ . Therefore,  $t_1 = t_2 = t_3 = 0$  modulo  $\pi$ . Let  $p_j = e^{it_j} r_j$ , then the  $p_j$  are non-zero real numbers. We have

$$T(x) = p_1 e^{-ikx} + p_2 + p_3 e^{ilx} = \varrho e^{i\vartheta},$$

so that, multiplying by  $e^{-i\vartheta}$  and taking real and imaginary parts, we have

$$(24) \quad p_1 \cos(\vartheta + kx) + p_3 \cos(\vartheta - lx) = \varrho - p_2 \cos \vartheta$$

$$(25) \quad p_1 \sin(\vartheta + kx) + p_3 \sin(\vartheta - lx) = -p_2 \sin \vartheta.$$

The computation

$$\frac{1}{2} \frac{d|T|^2}{dx}(x) = \Re\left(\overline{T(x)} \frac{dT}{dx}(x)\right) = \Re\left(\overline{T(x)}(-ikp_1 e^{-ikx} + ilp_3 e^{ilx})\right)$$

yields

$$(26) \quad kp_1 \sin(\vartheta + kx) - lp_3 \sin(\vartheta - lx) = 0.$$

Equations (25) and (26) yield  $p_1$  and  $p_3$  as linear functions of  $p_2$  because  $\sin(\vartheta + kx) \sin(\vartheta - lx) \neq 0$ ; otherwise, both factors would vanish, so that  $\vartheta = x = 0$  modulo  $\pi$  and  $x = y$  modulo  $2\pi$ . As  $\varrho \neq 0$ , Equation (24) has at most one solution in  $p_2$ .

(b) We are necessarily in Case 1 or 2 of Theorem 7.1(c), so that we may suppose that  $l = 1$ . Let us determine all trigonometric trinomials  $T$  that attain their maximum modulus at  $x$  with multiplicity 4 and satisfy  $T(x) = \varrho e^{i\vartheta}$ . Translating  $T$  by  $x$ , we may suppose that  $x = 0$ . Dividing  $T$  by  $e^{i\vartheta}$ : we may suppose that  $\vartheta = 0$ . As  $T$  attains its maximum modulus at 0 with multiplicity 4, we have  $s - 0 = 0$  and  $2t_1 - ks = 2t_2 = 2t_3 + s = 2\vartheta = 0$  modulo  $2\pi$ . Therefore,  $t_1 = t_2 = t_3 = 0$  modulo  $\pi$ . Let  $p_j = e^{it_j} r_j$ ; then the  $p_j$  are non-zero real numbers which satisfy the system

$$\begin{cases} p_1 + p_2 + p_3 = \varrho \\ k^2 p_1 p_2 + (k + 1)^2 p_1 p_3 + p_2 p_3 = 0, \end{cases}$$

that is

$$\begin{cases} p_2 = \varrho - p_1 - p_3 \\ (kp_1 - p_3)^2 = \varrho(k^2 p_1 + p_3). \end{cases}$$

This is the equation of a parabola.

**Remark 8.2.** The equality

$$(27) \quad \max_x |r_1 e^{i(t_1 + \lambda_1 x)} + r_2 e^{i(t_2 + \lambda_2 x)}| = r_1 + r_2$$

shows that the exposed points of the unit ball of the space  $\overline{\mathcal{C}_{\{\lambda_1, \lambda_2\}}}$  are the trigonometric monomials  $e^{i\alpha} e_{\lambda_1}$  and  $e^{i\alpha} e_{\lambda_2}$  with  $\alpha \in \mathbb{R}$  and that no trigonometric binomial is an extreme point of the unit ball of  $\mathcal{C}_\Lambda$ .

**Proof of Theorem 1.2 (a).** A linear functional on  $\mathcal{C}_\Lambda$  extends to a linear functional on the space of continuous functions, that is, to a measure  $\mu$ , with the same norm. A measure  $\mu$  attains its norm only at functions with constant modulus on its support. If  $\mu$  attains its norm on a trigonometric binomial, then it attains its norm on a trigonometric monomial, because this trigonometric binomial is a convex combination of two trigonometric monomials with same norm by (27). If  $\mu$  attains its norm on a trigonometric trinomial  $T$ , there are two cases by Theorem 7.1 (b, c).

