A NEW APPROACH TO SUBORDINATION RESULTS IN FREE PROBABILITY

By

S. T. BELINSCHI AND H. BERCOVICI*

Abstract. We show that the subordination results of D. Voiculescu and Ph. Biane can be deduced from a continuity property of fixed points for analytic functions.

1 Introduction

Consider Borel probability measures μ, ν on the real line $\mathbb{R}$, and the associated Cauchy transforms G_μ, G_ν defined on $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\, d\mu(t), \quad G_\nu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\, d\nu(t).$$

It was shown by Voiculescu [7] that there exists another probability measure, denoted $\mu \boxplus \nu$ and called the free additive convolution of μ and ν, such that

$$G_\mu^{-1}(z) + G_\nu^{-1}(z) = G_{\mu\boxplus\nu}^{-1}(z) + 1/z$$

for z in a domain of the form

$$\{z : |\Re z| < -\alpha\Im z, -\beta < \Im z < 0\},$$

where $\alpha, \beta > 0$. Here we use G_μ^{-1} for the inverse of G_μ as a function, i.e. $G_\mu(G_\mu^{-1}(z)) = z$.

An important property of free additive convolution is subordination: there exists an analytic function $\omega : \mathbb{C}^+ \to \mathbb{C}^+$ such that $G_{\mu\boxplus\nu} = G_\mu \circ \omega$ and

$$\lim_{y\uparrow\infty} \frac{\omega(iy)}{iy} = 1.$$

This was first shown under a genericity assumption by Voiculescu [9], and in full generality by Biane [2]. Biane also proved analogous results for free multiplicative

*The second author was supported in part by a grant from the National Science Foundation.

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 101 (2007)
DOI 10.1007/s11854-007-0013-1

convolutions. These results were approached from an abstract coalgebra point of view in [10, 11], and this approach provides subordination in even more general contexts.

Our purpose is to show that these subordination results can be viewed as pure complex analysis theorems. The main tool is the elementary observation that the Denjoy–Wolff point of an analytic selfmap f of the unit disk depends analytically on f, except in the neighborhood of a Moebius map. This observation, along with some background information, is presented in Section 2. The applications to free convolutions are derived in Sections 3 and 4.

2 Denjoy–Wolff Points

Denote by $\mathbb{D} = \{z : |z| < 1\}$ the unit disk in the complex plane, and let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be an analytic function. We recall that a point $w \in \overline{\mathbb{D}}$ is a **Denjoy–Wolff point** for f if either

(1) $w \in \mathbb{D}$ and $f(w) = w$; or

(2) $|w| = 1$, $\lim_{r\uparrow 1} f(rw) = w$, and

$$\lim_{r\uparrow 1} \frac{w - f(rw)}{(1-r)w} \leq 1.$$

The limit displayed above is called the Julia-Carathéodory derivative of f at w. Except for the identity map of $\mathbb{D}$, every f has a unique Denjoy–Wolff point. Moreover, this point is a limit of the iterates of f in most cases. The following result is due to Denjoy and Wolff; we refer the reader to [6] for an excellent exposition.

Theorem 2.1. *Assume that $f : \mathbb{D} \to \mathbb{D}$ is an analytic function with Denjoy–Wolff point w, and denote by*

$$f^{\circ n} = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$$

the composition of n copies of f. If f is not a conformal automorphism of $\mathbb{D}$, then the sequence $(f^{\circ n})_{n=1}^{\infty}$ converges to w uniformly on compact subsets of $\mathbb{D}$.

We consider now an open subset $\Omega \subset \mathbb{C}$, and an analytic function $g : \Omega \times \mathbb{D} \to \mathbb{D}$. We are interested in the dependence on λ of the Denjoy–Wolff point of the function $g_\lambda(z) = g(\lambda, z)$.

Lemma 2.2. *With the notation above, if there exists $\lambda_0 \in \Omega$ such that g_{λ_0} is a conformal automorphism, then $g(\lambda, z) = g(\lambda_0, z)$ for every $(\lambda, z) \in \Omega \times \mathbb{D}$.*

Proof. Replacing g_λ by $g_{\lambda_0}^{-1} \circ g_\lambda$, we may assume that g_{λ_0} is the identity map. Fix a point $z \in \partial\mathbb{D}$. By the Vitali–Montel theorem, there exists a sequence $r_n \uparrow 1$ such that $g(\lambda, r_n z)$ converges to an analytic function $h(\lambda)$ as $n \to \infty$. Clearly $|h(\lambda)| \leq 1$ for all $\lambda \in \mathbb{D}$, and $h(\lambda_0) = z$. By the maximum principle, $h(\lambda) = z$ for all $\lambda \in \Omega$. Now fix $\lambda \in \Omega$. We deduce that

$$\lim_{r \uparrow 1} g(\lambda, rz) = z$$

for almost every $z \in \partial\mathbb{D}$. The F. and M. Riesz theorem [4] now yields the desired conclusion. □

The analytic dependence of the Denjoy–Wolff point is now easy to deduce.

