A BOUNDARY VERSION OF
AHLFORS' LEMMA, LOCALLY COMPLETE
CONFORMAL METRICS AND CONFORMALLY INVARIANT
REFLECTION PRINCIPLES FOR ANALYTIC MAPS

By

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Abstract. A boundary version of Ahlfors' Lemma is established and used to show that the classical Schwarz–Carathéodory reflection principle for holomorphic functions has a purely conformal geometric formulation in terms of Riemannian metrics. This conformally invariant reflection principle generalizes naturally to analytic maps between Riemann surfaces and contains among other results a characterization of finite Blaschke products due to M. Heins.

1 Introduction

In this paper, we show that the classical Schwarz–Carathéodory reflection principle for holomorphic functions can be stated in a conformally invariant way exclusively in terms of conformal Riemannian metrics. As some applications of this conformally invariant reflection principle, we obtain

(i) sharper and localized versions of Heins' characterization of finite Blaschke products [15];

(ii) the limiting case of a hyperbolic reflection principle due to Fournier and Ruscheweyh [10, 11];

and

(iii) an extension of the Schwarz–Carathéodory reflection principle for analytic maps between Riemann surfaces, which complements the results of [22].

The proofs of these results use a mixture of methods from classical function theory and nonlinear elliptic PDE. A basic ingredient of our approach is a boundary

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version of the ubiquitous Ahlfors Lemma [1, 5, 8] (see Theorem 1.4 below), which might be of interest in its own right. This "boundary Ahlfors Lemma" applies to locally complete conformal Riemannian metrics. It is related to work of Yau [26] and Bland [3] in higher dimensional complex differential geometry and previous work on the Gaussian curvature equation $\Delta u = -\kappa(z)\, e^{2u}$; see, for instance, [2, 4]. However, although we discuss these connections in some detail, the emphasis of the present paper is on function-theoretic applications of the boundary Ahlfors Lemma.

Here is a quick outline of our work. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$. Recall that an open subarc of the unit circle $\partial\mathbb{D} := \{z \in \mathbb{C} : |z| = 1\}$ is an open connected subset of $\partial\mathbb{D}$.

Theorem 1.1. *Let Γ be an open subarc of the unit circle $\partial\mathbb{D}$ and let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then the following conditions are equivalent.*

(a) *For every $\xi \in \Gamma$,*

$$\lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = +\infty \, .$$

(b) *For every $\xi \in \Gamma$,*

$$\liminf_{z \to \xi} \left(1 - |z|^2\right) \frac{|f'(z)|}{1 - |f(z)|^2} > 0 \, .$$

(c) *For every $\xi \in \Gamma$,*

$$\lim_{z \to \xi} \left(1 - |z|^2\right) \frac{|f'(z)|}{1 - |f(z)|^2} = 1 \, .$$

(d) *For every $\xi \in \Gamma$,*

$$\lim_{z \to \xi} |f(z)| = 1 \, .$$

(e) *f has a holomorphic extension across the arc Γ with $f(\Gamma) \subset \partial\mathbb{D}$.*

Roughly speaking, Theorem 1.1 asserts that if the quantity

$$\lambda(z) := \frac{|f'(z)|}{1 - |f(z)|^2}$$

blows up at the boundary arc Γ, then it grows at least as fast as

$$\lambda_{\mathbb{D}}(z) := \frac{1}{1 - |z|^2} \tag{1.1}$$

at Γ and then, in fact, it blows up exactly as fast as $\lambda_{\mathbb{D}}(z)$ at Γ. This in turn is equivalent to $|f(z)| \to 1$ as z tends to Γ.

Remark 1.2. We note that the implications "(e) $\Rightarrow$ (d)" and "(c) $\Rightarrow$ (b) $\Rightarrow$ (a)" in Theorem 1.1 hold trivially (and in fact for *every* set $\Gamma \subseteq \partial\mathbb{D}$). Also, the direction "(d) $\Rightarrow$ (e)" in Theorem 1.1 is of course exactly the statement of the classical Schwarz–Carathéodory reflection principle for holomorphic self-maps of the unit disk (see for instance [7]), while "(e) $\Rightarrow$ (a)" is rather straightforward: just observe that condition (e) combined with $f(\mathbb{D}) \subseteq \mathbb{D}$ guarantees $f' \neq 0$ on Γ, so (a) follows immediately. Thus the major parts of the proof of Theorem 1.1 lie in the remaining implications "(a) $\Rightarrow$ (c)" and "(b) $\Rightarrow$ (d)".

Remark 1.3. Direction "(a) $\Rightarrow$ (c)" in Theorem 1.1 might be considered as a statement about conformal metrics. In fact, if f is an analytic self-map of the unit disk with property (a), then

$$\lambda(z)\,|dz| := \frac{|f'(z)|}{1-|f(z)|^2}\,|dz|$$

is the pullback of the Poincaré or hyperbolic metric $\lambda_{\mathbb{D}}(w)\,|dw|$ (see (1.1)) on the unit disk $\mathbb{D}$ under the holomorphic map $w = f(z)$ and is therefore a regular pseudo-metric in $\mathbb{D}$ itself with the same (Gaussian) curvature as $\lambda_{\mathbb{D}}(z)\,|dz|$ (see Section 2 for details). Thus the implication "(a) $\Rightarrow$ (c)" in Theorem 1.1 is a very special case of the following result combined with the Schwarz–Pick inequality (see Remark 1.7).

Theorem 1.4. *Let Γ be an open subarc of the unit circle $\partial\mathbb{D}$, let $\lambda(z)\,|dz|$ be a regular conformal pseudo-metric on the unit disk $\mathbb{D}$ with curvature bounded below by a negative constant $-c_\lambda$ and let $\mu(z)\,|dz|$ be a conformal pseudo-metric on $\mathbb{D}$ with curvature bounded above by a negative constant $-C_\mu$. If*

$$\lim_{z\to\xi} \lambda(z) = +\infty$$

for every point $\xi \in \Gamma$, then

$$\liminf_{z\to\xi} \frac{\lambda(z)}{\mu(z)} \geq \sqrt{\frac{C_\mu}{c_\lambda}} \tag{1.2}$$

for every $\xi \in \Gamma$.

We refer to Theorem 1.4 as the boundary Ahlfors Lemma.

Remarks 1.5.

(i) The classical Ahlfors Lemma [1] (for the unit disk) is contained in Theorem 1.4 as a special case. To check this, choose $\lambda(z)\,|dz| := \lambda_{\mathbb{D}}(z)\,|dz|$ and let $\mu(z)\,|dz|$ be a conformal pseudo-metric with curvature bounded above by -4. Then $\lambda(z)\,|dz|$ has constant curvature -4 and $\lambda(z) \to +\infty$ as $|z| \to 1$, so

$$\limsup_{|z|\to 1} \frac{\mu(z)}{\lambda_{\mathbb{D}}(z)} \leq 1$$

by (1.2). Since the function

$$\log^+\Big(\frac{\mu(z)}{\lambda_{\mathbb{D}}(z)}\Big) := \max\Big\{0, \log\Big(\frac{\mu(z)}{\lambda_{\mathbb{D}}(z)}\Big)\Big\}$$

is subharmonic in $\mathbb{D}$, we deduce $\mu \leq \lambda_{\mathbb{D}}$ in $\mathbb{D}$ from the maximum principle for subharmonic functions.

(ii) As already mentioned above, Theorem 1.4 is connected with work of Yau [26] and Bland [3] in higher dimensional complex geometry. This is discussed in more detail in Section 5 below.

(iii) The proof of Theorem 1.4 (given in Section 3) is based on constructing suitable solutions of the nonlinear elliptic PDE $\Delta u = -\kappa(z)\, e^{2u}$. It turns out that it suffices to assume that the curvatures of $\lambda(z)\,|dz|$ and $\mu(z)\,|dz|$ are bounded from below and above respectively near the boundary arc Γ, i.e., in $U \cap \mathbb{D}$, where $U \subseteq \mathbb{C}$ is an open neighborhood of Γ.

We now return to a discussion of Theorem 1.1.

Remark 1.6. It is indispensable to consider *unrestricted* limits in Theorem 1.1. For instance, the implication "(d) $\Rightarrow$ (e)" of Theorem 1.1 would be false under the weaker assumption

$$\angle \lim_{z\to\xi} |f(z)| = 1 \tag{1.3}$$

for every $\xi \in \Gamma$ (where $\angle$ indicates non-tangential approach to the unit circle). In fact, Heins [16] constructed an infinite Blaschke product f with the following remarkable properties. The zeros of f cluster exactly at $z = 1$, f' has an infinite number of zeros in the open interval $(0, 1)$ and f has the angular limit 1 at $z = 1$. In particular, (1.3) holds for every $\xi \in \partial\mathbb{D}$, but f has clearly no holomorphic extension to a neighborhood of $\partial\mathbb{D}$.

Another (explicit) example of a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ satisfying (1.3) for every $\xi \in \partial\mathbb{D}$, but without an analytic continuation across the unit circle can be produced as follows. A result of Frostman [12] asserts that a Blaschke product

$$f(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j}\, z} \tag{1.4}$$

has an angular limit of modulus 1 at a point $\xi \in \partial\mathbb{D}$ if and only if

$$\sum_{j=1}^{\infty} \frac{1-|z_j|}{|\xi - z_j|} < \infty \,. \tag{1.5}$$

Choosing $r_j = 1 - 1/j^3$, one sees easily that

$$\frac{1 + r_j^2 - (1-r_j)^2 j^4}{2r_j} = \cos\phi_j$$

for some $\phi_j \in (-\pi, \pi)$. By construction, the points $z_j := r_j e^{i\phi_j} \in \mathbb{D}$ tend to 1 and satisfy

$$\sum_{j=1}^{\infty} \frac{1-|z_j|}{|1-z_j|} = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty \,.$$

Thus condition (1.3) holds on the entire unit circle for the corresponding infinite Blaschke product (1.4), which obviously has no analytic continuation to a neighborhood of $z = 1$.

Remark 1.7. The implication "(c) $\Rightarrow$ (d)" in Theorem 1.1 can be considered as a boundary version of Schwarz' Lemma. Recall that the Schwarz Lemma or, rather, the Schwarz–Pick Lemma asserts that

$$\left(1 - |z|^2\right) \frac{|f'(z)|}{1 - |f(z)|^2} \le 1$$

for every analytic self-map f of $\mathbb{D}$ with equality at some point z *inside* $\mathbb{D}$ if and only if f is a conformal automorphism of $\mathbb{D}$. In particular, equality at some point z inside $\mathbb{D}$ implies that f has an analytic extension across the unit circle. Theorem 1.1 "(c) $\Rightarrow$ (d)" allows the weaker conclusion that f can be analytically continued at least across an open subarc of the unit circle provided equality holds on this boundary arc (in the sense of condition (c)). Somewhat surprisingly, it even suffices to assume one of the (apparently) weaker conditions (a) or (b). It is not known to us whether "(c) $\Rightarrow$ (d)" holds if Γ reduces to a single point. Example 4.4 below shows that (a) does not imply (d) in this case.

Remark 1.8 (The Schwarz–Carathéodory reflection principle)**.** As noted in Remark 1.2 above, the direction "(d) $\Rightarrow$ (e)" in Theorem 1.1 is simply the Schwarz–Carathéodory reflection principle for holomorphic self-maps of the unit disk. Recall that the Schwarz reflection principle [24] tells us that a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with a continuous extension $f : \mathbb{D} \cup \Gamma \to \mathbb{C}$ to an open subarc $\Gamma \subseteq \partial\mathbb{D}$ such that $f(\Gamma) \subset \partial\mathbb{D}$ has an analytic continuation across Γ. More precisely, f has a meromorphic continuation to the entire Riemann sphere $\hat{\mathbb{C}}$ except the complementary arc $\partial\mathbb{D} \setminus \Gamma$; and this extension is given by

$$f(z) := \frac{1}{\overline{f(1/\overline{z})}}\,, \quad z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}\,. \tag{1.6}$$

In Carathéodory's generalization [7] of the Schwarz reflection principle (implication "(d) $\Rightarrow$ (e)" in Theorem 1.1), not even the continuity of f on Γ is assumed. Indeed, it suffices to suppose that the *modulus* of f has a continuous extension to Γ with $|f(\Gamma)| = 1$ in order to guarantee the analytic continuability of f across Γ. Theorem 1.1 shows that we can use one of the conformally invariant conditions (a), (b) or (c) instead of condition (d) to obtain a formulation of the Schwarz–Carathéodory reflection principle entirely in terms of the Poincaré metric $\lambda_{\mathbb{D}}(z)\,|dz|$ on $\mathbb{D}$. In this form, the Schwarz–Carathéodory reflection principle admits various generalizations. For instance, when combined with the boundary Ahlfors Lemma (Theorem 1.4), it leads to a reflection principle in terms of more general conformal metrics than the Poincaré metric (see Theorem 4.3). Moreover, it carries easily over to analytic maps $f : S \to R$, where

(i) S is a domain in the complex plane with a free analytic boundary arc and R is a simply connected bordered Riemann surface

or

(ii) S is a bordered Riemann surface and R is a simply connected bordered Riemann surface.

We discuss a number of such extensions of Theorem 1.1, which complement the results in [22], in Section 4 and Section 6 below.