- $T$  attains its maximum modulus at a unique point modulo  $2\pi/d$ . Then the support of  $\mu$  has only one point modulo  $2\pi/d$ , so that  $\mu$  is a multiple of a Dirac measure and attains its norm on any trigonometric monomial;
- $T$  attains its maximum modulus at two points modulo  $2\pi/d$ .

Conversely, the trigonometric monomial  $e^{i\alpha} e_\lambda$  is exposed to the linear form

$$P \mapsto \frac{1}{2\pi} \int_0^{2\pi} P(x) e^{-i(\alpha + \lambda x)} dx.$$

A trigonometric trinomial  $T$  that attains its maximum modulus 1 at two points  $x_1^*$  and  $x_2^*$  modulo  $2\pi/d$  is exposed, by Lemma 8.1 (a), to any non-trivial convex combination of the unimodular multiples of Dirac measures  $\overline{T(x_1^*)} \delta_{x_1^*}$  and  $\overline{T(x_2^*)} \delta_{x_2^*}$ .

**Remark 8.3.** This is a complex counterpart to Lemma 2.3 in [21], dealing with the exposed points of the unit ball of the three-dimensional space spanned by the functions  $1, \cos x$  and  $\cos kx$  in the space of continuous functions.

**Proof of Theorem 1.2 (b).** Let  $K$  be the unit ball of  $\mathcal{C}_\Lambda$ . Straszewicz’s Theorem [26] asserts that the exposed points of  $K$  are dense in the set of its extreme points. Let  $P$  be a limit point of exposed points of  $K$ . If  $P$  is a trigonometric monomial,  $P$  is exposed. If  $P$  is a trigonometric binomial,  $P$  is not an extreme point of  $K$  by Remark 8.2. If  $P$  is a trigonometric trinomial, it is the limit point of trigonometric trinomials that attain their maximum modulus twice modulo  $2\pi/d$ , so that either  $P$  also attains its maximum modulus twice modulo  $2\pi/d$  or, by Rolle’s Theorem,  $P$  attains its maximum modulus with multiplicity 4. Let us prove that if a trigonometric trinomial  $T$  attains its maximum modulus with multiplicity 4 at a

point  $x$ , then  $T$  is an extreme point of  $K$ . Suppose that  $T$  is the midpoint of two points  $A$  and  $B$  in  $K$ . Then  $|A(x)| \leq 1$ ,  $|B(x)| \leq 1$  and  $(A(x) + B(x))/2 = T(x)$ , so that  $A(x) = B(x) = T(x)$ . Furthermore,

$$|T(x + h)| \leq \frac{|A(x + h)| + |B(x + h)|}{2} = 1 + \frac{h^2}{4} \left( \frac{d^2|A|}{dx^2}(x) + \frac{d^2|B|}{dx^2}(x) \right) + o(h^2),$$

so that, as  $|T(x + h)| = 1 + o(h^3)$ ,

$$\frac{d^2|A|}{dx^2}(x) \leq 0, \quad \frac{d^2|B|}{dx^2}(x) \leq 0 \quad \text{and} \quad \frac{d^2|A|}{dx^2}(x) + \frac{d^2|B|}{dx^2}(x) \geq 0$$

and therefore  $A$  and  $B$  also attain their maximum modulus with multiplicity 4 at  $x$ . As this implies that  $A$  and  $B$  are trigonometric trinomials, Lemma 8.1 (b) yields that  $T$ ,  $A$  and  $B$  lie on a parabola. This implies  $A = B = T$ . □

**Remark 8.4.** The set of extreme points of the unit ball of  $\mathcal{C}_\Lambda$  is not closed. For example, if  $\lambda_2$  lies between  $\lambda_1$  and  $\lambda_3$ , every absolutely convex combination of  $e_{\lambda_1}$  and  $e_{\lambda_3}$  is a limit point of exposed points.

**Remark 8.5.** If  $\lambda_2$  is between  $\lambda_1$  and  $\lambda_3$ , and  $\lambda_1 - \lambda_2$  is not a multiple of  $\lambda_3 - \lambda_2$  nor vice versa, then every extreme point of the unit ball of  $\mathcal{C}_\Lambda$  is exposed.

**Remark 8.6.** In particular, compare our description of the extreme points of the unit ball of  $\mathcal{C}_{\{0,1,2\}}$  with the characterisation given by K. M. Dyakonov in [7, Theorem 1]. He shows in his Example 1 that it is false that “in order to recognize the extreme points”, “one only needs to know ‘how often’ [the modulus of a trigonometric polynomial] takes the extremal value 1.” We show that it is true in our elementary framework with the exception of trigonometric binomials.