Theorem 2.3. *Let $g : \Omega \times \mathbb{D} \to \mathbb{D}$ be an analytic function such that the map $g_\lambda(z) = g(\lambda, z)$ is not a conformal automorphism of $\mathbb{D}$ for some (and hence all) $\lambda \in \Omega$. Denote by $\omega(\lambda)$ the Denjoy–Wolff point of g_λ. Then the function $\omega : \Omega \to \overline{\mathbb{D}}$ is analytic.*

Proof. Note that $\omega(\lambda)$ is indeed well-defined since none of the g_λ is the identity map. Since none of the g_λ are conformal automorphisms, Theorem 2.1 implies that

$$\omega(\lambda) = \lim_{n \to \infty} g_\lambda^{\circ n}(0),$$

where $g_\lambda^{\circ n}$ denotes, as before, the composition of n copies of g_λ. The analyticity of ω then follows from Montel's theorem. □

The function ω can take values in $\partial\mathbb{D}$ only if it is constant. If one of the functions g_λ has a fixed point in $\mathbb{D}$, then $\omega(\lambda)$ is the unique fixed point of g_λ for every λ. The preceding result can be reformulated replacing $\mathbb{D}$ by a conformally equivalent domain. We record the statement for further reference.

Theorem 2.4. *Consider a domain Δ conformally equivalent to $\mathbb{D}$ and an analytic function $g : \Omega \times \Delta \to \Delta$ such that for some $\lambda \in \Omega$, the map $g_\lambda(z) = g(\lambda, z)$ is not a conformal automorphism of Δ and has a fixed point in Δ. Then there exists an analytic function $\omega : \Omega \to \Delta$ such that $g(\lambda, \omega(\lambda)) = \omega(\lambda)$, $\lambda \in \Omega$. Moreover, for every $w \in \Delta$,*

$$\omega(\lambda) = \lim_{n \to \infty} g_\lambda^{\circ n}(w)$$

uniformly on compact subsets of G.

3 Subordination for Multiplicative Convolution

We start with an easy consequence of Theorem 2.3.

Theorem 3.1. *Consider two analytic functions* $f_1, f_2 : \mathbb{D} \to \overline{\mathbb{D}}$. *There exist unique analytic functions* $\omega_1, \omega_2 : \mathbb{D} \to \mathbb{D}$ *such that*

(1) $\omega_1(0) = \omega_2(0) = 0$,

(2) $\omega_1(\lambda) = \lambda f_2(\omega_2(\lambda))$, *and*

(3) $\omega_2(\lambda) = \lambda f_1(\omega_1(\lambda))$ *for all* $z \in \mathbb{D}$.

Proof. Combining conditions (2) and (3), we see that ω_1 must satisfy the equation

$$\omega_1(\lambda) = \lambda f_2(\lambda f_1(\omega_1(\lambda))), \quad \lambda \in \mathbb{D}.$$

Consider therefore the function $g : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ defined by

$$g(\lambda, z) = \lambda f_2(\lambda f_1(z)), \quad \lambda, z \in \mathbb{D}.$$

Note that $g_0(z) = g(0, z) = 0$ is certainly not a conformal automorphism. Theorem 2.3 then guarantees the existence of an analytic function $\omega_1 : \mathbb{D} \to \mathbb{D}$ such that $g(\lambda, \omega_1(\lambda)) = \omega_1(\lambda)$ for $\lambda \in \mathbb{D}$. (Note that the function ω_1 must satisfy $\omega_1(0) = 0$; hence it does not take values in $\partial\mathbb{D}$.) Setting $\omega_2(\lambda) = \lambda f_1(\omega_1(\lambda))$ yields the desired functions. The fact that ω_1, ω_2 are unique follows from the uniqueness of Denjoy–Wolff points. $\square$