Remark 1.9 (The Fournier–Ruscheweyh reflection principle)**.** Theorem 1.1 may also be considered as the limiting case of a recent reflection principle due to Fournier and Ruscheweyh [11] (see also [22]). This reflection principle asserts that a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ has an analytic continuation across an open subarc Γ of the unit circle with $f(\Gamma) \subseteq \mathbb{D}$ if for some constant $k > 0$ the free boundary condition

$$\lim_{z \to \xi} \frac{|f'(z)|}{1 - |f(z)|^2} = k\,, \tag{1.7}$$

holds for every $\xi \in \Gamma$. Note that condition (1.7) ensures that the analytic continuation of f maps the arc Γ onto an analytic arc which lies compactly inside the unit disk. Also, the continuation is carried out implicitly by solving a Riccati differential equation, and it is still an open problem to describe the possible types of singularities of the analytic continuation of f (see [11, 22]). The limit case

$k \to \infty$ of the free boundary condition (1.7) is handled by Theorem 1.1 and shows that f has a global (meromorphic) extension to $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ across the boundary arc Γ. This extension maps Γ onto a subarc of the unit circle and is given by the simple explicit formula (1.6). In particular, Theorem 1.1 combined with the Fournier–Ruscheweyh reflection principle give conformally invariant characterizations of holomorphic self-maps of the unit disk which have an analytic extension across a boundary arc Γ in the two cases that $f(\Gamma) \subset \mathbb{D}$ or $f(\Gamma) \subseteq \partial\mathbb{D}$.

We finally note another application of Theorem 1.1. The special case of Theorem 1.1 that the boundary arc Γ is the whole of $\partial\mathbb{D}$ leads to the following corollary.

Corollary 1.10. *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then the following conditions are equivalent.*

(a) $\displaystyle \lim_{|z|\to 1} \frac{|f'(z)|}{1-|f(z)|^2} = +\infty.$

(b) *f is a finite Blaschke product.*

In fact, by Theorem 1.1, condition (a) holds if and only if f has a meromorphic continuation to $\hat{\mathbb{C}}$ such that $|f(z)| = 1$ on $\partial\mathbb{D}$, i.e., if and only if f is a finite Blaschke product.

Remark 1.11. Corollary 1.10 is closely related to a result of Heins [15], who showed that an analytic self-map f of the unit disk is a finite Blaschke product if and only if

$$\lim_{|z|\to 1} \left(1-|z|^2\right) \frac{|f'(z)|}{1-|f(z)|^2} = 1 \,. \tag{1.8}$$

Theorem 1.1 might therefore be considered as a localized and sharper form of Heins' result, since it also applies to proper subarcs of the unit circle and allows to replace condition (1.8) by the (apparently) weaker condition (a) of Corollary 1.10. Heins' proof is based on a completely different but very interesting argument. He first shows that (1.8) implies f has finite valence and maps onto $\mathbb{D}$, so f is a proper self-map of the unit disk, i.e., a finite Blaschke product. This global argument cannot be used to prove the (local) statement of Theorem 1.1.

The present paper is organized as follows. In Section 2, we start off by recalling some basic facts about conformal metrics, including a version of Ahlfors' Lemma which is closely related to the well-known Ahlfors–Yau lemma [26]. Section 3 is devoted to the proof of the boundary Ahlfors Lemma (Theorem 1.4). The proof of the (remaining) implication "(b)⇒(d)" of Theorem 1.1 is given in Section 4, which also contains a further discussion of the assumptions of Theorem 1.1. In

Section 5, we extend Theorem 1.4 to conformal metrics defined on more general domains in $\mathbb{C}$ than the unit disk. For this purpose, we compare the notion of a locally complete conformal metric introduced by J. Bland [3] (for the more general situation of complex manifolds) with the condition that the density of the metric goes to infinity at the boundary. Based on the results of Section 5, we finally prove several extensions of Theorem 1.1 for analytic maps between Riemann surfaces in Section 6.

2 Conformal metrics and Ahlfors' lemma

We first give a short review of some basic facts about one-dimensional Kähler metrics, which we prefer to call conformal metrics. For more information, the reader is refered to [1, 14].

For brevity, we call every positive (resp. non-negative) upper semicontinuous function λ on a domain $G \subseteq \mathbb{C}$ a conformal metric (resp. conformal pseudo-metric) and denote it by $\lambda(z)\,|dz|$. We sometimes need to distinguish between the function λ and the induced pseudo-metric. In this case, we call λ the density of $\lambda(z)\,|dz|$. A conformal (pseudo-)metric $\lambda(z)\,|dz|$ on a domain $G \subseteq \mathbb{C}$ is said to be regular, if λ is of class C^2 in a neighborhood of every point where λ does not vanish. The (Gaussian) curvature of a regular conformal pseudo-metric $\lambda(z)\,|dz|$ is defined for every point $z \in G$ with $\lambda(z) > 0$ by

$$\kappa_\lambda(z) := -\frac{(\Delta \log \lambda)(z)}{\lambda(z)^2}\,.$$

Here, Δ is the Laplace operator. Inequalities such as $\kappa_\lambda \leq 0$ are thus interpreted as $\kappa_\lambda(z) \leq 0$ for all $z \in G \setminus Z_\lambda$, where $Z_\lambda := \{z \in G : \lambda(z) = 0\}$. Given a conformal metric $\lambda(z)\,|dz|$ on a domain $G \subseteq \mathbb{C}$, we can associate a distance function

$$\mathrm{d}_\lambda(z_0, z_1) := \inf_\gamma \int\limits_\gamma \lambda(z)\,|dz|\,,$$

where the infimum is taken over all rectifiable paths in G joining $z_0 \in G$ and $z_1 \in G$. This makes (G, d_λ) a metric space. We call $\lambda(z)\,|dz|$ complete (on G), if (G, d_λ) is a complete metric space.

A basic property of curvature is its absolute conformal invariance. This means that for a regular conformal pseudo-metric $\lambda(w)\,|dw|$ on a domain D and a holomorphic map $w = f(z)$ from another domain G to D, the pullback $(f^*\lambda)(z)\,|dz| := \lambda(w)\,|dw|$ is again a regular conformal pseudo-metric on G with curvature

$$\kappa_{f^*\lambda}(z) = \kappa_\lambda(f(z))\,.$$

When the domain $G \subset \mathbb{C}$ has at least two boundary points (i.e., when G is a hyperbolic domain) and $-a$ is a negative constant, G carries a unique complete regular conformal metric with constant curvature $-a$. This metric $\lambda(w)\,|dw|$ is obtained from the Poincaré metric $\lambda_{\mathbb{D}}(z)\,|dz|$ by means of a universal cover projection $\pi : \mathbb{D} \to G$ from

$$\lambda(\pi(z))\,|\pi'(z)| = \frac{2}{\sqrt{a}}\,\lambda_{\mathbb{D}}(z) = \frac{2}{\sqrt{a}}\frac{1}{1-|z|^2}\,.$$

We call $\lambda(w)\,|dw|$ the hyperbolic metric on G with curvature $-a$. Unless explicitly stated otherwise, we take the normalization $a = 4$ and call the corresponding metric *the* hyperbolic metric of G (with constant curvature -4). This metric is denoted by $\lambda_G(w)\,|dw|$.

The most important result about conformal metrics is Ahlfors' Lemma [1]. We make use of the following slight generalization of this result.

Theorem 2.1. *Let $\lambda(z)\,|dz|$ be a regular conformal metric on the unit disk $\mathbb{D}$ with curvature bounded below by a negative constant $-c_\lambda$ and let $\mu(z)\,|dz|$ be a regular conformal pseudo-metric on $\mathbb{D}$ with curvature bounded above by a negative constant $-C_\mu$. If*

$$\lim_{|z|\to 1} \lambda(z) = +\infty, \tag{2.1}$$

then

$$\lambda(z) \geq \sqrt{\frac{C_\mu}{c_\lambda}} \cdot \mu(z) \tag{2.2}$$

for every $z \in \mathbb{D}$. In particular, $\lambda(z)\,|dz|$ is a complete metric on $\mathbb{D}$.

The proof of Theorem 2.1 follows the same lines as the proof of the classical version of Ahlfors' Lemma. However, since Theorem 2.1 plays a major rôle in this paper, we include a proof for the convenience of the reader.

Proof. Fix a number $0 < r < 1$ and consider

$$\mu_r(z) := r\mu(rz)\,.$$

Then $\mu_r(z)\,|dz|$ is a conformal pseudo-metric with curvature bounded above by $-C_\mu$. Since $\mu_r(z)$ is bounded from above in $\mathbb{D}$ and $\lambda(z) \to +\infty$ as $|z| \to 1$, the function

$$\sigma(z) := \log\frac{\lambda(z)}{\mu_r(z)}\,, \qquad z \in \mathbb{D}\,,$$

attains a minimum value at some point $z_0 \in \mathbb{D}$. Note that $\mu_r(z_0) \neq 0$, so σ is of class C^2 in a neighborhood of z_0; and we obtain $0 \leq \Delta\sigma(z_0) = -\kappa_\lambda(z_0)\,\lambda(z_0)^2 + \kappa_{\mu_r}(z_0)\,\mu_r(z_0)^2$. This implies $\kappa_\lambda(z_0) < 0$, and we can write

$$\begin{aligned} 0 \leq& \Delta\sigma(z_0) = -\kappa_\lambda(z_0)\,\lambda(z_0)^2 + \kappa_{\mu_r}(z_0)\,\mu_r(z_0)^2 \\ =& -\kappa_\lambda(z_0)\,\lambda(z_0)^2 \left(1 - \frac{\kappa_{\mu_r}(z_0)}{\kappa_\lambda(z_0)}\,\frac{\mu_r(z_0)^2}{\lambda(z_0)^2}\right). \end{aligned}$$

Thus

$$\inf_{z \in \mathbb{D}} \frac{\lambda(z)}{\mu_r(z)} = \inf_{z \in \mathbb{D}} e^{\sigma(z)} = e^{\sigma(z_0)} = \frac{\lambda(z_0)}{\mu_r(z_0)} \geq \sqrt{\frac{\kappa_{\mu_r}(z_0)}{\kappa_\lambda(z_0)}} \geq \sqrt{\frac{C_\mu}{c_\lambda}}\,.$$

Inequality (2.2) follows by letting $r \to 1$.

In order to show $\lambda(z)\,|dz|$ is a complete conformal metric on $\mathbb{D}$, we set $\mu(z)\,|dz| = \lambda_{\mathbb{D}}(z)\,|dz|$. Then, in view of (2.2), there is a constant $\alpha > 0$ such that

$$\mathrm{d}_\lambda(z, z_0) \geq \alpha \cdot \mathrm{d}_{\mathbb{D}}(z, z_0)$$

for any $z, z_0 \in \mathbb{D}$. Here, $\mathrm{d}_{\mathbb{D}}$ denotes the hyperbolic distance induced by the Poincaré metric $\lambda_{\mathbb{D}}(z)\,|dz|$. Thus every Cauchy sequence $\{z_n\}$ in $(\mathbb{D}, \mathrm{d}_\lambda)$ is also a Cauchy sequence in the complete metric space $(\mathbb{D}, \mathrm{d}_{\mathbb{D}})$ and therefore convergent with limit z_0 in $\mathbb{D}$. But the topologies of $(\mathbb{D}, \mathrm{d}_{\mathbb{D}})$ and $(\mathbb{D}, \mathrm{d}_\lambda)$ are compatible, so the sequence $\{z_n\}$ converges to z_0 also in $(\mathbb{D}, \mathrm{d}_\lambda)$. □

Remarks 2.2.

(a) If $\lambda(z)\,|dz| = \lambda_{\mathbb{D}}(z)\,|dz|$ is the hyperbolic metric in $\mathbb{D}$ and if $\mu(z)\,|dz|$ is a pseudo-metric in $\mathbb{D}$ with curvature bounded above by -4, then Theorem 2.1 is the classical Ahlfors Lemma [1].

(b) Theorem 2.1 is related to Yau's celebrated extension of Ahlfors' Lemma for Kähler manifolds (see [26]). In the very special case of the unit disk $\mathbb{D} \subset \mathbb{C}$, Yau's version of Ahlfors' Lemma differs from Theorem 2.1 only in the assumptions on the conformal metric $\lambda(z)\,|dz|$. Yau requires that $\lambda(z)\,|dz|$ be a complete conformal metric, whereas Theorem 2.1 assumes the boundary condition (2.1) for the density λ and yields the completeness of $\lambda(z)\,|dz|$ as a corollary to the estimate (2.2). Thus, Theorem 2.1 and Yau's lemma (for the unit disk) are equivalent. We note that Yau's lemma remains valid for arbitrary hyperbolic domains (and way beyond), while Theorem 2.1 can be extended only to domains with sufficiently nice boundary. This is discussed in Section 5.

(c) The regularity assumption on the pseudo-metric $\mu(z)\,|dz|$ in Theorem 2.1 is not essential. In fact, it suffices to assume that $\mu(z)\,|dz|$ is an upper semicontinuous pseudo-metric on $\mathbb{D}$ with generalized curvature

$$\underline{\kappa}_\mu(z) := -\frac{\underline{\Delta} \log \mu(z)}{\mu(z)^2}$$

bounded above by a negative constant. Here

$$\underline{\Delta} \log \mu(z) := \liminf_{r \to 0} \frac{4}{r^2} \left\{ \frac{1}{2\pi} \int\limits_0^{2\pi} \log \mu(z + re^{it})\, dt - \log \mu(z) \right\}$$

denotes the generalized lower Laplacian of $\log \mu$, which is defined for every $z \in \mathbb{D}$ with $\mu(z) > 0$. Accordingly, when we speak in the following of a conformal pseudo-metric $\mu(z)\,|dz|$ with curvature κ_μ, we always mean that $\mu(z)\,|dz|$ is upper semicontinuous with generalized curvature $\underline{\kappa}_\mu$.