**Remark 8.7.** This is a complex counterpart to [16], dealing with the extreme points of the unit ball of the three-dimensional space spanned by the functions 1,  $x^n$  and  $x^m$  in the space of real valued continuous functions on  $[-1, 1]$ .

## 9 Dependence of the maximum modulus on the arguments

We wish to study how the maximum modulus of a trigonometric trinomial depends on the phase of its coefficients. We use the following formula, established in [4], which gives an expression for the directional derivative of a maximum function. Elementary properties of maximum functions are addressed in [20, Part Two, Problems 223–226].

**N. G. Chebotarëv’s formula** ([6, Theorem VI.3.2, (3.6)]). *Let  $I \subset \mathbb{R}$  be an open interval and  $K$  a compact space. Let  $f(t, x)$  be defined and continuous on  $I \times K$  along with  $\frac{\partial f}{\partial t}(t, x)$ . Let*

$$f^*(t) = \max_{x \in K} f(t, x).$$

Then for each  $t \in I$ ,

$$(28) \quad f^*(t + h) = f^*(t) + \max_{f(t,x)=f^*(t)} \left( h \frac{\partial f}{\partial t}(t, x) \right) + o(h).$$

**Proposition 9.1.** *Let  $k$  and  $l$  be positive coprime integers and  $r_1, r_2$  and  $r_3$  positive real numbers. Then*

$$\max_x |r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}|$$

is an even  $2\pi/(k+l)$ -periodic function of  $t$  which decreases strictly on  $[0, \pi/(k+l)]$ . In particular,

$$\min_t \max_x |r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}| = \max_x |r_1 e^{-ikx} + r_2 e^{i\pi/(k+l)} + r_3 e^{ilx}|.$$

**Proof.** Let

$$(29) \quad f(t, x) = |r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}|^2.$$

By (10) and (11),  $f^*$  is an even  $2\pi/(k+l)$ -periodic function.

Let  $t \in ]0, \pi/(k+l)[$ , and choose  $x^*$  such that  $f(t, x^*) = f^*(t)$ . Then  $x^* \in [-t/k, t/l]$  by Proposition 5.1, so that

$$\frac{1}{2r_2} \frac{\partial f}{\partial t}(t, x^*) = -r_1 \sin(t + kx^*) - r_3 \sin(t - lx^*) < 0$$

because  $t + kx^* \in [0, (k+l)t/l]$  and  $t - lx^* \in [0, (k+l)t/k]$  do not vanish simultaneously. By Formula (28),  $f^*$  decreases strictly on  $[0, \pi/(k+l)]$ .

**Proposition 9.2.** *Let  $k$  and  $l$  be positive coprime integers and  $r_1, r_2$  and  $r_3$  positive real numbers. Then*

$$(30) \quad \frac{\max_x |r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}|}{|r_1 + r_2 e^{it} + r_3|}$$

is an increasing function of  $t \in [0, \pi/(k+l)]$ . If  $kr_1 = lr_3$ , it is constantly equal to 1; otherwise, it is strictly increasing.

**Proof.** Let  $f(t, x)$  be as in (29). Then the expression (30) is  $g^*(t)^{1/2}$  with

$$g(t, x) = \frac{f(t, x)}{f(t, 0)}.$$

If  $kr_1 = lr_3$ , then  $f(t, 0) = f^*(t)$ , so that  $g^*(t) = 1$ . As shown at the beginning of Section 6, we may suppose without loss of generality that  $kr_1 < lr_3$ . Let  $t \in ]0, \pi/(k + l)[$  and choose  $x^*$  such that  $f(t, x^*) = f^*(t)$ . Then  $x^* \in ]0, t/l[$  by Propositions 5.1 and 6.1, and

$$\begin{aligned} \frac{f(t, 0)^2}{2r_2} \frac{\partial g}{\partial t}(t, x^*) &= \frac{1}{2r_2} \left( \frac{\partial f}{\partial t}(t, x^*)f(t, 0) - f(t, x^*) \frac{\partial f}{\partial t}(t, 0) \right) \\ &= (-r_1 \sin(t + kx^*) - r_3 \sin(t - lx^*))f(t, 0) + f^*(t)(r_1 + r_3) \sin t \\ &= h(0)f^*(t) - h(x^*)f(t, 0) \end{aligned}$$

with

$$h(x) = r_1 \sin(t + kx) + r_3 \sin(t - lx).$$

Let us show that  $h$  is strictly decreasing on  $[0, t/l]$ . In fact, if  $x \in ]0, t/l]$ ,

$$\frac{dh}{dx}(x) = kr_1 \cos(t + kx) - lr_3 \cos(t - lx) < (kr_1 - lr_3) \cos(t - lx) < 0.$$

As  $f^*(t) > f(t, 0)$  and  $h(0) > h(x^*)$ ,  $(\partial g/\partial t)(t, x^*) > 0$ . By N. G. Chebotarëv's formula,  $g^*$  increases strictly on  $[0, \pi/(k + l)]$ .

