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ORIGINAL PAPER

Mean–variance vs trend–risk portfolio selection

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Abstract

In this paper, we provide an alternative trend (time)-dependent risk measure to Ruttiens' accrued returns variability (Ruttiens in Comput Econ 41:407–424, 2013). We propose to adjust the calculation procedure to achieve an alternative risk measure. Our modification eliminates static mean component and it is based on the deviation of squared dispersions, which reflects the trend (time factor) precisely. Moreover, we also present a new perspective on dependency measures and we apply a PCA to a new correlation matrix in order to determine a parametric and nonparametric return approximation. In addition, the two-phase portfolio selection strategy is considered, where the mean–variance portfolio selection strategies represent the first optimization. The second one is the minimization of deviations from their trend leading to identical mean and final wealth. Finally, an empirical analysis verify the property and benefit of portfolio selection strategies based on these trend-dependent measures. In particular, the ex-post results show that applying the modified measure allows us to reduce the risk with respect to the trend of several portfolio strategies.

Keywords Dependency measure · Risk measure · Volatility · Portfolio selection · Trend analysis

Mathematics Subject Classification 62H25 · 91G10 · 91G70

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1 Introduction

In a portfolio optimization theory, a rationally minded investor focuses on maximizing potential profit with respect to possible portfolio losses (risk incurred). In general, the question of how to adequately estimate and evaluate the risk of a large-scale portfolio or a single asset remains an important problem in financial risk management (Hallerbach and Spronk 2002). In the existing literature, return time series are generally identified as specifically distributed random variables with fat tails and higher peaks (Fama 1965; Mandelbrot 1963; Ortobelli et al. 2017). Decades ago, Mandelbrot (1963) analysed an empirical distribution of equity returns, proved that the assumption of normality made in the then existing financial theories (e.g. Markowitz 1952; Tobin 1958) was a misconception, and explained the distribution of returns by a stable distribution. Recently, this finding was pursued by Rachev and Mittnik (2000), Rachev et al. (2008), and Ortobelli and Tichý (2015).

Since the beginning of the modern portfolio theory presented by Markowitz, portfolio selection has depended on a mean and a risk represented by the variance of the historical joint distribution of all asset returns (Markowitz 1952). However, the variance examines and penalizes both-side deviations and so it does not correspond to rational thinking about financial exposure. In the measurement of investment risk, it is more relevant to use semivariance or semideviation, which takes into account only downside deviations (losses) (Markowitz 1959). Even during the last dozen years, numerous variants of portfolio risk measures have been proposed in the literature (Artzner et al. 1999; Rockafellar and Uryasev 2002; Szegö 2002; Rachev et al. 2008).

A specific group of risk measures evaluates possible losses based on the quantiles of the return distribution, such as Value-at-Risk (VaR) or Conditional Value-at-Risk (CVaR) (Duffie and Pan 1997; Altman et al. 1998; Jorion 2000; Rockafellar and Uryasev 2002). Furthermore, the coherent risk measures presented in Artzner et al. (1999) and their subsequent adjustments (Delbaen et al. 1998; Miller and Ruszczyński 2008, and references therein) were designed to detect the amount of capital requirement by expressing the riskiness of the financial portfolio numerically. The above-mentioned risk measurement approaches are established on the basis of historical returns, which essentially eliminates any time impact.

Therefore, a significant problem related to these risk measures is the incorporation of the time factor, which plays an important role in the financial area. In this context, Ruttiens (2013) proved that for the statistical measures used on resorted historical data, the time factor is not negligible, and proposed alternative dynamic risk measures. Recently, Ortobelli et al. (2017) provided an empirical application of such a time-dependent risk measure in the portfolio selection framework. Even though Ruttiens provides a suitable time-compromising approach to measure risk, we modify its original formulation into the trend analysis framework.