(d) (The case of equality in (2.2).)

If equality holds in (2.2) for some point $z \in \mathbb{D}$, then equality holds for every point $z \in \mathbb{D}$. To see this (cf. [14, 19, 23]), note that the function

$$u(z) := \log \frac{\mu(z)}{\lambda(z)} - \log \sqrt{\frac{c_\lambda}{C_\mu}}$$

is non-positive and satisfies

$$\begin{aligned} \Delta u(z) &= -\kappa_\mu(z)\mu(z)^2 + \kappa_\lambda(z)\lambda(z)^2 \geq C_\mu \mu(z)^2 - c_\lambda \lambda(z)^2 \\ &= c_\lambda \lambda(z)^2 \big(e^{2u(z)} - 1\big) \\ &\geq 2c_\lambda \lambda(z)^2 u(z) \end{aligned}$$

in the open set $D := \{z \in \mathbb{D} : \mu(z) > 0\}$. Thus, if $u(a) = 0$ for some $a \in \mathbb{D}$, the function u restricted to the component D' of D which contains the point a has a non-negative maximum at a. By the strong maximum principle of E. Hopf (cf. [13, Thm. 3.5]), u has to be constant in D', so μ/λ is constant in $\mathbb{D}$. Again, this uniqueness result holds also for upper semicontinuous pseudo-metrics $\mu(z)\,|dz|$ with generalized curvature $\underline{\kappa}_\mu \leq -C_\mu < 0$. One just needs to use Calabi's extension of the Hopf maximum principle, see [6] (and [19]).

(e) The assumptions on the curvature of the metrics $\lambda(z)\,|dz|$ and $\mu(z)\,|dz|$ in Theorem 2.1 are quite natural and cannot be omitted. This is illustrated with the following examples.

Example 2.3. Let $\lambda(z) = \sqrt{\lambda_{\mathbb{D}}(z)}$ and $\mu(z) = \lambda_{\mathbb{D}}(z)$ for $z \in \mathbb{D}$. Then $\kappa_\lambda(z) \leq -2$ for $z \in \mathbb{D}$ and $\kappa_\lambda(z) \to -\infty$ when $|z| \to 1$. Obviously, there is no positive constant C such that $\lambda(z) > C\mu(z)$ for all $z \in \mathbb{D}$.

Example 2.4. Consider $\lambda(z) = \lambda_{\mathbb{D}}(z)$ and $\mu(z) = \exp(\lambda_{\mathbb{D}}(z))$. Then $\kappa_\mu(z) < 0$ for $z \in \mathbb{D}$, and $\kappa_\mu(z) \to 0$ as $|z| \to 1$. Again, there exists no constant $C > 0$ such that $\lambda_{\mathbb{D}}(z) \geq C \exp(\lambda_{\mathbb{D}}(z))$ for all $z \in \mathbb{D}$.

3 Proof of the boundary Ahlfors Lemma

Before we give the proof of Theorem 1.4, we sketch its main idea. For this purpose suppose, that $\lambda(z)\,|dz|$ is a regular conformal metric which tends to $+\infty$ on a proper open subarc Γ of the unit circle. As becomes clear later, we may further assume without loss of generality that

$$m_\lambda := \inf_{z\in\mathbb{D}} \lambda(z) > 0. \tag{3.1}$$

The strategy is to 'modify' $\lambda(z)\,|dz|$ appropriately so that Theorem 2.1 applies. We make the Ansatz

$$\Lambda(z)\,|dz| := \lambda(z)\sigma(z)\,|dz|$$

and wish to choose the auxiliary conformal metric $\sigma(z)\,|dz|$ such that

$$\lim_{z\to\xi} \sigma(z) = +\infty \ \text{ for } \xi \in \partial\mathbb{D}\setminus\Gamma \quad \text{ and } \quad \liminf_{z\to\xi} \sigma(z) > 0 \ \text{ for } \xi \in \Gamma\,. \tag{3.2}$$

This ensures, in view of (3.1), that $\Lambda(z) \to +\infty$ on the entire unit circle. On the other hand, (3.2) implies that $\kappa_\sigma(z)$ is negative at least at some point $z_0 \in \mathbb{D}$, because otherwise $\log\sigma$ is superharmonic in $\mathbb{D}$, which contradicts (3.2). Hence

$$c_\sigma := -\inf_{z\in\mathbb{D}} \kappa_\sigma(z) > 0\,. \tag{3.3}$$

For the curvature of $\Lambda(z)\,|dz|$, we find

$$\kappa_\Lambda(z) = \frac{\kappa_\lambda(z)}{\sigma(z)^2} + \frac{\kappa_\sigma(z)}{\lambda(z)^2} \geq -\frac{c_\lambda}{\sigma(z)^2} + \frac{\kappa_\sigma(z)}{\lambda(z)^2} \geq -\frac{c_\lambda}{\sigma(z)^2} - \frac{c_\sigma}{m_\lambda^2}\,.$$

In order to be able to apply Theorem 2.1, we need a lower bound for this curvature. In view of (3.3), we therefore require

$$m_\sigma := \inf_{z\in\mathbb{D}} \sigma(z) > 0\,.$$

Thus we have

$$\kappa_\Lambda(z) \geq -c_\Lambda \quad \text{for } c_\Lambda := \frac{c_\lambda}{m_\sigma^2} + \frac{c_\sigma}{m_\lambda^2}\,,$$

and Theorem 2.1 gives

$$\frac{\lambda(z)}{\mu(z)} = \frac{\Lambda(z)}{\mu(z)}\frac{1}{\sigma(z)} \geq \sqrt{\frac{C_\mu}{c_\Lambda}}\frac{1}{\sigma(z)} = \sqrt{\frac{C_\mu}{c_\lambda + c_\sigma\dfrac{m_\sigma^2}{m_\lambda^2}}}\frac{m_\sigma}{\sigma(z)}$$

for $z \in \mathbb{D}$. The right side of the last inequality, however, is less than

$$\sqrt{\frac{C_\mu}{c_\lambda + c_\sigma \dfrac{m_\sigma^2}{m_\lambda^2}}},$$

which in turn is (even strictly) less than the bound $\sqrt{C_\mu / c_\lambda}$, we want to establish; see (1.2). To arrive at this desired bound, we would have to put $c_\sigma = 0$, which, as we have seen above, is not possible because of (3.2).

We sidestep this problem by carefully choosing a sequence of conformal metrics $\sigma_n(z)\,|dz|$ with negative curvature κ_{σ_n} tending to 0 and having the appropriate boundary behavior (3.2). Each of these metrics $\sigma_n(z)\,|dz|$ is constructed as the solution to a boundary value problem with degenerate Dirichlet data for the Gaussian curvature equation.

Theorem 3.1. *Let I be a proper open subarc of the unit circle $\partial\mathbb{D}$ such that $\overline{I} \neq \partial\mathbb{D}$ and let $k : \mathbb{D} \to [a, M]$, $0 < a < M < \infty$, be a locally Hölder continuous function with exponent α, $0 < \alpha \leq 1$. Then there is a C^2-solution $u : \mathbb{D} \to \mathbb{R}$ to*

$$\Delta u = k(z)\, e^{2u} \quad \textit{in } \mathbb{D} \tag{3.4}$$

such that

$$\lim_{z \to \xi} u(z) = \begin{cases} +\infty & \textit{for } \xi \in \partial\mathbb{D} \backslash \overline{I} \\ 0 & \textit{for } \xi \in I \end{cases} \tag{3.5}$$

and

$$\liminf_{z \to \xi} u(z) \geq 0 \quad \textit{if } \xi \in \overline{I} \backslash I\,. \tag{3.6}$$

In the proof of Theorem 3.1, we make repeated use of the following facts about the Gaussian curvature equation (3.4). We adopt standard notation, so $C(\overline{\Omega})$ is the set of real-valued continuous functions on the set $\overline{\Omega} \subset \mathbb{C}$ and $C^k(\Omega)$ is the set of real-valued functions having all derivatives of order $\leq k$ continuous in the open set $\Omega \subseteq \mathbb{C}$.

(CE1) (Comparison principle, see [13, Theorem 10.1])

If $u_1, u_2 \in C^2(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ are two solutions to (3.4) and if $u_1 \leq u_2$ on the boundary $\partial\Omega$ of a domain $\Omega \subseteq \mathbb{D}$, then $u_1 \leq u_2$ in $\overline{\Omega}$. For this fact $k(z) \geq 0$ in $\mathbb{D}$ would suffice.

(CE2) Let $\Omega \subseteq \mathbb{D}$ be a regular domain with Green's function[1] g_Ω. If $k : \mathbb{D} \to \mathbb{R}$ is bounded and locally Hölder continuous with exponent $\alpha, 0 < \alpha \leq 1$, and u is a bounded and integrable function on Ω, then

$$v(z) := -\frac{1}{2\pi} \iint\limits_{\Omega} g_\Omega(z,\zeta)\, k(\zeta)\, e^{2u(\zeta)}\, d\sigma_\zeta, \quad z \in \Omega\,,$$

σ_ζ denotes two-dimensional Lebesgue measure, belongs to $C^1(\Omega) \cap C(\overline{\Omega})$ and $v \equiv 0$ on $\partial\Omega$. If, in addition, u is locally Hölder continuous with exponent β, $0 < \beta \leq 1$, then $v \in C^2(\Omega)$ and $\Delta v = k(z)\, e^{2u}$ in Ω (see [13, p. 53-54] and [9, p. 241]).

Combining (CE1) and (CE2), we obtain

(CE3) Let $k : \mathbb{D} \to \mathbb{R}$ be bounded, non-negative and locally Hölder continuous with exponent $\alpha, 0 < \alpha \leq 1$. If $u : \mathbb{D} \to \mathbb{R}$ is a C^2-solution to (3.4) and u is continuous on the closure $\overline{\Omega}$ of a regular domain $\Omega \subseteq \mathbb{D}$, then the integral formula

$$u(z) = h(z) - \frac{1}{2\pi} \iint\limits_{\Omega} g_\Omega(z,\zeta)\, k(\zeta)\, e^{2u(\zeta)}\, d\sigma_\zeta \tag{3.7}$$

holds for every $z \in \Omega$. Here, h is harmonic in Ω and continuous on $\overline{\Omega}$ with boundary values u, i.e.,

$$h\big|_{\partial\Omega} \equiv u\big|_{\partial\Omega}\,.$$

Conversely, if u is a locally integrable and bounded function on a regular domain $\Omega \subseteq \mathbb{D}$ satisfying (3.7) for some harmonic function h in Ω which is continuous on $\overline{\Omega}$, then u belongs to $C^2(\Omega) \cap C(\overline{\Omega})$ and is a solution to $\Delta u = k(z)\, e^{2u}$ in Ω with $h \equiv u$ on $\partial\Omega$.

We also need another refinement of Ahlfors' Lemma, due to D. Minda [20], which is provided in the next lemma.

Lemma 3.2. *Let Ω be a domain in the complex plane which carries a hyperbolic metric $\lambda(z)\,|dz|$ with constant curvature $-a' < 0$. If $\nu(z)\,|dz|$ is a conformal pseudo-metric in Ω and if for every point $z_0 \in \Omega$ either*

(i) $\nu(z_0) \leq \lambda(z_0)$

or

(ii) $\nu(z_0) > 0$, *ν is of class C^2 in a neighborhood $U \subseteq \Omega$ of z_0 and $\kappa_\nu(z) \leq -a'$ for all $z \in U$,*

[1] We use the function-theoretic convention. Thus Green's function is always non-negative.

then $\nu(z) \le \lambda(z)$ *for every* $z \in \Omega$.

Proof of Theorem 3.1. Let us denote the two boundary points of I by $e^{i\varphi_1}$ and $e^{i\varphi_2}$, i.e., $\overline{I} \backslash I = \{e^{i\varphi_1}, e^{i\varphi_2}\}$, where $e^{i\varphi_1}$ is the right-hand endpoint of I and $e^{i\varphi_2}$ the left-hand endpoint (in counterclockwise direction). We define for each integer $n \ge 1$ a continuous function $f_n : \partial\mathbb{D} \to \mathbb{R}$ by

(i) $f_n \equiv 0$ on $\overline{I}$,

(ii) $f_n \equiv n$ on $\partial\mathbb{D} \backslash I_n$, where

$$I_n := I \cup \left\{e^{it} : t \in (\varphi_1 - \tfrac{1}{n}, \varphi_1]\right\} \cup \left\{e^{it} : t \in [\varphi_2, \varphi_2 + \tfrac{1}{n})\right\},$$

(iii) f_n is linear on $\left\{e^{it} : t \in (\varphi_1 - \frac{1}{n}, \varphi_1)\right\}$ and $\left\{e^{it} : t \in (\varphi_2, \varphi_2 + \frac{1}{n})\right\}$,

see Figure 1.