It is possible to describe the decrease of the maximum modulus of a trigonometric trinomial independently of the  $r$ 's as follows.

**Proposition 9.3.** *Let  $k$  and  $l$  be positive coprime integers and  $r_1, r_2$  and  $r_3$  positive real numbers. Let  $0 \leq t' < t \leq \pi/(k + l)$ . Then*

$$(31) \quad \max_x |r_1 e^{-ikx} + r_2 e^{it'} + r_3 e^{ilx}| \leq \frac{\cos(t'/2)}{\cos(t/2)} \max_x |r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}|,$$

with equality if and only if  $r_1 : r_2 : r_3 = l : k + l : k$ .

**Proof.** Applying Proposition 9.2, we have

$$\begin{aligned} \frac{|r_1 + r_2 e^{it'} + r_3|^2}{|r_1 + r_2 e^{it} + r_3|^2} &= 1 + \frac{2r_2(r_1 + r_3)(\cos t' - \cos t)}{(r_1 + r_3)^2 + 2r_2(r_1 + r_3) \cos t + r_2^2} \\ &= 1 + \frac{\cos t' - \cos t}{\cos t + (r_2^2 + (r_1 + r_3)^2)/2r_2(r_1 + r_3)} \\ &\leq 1 + \frac{\cos t' - \cos t}{\cos t + 1} = \frac{\cos t' + 1}{\cos t + 1} \end{aligned}$$

by the arithmetic-geometric mean inequality, with equality if and only if  $r_2 = r_1 + r_3$ . Thus, (31) holds, with equality if and only if  $kr_1 = lr_3$  and  $r_2 = r_1 + r_3$ .

We may now find the minimum of the maximum modulus of a trigonometric trinomial with given spectrum, Fourier coefficient arguments and moduli sum. Proposition 9.3 yields with  $t' = 0$

**Corollary 9.4.** *Let  $k$  and  $l$  be positive coprime integers and  $r_1, r_2$  and  $r_3$  positive real numbers. Let  $t \in ]0, \pi/(k + l)]$ . Then*

$$\frac{\max_x |r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}|}{r_1 + r_2 + r_3} \geq \cos(t/2),$$

with equality if and only if  $r_1 : r_2 : r_3 = l : k + l : k$ .

**Remark 9.5.** There is a shortcut proof to Corollary 9.4:

$$\begin{aligned} \frac{\max_x |r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}|}{r_1 + r_2 + r_3} &\geq \frac{|r_1 + r_2 e^{it} + r_3|}{r_1 + r_2 + r_3} \\ &= \sqrt{1 - \frac{4(r_1 + r_3)r_2}{(r_1 + r_2 + r_3)^2} \sin^2(t/2)} \\ &\geq \sqrt{1 - \sin^2(t/2)} = \cos(t/2), \end{aligned}$$

and equality holds if and only if  $|r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}|$  is maximal for  $x = 0$  and  $r_1 + r_3 = r_2$ .

## 10 The norm of unimodular relative Fourier multipliers

We may now compute the norm of unimodular relative Fourier multipliers.

**Corollary 10.1.** *Let  $k$  and  $l$  be two positive coprime integers and let  $t \in [0, \pi/(k + l)]$ . Let  $M$  be the relative Fourier multiplier  $(0, t, 0)$  that maps the element*

$$(32) \quad r_1 e^{iu_1} e_{-k} + r_2 e^{iu_2} e_0 + r_3 e^{iu_3} e_l$$

of the normed space  $\mathcal{C}_{\{-k,0,l\}}$  on

$$r_1 e^{iu_1} e_{-k} + r_2 e^{i(t+u_2)} e_0 + r_3 e^{iu_3} e_l.$$

Then  $M$  has norm  $\cos(\pi/2(k+l) - t/2) / \cos(\pi/2(k+l))$  and attains its norm exactly at elements of the form (32) with  $r_1 : r_2 : r_3 = l : k + l : k$  and