The subordination result for free multiplicative convolution on the unit circle is essentially a reformulation of the preceding result. Recall from [8] that, given probability measures μ, ν on $\partial\mathbb{D}$, one defines a probability measure $\mu \boxtimes \nu$, called the free multiplicative convolution of μ and ν. When μ and ν have nonzero first moments, $\mu \boxtimes \nu$ can be calculated as follows. Define analytic functions $\psi_\mu : \mathbb{D} \to \mathbb{C}$ and $\eta_\mu : \mathbb{D} \to \mathbb{D}$ by

$$\psi_\mu(\lambda) = \int \frac{t\lambda}{1 - t\lambda}\, d\mu(t), \quad \eta_\mu(\lambda) = \frac{\psi_\mu(\lambda)}{1 + \psi_\mu(\lambda)}, \quad \lambda \in \mathbb{D}.$$

We have $\eta_\mu(0) = 0$, and η_μ is invertible near $\lambda = 0$ because $\eta'_\mu(0)$ equals the first moment of μ. The measure $\mu \boxtimes \nu$ is uniquely determined by the requirement that

$$z\eta^{-1}_{\mu\boxtimes\nu}(z) = \eta^{-1}_\mu(z)\eta^{-1}_\nu(z) \tag{3.1}$$

for z close to zero. The following result is due to Biane [2]. The original proof is combinatorial.

Theorem 3.2. *Given Borel probability measures μ, ν on $\partial\mathbb{D}$, there exist analytic functions $\omega_1, \omega_2 : \mathbb{D} \to \mathbb{D}$ such that*
(1) $\omega_1(0) = \omega_2(0) = 0$,
(2) $\psi_{\mu\boxtimes\nu}(\lambda) = \psi_\mu(\omega_1(\lambda)) = \psi_\nu(\omega_2(\lambda))$, *and*
(3) $\omega_1(\lambda)\omega_2(\lambda) = \lambda\eta_{\mu\boxtimes\nu}(\lambda)$ *for all* $\lambda \in \mathbb{D}$.

Proof. As noted by Biane, it suffices to prove the theorem in case μ and ν have nonzero first moments. Since $\eta_\mu(0) = 0$, we can write $\eta_\mu(\lambda) = \lambda f_1(\lambda)$, where $f_1 : \mathbb{D} \to \overline{\mathbb{D}}$ is analytic. Analogously, $\eta_\nu(\lambda) = \lambda f_2(\lambda)$. The assumption about μ, ν implies that we can define analytic functions ω_1, ω_2 in a neighborhood of zero by setting

$$\omega_1(\lambda) = \eta_\mu^{-1}(\eta_{\mu\boxtimes\nu}(\lambda)), \quad \omega_2(\lambda) = \eta_\nu^{-1}(\eta_{\mu\boxtimes\nu}(\lambda)).$$

Replacing z by $\eta_{\mu\boxtimes\nu}(\lambda)$ in (3.1), we obtain

$$\lambda\eta_{\mu\boxtimes\nu}(\lambda) = \omega_1(\lambda)\omega_2(\lambda)$$

for λ in some neighborhood of zero. We also have

$$\omega_1(\lambda) = \frac{\lambda\eta_{\mu\boxtimes\nu}(\lambda)}{\omega_2(\lambda)} = \frac{\lambda\eta_\nu(\omega_2(\lambda))}{\omega_2(\lambda)} = \lambda f_2(\omega_2(\lambda)).$$

Analogously, $\omega_2(\lambda) = \lambda f_1(\omega_1(\lambda))$ in some neighborhood of zero. The uniqueness of Denjoy–Wolff points shows that ω_1, ω_2 coincide in a neighborhood of zero with the unique analytic functions satisfying the conclusion of Theorem 3.1. In other words, ω_1 and ω_2 can be continued analytically to the entire unit disk. Moreover, the equalities (2) and (3), which are true in a neighborhood of zero, extend to $\mathbb{D}$ by unique continuation. □