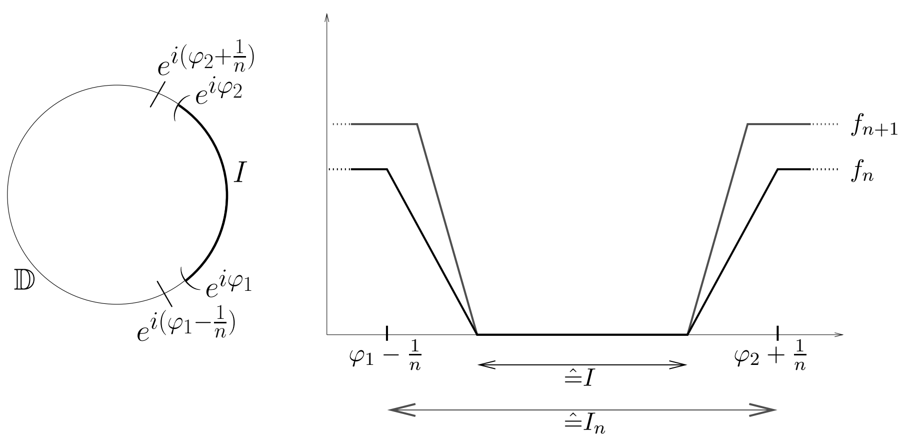

Figure 1.

The standard theory for non-linear elliptic PDEs guarantees (see [9, p. 286] or [13, p. 250]) that there is a unique real-valued solution $u_n \in C^2(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ to the boundary value problem

$$\begin{aligned} \Delta u &= k(z)\, e^{2u} && \text{in } \mathbb{D} \\ u &= f_n && \text{on } \partial\mathbb{D}\,. \end{aligned}$$

We are going to show that the sequence $\{u_n\}$ converges in $\mathbb{D}$ to a C^2-function $u : \mathbb{D} \to \mathbb{R}$, which is a solution to (3.4) and satisfies the boundary conditions (3.5) and (3.6).

Since the boundary functions f_n form a monotonically increasing sequence on $\partial\mathbb{D}$, the comparison principle (CE1) shows that the corresponding solutions u_n

also form a monotonically increasing sequence in $\mathbb{D}$. In addition, $\{u_n\}$ is locally bounded in $\mathbb{D}$. To see this, we need only note that $e^{u_n(z)}\,|dz|$ is a regular conformal metric on $\mathbb{D}$ with curvature bounded above by $-a<0$. Thus Theorem 2.1 implies that $u_n(z)\le v(z)$ for every $z\in\mathbb{D}$ and every $n\ge 1$, where $e^{v(z)}\,|dz|$ is the hyperbolic metric on $\mathbb{D}$ with curvature $-a$, i.e.,

$$v(z) := \log\left(\frac{2}{\sqrt{a}}\,\frac{1}{1-|z|^2}\right). \tag{3.8}$$

Hence $\{u_n\}$ converges monotonically in $\mathbb{D}$ to a locally bounded and locally integrable function $u:\mathbb{D}\to\mathbb{R}$.

To show that u is C^2 in $\mathbb{D}$ and also a solution to (3.4), fix $0<\varrho<1$ and let $\Omega=K_\varrho(0)$.[2] We choose continuous functions $h_n:\overline{\Omega}\to\mathbb{R}$ which are harmonic in Ω and having boundary values u_n. Then by (CE3),

$$u_n(z) = h_n(z) - \frac{1}{2\pi}\iint\limits_{\Omega} g_\Omega(z,\zeta)\,k(\zeta)\,e^{2u_n(\zeta)}\,d\sigma_\zeta\,, \quad z\in\Omega\,.$$

Let

$$V := \max_{z\in\overline{\Omega}} v(z)\,,$$

where the function v is defined in (3.8). Then

$$h_n(z) \le V + \frac{1}{2\pi}\,M\,e^{2V}\iint\limits_{\Omega} g_\Omega(z,\zeta)\,d\sigma_\zeta \le C_1\,;$$

see [9, p. 241]. Therefore, $\{h_n\}$ is a monotonically increasing and bounded sequence of harmonic functions in Ω, so Harnack's theorem shows that $\{h_n\}$ converges locally uniformly in Ω to a harmonic function $h:\Omega\to\mathbb{R}$. Thus, by monotone convergence,

$$u(z) = h(z) - \frac{1}{2\pi}\iint\limits_{\Omega} g_\Omega(z,\zeta)\,k(\zeta)\,e^{2u(\zeta)}\,d\sigma_\zeta\,, \quad z\in\Omega\,.$$

Since u is bounded and integrable in Ω, (CE3) shows that u is C^2 in Ω and a solution to $\Delta u = k(z)e^{2u}$ there.

It remains to study the boundary behavior of u. This is easy for ξ off the arc I but a little more subtle for $\xi\in I$.

(i) Fix $\xi\in\partial\mathbb{D}\backslash\overline{I}$. By our choice of the subarcs I_n, there exists an integer N such that $\xi\notin I_n$ for all $n\ge N$. To establish (3.5), assume to the contrary that we can find a sequence $\{z_j\}\subset\mathbb{D}$ which converges to ξ and a constant $0<C_2<\infty$

[2] We denote by $K_r(z_0)$ the euclidean disk about $z_0\in\mathbb{C}$ with radius $r>0$.

such that $u(z_j) < C_2$ for all j. Now choose an integer $m > \max\{N, C_2\}$. Since $u_m(z_j) \to m$ as $j \to \infty$, there is an integer J such that $u_m(z_j) > C_2$ for all $j > J$. But then the monotonicity of $\{u_n\}$ yields $u(z_j) \geq C_2$ for all $j > J$, and the contradiction is apparent. Thus

$$\lim_{z \to \xi} u(z) = +\infty$$

for every $\xi \in \partial\mathbb{D} \backslash \overline{I}$, as desired.

(ii) Pick $\xi \in I$. Since I is open, we can find ε, $0 < \varepsilon < 1$, such that $K_\varepsilon(\xi) \cap \partial\mathbb{D} \subset I$. In order to prove

$$\lim_{z \to \xi} u(z) = 0,$$

we first show that $\{u_n\}$ is bounded on $\overline{\Omega}$, where $\Omega = K_{\varepsilon/2}(\xi) \cap \mathbb{D}$. For that, let $\nu(z)\,|dz|$ be a conformal metric on $K_\varepsilon(\xi)$ with constant curvature $-a' < 0$, where

$$a' := \min\{a, 4\} > 0$$

and $\nu(z) > 1$ for $z \in K_\varepsilon(\xi)$. Recall that a denotes a lower bound of the function k. For example, we can choose

$$\nu(z)\,|dz| = \frac{1}{\varepsilon\left(1 - \dfrac{a'}{4\varepsilon^2}\,|z-\xi|^2\right)}\,|dz|\,.$$

For each integer $n \geq 1$, we consider the upper semicontinuous conformal metric $\mu_n(z)\,|dz|$ on $K_\varepsilon(\xi)$ defined by

$$\mu_n(z) = \begin{cases} e^{u_n(z)} & \text{for } z \in K_\varepsilon(\xi) \cap \mathbb{D} \\ \nu(z) & \text{for } z \in K_\varepsilon(\xi) \backslash \mathbb{D}\,. \end{cases}$$

We claim that $\mu_n(z) \leq \lambda(z)$, where $\lambda(z)\,|dz|$ denotes the hyperbolic metric on $K_\varepsilon(\xi)$ with constant curvature $-a'$. This is a consequence of Lemma 3.2, because

$$\kappa_{\mu_n}(z) = \kappa_{e^{u_n}}(z) = -k(z) \leq -a \leq -a' \quad \text{for } z \in K_\varepsilon(\xi) \cap \mathbb{D}$$

and

$$\mu_n(z) = \nu(z) \leq \lambda(z) \quad \text{in } K_\varepsilon(\xi) \backslash \mathbb{D}$$

(note that the latter inequality follows, for instance, from Theorem 2.1). Thus $\mu_n(z) \leq \lambda(z)$ for $z \in K_\varepsilon(\xi)$ and all n, so

$$u_n(z) \leq \max_{z \in \overline{\Omega}} \log \lambda(z) =: C_3$$

for $z \in \overline{\Omega}$ and all n.

As every u_n is a C^2-solution to $\Delta u = k(z)e^{2u}$ in Ω and continuous on $\overline{\Omega}$, we can use once more the integral formula for u_n, that is,

$$u_n(z) = h_n(z) - \frac{1}{2\pi} \iint\limits_{\Omega} g_\Omega(z,\zeta)\, k(\zeta)\, e^{2u_n(\zeta)}\, d\sigma_\zeta\,, \quad z \in \Omega\,, \tag{3.9}$$

where h_n is harmonic in Ω and continuous on $\overline{\Omega}$ with boundary values u_n. In particular, $\{h_n\}$ is bounded on Ω and $h_n \equiv 0$ on $\gamma := \overline{\Omega} \cap \partial\mathbb{D}$ for every integer $n \geq 1$. Letting $n \to \infty$ in (3.9) yields

$$u(z) = h(z) - \frac{1}{2\pi} \iint\limits_{\Omega} g_\Omega(z,\zeta)\, k(\zeta)\, e^{2u(\zeta)}\, d\sigma_\zeta$$

for $z \in \Omega$, where

$$h = \lim_{n\to\infty} h_n$$

is a harmonic function on Ω. In view of (CE2), it remains to verify

$$\lim_{z\to\xi} h(z) = 0 \tag{3.10}$$

in order to prove $\lim\limits_{z\to\xi} u(z) = 0$.

To justify (3.10), we reflect each h_n across γ and call the resulting new harmonic function H_n. Then $\{H_n\}$ is a bounded sequence of harmonic functions in a domain G containing $\Omega \cup \gamma$. Obviously, $\{H_n\}$ converges locally uniformly in G to a harmonic function H which vanishes on γ. Since $H\big|_\Omega \equiv h$, we get (3.10) as required.

(iii) Lastly, by construction,

$$\liminf_{z\to\xi} u(z) \geq 0$$

for $\xi \in \overline{I}\backslash I$. $\square$

Proof of Theorem 1.4. We first prove Theorem 1.4 under the additional assumption that

$$m_\lambda := \inf_{z\in\mathbb{D}} \lambda(z) > 0\,.$$

Fix $\xi_0 \in \Gamma$ and let I be an open subarc of Γ such that $\xi_0 \in I$, $\overline{I} \subset \Gamma$ and $\overline{I} \neq \partial\mathbb{D}$. Then by Theorem 3.1, there exists a sequence $\{\,\sigma_n(z)\,|dz|\,\}$, $n \geq 1$, of regular conformal metrics in $\mathbb{D}$ with

(i) $\kappa_{\sigma_n} \equiv -1/n$,

(ii) $\displaystyle\lim_{z\to\xi} \sigma_n(z) = \begin{cases} +\infty & \text{for } \xi \in \partial\mathbb{D}\backslash\overline{I} \\ 1 & \text{for } \xi \in I \end{cases}$,

and

(iii) $\displaystyle\liminf_{z\to\xi} \sigma_n(z) \geq 1$ for $\xi \in \overline{I}\backslash I$.

To show that

$$m_{\sigma_n} := \inf_{z\in\mathbb{D}} \sigma_n(z) > 0,$$

we consider on $\mathbb{D}$ the metric

$$\mu_n(z)\,|dz| = \frac{2\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)}{1-\left(\sqrt{n+1}-\sqrt{n}\right)^2|z|^2}\,|dz|\,,$$

which has constant curvature $-1/n$ and boundary values $\equiv 1$. Further, $\mu_n(z)$ attains its minimum value m_{μ_n} at $z = 0$, i.e.,

$$m_{\mu_n} = 2\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)\,. \tag{3.11}$$

According to the comparison principle, $\sigma_n \geq \mu_n$ in $\mathbb{D}$, so

$$m_{\sigma_n} \geq m_{\mu_n}\,.$$

In addition,

$$\lim_{n\to\infty} m_{\sigma_n} = 1 \tag{3.12}$$

by (3.11) and the fact that $m_{\sigma_n} \leq 1$.

We now define the conformal metrics $\Lambda_n(z)\,|dz|$ by

$$\Lambda_n(z) := \lambda(z)\,\sigma_n(z)\,.$$

By construction, each $\Lambda_n(z)\,|dz|$ fulfills the hypotheses of Theorem 2.1, i.e.,

$$\lim_{|z|\to 1} \Lambda_n(z) = +\infty \quad \text{and} \quad \kappa_{\Lambda_n}(z) \geq -\left(\frac{c_\lambda}{m_{\sigma_n}^2} + \frac{1}{n}\,\frac{1}{m_\lambda^2}\right).$$

Thus we obtain

$$\frac{\lambda(z)}{\mu(z)} \geq \frac{\sqrt{C_\mu}}{\sqrt{c_\lambda + \dfrac{1}{n}\,\dfrac{m_{\sigma_n}^2}{m_\lambda^2}}}\;\frac{m_{\sigma_n}}{\sigma_n(z)}\,, \quad n = 1, 2, \ldots$$

for every $z \in \mathbb{D}$. Consequently, we have

$$\liminf_{z \to \xi_0} \frac{\lambda(z)}{\mu(z)} \geq \frac{\sqrt{C_\mu}}{\sqrt{c_\lambda + \dfrac{1}{n} \, \dfrac{m_{\sigma_n}^2}{m_\lambda^2}}} \, m_{\sigma_n} \, , \quad n = 1, 2, \ldots ,$$

and the desired conclusion follows by letting $n \to \infty$ and taking condition (3.12) into account.