$$-lu_1 + (k + l)u_2 - ku_3 = \pi \pmod{2\pi}.$$

**Proof.** This follows from Proposition 9.3 and the concavity of  $\cos$  on  $[0, \pi/2]$ .

**Remark 10.2.** This corollary enables us to guess how to lift  $M$  to an operator that acts by convolution with a measure  $\mu$ . Note that  $\mu$  is a Hahn-Banach extension of the linear form  $f \mapsto Mf(0)$ . The relative multiplier  $M$  is an isometry if and only if  $t = 0$  and  $\mu$  is the Dirac measure in 0. Otherwise,  $t \neq 0$ ; the proof of Theorem 1.2 (a) in Section 8 shows that  $\mu$  is a linear combination  $\alpha\delta_y + \beta\delta_w$  of two Dirac measures such that the norm of  $M$  is  $|\alpha| + |\beta|$ . Let

$$f(x) = le^{-ikx} + (k+l)e^{i\pi/(k+l)} + ke^{ilx}.$$

$M$  attains its norm at  $f$ ,  $f$  attains its maximum modulus at 0 and  $2m\pi/(k+l)$ , and  $Mf$  attains its maximum modulus at  $2m\pi/(k+l)$ , where  $m$  is the inverse of  $l$  modulo  $k+l$ . As

$$\begin{aligned} (|\alpha| + |\beta|) \max_x |f(x)| &= \max_x |Mf(x)| \\ &= |\mu * f(2m\pi/(k+l))| \\ &= |\alpha f(2m\pi/(k+l) - y) + \beta f(2m\pi/(k+l) - w)|, \end{aligned}$$

we must choose  $\{y, w\} = \{0, 2m\pi/(k+l)\}$ . A computation then yields

$$\mu = e^{it/2} \frac{\sin(\pi/(k+l) - t/2)}{\sin(\pi/(k+l))} \delta_0 + e^{i(t/2 + \pi/(k+l))} \frac{\sin(t/2)}{\sin(\pi/(k+l))} \delta_{2m\pi/(k+l)}.$$

If  $k = l = 1$ , this is a special case of a formula appearing in [10, proof of Prop. 1]. Consult [25] on this issue.

### 11 The Sidon constant of integer sets

Let us study the maximum modulus of a trigonometric trinomial with given spectrum and Fourier coefficient moduli sum. The following result is an immediate consequence of Corollary 9.4.

**Proposition 11.1.** *Let  $k$  and  $l$  be positive coprime integers; let  $r_1, r_2$  and  $r_3$  be positive real numbers and let  $t \in [0, \pi/(k+l)]$ . Then*

$$\max_x |r_1 e^{-ikx} + r_2 e^{it} + r_3 e^{ilx}| \geq \cos(\pi/2(k+l)) \cdot (r_1 + r_2 + r_3)$$

with equality if and only if  $r_1 : r_2 : r_3 = l : k+l : k$  and  $t = \pi/(k+l)$ .

This means that the Sidon constant of  $\{-k, 0, l\}$  equals  $\sec(\pi/2(k+l))$ .

The Sidon constant of integer sets was previously known only in the following three instances,

- The equality

$$\max_x |r_1 e^{i(t_1 + \lambda_1 x)} + r_2 e^{i(t_2 + \lambda_2 x)}| = r_1 + r_2$$

shows that the Sidon constant of sets with one or two elements is 1.

- The Sidon constant of  $\{-1, 0, 1\}$  is  $\sqrt{2}$  and is attained for  $e_{-1} + 2i + e_1$ . Let us give the original argument: if  $f(x) = |r_1 e^{-ix} + r_2 e^{it} + r_3 e^{ix}|^2$ , the parallelogram identity and the arithmetic-quadratic inequality yield

$$\begin{aligned} \max_x f(x) &\geq \max_x \frac{f(x) + f(x + \pi)}{2} \\ &= \max_x \frac{|r_1 e^{-ix} + r_3 e^{ix} + r_2 e^{it}|^2 + |r_1 e^{-ix} + r_3 e^{ix} - r_2 e^{it}|^2}{2} \\ &= \max_x (|r_1 e^{-ix} + r_3 e^{ix}|^2 + |r_2 e^{it}|^2) \\ &= (r_1 + r_3)^2 + r_2^2 \geq \frac{(r_1 + r_2 + r_3)^2}{2}. \end{aligned}$$

- The Sidon constant of  $\{0, 1, 2, 3, 4\}$  is 2 and it is attained for  $1 + 2e_1 + 2e_2 - 2e_3 + e_4$ .