We now pass to free multiplicative convolution of measures on $\mathbb{R}_+ = [0, +\infty)$. Given a Borel probability measure μ on $\mathbb{R}_+$, one defines the functions ψ_μ and η_μ by the same formulas used for measures on $\partial\mathbb{D}$, but now for $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$. The function η_μ satisfies the conditions

$$\eta_\mu(\overline{\lambda}) = \overline{\eta_\mu(\lambda)}, \quad \eta_\mu(\lambda) \in \mathbb{C}^+, \quad \arg \eta_\mu(\lambda) \geq \arg \lambda \quad \text{for } \lambda \in \mathbb{C}^+,$$

and

$$\eta_\mu(0-) = \lim_{x \uparrow 0} \eta_\mu(x) = 0.$$

If μ and ν are probability measures on $\mathbb{R}_+$ different from the Dirac measure δ_0 at zero, (3.1) again holds, this time in an open subset of $\mathbb{C}$ containing some interval of the form $(-\alpha, 0)$ with $\alpha > 0$. It is convenient to use, in place of $\mathbb{C} \setminus \mathbb{R}_+$, the domain

$$\mathbb{S} = \{x + iy : x \in \mathbb{R}, y \in (-\pi, \pi)\},$$

which is conformally equivalent via the map $u : \mathbb{S} \to \mathbb{C} \setminus \mathbb{R}_+$ defined by

$$u(\lambda) = -e^{-\lambda}, \quad \lambda \in \mathbb{S}.$$

The following result was first proved by Biane [2].

Theorem 3.3. *Given probability measures μ, ν on $\mathbb{R}_+$, both different from δ_0, there exist unique analytic functions $\omega_1, \omega_2 : \mathbb{C} \setminus \mathbb{R}_+ \to \mathbb{C} \setminus \mathbb{R}_+$ such that*

(1) $\omega_1(0-) = \omega_2(0-) = 0$;

(2) *for every $\lambda \in \mathbb{C}^+$, we have $\omega_j(\overline{\lambda}) = \overline{\omega_j(\lambda)}$, $\omega_j(\lambda) \in \mathbb{C}^+$, and $\arg \omega_j(\lambda) \geq \arg \lambda$ for $j = 1, 2$;*

(3) $\eta_{\mu \boxtimes \nu}(\lambda) = \eta_\mu(\omega_1(\lambda)) = \eta_\nu(\omega_2(\lambda))$ *for $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$; and*

(4) $\omega_1(\lambda)\omega_2(\lambda) = \lambda \eta_{\mu \boxtimes \nu}(\lambda)$.

Proof. As in the proof of Theorem 3.2, relation (3.1) yields analytic functions

$$\omega_1 = \eta_\mu^{-1} \circ \eta_{\mu \boxtimes \nu}, \quad \omega_2 = \eta_\nu^{-1} \circ \eta_{\mu \boxtimes \nu}$$

defined in some open subset of $\mathbb{C} \setminus \mathbb{R}_+$, symmetric relative to $\mathbb{R}$, satisfying relations (3) and (4) in that neighborhood. We need to show that these functions have continuations with the required properties. Consider the functions

$$h_1 = u^{-1} \circ \eta_\mu \circ u, \quad h_2 = u^{-1} \circ \eta_\nu \circ u.$$

These are analytic selfmaps of $\mathbb{S}$ such that for every $\lambda \in \mathbb{S} \cap \mathbb{C}^+$, we have $h_j(\lambda) \in \mathbb{C}^+$, $h_j(\overline{\lambda}) = \overline{h_j(\lambda)}$, and $0 \leq \Im h_j(\lambda) \leq \Im \lambda$. The existence of ω_1 and ω_2 amounts to the existence of analytic selfmaps u_1, u_2 of $\mathbb{S}$ such that for every $\lambda \in \mathbb{S} \cap \mathbb{C}^+$, we have

$$u_j(\lambda) \in \mathbb{C}^+, \quad \Im u_j(\lambda) \leq \Im \lambda,$$

and

$$u_1(\lambda) + u_2(\lambda) = \lambda + h_j(u_j(\lambda)), \quad j = 1, 2.$$

Writing $v_j(\lambda) = u_j(\lambda) - \lambda$, $f_j(\lambda) = h_j(\lambda) - \lambda$ for $j = 1, 2$, we see that

$$v_1(\lambda) = u_1(\lambda) - \lambda = h_2(u_2(\lambda)) - u_2(\lambda) = f_2(u_2(\lambda)) = f_2(\lambda + v_2(\lambda));$$