We now turn to the general case of Theorem 1.4. Fix a point $\xi_0 \in \Gamma$. Since $\lambda(z) \to +\infty$ as $z \to \xi_0$, there is an open subarc Γ' of Γ with $\xi_0 \in \Gamma'$ and $\overline{\Gamma'} \subsetneq \partial\mathbb{D}$ and a *Dini* smooth Jordan domain[3] $G \subseteq \mathbb{D}$ such that $\Gamma' = \partial G \cap \partial \mathbb{D} \subset \Gamma$, $\lambda(z) \geq c_1$ in G for some positive constant $c_1 > 0$ and $\kappa_\lambda(z) \geq -c_\lambda$ for $z \in G$ as well as $\kappa_\mu(z) \leq -C_\mu$ for $z \in G$. Let Ψ be a conformal map from $\mathbb{D}$ onto G. Then $\Psi : \mathbb{D} \to G$ extends to a homeomorphism of the closures $\overline{\mathbb{D}}$ and $\overline{G}$, which we continue to denote by Ψ. We also note that Ψ' has a continuous and non-vanishing extension to $\overline{\mathbb{D}}$, again denoted by Ψ' (see [21]). Thus there are positive constants c_2 and c_3 such that $c_3 \geq |\Psi'(z)| \geq c_2 > 0$ for all $z \in \overline{\mathbb{D}}$. Next we pull back the pseudo-metrics $\lambda(z)\,|dz|$ and $\mu(z)\,|dz|$ via $z = \Psi(u)$ to the pseudo-metrics $(\Psi^*\lambda)(u)\,|du|$ and $(\Psi^*\mu)(u)\,|du|$, respectively, on $\mathbb{D}$. These new pseudo-metrics have the properties

(i) $(\Psi^*\lambda)(u) \geq c_1 \cdot c_2 > 0$ in $\mathbb{D}$;

(ii) $\kappa_{\Psi^*\lambda}(z) \geq -c_\lambda$ for $z \in \mathbb{D}$;

and

(iii) $\kappa_{\Psi^*\mu}(z) \leq -C_\mu$ for $z \in \mathbb{D}$.

Thus we are exactly in the situation of the special case of Theorem 1.4 which we have considered above in the first part of the proof, so

$$\liminf_{z \to \xi_0} \frac{\lambda(z)}{\mu(z)} = \liminf_{u \to \Psi^{-1}(\xi_0)} \frac{\lambda(\Psi(u))\,|\Psi'(u)|}{\mu(\Psi(u))\,|\Psi'(u)|} \geq \sqrt{\frac{C_\mu}{c_\lambda}} \, . \qquad \square$$

Remark 3.3. The above proof can easily be modified to give a short (and elementary) proof of the implication "(a) $\Rightarrow$ (b)" of Theorem 1.1. Indeed, if $\lambda(z)\,|dz|$ is a regular conformal pseudo-metric in $\mathbb{D}$ with curvature bounded below such that $\lambda(z) \to +\infty$ for $z \to \xi \in \Gamma$, where Γ is a proper open subarc of $\partial\mathbb{D}$, then we may again assume that $\lambda(z) \geq c > 0$ in $\mathbb{D}$ for some $c > 0$. Further, we can suppose without loss of generality that Γ is a proper open subarc of $\partial\mathbb{D}$ with $\overline{\Gamma} \neq \partial\mathbb{D}$, $\Gamma \supset \{e^{i\phi} : -\pi/2 \leq \phi \leq \pi/2\}$ and

$$\limsup_{z \to \xi} \lambda(z) < +\infty \qquad \text{for every } \xi \in \partial\mathbb{D} \backslash \overline{\Gamma} \, .$$

[3]See [21].

Fix a point $\xi_0 \in \Gamma$. Then, in view of our extra assumptions, we can find a point $\omega_0 \in \partial\mathbb{D}$ such that

(i) $\limsup_{z \to \xi_0} \lambda(\omega_0 z) < +\infty$, and

(ii) $\lim_{z \to \xi} \lambda(z)\lambda(\omega_0 z) = +\infty$ for every $\xi \in \partial\mathbb{D}$.

Thus the curvature of the conformal metric

$$\Lambda(z)\,|dz| = \lambda(z)\,\lambda(\omega_0 z)\,|dz|$$

is bounded below by some constant, and we can apply Theorem 2.1 to $\Lambda(z)\,|dz|$ and $\mu(z)\,|dz|$. This gives

$$\liminf_{z \to \xi_0} \frac{\Lambda(z)}{\mu(z)} > 0\,,$$

so

$$\liminf_{z \to \xi_0} \frac{\lambda(z)}{\mu(z)} = \liminf_{z \to \xi_0} \frac{\Lambda(z)}{\mu(z)}\,\frac{1}{\lambda(\omega_0 z)} > 0\,.$$

4 Theorem 1.1: Proof, extension and further discussion

In view of Remark 1.2 and Remark 1.3, we are left to prove the implication "(b) $\Rightarrow$ (d)" in Theorem 1.1. We break up the proof into the following two lemmas.

Lemma 4.1. *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$ and let Γ be an open subarc of the unit circle $\partial\mathbb{D}$. If*

$$\liminf_{z \to \xi} \left(1 - |z|^2\right) \frac{|f'(z)|}{1 - |f(z)|^2} > 0 \tag{4.1}$$

for every $\xi \in \Gamma$, then the meromorphic function

$$g(z) := \frac{z\,f'(z)}{f(z)}$$

has a meromorphic extension to a neighborhood of Γ, which is real on Γ.

Proof. Since we assume $f(0) = 0$, we have $|f(z)| \leq |z|$ by Schwarz' Lemma. As a consequence, (4.1) implies

$$\liminf_{z \to \xi} |f'(z)| = \liminf_{z \to \xi} \left(1 - |z|^2\right) \frac{|f'(z)|}{1 - |f(z)|^2} \underbrace{\frac{1 - |f(z)|^2}{1 - |z|^2}}_{\geq 1} > 0$$

for every $\xi \in \Gamma$. Thus, for fixed $\xi_0 \in \Gamma$, there is a constant $\delta > 0$ such that $|f'(z)| \geq \delta$ for all $z \in \mathbb{D}$ sufficiently close to ξ_0, say for all $\xi \in \mathbb{D} \cap K_\varepsilon(\xi_0)$, where $K_\varepsilon(\xi_0)$ is the open disk around ξ_0 with radius $\varepsilon > 0$. This implies that the meromorphic function

$$h(z) := \frac{f(z)}{zf'(z)}$$

is in fact holomorphic and bounded in $\mathbb{D} \cap K_\varepsilon(\xi_0)$. We are going to show that $h(z)$ has a holomorphic extension to the disk $K_\varepsilon(\xi_0)$, which is real on $\Gamma \cap K_\varepsilon(\xi_0)$.

Pick a point $\xi_1 \in \Gamma \cap K_\varepsilon(\xi_0)$. There are two cases to be considered:

$$\liminf_{z \to \xi_1} \frac{1 - |f(z)|}{1 - |z|} = +\infty \tag{4.2}$$

and

$$\liminf_{z \to \xi_1} \frac{1 - |f(z)|}{1 - |z|} < +\infty \, . \tag{4.3}$$

If (4.2) holds, then

$$\liminf_{z \to \xi_1} |f'(z)| = \liminf_{z \to \xi_1} \left(1 - |z|^2\right) \frac{|f'(z)|}{1 - |f(z)|^2} \cdot \frac{1 - |f(z)|}{1 - |z|} \cdot \frac{1 + |f(z)|}{1 + |z|} = +\infty \, ,$$

whereas for the case (4.3), the Julia–Wolff–Carathéodory lemma (see for instance [25]) applies and shows that the angular limits

$$\angle \lim_{z \to \xi_1} f(z) =: f(\xi_1) \in \partial\mathbb{D} \, , \qquad \angle \lim_{z \to \xi_1} f'(z) =: f'(\xi_1) \in \mathbb{C}$$

exist and that

$$\frac{\xi_1 \, f'(\xi_1)}{f(\xi_1)} \in [1, +\infty) \, .$$

Thus, in any case, the bounded holomorphic function $h : \mathbb{D} \cap K_\varepsilon(\xi_0) \to \mathbb{C}$ has for every $\xi_1 \in \Gamma \cap K_\varepsilon(\xi_0)$ an angular limit $h(\xi_1)$ with $\operatorname{Im} h(\xi_1) = 0$. From the classical Schwarz–Carathéodory reflection principle (see, for instance, [17, p. 87]), we conclude that h has a holomorphic extension to $K_\varepsilon(\xi_0)$, which is real on $\Gamma \cap K_\varepsilon(\xi_0)$. $\square$

We are now in a position to prove the implication "(b) ⇒ (d)" in Theorem 1.1.

Lemma 4.2. *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function and let Γ be an open subarc of the unit circle $\partial\mathbb{D}$. If*

$$\liminf_{z \to \xi} \left(1 - |z|^2\right) \frac{|f'(z)|}{1 - |f(z)|^2} > 0 \tag{4.4}$$

for every $\xi \in \Gamma$, then f has an analytic extension across Γ and

$$\lim_{z \to \xi} |f(z)| = 1$$

for every $\xi \in \Gamma$.

Proof. Since

$$\left(1-|z|^2\right)\,\frac{|f'(z)|}{1-|f(z)|^2}$$

is invariant under pre- and postcomposition with conformal automorphisms of $\mathbb{D}$, we may assume $f(0)=0$. Therefore, by Lemma 4.1,

$$g(z) := \frac{z\,f'(z)}{f(z)} \tag{4.5}$$

has a *meromorphic* extension to a neighborhood of Γ, which is real on Γ.

(a) First we show that g has in fact a *holomorphic* extension to Γ. Suppose to the contrary that g has a pole of order $N \geq 1$ at a point $\xi \in \Gamma$. We may take, without loss of generality, $\xi = -1$. Thus

$$g(z) = \frac{h(z)}{(1+z)^N}\,, \tag{4.6}$$

where h is holomorphic at $z=-1$ and $h(-1) \neq 0$. Since g is real on Γ, we have

$$\overline{g(1/\overline{z})} = g(z) \qquad \text{on } \Gamma\,,$$

so

$$\overline{h(1/\overline{z})} = \frac{h(z)}{z^N} \qquad \text{on } \Gamma\,.$$

In particular, $\overline{h(-1)} = (-1)^N h(-1)$, that is,

$$h(-1) \in \begin{cases} \mathbb{R}\setminus\{0\} & \text{if } N \text{ is even} \\ i\mathbb{R}\setminus\{0\} & \text{if } N \text{ is odd}\,. \end{cases}$$

Solving the ODE (4.5) for f, we have in $\mathbb{D}_- := \{z \in \mathbb{D} : \operatorname{Re} z < 0\}$

$$\begin{aligned} f(z) &= \exp\left(\int_{z_0}^{z} \frac{g(u)}{u}\,du\right) \\ &= \exp\biggl(\frac{-h(-1)}{(1-N)(1+z)^{N-1}} + \frac{c_2}{(2-N)(1+z)^{N-2}} + \cdots + \frac{c_{N-1}}{-(1+z)} \\ &\qquad + c_N \log(1+z) + \cdots\biggr)\,, \end{aligned}$$

where z_0 is a point in $\mathbb{D}_-$ near -1 and $\log$ is the principal branch of the logarithm.

In order to exclude the possibility that g has a pole at $z=-1$, we distinguish between the three cases: $N \geq 2$ even, $N \geq 3$ odd, and $N=1$.

(i) $N \geq 2$ even. We approach $z = -1$ radially to arrive at a contradiction.

Since $h(-1) \in \mathbb{R} \backslash \{0\}$ in this case, we find for x in the interval $(-1, 0)$

$$|f(x)| = \exp\left(\frac{-h(-1)}{(1-N)(1+x)^{N-1}} + \cdots + \operatorname{Re}(c_N)\ln(1+x) + \cdots\right),$$

so that

$$\lim_{x \to -1} |f(x)| = \begin{cases} +\infty & \text{if } h(-1) > 0 \\ 0 & \text{if } h(-1) < 0\,. \end{cases}$$

Both possibilities contradict our assumptions. If $h(-1) > 0$, then the assumption $f(\mathbb{D}) \subset \mathbb{D}$ is contradicted. If $h(-1) < 0$, then $|f(x)| \to 0$ as $x \to -1$ implies $|f'(x)| \to 0$ as $x \to -1$, which contradicts (4.4).

(ii) $N \geq 3$ odd. We let z tend to -1 on a suitable ray.

Choose $\eta := e^{i\pi/(2(N-1))}$ and set $\zeta_r = -1 + r\eta$ for $r \in (0, 1)$. Then

$$|f(\zeta_r)| = \exp\left(\frac{ih(-1)}{(1-N)r^{N-1}} + \cdots + \operatorname{Re}\big(c_N \log(r\eta)\big) + \cdots\right),$$

because $h(-1) \in i\mathbb{R} \backslash \{0\}$. Thus

$$\lim_{r \to 0} |f(\zeta_r)| = \begin{cases} +\infty & \text{if } (-i) \cdot h(-1) > 0 \\ 0 & \text{if } (-i) \cdot h(-1) < 0\,. \end{cases}$$

As before, we conclude that neither case is possible.

(iii) $N = 1$. We approach $z = -1$ on a certain horocycle.