These results were obtained by D. J. Newman (see [24].) The fact that the Sidon constant of sets of three integers cannot be 1 had been noted with pairwise different proofs in [24, 3, 13].

**Remark 11.2.** The real algebraic counterpart is better understood: the maximal absolute value of a real algebraic polynomial of degree at most  $n$  with given coefficient absolute value sum is minimal for multiples of the  $n$ th Chebyshev polynomial  $T_n$  (look up the last paragraph of [8]). As the sum of the absolute values of  $T_n$ 's coefficients is the integer  $t_n$  nearest to  $(1 + \sqrt{2})^n / 2$ , we have for real  $a_0, a_1, \dots, a_n$

$$\max_{x \in [-1, 1]} |a_0 + a_1 x + \dots + a_n x^n| \geq t_n^{-1} (|a_0| + |a_1| + \dots + |a_n|).$$

The following estimates for the Sidon constant of large integer sets are known.

- E. Beller and D. J. Newman [2] showed that the Sidon constant of  $\{0, 1, \dots, n\}$  is equivalent to  $\sqrt{n}$ .
- (Hadamard sets.) Let  $q > 1$  and suppose that the sequence  $(\lambda_j)_{j \geq 1}$  grows with geometric ratio  $q$ :  $|\lambda_{j+1}| \geq q|\lambda_j|$  for every  $j$ . Then the Sidon constant of  $\{\lambda_1, \lambda_2, \dots\}$  is finite; it is at most 4.27 if  $q \geq 2$  (see [11]), at most 2 if  $q \geq 3$  (see [15]), and at most  $1 + \pi^2 / (2q^2 - 2 - \pi^2)$  if  $q > \sqrt{1 + \pi^2 / 2}$  (see [18, Corollary 9.4] or the updated [19, Corollary 10.2.1]).

Our computations show that the last estimate of the Sidon constant has the right order in  $q^{-1}$  for geometric progressions.



**Proposition 11.3.** *Let  $C$  be the Sidon constant of the geometric progression  $\{1, q, q^2, \dots\}$ , where  $q \geq 3$  is an integer. Then*

$$1 + \pi^2/8(q+1)^2 \leq \sec(\pi/2(q+1)) \leq C \leq 1 + \pi^2/(2q^2 - 2 - \pi^2).$$

One initial motivation for this work was to decide whether there are sets  $\{\lambda_j\}_{j \geq 1}$  with  $|\lambda_{j+1}| \geq q|\lambda_j|$  whose Sidon constant is arbitrarily close to 1 and to find evidence among sets with three elements. That there are such sets, arbitrarily large albeit finite, may in fact be proved by the method of Riesz products in [12, Appendix V, §1.II]; see also [19, Proposition 13.1.3]. The case of infinite sets remains open.

A second motivation was to show that the real and complex unconditional constants of the basis  $(e_{\lambda_1}, e_{\lambda_2}, e_{\lambda_3})$  of  $\mathcal{C}_\Lambda$  are different; however, it turns out that they coincide, and it remains an open question whether they may be different for larger sets. The **real unconditional constant** of  $(e_{\lambda_1}, e_{\lambda_2}, e_{\lambda_3})$  is the maximum of the norm of the eight unimodular relative Fourier multipliers  $(t_1, t_2, t_3)$  such that  $t_k = 0$  modulo  $\pi$ . Let  $i, j, k$  be a permutation of 1, 2, 3 such that the power of 2 in  $\lambda_i - \lambda_k$  and in  $\lambda_j - \lambda_k$  are equal. Lemma 2.1 shows that the four relative multipliers satisfying  $t_i = t_j$  modulo  $2\pi$  are isometries and that the norm of any of the four others, satisfying  $t_i \neq t_j$  modulo  $2\pi$ , gives the real unconditional constant. In general, the complex unconditional constant is bounded by  $\pi/2$  times the real unconditional constant, as proved in [23]; in our case, they are equal.

**Corollary 11.4.** *The complex unconditional constant of the basis  $(e_{\lambda_1}, e_{\lambda_2}, e_{\lambda_3})$  of  $\mathcal{C}_\Lambda$  is equal to its real unconditional constant.*

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