The major contribution of this paper is to provide a time-dependent risk (dispersion) measure which is an alternative to the “accrued returns variability”

(ARV) introduced by Ruttiens (see Ruttiens 2013). Moreover, we enhance the existing literature focused on portfolio theory and trend-dependent risk measures (Markowitz 1952; Rockafellar and Uryasev 2002; Szegö 2002; Rachev et al. 2008; Ortobelli et al. 2019). In the original work of Ruttiens, the ARV is calculated as the standard deviation of spreads between cumulative returns and a trend risk free variation of these returns at each time leading to the same final cumulative returns (or the final value of a portfolio). Rather, the definition of the non-volatile linear alternative formulation could correspond to the analysis of trend dispersion. The mathematical formulation of the presented dynamic risk measure can be more precise when we consider the mean of squared spreads of cumulative return series with respect to the non-volatile benchmark. Therefore, our "modified accrued returns variability" (modARV) is more accurate in the portfolio optimization framework while minimizing distortion with respect to the predefined zero-risk linear trend. Additionally, we define dependency measures, such as covariance and correlation, derived from the proposed modifications that can be used in the principal component analysis (PCA) framework.

Furthermore, we apply a newly proposed trend–risk measures and related dependency matrices in various mean–variance portfolio selection strategies to demonstrate their properties in the portfolio theory. In particular, to evaluate the effect of trend-dependent risk measures, we compare the mean–variance optimization strategy with a compounded double optimization strategy. This double optimization strategy consists of two steps. In the first step, we fit optimal portfolios of the mean–variance efficient frontier. In the second step, we determine the minimum modified Ruttiens risk measure fixing the expected mean and the final wealth of the optimal mean–variance portfolios in order to minimize the deviations with respect to the trend. To reduce the dimensionality of the portfolio, we use parametric and nonparametric returns approximation techniques with PCA applied to linear or trend-dependent correlation matrices (Ruppert and Wand 1994; Ortobelli et al. 2019; Kouaissah and Hocine 2021). According to this empirical analysis, the newly proposed approach leads to the mitigation of shortcomings and improves the ex-post portfolio statistics compared to the mean–variance scenarios.

This paper is structured as follows. In Sect. 2, we discuss the trend–risk and trend-dependency measures based on ARV. In Sect. 3, we discuss parametric and nonparametric return approximation techniques based on the PCA. Section 4 presents the empirical analysis. Conclusions are summarized in Sect. 5.

2 Theoretical aspects of trend–risk measures

The purpose of this section is to discuss the trend–risk measure called "accrued returns variability" (ARV) presented by Ruttiens and to provide our comprehensive modification that leads to a more precise and accurate computation in the portfolio optimization process.[1]

[1] We also refer to the accrued returns variability as trend-dependent risk measure, time-dependent risk measure, or dynamic risk measure.

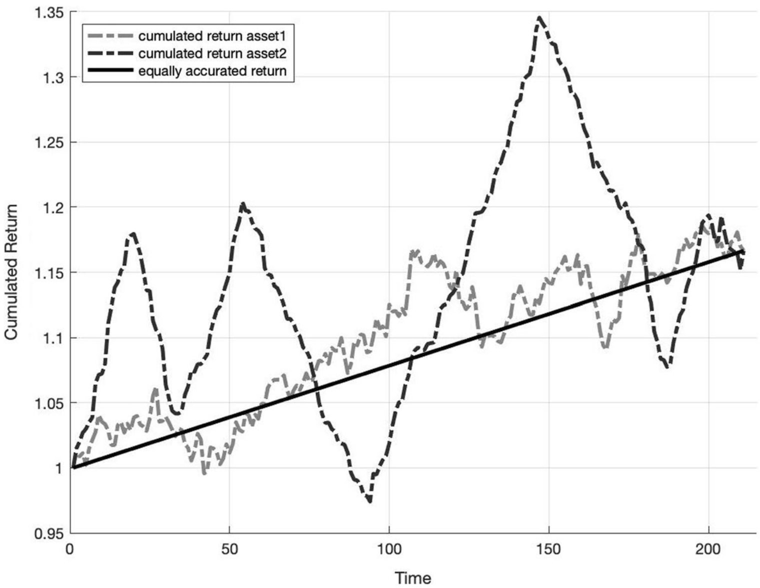

Fig. 1 Evolution of cumulative returns of two portfolios compared to the linear trend line

First, let us begin with an example that helps to understand the main reason why measuring the risk using cumulative return series alongside the classical statistics of returns (variance, standard deviation, or semi-deviation). Basically, classical statistics measure the one-sided or two-sided risk associated with each return series as the mean of the deviations from the average. An important issue in financial time series arises from considering static types of statistics in the computation to determine the individual deviations with the time-dependent variable. Due to the constancy of the mean value over time, alternative penalty methods have been proposed to emphasize the time dependence of the returns (Fastrich et al. 2015).