Here, the function f takes the form

$$f(z) = \exp(-i\gamma \log(1+z)) \cdot \exp \tilde{h}(z)\,,$$

where $\gamma = -i\, h(-1) \in \mathbb{R} \backslash \{0\}$ and $\tilde{h}$ is a holomorphic function in a neighborhood of $z = -1$. Hence

$$|f(z)| = \exp(\gamma \arg(1+z)) \exp\left(\operatorname{Re}\tilde{h}(z)\right).$$

This, combined with $f(\mathbb{D}) \subset \mathbb{D}$, implies that

$$\operatorname{Re}\tilde{h}(-1) \leq 0\,.$$

Let

$$z_\phi = -\frac{1}{2} + \frac{1}{2}e^{i\phi} \quad \text{with } \phi \in \begin{cases} \left(-\pi, -\frac{\pi}{2}\right) & \text{if } \gamma > 0 \\ \left(\frac{\pi}{2}, \pi\right) & \text{if } \gamma < 0\,. \end{cases}$$

If $\gamma > 0$, then using

$$\lim_{\phi \searrow -\pi} |f(z_\phi)| \le \exp\left(-\gamma \frac{\pi}{2}\right) < 1$$

gives

$$\begin{aligned}\liminf_{\phi \to -\pi} (1-|z_\phi|^2) \frac{|f'(z_\phi)|}{1-|f(z_\phi)|^2} \\ &= \liminf_{\phi \to -\pi} \frac{1-|z_\phi|^2}{|1+z_\phi|^2} \frac{|f(z_\phi)| \left| -i\gamma\,(1+z_\phi) + (1+z_\phi)^2 \tilde{h}'(z_\phi)\right|}{1-|f(z_\phi)|^2} \\ &= 0\,.\end{aligned}$$

If $\gamma < 0$, then

$$\lim_{\phi \nearrow \pi} |f(z_\phi)| \le \exp\left(\gamma \frac{\pi}{2}\right) < 1;$$

and we obtain as before

$$\liminf_{\phi \to \pi} (1-|z_\phi|^2) \frac{|f'(z_\phi)|}{1-|f(z_\phi)|^2} = 0\,.$$

Thus both possibilities contradict (4.4).

All in all, we conclude that g is holomorphic at $z = -1$.

(b) In part (a), we proved that g is holomorphic in a neighborhood of every point $\xi \in \Gamma$. Thus, as a solution of the complex ODE $y' = (g(z)/z)\,y$, the function f has also a holomorphic extension to a neighborhood of Γ. In particular, $\lim_{z \to \xi} |f(z)| =: q \in [0,1]$ exists. If $q < 1$, then (4.4) would be violated, so q has to be equal to 1. □

In order to highlight that Theorem 1.1 is for the most part a statement about conformal metrics, we now combine it with the boundary Ahlfors Lemma to prove the following generalization of the statements "(a) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (e)" of Theorem 1.1. This shows that one may replace the Poincaré metric $\lambda_{\mathbb{D}}(z)\,|dz|$ in Theorem 1.1 by more general conformal pseudo-metrics.

Theorem 4.3. *Let Γ be an open subarc of the unit circle $\partial\mathbb{D}$, let $w = f(z)$ be an analytic self-map of the unit disk $\mathbb{D}$ and let $\mu(z)\,|dz|$ be a conformal pseudo-metric on the unit disk $\mathbb{D}$ with curvature $\kappa_\mu \le -C_\mu$ for some positive constant C_μ and*

$$\lim_{z \to \xi} \mu(z) = +\infty$$

for every $\xi \in \Gamma$. Then the following conditions are equivalent.

(a') *There exists a regular conformal pseudo-metric* $\lambda(z)\,|dz|$ *on* $\mathbb{D}$ *with curvature* κ_λ *satisfying* $-c_\lambda \leq \kappa_\lambda \leq -C_\lambda$ *for positive constants* c_λ *and* C_λ *such that*

$$\lim_{z\to\xi} \lambda(f(z))\,|f'(z)| = +\infty$$

for every $\xi \in \Gamma$.

(c') *There exists a regular conformal pseudo-metric* $\lambda(z)\,|dz|$ *on* $\mathbb{D}$ *with curvature* κ_λ *satisfying* $-c_\lambda \leq \kappa_\lambda \leq -C_\lambda$ *for positive constants* c_λ *and* C_λ *such that*

$$\liminf_{z\to\xi} \frac{\lambda(f(z))\,|f'(z)|}{\mu(z)} \geq \sqrt{\frac{C_\mu}{c_\lambda}}$$

for every $\xi \in \Gamma$.

(e') *The function* f *has a holomorphic extension across the boundary arc* Γ *with* $f(\Gamma) \subset \partial\mathbb{D}$.

Proof. (a')⇒(c') This is the boundary Ahlfors Lemma (Theorem 1.4).

(c') ⇒(e') Just note that Theorem 2.1 gives

$$\lambda_{\mathbb{D}}(z) \geq \sqrt{\frac{C_\lambda}{4}}\,\lambda(z)\,.$$

So the fact that $\mu(z) \to +\infty$ as $z \to \xi \in \Gamma$ implies

$$\lim_{z\to\xi} \lambda_{\mathbb{D}}(f(z))\,|f'(z)| \geq \sqrt{\frac{C_\lambda}{4}}\,\lim_{z\to\xi} \lambda(f(z))\,|f'(z)| = +\infty,$$

and (e') follows from Theorem 1.1.

(e') ⇒(a') Choose $\lambda(z)\,|dz| = \lambda_{\mathbb{D}}(z)\,|dz|$ and apply Theorem 1.1 "(e)⇒(a)". □

We close this section with a discussion of two examples which demonstrate that Theorem 1.1 does not hold in general, when the arc Γ reduces to a singleton.

Example 4.4. The holomorphic function $f:\mathbb{D}\to\mathbb{D}$ with

$$f(z) = \frac{\sqrt{1-z}}{\sqrt{1-z}+\sqrt{1+z}}$$

obviously has no analytic extension to any neighborhood of $z=1$. However, condition (a) of Theorem 1.1 is satisfied at $z=1$, since

$$\lim_{z\to 1} |f'(z)| = +\infty \quad \text{and} \quad \lim_{z\to 1} |f(z)| = 0\,.$$

Hence, the implications "(a)⇒(e)" and "(a)⇒(d)" both fail if Γ is a single point.

Example 4.5. The function $f : \mathbb{D} \to \mathbb{D}$ defined by

$$f(z) = T^{-1}(\sqrt{T(z)}\,)\,,$$

where $T(z) = (1+z)/(1-z)$, fulfills condition (d) of Theorem 1.1 at $z = 1$ although it has no holomorphic extension to any neighborhood of $z = 1$. Further, f meets condition (a) of Theorem 1.1; but it does not satisfy condition (b), since for the points

$$z_\varphi = \frac{1}{2} + \frac{1}{2}e^{i\varphi},$$

we have

$$\lim_{\varphi \to 0}(1 - |z_\varphi|^2)\frac{|f'(z_\varphi)|}{1 - |f(z_\varphi)|^2} = 0\,.$$

In particular,

$$\liminf_{z \to 1}(1 - |z|^2)\frac{|f'(z)|}{1 - |f(z)|^2} = 0\,.$$

Thus the implications "(d)⇒(b)", "(a)⇒(b)" and "(d)⇒(e)" of Theorem 1.1 are no longer true if Γ consists of a single point.

5 Locally complete conformal metrics on smoothly bounded domains

In this section, we extend the boundary Ahlfors Lemma (Theorem 1.4) to domains Ω in the complex plane whose boundary contains a sufficiently smooth subset Γ. As we shall see, this generalization is closely linked with (a very special case of) a boundary Schwarz Lemma due to Bland [3], in which so-called locally complete metrics play an important rôle. Our extended boundary Ahlfors Lemma might be considered as a converse of Bland's boundary Schwarz Lemma. Combining both yields a characterization of locally complete conformal metrics $\lambda(z)\,|dz|$ on smooth domains in terms of the boundary behavior of the density λ.

Let us begin with defining what we mean by a smooth boundary subset of a domain in the complex plane. We call a Jordan domain G (i.e., a domain bounded by a Jordan curve in $\mathbb{C}$) smooth, if there is a conformal map ϕ from $\mathbb{D}$ onto G such that $|\phi'|$ extends continuously to $\partial\mathbb{D}$ with $|\phi'| \neq 0$ on $\partial\mathbb{D}$. By Carathéodory's extension theorem, this conformal map ϕ extends to a homeomorphism of the closures $\overline{\mathbb{D}}$ and $\overline{G}$. Examples of smooth Jordan domains are given by domains bounded by a Dini–smooth Jordan curve; see [21, Theorem 3.5].

Definition 5.1. Let Ω be a subdomain of the complex plane $\mathbb{C}$. A subset Γ of the boundary of Ω is called **smooth**, if for every point $\xi \in \Gamma$ there exists a smooth Jordan domain $G \subseteq \Omega$ and an open neighborhood $U \subseteq \mathbb{C}$ of ξ such that $\xi \in \partial G \cap U \subseteq \Gamma$.

We can now state the following generalization of Theorem 1.4.

Theorem 5.1 (Boundary Ahlfors Lemma for smooth boundary sets)**.** *Let $\Omega \subseteq \mathbb{C}$ be a domain and let Γ be a smooth subset of $\partial\Omega$. Further, let $\lambda(z)\,|dz|$ be a regular conformal pseudo-metric on Ω with $\kappa_\lambda \geq -c_\lambda$, and let $\mu(z)\,|dz|$ be a conformal pseudo-metric on Ω with $\kappa_\mu \leq -C_\mu$ for some positive constants c_λ and C_μ. If*

$$\lim_{z\to\xi} \lambda(z) = +\infty \tag{5.1}$$

for every $\xi \in \Gamma$, then

$$\liminf_{z\to\xi} \frac{\lambda(z)}{\mu(z)} \geq \sqrt{\frac{C_\mu}{c_\lambda}} \tag{5.2}$$

for every $\xi \in \Gamma$. In particular,

$$\lim_{z\to\xi} \mathrm{d}_\lambda(z_0, z) = +\infty \tag{5.3}$$

for every $\xi \in \Gamma$ and every point $z_0 \in \Omega$.

Proof. Pick a point $\xi_0 \in \Gamma$. Then there exists a neighborhood $U \subseteq \mathbb{C}$ of ξ_0 and a smooth Jordan domain $G \subseteq \Omega$ such that $\xi_0 \in \partial G \cap U \subseteq \Gamma$. Let ϕ be a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{G}$, conformal in $\mathbb{D}$, such that $|\phi'|$ is continuous and nonvanishing on $\partial\mathbb{D}$. Let $\xi_0' = \phi^{-1}(\xi_0)$. Then there is an open subarc $\Gamma' \subseteq \partial\mathbb{D}$ such that $\phi(\Gamma') \subseteq \partial G \cap U \subseteq \Gamma$. We can apply Theorem 1.4 to the pullbacks $\phi^*\mu(u)\,|du| = \mu(\phi(u))\,|\phi'(u)|\,|du|$ and $\phi^*\lambda(u)\,|du| = \lambda(\phi(u))\,|\phi'(u)|\,|du|$ since

(i) $\kappa_{\phi^*\mu}(u) = \kappa_\mu(\phi(u)) \leq -C_\mu$,

(ii) $\kappa_{\phi^*\lambda}(u) = \kappa_\lambda(\phi(u)) \geq -c_\lambda$,

and

(iii) $\lim\limits_{u\to\xi'} \phi^*\lambda(u) = \lim\limits_{u\to\xi'} \lambda(\phi(u))\,|\phi'(u)| = +\infty$ for every $\xi' \in \Gamma'$

in view of $|\phi'(\xi')| > 0$ for each $\xi' \in \Gamma'$. Thus

$$\liminf_{z\to\xi_0} \frac{\lambda(z)}{\mu(z)} = \liminf_{u\to\xi_0'} \frac{\lambda(\phi(u))\,|\phi'(u)|}{\mu(\phi(u))\,|\phi'(u)|} = \liminf_{u\to\xi_0'} \frac{\phi^*\lambda(u)}{\phi^*\mu(u)} \geq \sqrt{\frac{C_\mu}{c_\lambda}}$$

for every $\xi_0 \in \Gamma$. This estimate easily implies (5.3) for every $\xi \in \Gamma$ and every $z_0 \in \Omega$. In fact, fix a point $\xi_0 \in \Gamma$ and a point $z_0 \in \Omega$. As Γ is smooth, there is a point $\xi_1 \in \Gamma \setminus \{\xi_0\}$. Now let $\mu(z)\,|dz|$ be the hyperbolic metric of the twice-punctured plane $\mathbb{C}'' := \mathbb{C} \setminus \{\xi_0, \xi_1\}$. Thus $\mu(z)\,|dz|$ has constant curvature -4 and is complete on $\mathbb{C}''$. Let $\varepsilon := \min\{|\xi_0 - \xi_1|, |z_0 - \xi_0|\} > 0$. Then $K_\varepsilon(\xi_0) \setminus \{\xi_0\} \subset \mathbb{C}''$ and $z_0 \notin K_\varepsilon(\xi_0)$. After shrinking $\varepsilon > 0$ if necessary, estimate (5.2) yields the inequality $\lambda(z) \geq c\,\mu(z)$ for every $z \in K_\varepsilon(\xi_0) \cap \Omega$ and some constant $c > 0$. Thus

$$\mathrm{d}_\lambda(z, z_0) \geq c \cdot \min_{|w - \xi_0| = \varepsilon} \mathrm{d}_\mu(w, z) \quad (z \in K_\varepsilon(\xi_0) \cap \Omega)\,.$$

Since $\mu(z)\,|dz|$ is complete in $\mathbb{C} \setminus \{\xi_0, \xi_1\}$, we infer that $\mathrm{d}_\lambda(z, z_0) \to +\infty$ as $z \to \xi_0$ in Ω. □

When Ω is a smooth domain, so the entire boundary $\Gamma = \partial\Omega$ is a smooth boundary set, and $\lambda(z)\,|dz|$ is a regular conformal metric on Ω, Theorem 5.1 implies that $\lambda(z)\,|dz|$ is complete on Ω. This is a consequence of the Hopf–Rinow theorem, which ensures that a conformal metric $\lambda(z)\,|dz|$ on a domain Ω is complete if and only if (5.3) holds for every $\xi \in \partial\Omega$ and some (and then for every) point $z_0 \in \Omega$. The latter characterization of complete conformal metrics in terms of the boundary behavior of the associated distance function d_λ can be localized quite easily.