and this leads to the fixed point equation

$$v_1(\lambda) = f_2(\lambda + f_1(\lambda + v_1(\lambda))).$$

It is then natural to consider the function

$$g(\lambda, z) = f_2(\lambda + f_1(\lambda + z)).$$

This function is not defined on the entire product $\mathbb{S} \times \mathbb{S}$. However, $g(\lambda, z) \in \mathbb{S} \cap \mathbb{C}^-$ provided that $\lambda \in \mathbb{S} \cap \mathbb{C}^+$ and $z \in \mathbb{S} \cap \mathbb{C}^-$. To verify this, fix $\lambda \in \mathbb{S} \cap \mathbb{C}^+$ and $z \in \mathbb{S} \cap \mathbb{C}^-$. If $\Im(\lambda + z) \geq 0$, then $-\Im(\lambda + z) \leq \Im f_1(\lambda + z) \leq 0$, so that $0 \leq -\Im z \leq \Im(\lambda + f_1(\lambda + z)) \leq \Im \lambda$. Thus $\lambda + f_1(\lambda + z) \in \mathbb{S} \cap \mathbb{C}^+$, and therefore $g(\lambda, z) \in \mathbb{S} \cap \mathbb{C}^-$ and

$$0 \geq \Im g(\lambda, z) \geq -\Im \lambda.$$

Similarly, if $\Im(\lambda + z) < 0$, then $0 \leq \Im f_1(\lambda + z) \leq -\Im(\lambda + z)$, so that $\Im \lambda \leq \Im(\lambda + f_1(\lambda + z)) \leq -\Im z$. Again we conclude that $\lambda + f_1(\lambda + z) \in \mathbb{S} \cap \mathbb{C}^+$ and

$$0 \geq \Im g(\lambda, z) \geq \Im z.$$

The local existence of the function ω_1 indicates that the equation $g_\lambda(z) = z$ has a solution for some value of $\lambda \in \mathbb{S} \cap \mathbb{C}^+$. Theorem 2.4 then yields an analytic function $v_1 : \mathbb{S} \cap \mathbb{C}^+ \to \mathbb{S} \cap \mathbb{C}^-$ satisfying

$$g(\lambda, v_1(\lambda)) = v_1(\lambda), \quad \lambda \in \mathbb{S} \cap \mathbb{C}^+.$$

The inequalities obtained above also show inductively that

$$0 \geq \Im g_\lambda^{\circ n}(-\lambda) \geq -\Im \lambda,$$

thus establishing the inequalities $0 \geq \Im v_1(\lambda) \geq -\Im \lambda$ for $\lambda \in \mathbb{S} \cap \mathbb{C}^+$. The function $u_1(\lambda) = \lambda + v_1(\lambda)$ then maps $\mathbb{S} \cap \mathbb{C}^+$ to itself; and setting

$$u_2(\lambda) = \lambda + f_1(u_1(\lambda)) = \lambda + h_1(u_1(\lambda)) - u_1(\lambda),$$

we obtain functions satisfying our requirements in $\mathbb{S} \cap \mathbb{C}^+$. Since

$$0 \leq \Im u_j(\lambda) \leq \Im \lambda, \quad \lambda \in \mathbb{S} \cap \mathbb{C}^+, j = 1, 2,$$

it follows that all the limit values of u_j on the real line are real or infinite. Seidel's version of the Schwarz reflection principle [3] allows us to extend u_j to $\mathbb{S}$ in such a way that $u_j(\overline{\lambda}) = \overline{u_j(\lambda)}$ for all λ. This establishes the existence of u_1, u_2 and therefore of ω_1, ω_2. The uniqueness of these functions follows from the uniqueness of Denjoy–Wolff points. □

4 Subordination for Additive Convolution

For a Borel probability measure μ on the real line, we write $F_\mu(\lambda) = 1/G_\mu(\lambda)$, $\lambda \in \mathbb{C}^+$. As observed in [5] and [1],

$$\Im F_\mu(\lambda) \geq \Im \lambda, \quad \lambda \in \mathbb{C}^+,$$

and

$$\lim_{y\uparrow\infty}\frac{F_\mu(iy)}{iy}=1.$$

This implies the invertibility of the function F_μ in a region where $|\lambda|=|x+iy|$ is sufficiently large, provided that x/y remains bounded. The defining equation of free additive convolution is now

$$F_\mu^{-1}(z)+F_\nu^{-1}(z)=z+F_{\mu\boxplus\nu}^{-1}(z).$$

The earliest subordination result in this area is the following theorem, first proved in the generic case in [9] and in full generality in [2].