To illustrate the idea of a time-dependent (trend-dependent) risk measure, consider two identical portfolio return series, but ordered differently. Both series of returns lead to the same final cumulative value as captured in Fig. 1. The commonly analysed statistics, e.g. mean, variance, standard deviation, skewness, and kurtosis, are identical for both series. However, from the investor's perspective, the ways to obtain the final cumulative return are not equal. The cumulative return of *portfolio2* looks more volatile. Rationally, we should compare the cumulative returns with the equally accrued return leading to the same cumulative return, which incorporates the impact of time. The non-volatile alternative investment (zero-risk linear trend) can also be seen as a benchmark.

It is apparent that Ruttiens defined ARV as the standard deviation of the positive and negative distances between the cumulative returns at a specific time and the analogous ones with a zero volatility linear trend.

Based on this concept, let us assume the portfolio returns are $x'r$, where the vector of portfolio weights is denoted by $x = [x_1, \dots, x_z]$ and the vector of returns is $r = [r_1, r_2, \dots, r_z]$ with x_i, respectively, r_i, the weight and return of the ith asset for $i = 1, 2, \dots, z$. Suppose that we have T observations of the portfolio returns. Then we denote the tth observation of the portfolio by $x'r_{(t)}$. Assume the cumulative portfolio return is $c_{x'r}$. Then its tth observation is given by $c_{x'r,t} = c_{x'r,t-1}(1 + x'r_{(t)})$, for $t = 1, \dots, T$. At the same time, the tth observation of the equally accrued return $e_{x'r}$ is $e_{x'r,t} = c_{x'r,0} + \frac{t}{T}(c_{x'r,T} - c_{x'r,0})$ and $c_{x'r,0}$ represents the initial investment. Recall that the slope of the linear line depends on the original series of the returns and may have an increasing or a decreasing tendency. In addition, if at least one value in the original series is changed, the linear trend changes as well. Furthermore, the mean of the differences $m_{x'r} = E(c_{x'r} - e_{x'r})$ is approximated by its empirical mean $\frac{1}{T}\sum_{t=1}^{T}(c_{x'r,t} - e_{x'r,t})$. According to Ruttiens (2013), $ARV(x'r)$ is the standard deviation of $c_{x'r} - e_{x'r}$ and is approximated by its empirical standard deviation:

$$ARV(x'r) \cong \sqrt{\frac{1}{T}\sum_{t=1}^{T}[(c_{x'r,t} - e_{x'r,t}) - m_{x'r}]^2}. \tag{1}$$

Observe that the ARV is a dynamic measure that is sensitive to the order of the return observations over time. Thus, the ARV measure provides better perspectives for measuring the risk of portfolios of financial series than do the classical static risk measures (such as the variance). Clearly, in portfolio theory, a rational investor generally prefers to minimize the so called *trend risk*, which is formally defined as follows.

Definition 1 We call trend risk the variability around a defined trend.

The basic idea of Ruttiens' ARV measure is to control the trend risk while maintaining the cumulative returns close to the desired trend throughout the investment period. Unfortunately, Ruttiens' ARV may lead to an inaccurate valuation of the trend risk, because ARV measures the distance between the cumulative returns and the trend line $e_{x'r}$ summed with the average of the deviations $m_{x'r}$, as illustrated in Fig. 2.

To mitigate this drawback, we present a modified risk measurement with a refinement of the computation and greater usefulness in the portfolio optimization framework. Improving on Ruttiens (2013), we suggest using the second moment of the deviations $y_{x'r} = c_{x'r} - e_{x'r}$, instead of its standard deviation. Thus, we define a modification of the original ARV measure given by $ARV_{\text{mod}}(x'r) = E(y^2_{x'r})$, which is approximated by the empirical second moment of the deviations:

$$ARV_{\text{mod}}(x'r) \cong \frac{1}{T}\sum_{t=1}^{T} y^2_{x'r,t}, \tag{2}$$

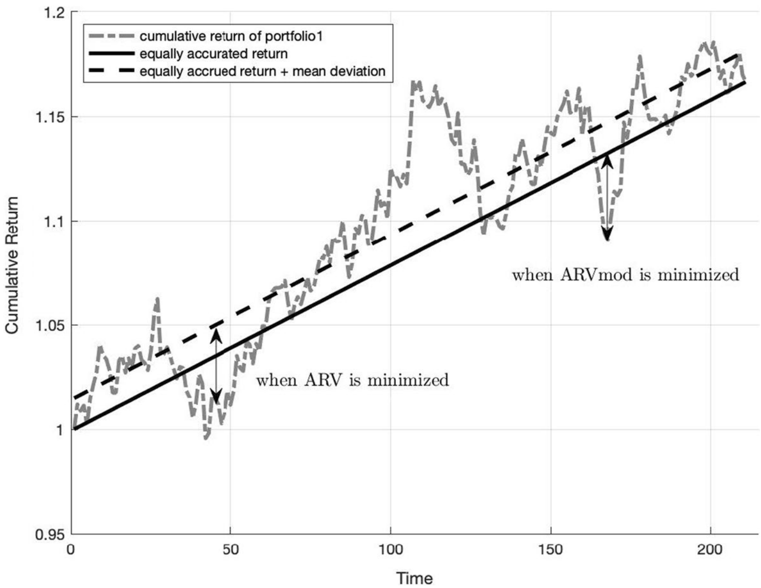

Fig. 2 Impact of ARV and ARV_{mod} on the portfolio optimization task

where $ARV_{\text{mod}}(x'r) \geq 0$ similarly to $ARV(x'r)$. It is worth noting that these two trend-dependent measures are highly correlated.

Proposition 1 *When ARV_{mod} equals zero, then the cumulative volatility equals the zero volatility trend line (that is, $ARV_{\text{mod}}(x'r) = 0 \Leftrightarrow c_{x'r} = e_{x'r}$), which is required for the elimination of the volatility around the trend line. This is in contrast to the original formulation of Ruttiens, for which ARV equals zero implies the cumulative returns essentially equal the zero volatility trend line plus the average of the deviations (that is, $ARV(x'r) = 0$ if and only if $c_{x'r} - e_{x'r} = E(y_{x'r})$, where generally $E(y_{x'r}) \neq 0$).*

Remark 1 Observe that the expected deviation constant function is different from the linear trend function or the equally accrued return $e_{x'r}$ in Ruttiens' work, resulting in an inaccuracy in the computation. Thus, when we minimize $ARV_{\text{mod}}(x'r) = E(y^2_{x'r})$, the cumulative return portfolio tends to the zero volatility trend line (the equally accrued return $e_{x'r}$ is fixed as an optimal wealth path and, therefore, $c_{x'r} \to e_{x'r}$). In contrast, considering the minimization of ARV causes a shift of the optimal path by the value of the mean deviation $m_{x'r}$, which can lead to a distortion, since $c_{x'r} \to (e_{x'r} + m_{x'r})$. This situation is also presented in Fig. 2.

Ruttiens' ARV still offers a usable alternative to measuring risk based on the transformation of the original series to the spread series. Recall that our modification is concentrated on the elimination of trend deviations and mitigating the imperfection mentioned in the computation (optimization) framework. Due to the modification presented, the results of a decision-making process that includes ARV or $\mathrm{ARV}_{\mathrm{mod}}$ are not considerably different but, within the theoretical concept examined, more accurate. This measure can also be assigned to the category of trend-deviation measures. In general, we work with the linear trend, though in the financial sphere either an exponential or non-linear trend that replicates a specified index could be desirable.

2.1 Dependency measures

As one of the objectives of this study is to evaluate the effect in the classical Markowitz mean–variance portfolio analysis, the portfolio risk is logically measured through a variance (Markowitz 1952). To calculate this, a dependency structure, expressed by a variance–covariance matrix $\Sigma = [\sigma_{i,j}]$, between all combinations of assets, is essential. Mathematically, the covariance between the ith and the jth return is given by

$$\sigma_{i,j} = E[(r_{i,t} - \mu_i)(r_{j,t} - \mu_j)], \tag{3}$$

where μ_i is the mean of the ith returns. Of course, variance and covariance are static types of measures.