Definition 5.2 (cf. [3]). A conformal pseudo-metric $\lambda(z)\,|dz|$ on a domain $\Omega \subset \mathbb{C}$ is called **locally complete near a subset** Γ of the boundary of Ω if

$$\lim_{z \to \xi} \mathrm{d}_\lambda(z, z_0) = +\infty \tag{5.4}$$

for every $\xi \in \Gamma$ and some (and then for every) point $z_0 \in \Omega$.

According to this definition, Theorem 5.1 simply says that a pseudo-metric $\lambda(z)\,|dz|$ whose density $\lambda(z)$ blows up at a boundary set Γ in the sense of condition (5.1) is locally complete near Γ provided the curvature of $\lambda(z)\,|dz|$ is bounded below and the boundary set Γ is smooth. The curvature condition cannot be dropped completely, as Example 2.3 shows. Moreover, one also needs some assumptions on the boundary set Γ. This is illustrated with the next example, which shows that rough boundary subsets Γ such as isolated points lack sufficient 'influence' on the speed at which the pseudo-metric tends to $+\infty$ at Γ.

Example 5.2. Consider on the punctured unit disk $\mathbb{D}\backslash\{0\}$ the metric $\lambda(z)\,|dz|$ with

$$\lambda(z) = \frac{\sqrt{1 + |z|^{1/3}}}{|z|^{5/6}} \frac{1}{1 - |z|^2}\,.$$

Then (5.1) holds for every $\xi \in \partial\mathbb{D} \cup \{0\}$, but $\lambda(z)\,|dz|$ is clearly not complete near $z = 0$. Note that $-2 \leq \kappa_\lambda(z) \leq -1/18$ for $z \in \mathbb{D}\backslash\{0\}$.

What about the converse of Theorem 5.1 ? Does the density $\lambda(z)$ of a regular conformal metric $\lambda(z)\,|dz|$ which is locally complete at a subset Γ of the boundary tend to $+\infty$ at Γ ? Without a lower bound on the curvature of $\lambda(z)\,|dz|$ the answer is no, even for real analytic boundary sets.

Example 5.3. The metric

$$\lambda(z)\,|dz| = \exp\left[\frac{1}{1-|z|}\left(\sin\frac{1}{1-|z|}+1\right)+1-|z|\right]\,|dz|$$

is complete on $\mathbb{D}$, but condition (5.1) clearly does not hold. To check completeness, we observe that a routine calculation shows that

$$\int\limits_0^r \lambda(z)\,|dz| \approx -\log(1-r) \quad (r \to 1),$$

so that $\mathrm{d}_\lambda(0,z) \to +\infty$ as $|z| \to 1$. The curvature of the metric $\lambda(z)\,|dz|$ is bounded from above but not from below on the unit disk.

However, if a regular conformal metric $\lambda(z)\,|dz|$ is locally complete near a smooth boundary set Γ *and* has curvature bounded from below, then the density $\lambda(z)$ indeed blows up at Γ, i.e., condition (5.1) holds for every $\xi \in \Gamma$. As we see in Corollary 5.6 below, this is a straightforward consequence of the following boundary Schwarz Lemma of Bland, see [3], which therefore might be viewed as a counterpart to Theorem 5.1.

Theorem 5.4 (Bland's boundary Schwarz Lemma)**.** *Let $\Omega \subseteq \mathbb{C}$ be a domain and let Γ be a smooth subset of $\partial\Omega$. Further, let $\lambda(z)\,|dz|$ be a regular conformal metric on Ω with $\kappa_\lambda \geq -c_\lambda$, and let $\mu(z)\,|dz|$ be a regular conformal pseudo-metric on Ω with $\kappa_\mu \leq -C_\mu$ for some positive constants c_λ and C_μ. If $\lambda(z)\,|dz|$ is locally complete near Γ, then*

$$\liminf_{z\to\xi}\frac{\lambda(z)}{\mu(z)} \geq \sqrt{\frac{C_\mu}{c_\lambda}}$$

for every $\xi \in \Gamma$.

Remark 5.5. Theorem 5.4 is just a very special case of Bland's boundary Schwarz Lemma (which in its original form applies to higher dimensional situations). Note that Bland's boundary Schwarz Lemma is formulated for regular conformal metrics $\lambda(z)\,|dz|$ and regular conformal pseudo-metrics $\mu(z)\,|dz|$, while in Theorem 5.1, $\lambda(z)\,|dz|$ is a regular conformal pseudo-metric and the regularity of $\mu(z)\,|dz|$ is of no importance.

By now, we have in hand enough technology to establish the following characterization of locally complete regular conformal metrics in terms of the boundary behavior of their density functions for *smooth* boundary sets.

Corollary 5.6. *Let $\Omega \subseteq \mathbb{C}$ be a domain, let Γ be a smooth subset of $\partial\Omega$ and let $\lambda(z)\,|dz|$ be a regular conformal metric on Ω with $\kappa_\lambda \geq -c_\lambda$ for some positive constant c_λ. Then the following are equivalent:*

(a) $\lambda(z)\,|dz|$ *is locally complete near* Γ*;*

(b) $\lim\limits_{z\to\xi} \lambda(z) = +\infty$ *for every* $\xi \in \Gamma$.

Proof. The domain Ω has at least two boundary points, since it has a smooth boundary set Γ. Thus Ω carries a complete regular conformal metric $\lambda_\Omega(z)\,|dz|$ with constant negative curvature. Theorem 5.1 applied to $\mu(z)\,|dz| := \lambda_\Omega(z)\,|dz|$ yields implication "(b) $\Rightarrow$ (a)". The converse implication follows from Theorem 5.4 applied to $\mu(z)\,|dz| = \lambda_\Omega(z)\,|dz|$, using the well-known fact that $\lambda_\Omega(z) \to +\infty$ as $z \to \xi$ for every $\xi \in \partial\Omega$ (see, for instance, [14]). $\square$

Problem 5.7. *What are the minimal regularity conditions on the boundary set* Γ *such that the two conditions* (a) *and* (b) *in Corollary 5.6 are still equivalent ?*

6 Reflection principles for analytic maps between Riemann surfaces

In this final section, we extend Theorem 1.1 to analytic maps between Riemann surfaces.

We say a Riemann surface R has analytic boundary if it sits inside a compact bordered Riemann surface R' with border $\partial R' \neq \emptyset$ such that $R = R' \setminus \partial R'$. In this case, R' and $\partial R'$ are uniquely determined by R. We call $\partial R'$ the analytic boundary of R and denote it by ∂R. Then $R \cup \partial R$ is a compact bordered Riemann surface. Notice that ∂R is a (not necessarily connected) real analytic manifold of (real) dimension 1. Now let $f : \mathbb{D} \to R$ be an analytic map. We say that f has an analytic extension across an open subarc Γ of $\partial\mathbb{D}$ with $f(\Gamma) \subseteq \partial R$, if there exists an analytic map F defined on a neighborhood $U \subseteq \mathbb{C}$ of Γ, which maps into the Schottky double $\mathcal{R} = R \cup \partial R \cup R^*$ such that $F(\Gamma) \subseteq \partial R$ and $F = f$ in $\mathbb{D} \cap U$. Here R^* denotes the mirror of R.

Theorem 6.1. *Let Γ be an open subarc of $\partial\mathbb{D}$, let R be a simply connected Riemann surface with analytic boundary ∂R, let $\lambda(w)\,|dw|$ be a complete regular conformal metric on R with curvature $-c_\lambda \leq \kappa_\lambda \leq -C_\lambda$ for some positive constants*

c_λ and C_λ, and let $f : \mathbb{D} \to R$ be an analytic map. Then f has an analytic extension across Γ with $f(\Gamma) \subset \partial R$ if and only if

$$\lim_{z \to \xi} \lambda(f(z))\,|f'(z)| = +\infty \tag{6.1}$$

for every $\xi \in \Gamma$.

Remarks 6.2.

(a) Note that $\lambda(f(z))\,|f'(z)|$ is the density of the pullback of the metric $\lambda(w)\,|dw|$ under the map f, so $f^*\lambda(z)\,|dz| = \lambda(f(z))\,|f'(z)|\,|dz|$ is a conformal pseudo-metric on $\mathbb{D}$ and therefore a well-defined function on $\mathbb{D}$.

(b) For the only if part of Theorem 6.1, it suffices to assume that $\lambda(w)\,|dw|$ is a complete regular conformal metric with curvature bounded below. However, these assumptions cannot be weakened further. For instance, take $R = \mathbb{D}$, $\lambda(w)\,|dw| = |dw|$ and any function holomorphic in a neighborhood of a point of the unit circle. Then $\lambda(w)\,|dw|$ is not complete on $\mathbb{D}$ and has curvature 0, but (6.1) does not hold. For $R = \mathbb{D}$, $\lambda(w)\,|dw|$ as in Example 5.3 and $f = \mathrm{id}$, the metric $\lambda(w)\,|dw|$ is complete on $\mathbb{D}$, but the curvature of $\lambda(w)\,|dw|$ is not bounded below and the boundary condition (6.1) does not hold.

(c) For the if part, it is enough that R carry a conformal pseudo-metric $\lambda(w)\,|dw|$ on R with curvature bounded above by a negative constant.

(d) In Theorem 6.1, it is essential that R be simply connected, as the following example shows. Let R be the annulus $\{z \in \mathbb{C} : e^{-1} < |z| < 1\}$ and let $\lambda(w)\,|dw|$ be the hyperbolic metric on R. Then

$$f(z) = \exp\left(\frac{i}{\pi}\log\left(i\frac{1+z}{1-z}\right)\right)$$

defines a universal covering map $f : \mathbb{D} \to R$, so $\lambda(f(z))\,|f'(z)| = \lambda_{\mathbb{D}}(z)$. Thus (6.1) holds for every $\xi \in \partial\mathbb{D}$, but f cannot be continued analytically across $\partial\mathbb{D}$, as f does not even have a continuous extension to the point $z = 1$.

For the proof of Theorem 6.1, we need the following lemma, which seems rather obvious at first glance. Since we have no exact reference and the argument contains a subtlety, we include the proof.

Lemma 6.3. *Let R be a simply connected Riemann surface with analytic boundary $\partial R \neq \emptyset$. Then R is hyperbolic and there is a conformal map π from $\mathbb{D}$ onto R, which has an analytic extension to a conformal map of a neighborhood of $\overline{\mathbb{D}}$ such that $\pi(\partial\mathbb{D}) = \partial R$.*

Proof. Let $\mathcal{R} = R \cup \partial R \cup R^*$ denote the Schottky double of R and let $\pi_{\mathcal{R}} : X \to \mathcal{R}$ be a universal cover projection, where $X = \mathbb{D}, \mathbb{C}$ or the Riemann sphere $\hat{\mathbb{C}}$. The Schottky double $\mathcal{R}$ is a compact Riemann surface without border. It is easy to check that $\mathcal{R}$ is also simply connected. Indeed, let $w_0 \in R$ be an arbitrary point and fix $z_0 \in X$ such that $\pi_{\mathcal{R}}(z_0) = w_0$. Then there exists a branch g of the inverse of $\pi_{\mathcal{R}}$ on the simply connected domain $R \subset \mathcal{R}$ such that $g(w_0) = z_0$ and $\pi_{\mathcal{R}} \circ g = \mathrm{id}\big|_R$. As ∂R is compact, g has an analytic extension to an open set containing $R \cup \partial R$. Since R^* is simply connected, this implies that g has an analytic continuation to $\mathcal{R}$. The fact that $g(\mathcal{R})$ is open and compact yields that $X = \hat{\mathbb{C}}$, i.e., $g(\mathcal{R}) = \hat{\mathbb{C}}$, so $\mathcal{R}$ is a simply connected compact surface and $\pi_{\mathcal{R}}$ is a conformal map. Let $D = \pi_{\mathcal{R}}^{-1}(R)$. Then D is a simply connected domain on $\hat{\mathbb{C}}$, and ∂D is a compact and real analytic one dimensional submanifold of $\hat{\mathbb{C}}$. The topology of the sphere $\hat{\mathbb{C}}$ also forces ∂D to be also connected. Therefore, ∂D is real analytic homeomorphic to the unit circle. This seems obvious, but is surprisingly difficult to prove (see, for instance, [18, Theorem 1]). Consequently, D is bounded by an analytic Jordan curve, and there is a conformal map Ψ defined on a neighborhood of $\overline{\mathbb{D}}$ which maps $\mathbb{D}$ onto D and $\partial\mathbb{D}$ homeomorphically onto ∂D. Finally, $\pi := \pi_{\mathcal{R}} \circ \Psi$ is a conformal map defined on a neighborhood of $\overline{\mathbb{D}}$ such that $\pi(\partial\mathbb{D}) = \partial R$. □

Proof of Theorem 6.1. Let $\pi : \mathbb{D} \to R$ be the conformal map of Lemma 6.3. We pull the metric $\lambda(w)\,|dw|$ back to the unit disk using $w = \pi(u)$ and get a *complete* regular conformal metric

$$\nu(u)\,|du| := \pi^*\lambda(u)\,|du| = \lambda(\pi(u))\,|\pi'(u)|\,|du|\,.$$

Since $\kappa_\nu \geq -c_\lambda$, Corollary 5.6 implies

$$\lim_{|u|\to 1} \nu(u) = +\infty\,.$$

Define a holomorphic function $g : \mathbb{D} \to \mathbb{D}$ by $g(z) := (\pi^{-1} \circ f)(z)$. Then

$$\lambda(f(z))\,|f'(z)| = \nu(g(z))\,|g'(z)| \quad (z \in \mathbb{D})\,. \tag{6.2}$$

We now prove the only if part of Theorem 6.1 and therefore assume that $f : \mathbb{D} \to R$ has an analytic continuation across Γ with $f(\Gamma) \subseteq \partial R$. Then, by Lemma 6.3, $g : \mathbb{D} \to \mathbb{D}$ also has an analytic extension across Γ with $g(\Gamma) \subset \partial\mathbb{D}$. In particular, $g' \neq 0$ on Γ and $\nu(g(z)) \to +\infty$ as $z \to \Gamma$, which combined with (6.2) gives (6.1).