Theorem 4.1. *Given Borel probability measures μ,ν on $\mathbb{R}$, there exist unique functions $\omega_1,\omega_2:\mathbb{C}^+\to\mathbb{C}^+$ such that*

(1) $\Im\omega_j(\lambda)\geq\Im\lambda$ *for* $\lambda\in\mathbb{C}^+$, *and*

$$\lim_{y\uparrow\infty}\frac{\omega_j(iy)}{iy}=1,\quad j=1,2;$$

(2) $F_{\mu\boxplus\nu}(\lambda)=F_\mu(\omega_1(\lambda))=F_\nu(\omega_2(\lambda))$; *and*

(3) $\omega_1(\lambda)+\omega_2(\lambda)=\lambda+F_{\mu\boxplus\nu}(\lambda)$ *for all* $\lambda\in\mathbb{C}^+$.

Proof. The analytic functions $\omega_1=F_\mu^{-1}\circ F_{\mu\boxplus\nu}$ and $\omega_2=F_\nu^{-1}\circ F_{\mu\boxplus\nu}$ are defined for $\lambda=iy$ if y is sufficiently large and satisfy

$$\lim_{y\uparrow\infty}\frac{\omega_j(iy)}{iy}=1,\quad j=1,2,$$

as well as conditions (2) and (3) in some open set. It remains to prove that these functions can be continued analytically to $\mathbb{C}^+$ and that the continuations satisfy the first condition in (1). Set $h_1(\lambda)=F_\mu(\lambda)-\lambda, h_2(\lambda)=F_\nu(\lambda)-\lambda$. The analytic functions h_1,h_2 have nonnegative imaginary part in $\mathbb{C}^+$. We combine (2) and (3) to obtain the fixed point equation

$$\omega_1(\lambda)=\lambda+h_2(\omega_2(\lambda))=\lambda+h_2(\lambda+h_1(\omega_1(\lambda))).$$

This leads to the function $g:\mathbb{C}^+\times\mathbb{C}^+\to\mathbb{C}^+$ defined by

$$g(\lambda,z)=\lambda+h_2(\lambda+h_1(z)),\quad \lambda,z\in\mathbb{C}^+.$$

The local existence of the functions ω_j implies that some of the functions $g_\lambda(z)=g(\lambda,z)$ have a fixed point. Therefore, Theorem 2.4 yields a globally defined function ω_1 such that $g(\lambda,\omega_1(\lambda))=\omega_1(\lambda)$. Note that

$$\Im\omega_1(\lambda)=\Im\lambda+\Im h_2(\lambda+h_1(\omega_1(\lambda)))\geq\Im\lambda,\quad \lambda\in\mathbb{C}^+.$$

The second function is obtained simply as $\omega_2(\lambda)=\lambda+h_1(\omega_1(\lambda))$. The uniqueness of the functions ω_j follows from the uniqueness of Denjoy–Wolff points. $\square$

References

[1] H. Bercovici and D. Voiculescu, *Free convolution of measures with unbounded support*, Indiana Univ. Math. J. **42** (1993), 733–773.

[2] Ph. Biane, *Processes with free increments*, Math. Z. **227** (1998), 143–174.

[3] E. F. Collingwood and A. J. Lohwater, *The Theory of Cluster Sets*, Cambridge University Press, Cambridge, 1966.

[4] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, N.J., 1962.

[5] H. Maassen, *Addition of freely independent random variables*, J. Funct. Anal. **106** (1992), 409–438.

[6] J. Shapiro, *Composition Operators and Classical Function Theory*, Springer Verlag, New York, 1993.

[7] D. Voiculescu, *Addition of certain noncommuting random variables*, J. Funct. Anal. **66** (1986), 323–346.

[8] D. Voiculescu, *Multiplication of certain noncommuting random variables*, J. Operator Theory **18** (1987), 223–235.

[9] D. Voiculescu, *The analogues of entropy and of Fisher's information measure in free probability. I.*, Comm. Math. Phys. **155** (1993), 71–92.

[10] D. Voiculescu, *The coalgebra of the free difference quotient and free probability*, Internat. Math. Res. Notices **2000**, 79–106.

[11] D. Voiculescu, *Analytic subordination consequences of free Markovianity*, Indiana Univ. Math. J. **51** (2002), 1161–1166.

S. T. Belinschi
INSTITUTE OF MATHEMATICS
Simion Stoilow OF THE ROMANIAN ACADEMY
P. O. BOX 1-764
RO-014700 BUCHAREST, ROMANIA
and
DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO N2L 3G1, CANADA
email: sbelinsc@math.uwaterloo.ca

H. Bercovici
MATHEMATICS DEPARTMENT
INDIANA UNIVERSITY
BLOOMINGTON, IN 47405, USA
email: bercovic@indiana.edu

(Received May 11, 2006)