Therefore, to evaluate a time dependent measure, we can consider the deviations between the cumulative returns and the zero volatile trend line. In particular, the corresponding Ruttiens covariance structure of the spread series of $(c_i - e_i)$ associated with the ith asset return is formulated as follows:

$$\sigma_{i,j}^{Rutt} = E[(c_i - e_i) - m_i][(c_j - e_j) - m_j], \tag{4}$$

where m_i is the expected value of $(c_i - e_i)$. According to Ruttiens (2013), $\sigma_{i,j}^{Rutt}$ is generally higher than the static covariance $\sigma_{i,j}$.

Similarly to the Ruttiens covariance in (4), we propose the modified covariance $\sigma_{i,j}^{mod\,Rutt}$ between the ith and the jth return series based on the spreads. Again, we eliminate the mean component m_i in the calculation so as to take into account only the trend. From the previous assumption, we obtain $\sigma_{i,j}^{mod\,Rutt}$ as follows:

$$\sigma_{i,j}^{mod\,Rutt} = E[(c_i - e_i)(c_j - e_j)], \tag{5}$$

where the properties of this dependency measure are similar to those of $\sigma_{i,j}^{Rutt}$.

According to this concept, the path-dependent risk measure of the portfolio $ARV(x'r) = x'\Sigma^{Rutt}x$ and $ARV_{\mathrm{mod}}(x'r) = x'\Sigma^{mod\,Rutt}x$, where $\Sigma^{Rutt} = [\sigma_{i,j}^{Rutt}]$ and $\Sigma^{mod\,Rutt} = [\sigma_{i,j}^{mod\,Rutt}]$. To express the dependency structure, linear correlation coefficients, such as those of Pearson or Kendall, are widely used in financial modelling. Many types of correlation measures assess the dependency structure between financial variables, but only some of them are appropriate for reducing the dimensionality with a PCA type of composition or to evaluate the dispersion statistic of portfolios (Ortobelli and Tichý 2015).

The general one is the linear Pearson correlation (PC) formulated as

$$\rho_{r_i,r_j} = \frac{E[(r_i - \mu_i)(r_j - \mu_j)]}{\sigma_{r_i}\sigma_{r_j}}, \tag{6}$$

where $\sigma_{r_i} = \sqrt{E(r_i - \mu_i)^2}$.

An alternative correlation measure can be used when we assume that the returns are in the domain of attraction of an α-stable sub-Gaussian distributed $S_\alpha(\gamma, \beta, \mu)$ random variable (Rachev and Mittnik 2000), characterized by its index of stability $\alpha \in (0, 2]$, its asymmetry parameter $\beta \in [-1, 1]$, its dispersion parameter $\gamma > 0$, and its location parameter μ. Thus, we can estimate the stable correlation measure (SC) as

$$\rho^{Stable}_{r_i,r_j} = \frac{E[(r_i - \mu_i)\mathrm{sign}(r_j - \mu_j)]}{2E(\mid r_i - \mu_i \mid)} + \frac{E[(r_j - \mu_j)\mathrm{sign}(r_i - \mu_i)]}{2E(\mid r_j - \mu_j \mid)}, \tag{7}$$

for all α-stable sub-Gaussian distributed returns with a finite mean (i.e. $\alpha > 1$) (Ortobelli and Tichý 2015).

Since we use the covariance between random variables with respect to their trend in formulas (4) and (5), we can define the following alternative Ruttiens correlation (RC) under the same assumptions:

$$\rho^{Rutt}_{r_i,r_j} = \frac{E\left\{\left[\left(c_i - e_i\right) - m_i\right]\left[\left(c_j - e_j\right) - m_j\right]\right\}}{\sigma_{\left(c_i - e_i\right)}\sigma_{\left(c_j - e_j\right)}}, \tag{8}$$

where the standard deviation $\sigma_{(c_i - e_i)} = \sqrt{E[(c_i - e_i) - m_i]^2}$. Thus, our proposed alternative, called the modified Ruttiens correlation (MRC), is formulated as

$$\rho^{mod\,Rutt}_{r_i,r_j} = \frac{E[(c_i - e_i)(c_j - e_j)]}{\nu_{(c_i - e_i)}\nu_{(c_j - e_j)}}, \tag{9}$$

where the standard deviation of $(c_i - e_i)$ is given by $\nu_{(c_i - e_i)} = \sqrt{E(y_i^2)}$.