In order to establish the if part of Theorem 6.1, we assume that $f : \mathbb{D} \to R$ satisfies the boundary condition (6.1). Then, in view of (6.2), we have

$$\lim_{z\to\xi} \nu(g(z))\,|g'(z)| = +\infty$$

for every $\xi \in \Gamma$. The curvature condition $\kappa_\nu \leq -C_\lambda < 0$ makes it possible to apply Theorem 2.1, which shows $\nu \leq c\,\lambda_{\mathbb{D}}$ for some constant $c > 0$. Therefore $\lambda_{\mathbb{D}}(g(z))\,|g'(z)| \to +\infty$ on Γ. In view of Theorem 1.1 this forces g to have an analytic extension across Γ, so $f = \pi \circ g$ has an analytic extension across Γ by Lemma 6.3. □

In the next step we replace the unit disk $\mathbb{D}$ in Theorem 6.1 by the 'interior' S_0 of a bordered Riemann surface $S = S_0 \cup \partial S$. For this purpose, we first clarify what (6.1) means for an analytic map between S_0 and R.

Remark 6.4. Let $S = S_0 \cup \partial S$ be a bordered Riemann surface and let $\lambda(z)\,|dz|$ be a pseudo-metric on S_0. Further, let $\{\varphi_\alpha : U_\alpha \to \mathbb{C}\}$ be the family of charts of S and let $\lambda_\alpha(u)\,|du|$ be the pseudo-metric $\lambda(z)\,|dz|$ in the local parameter φ_α restricted to S_0. Then, if $\xi \in \partial S$ belongs to U_α and

$$\lim_{z \to \xi} \lambda_\alpha(\varphi_\alpha(z)) = +\infty\,,$$

then

$$\lim_{z \to \xi} \lambda_{\alpha'}(\varphi_{\alpha'}(z)) = +\infty$$

for any other chart $\varphi_{\alpha'}$ with $\xi \in U_{\alpha'}$. This enables us to say that a pseudo-metric tends to $+\infty$ at a point ξ of the border ∂S, and we write $\lambda(z)\,|dz| \to +\infty$ as $z \to \xi$ in this case.

Taking the preceding remark into account, we see from the proof of Theorem 6.1 that the complete regular conformal metric $\lambda(w)\,|dw|$ in Theorem 6.1 can be replaced by a regular conformal pseudo-metric satisfying $\lambda(w)\,|dw| \to +\infty$ as $w \to \partial R$. We are now in a position to generalize Theorem 6.1 in the desired direction.

Theorem 6.5. *Let $S = S_0 \cup \partial S$ be a bordered Riemann surface; let R be a simply connected Riemann surface with analytic boundary ∂R; let $\lambda(w)\,|dw|$ be a regular conformal pseudo-metric $\lambda(w)\,|dw|$ on R with curvature bounded below and above by negative constants $-c_\lambda$ and $-C_\lambda$, respectively, and*

$$\lim_{w \to \tau} \lambda(w)\,|dw| = +\infty$$

for every $\tau \in \partial R$; and let $f : S_0 \to R$ be an analytic map. If ξ_0 is a point of the border of S, then the following statements are equivalent.

(i) *f has an analytic continuation to a neighborhood U of ξ_0 in the Schottky double $\mathcal{S}$ such that $f(U \cap \partial S) \subseteq \partial R$.*

(ii) *There is a neighborhood* $U \subseteq \mathcal{S}$ *of* ξ_0 *such that*

$$\lim_{z\to\xi} \lambda(f(z))\,|f'(z)|\,|dz| = +\infty$$

for every point $\xi \in U \cap \partial S$.

Proof. (i)⇒(ii): Let f have an analytic continuation to a neighborhood U of ξ_0 in $\mathcal{S}$. We may assume U lies in a parameter neighborhood $U_\alpha \cup U_\alpha^*$ of $\mathcal{S}$. Then there exists a simply connected domain V in U with $\xi_0 \in V$ such that $\overline{V} \subset U$ and a local parameter $\hat{\varphi}_\alpha : U_\alpha \cup U_\alpha^* \to \mathbb{C}$, $\hat{\varphi}_\alpha(z) = u$, such that $\hat{\varphi}_\alpha(V \cap S_0) = \mathbb{D}$. Let $\Gamma := V \cap \partial S$ and $I := \hat{\varphi}_\alpha(\Gamma)$. Then I is an open subarc of $\partial\mathbb{D}$, and the function $\hat{f} := f \circ \hat{\varphi}_\alpha^{-1} : \mathbb{D} \to R$ has an analytic continuation across I. By Theorem 6.1,

$$\lim_{u\to\eta} \lambda(\hat{f}(u))\,|\hat{f}'(u)| = +\infty$$

for every $\eta \in I$, so

$$\lim_{z\to\xi} \lambda(f(z))\,|f'(z)|\,|dz| = +\infty$$

for every $\xi \in \Gamma$.

(ii)⇒(i): Maintaining the same notation as in the proof of "(i) ⇒ (ii)", we see that

$$\lim_{z\to\xi} \lambda(f(z))\,|f'(z)|\,|dz| = +\infty$$

for every $\xi \in \Gamma$ gives

$$\lim_{u\to\eta} \lambda(\hat{f}(u))\,|\hat{f}'(u)| = +\infty$$

for every $\eta \in I$. By Theorem 6.1, the function $\hat{f}$ has an analytic extension across I. Thus $f = \hat{\varphi}_\alpha \circ \hat{f}$ has an analytic continuation across Γ and consequently to a whole neighborhood of ξ_0. □

Our next result is a global extension of Theorem 6.5.

Corollary 6.6. *Let* S *and* R *be simply connected Riemann surfaces with analytic boundaries* ∂S *and* ∂R*, respectively; let* Γ *be an open and connected subset of* ∂S*; and let* R *carry a complete regular conformal metric* $\lambda(w)\,|dw|$ *with curvature bounded below and above by negative constants* $-c_\lambda$ *and* $-C_\lambda$*, respectively. Further, let* $f : S \to R$ *be an analytic map. Then the following conditions are equivalent.*

(i) f *has an analytic extension across* Γ *such that* $f(\Gamma) \subset \partial R$.

(ii) *For every* $\xi \in \Gamma$,

$$\lim_{z\to\xi} \lambda(f(z))\,|f'(z)|\,|dz| = +\infty\,.$$

(iii) *If* $\mu(z)\,|dz|$ *is a conformal pseudo-metric on* S *whose curvature is bounded from above by a negative constant* $-C_\mu$ *and*

$$\lim_{z\to\xi}\mu(z)\,|dz| = +\infty$$

for every $\xi\in\Gamma$*, then*

$$\liminf_{z\to\xi}\frac{\lambda(f(z))\,|f'(z)|}{\mu(z)} \geq \sqrt{\frac{C_\mu}{c_\lambda}} \tag{6.3}$$

for every ξ *in* Γ.

Note that $\lambda(f(z))\,|f'(z)|/\mu(z)$ in (6.3) is the quotient of two conformal pseudo-metrics on S. Since $\mu(z)\,|dz|\to+\infty$ as $z\to\Gamma$, this quotient is therefore a well-defined function on the surface S at least near Γ.

Proof. Let $\pi_S:\mathbb{D}\to S$, $z=\pi_S(u)$, and $\pi_R:\mathbb{D}\to R$, $w=\pi_R(v)$, be the conformal maps described in Lemma 6.3 and let $I=\pi_S^{-1}(\Gamma)\subseteq\partial\mathbb{D}$. Further, define the analytic map $g:\mathbb{D}\to S$ by $g=f\circ\pi_S$ and the holomorphic function $h:\mathbb{D}\to\mathbb{D}$ by $h=\pi_R^{-1}\circ f\circ\pi_S$.

(i)⇒(ii): Suppose f has an analytic extension across Γ with $f(\Gamma)\subseteq\partial R$. Then by Lemma 6.3, the function g has an analytic extension across I as well, so Theorem 6.1 applied to the analytic map g yields

$$\lim_{u\to\eta}\lambda(g(u))\,|g'(u)| = +\infty$$

for every $\eta\in I$. This implies

$$\lim_{z\to\xi}\lambda(f(z))\,|f'(z)|\,|dz| = +\infty$$

for every $\xi\in\Gamma$.

(ii) ⇒ (iii): Let $\mu(z)\,|dz|$ be a conformal pseudo-metric on S with curvature $\kappa_\mu\leq -C_\mu<0$. Pulling back $\mu(z)\,|dz|$ via π_S, i.e., $(\pi_S^*\,\mu)(u)\,|du| = \mu(\pi_S(u))\,|\pi_S'(u)|\,|du|$, gives a pseudo-metric on $\mathbb{D}$ with curvature $\kappa_{\pi_S^*\mu}\leq -C_\mu$. From Theorem 1.4 follows

$$\liminf_{z\to\xi}\frac{\lambda(f(z))\,|f'(z)|}{\mu(z)} = \liminf_{u\to\pi_S^{-1}(\xi)}\frac{\lambda(g(u))\,|g'(u)|}{\pi_S^*\,\mu(u)} \geq \sqrt{\frac{C_\mu}{c_\lambda}}$$

for every $\xi\in\Gamma$.

(iii)⇒(i): Define on $\mathbb{D}$ the regular conformal metric $\pi_R^*\lambda(v):=\lambda(\pi_R(v))\,|\pi_R'(v)|$ with curvature $\kappa_{\pi_R^*\lambda}\geq -c_\lambda$. Note that $\pi_R^*\,\lambda(h(u))\,|h'(u)| = \lambda(g(u))\,|g'(u)|$ for $u\in\mathbb{D}$. Hence, by assumption,

$$\liminf_{u\to\eta}\frac{\pi_R^*\,\lambda(h(u))\,|h'(u)|}{\pi_S^*\,\mu(u)} \geq \sqrt{\frac{C_\mu}{c_\lambda}}$$

for every $\eta \in I$. Since

$$\lim_{u \to \eta} \pi_S^* \, \mu(u) = +\infty$$

for every $\eta \in I$, we obtain in view of Theorem 2.1

$$\lim_{u \to \eta} \lambda_{\mathbb{D}}(h(u)) \, |h'(u)| \geq \sqrt{\frac{C_\lambda}{4}} \cdot \lim_{u \to \eta} \pi_R^* \lambda(h(u)) \, |h'(u)| = +\infty$$

for every $\eta \in I$. Now h has an analytic extension across I with $h(I) \subseteq \partial\mathbb{D}$ by Theorem 1.1. Consequently, Lemma 6.3 gives that f has an analytic extension across Γ with $f(\Gamma) \subseteq \partial R$. $\square$

We conclude this paper by noting that the characterization of locally complete regular conformal metrics in terms of the boundary behavior of the density in Corollary 5.6 carries over to Riemann surfaces. We limit ourselves to stating the result for bordered Riemann surfaces and omit the proof, which is similar to that of Corollary 5.6.

Proposition 6.7. *Let $S = S_0 \cup \partial S$ be a bordered Riemann surface, let Γ be an open connected subset of the border ∂S and let $\lambda(z) \, |dz|$ be a regular conformal metric on S_0 with curvature bounded from below. Then the following are equivalent:*

(a) $\lambda(z) \, |dz| \to +\infty$ *as* $z \to \xi$ *for every* $\xi \in \Gamma$*;*

(b) $\lambda(z) \, |dz|$ *is locally complete near* Γ*, i.e., for any* $z_0 \in S$ *and for every* $\xi \in \Gamma$*,*

$$\lim_{z \to \xi} \mathrm{d}_\lambda(z, z_0) = +\infty \, .$$

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