3 Approximation of returns with parametric and nonparametric regressions

In this section, we present essential aspects of return approximation techniques using parametric and nonparametric regressions and principal component analysis (PCA). In particular, we reduce a large-scale portfolio complexity using a multifactor model that incorporates an adequate number of factors (not too large) such that the explained variability is significantly non-zero (Ortobelli et al. 2019). For this purpose, we consider the PCA of the correlation matrix of asset returns to determine the main factors. Similarly to the previous literature (e.g. Ortobelli and Tichý 2015; Kouaissah and Hocine 2021), to identify the main s factors (principal components), we apply a PCA to both the Pearson correlation and our proposed trend-dependent correlation matrix, respectively. Moreover, we consider the following linear multifactor model:

$$r_i = b_{i,0} + \sum_{j=1}^{s} b_{i,j} f_j + \varepsilon_i, \quad \text{for } i = 1, \dots, z, \tag{10}$$

where $b_{i,0}$ is the fixed constant for asset $i = 1, \dots, z$, $b_{i,j}$ is the coefficient related to factor f_j, s is the number of factors, and ε_i is the error part. Generally, an ordinary least squares (OLS) estimator is usually used to estimate the coefficients of the parametric linear regression problem (Fan et al. 2008). Using parametric regression, we replace the original z correlated return series $\{r_i\}_{i=1}^{z}$ with z uncorrelated time series $\{g_i\}_{i=1}^{z}$ obtained from the PCA, where each r_i follows a linear function of g_i. Dimensionality is reduced by using only a few factors that express a large part of the total variability, while the remaining factors form the error part. Thus, the linear multifactor model of returns series that is estimated by the OLS method is formulated as

$$\hat{r}_i = b_{i,0} + \sum_{j=1}^{s} b_{i,j} f_j + \sum_{j=s+1}^{z} b_{i,j} f_j = b_{i,0} + \sum_{j=1}^{s} b_{i,j} f_j + \epsilon_i, \tag{11}$$

where $(\hat{r}_i)$ is the approximate gross return of asset i, $b_{i,0}$ is the fixed constant for the ith asset, $b_{i,j}$ is the coefficient of the factor f_j, s is the number of factors, and ϵ_i is the error part of asset i.

The OLS estimator works well if the initial series is normally distributed, but this is not obvious for a return series (Rachev and Mittnik 2000). This strong assumption may lead to a misleading final estimate when applied to financial data. Recently, Ortobelli et al. (2019) proposed a nonparametric approximation to reduce the dimensionality of a large-scale portfolio, where the factors are again determined applying a PCA to the linear or another type of correlation structure and the $s+1$th factor B_{s+1}, that is,

$$r_i = E(r_i \mid f_1, \dots, f_s, B_{s+1}) + \varepsilon = \tau(f) + \varepsilon, \tag{12}$$

where $(f_1, \dots, f_s)$ represents s uncorrelated factors and B_{s+1} is a benchmark index. In this paper, we use the market upper stochastic bound defined as $B_{s+1} = \max_i(r_i)$ that gives the maximum possible return among the available assets at any time. The nonparametric regression model is consistent even if the returns commonly follow the stable distribution (Rachev and Mittnik 2000; Nolan and Ojeda-Revah 2013). The recently preferred tool for the estimation of the function $\tau(f)$ is the estimator based on locally weighted least squares (hereinafter only RW) proposed by Ruppert and Wand (1994). To estimate the regression function $\tau(f)$ by RW, we have to estimate the parameter a by optimizing the following problem:

$$\min_{a,b} \sum_{t=1}^{T} \left\{ r_i - a - b^T \left(f_{(t)} - f \right) \right\}^2 K_H \left(f_{(t)} - f \right), \tag{13}$$

where $f_{(t)}$ is the tth observation of the vector of factor f, K_H is a multivariate kernel estimator of an $s \times s$ symmetric positive definite matrix H that depends on the sample size T.

For the multivariate kernel estimator K_H, Scott (2015) suggests employing the s-dimensional multivariate Gaussian density with variance–covariance bandwidth $H = diag(h_1, \dots, h_s)$:

$$\text{Scott's rule in } \mathbb{R}^s : \hat{h}_i = \hat{\sigma}_i T^{-1/(s+4)}, i = 1, \dots, s, \tag{14}$$

where $\hat{\sigma}_i$ means the estimated standard deviation of variable f_i, and T is the number of observations. More details and a discussion of non-parametric regression are provided in Ruppert and Wand (1994); Bowman and Azzalini (1997); Scott (2015); Ortobelli et al. (2019). The empirical analysis of this paper uses the normal kernel function and the bandwidth selection method suggested by Scott (2015).

4 Empirical analysis of the portfolio selection strategy

In this empirical section, we analyse ex-post and ex-ante portfolio statistics of various portfolio selection strategies using approximated returns from factors by parametric and nonparametric regression models. In particular, we use a new framework that integrates both the trend–risk measures and trend-dependency structure introduced in Sect. 2 into several variations of portfolio optimization scenarios. These scenarios are basically derived from the mean–variance portfolio model, where we assume 40 various risk-averse strategies that ultimately form an efficient frontier (Markowitz 1952). Furthermore, we also propose a new portfolio selection framework with a double optimization process and a trend-correlation PCA. The advantage of the double optimization consists of setting a benchmark portfolio statistics in the first part and then finding the effective (less trend-risky) way of how to achieve at least the same level of profitability. According to the results obtained, we show the effect of different strategies on the wealth paths, return statistics, and diversification measures of particular portfolios.

In this analysis, we consider a dataset consisting of daily observations of 182 U.S. stocks, which were included as components of the S &P 100 index during the period from 2 January 2002 to 31 December 2021, which means 5036 daily observations in total. In addition, we monitored changes in the composition of the index that occurred regularly every three months and created a composition matrix. It follows that we are able to capture all the changes in the data-set to make it dynamic in time and eliminate the impact of survivorship bias. This fact makes the analysis more realistic, accurate, and credible. All data were downloaded from Thomson Reuters Datastream.

4.1 Ex-post comparison according to nonparametric regression among different portfolio strategies

To illustrate the effect using an ex-post empirical analysis, we assume that each portfolio is re-optimized at monthly intervals (every 21 days) based on a one-year moving window of historical observations (252 days). Furthermore, we assume that short sales are not allowed and the initial wealth of the portfolio $W_0 = 1$ is first invested on 2 January 2003.[2] In order to investigate the effect on portfolios on the efficient frontier, we decided to examine 40 portfolio strategies, which differ in risk attitudes and required minimum value of expected return while maintaining equidistant intervals between them.[3] For all 40 mean–variance strategies, we find the composition of the optimal portfolio, where the following steps are performed:

Step 1 In the first scenario, apply the PCA to the classical Pearson correlation matrix (i.e. PC), select 17 principal components (which explain at least 85% of the variability) and then use the OLS estimator or RW estimator to approximate the returns (i.e. OLS-Pearson or RW-Pearson). In the second scenario, apply the PCA to the stable conditional correlation matrix of returns (i.e. SC) and simultaneously to one of the Ruttiens (i.e. RC) or modRuttiens (i.e. MRC) correlation matrices. Then approximate the returns using the nonparametric RW estimator (i.e. RW-Ruttiens, respectively RW-modRuttiens). In this scenario, we select 3 factors of a particular PCA on trend-dependent correlations (explaining at least 90% of the particular variability), 13 factors from a stable PCA (explaining at least 60% of the stable variability), and factor B_{s+1} as a max benchmark. Thus, both strategies consist of 17 factors to approximate returns (see, e.g. Ortobelli et al. 2019).[4]

Step 2 Determine the optimal weights for all 40 portfolio strategies and create the efficient frontier. In particular, the simple mean–variance (mean–variability) model consists of determining the optimal vector of asset weights x for the first strategy while minimizing the global variance of the portfolio (GMV) as the following quadratic optimization problem:

$$\begin{aligned} \min(x'\Sigma x) \\ x'\varphi = 1 \\ 0 \leq x_i \leq 1;\, i = 1, \dots, z, \end{aligned} \tag{15}$$

[2] All observations in 2002 are used for the initial optimization, therefore the total investment period is reduced by one year.

[3] While the first strategy (Strategy 1) is a global minimum variance portfolio (GMV), the last one (Strategy 40) is a maximum expected return portfolio (MER). Therefore, for the rest of the strategies, we compute the lower bound of the expected return $M \in (x'_{GMV}r, x'_{MER}r)$ with the equidistant difference d calculated as $d = \frac{(x'_{MER}r - x'_{GMV}r)}{N-1}$, where N is the number of strategies. Recall that in this empirical analysis we use $N = 40$.

[4] According to our preliminary analysis, even when we include more factors explaining the given variability, the composition of the portfolios is not substantially different. In general, the portfolio statistics do not change significantly.

where Σ is the classical variance–covariance matrix and $\varphi' = (1, 1, \dots, 1)$ is a vector of ones. Then, determine the vector x for the last strategy (i.e. 40th) solving the optimization problem in order to maximize the mean expected return of the portfolio $\max E(x'r)$, while the conditions of the model remain the same as in (15). Furthermore, compute the individual vectors of weights x for the inner portfolios of the efficient frontier. To do so, minimize again the variance of the portfolio formulated as $\min(x'\Sigma x)$, while achieving a predetermined lower bound of expected return M. Thus, add a condition $E(x'r) \geq M$ to model (15).

In contrast, the double optimization strategy is further composed of the second optimization process, which considers the minimization of the modified Ruttiens risk measure ARV_{mod} formulated in Equation (2). To define the zero volatility trend line for each portfolio $e_{x'r}$, we use the first mean–variance (MV) optimization (15). Employing this assumption, compute $e_{x'r,t} = W_0 + \frac{t}{T}(W^{MV}_{x'r,T} - W_0)$, where $W_0 = 1$. As previously assumed, the condition in the optimization model is to maintain the required expected return of the portfolio S_M in each step computed from the MV model. Therefore, based on the formulation in Sect. 2.1, we have to solve the following quadratic optimization problem:

$$\begin{aligned} \min x'\Sigma^{mod\,Rutt} x \\ x' r_{[0,T]} = S_M \\ x'\varphi = 1 \\ 0 \leq x_i \leq 1;\ i = 1, \dots, z. \end{aligned} \tag{16}$$

where $r_{[0,T]}$ is the gross return vector observed during the last window of observations [0, T] (which is, in our analysis, one year, i.e. 252 daily returns). By adding this part into the algorithm, we should reduce another type of variability of the mean–variance selection strategy, achieving at least the same expected return and final wealth with a different path that reduces the deviation between cumulative portfolio returns and the zero volatile trend line.

Step 3 Compute the ex-post final wealth for each kth re-calibration interval as follows:

$$W_{t_{k+1}} = W_{t_k}(x' r^{ex-post}_{t_{k+1}}), \tag{17}$$

where $r^{ex-post}_{t_{k+1}}$ is the vector of gross returns between time t_k and t_{k+1}, meaning $t_{k+1} = t_k + \varrho$ where $\varrho = 21$.

The entire algorithm (Steps 1 to 3) is repeated until daily return observations are available.

The results of our analysis are reported in Tables 1, 2, 3 and Figs. 3, 4, 5, and 6. Tables 1, 2, and 3 report important portfolio statistics of ex-post returns, i.e. mean (%), standard deviation (%), skewness, kurtosis, VaR5% (%), CVaR5% (%), selected performance measures,[5] i.e. SR (%), TDR1, and TDR2, and final wealth for the

[5] In order to measure the performance of the portfolio, we selected the usually used Sharpe ratio (SR) (Sharpe 1994; Biglova et al. 2004). This indicator was chosen due to its explanatory power based on the entire distribution of returns. The Sharpe ratio expresses the expected excess return for the unity of risk measured as standard deviation calculated as $SR = \frac{E(x'r - r_b)}{\sigma_{x'r}}$, where $\sigma_{x'r}$ denotes the standard deviation of

scenarios proposed and all portfolio strategies (Ortobelli et al. 2017; Malavasi et al. 2020).

Table 1 illustrates the results of classical mean–variance portfolio selection strategies on ex-post approximated returns using PCA on the Pearson correlation matrix with parametric OLS and nonparametric RW regression models. It is evident that for the strategies with minimal risk and maximal expected returns located at the beginning and at the end, the OLS slightly outperforms the RW regression model in the profitability statistics, i.e. mean and final wealth. In contrast, strategies in the middle with RW approximation of returns lead to higher ex-post mean and final wealth. However, by analysing the risk measures of particular strategies, we observe that a significantly lower variability (standard deviation, VaR5%, or CVaR5%) of ex-post portfolio returns is achieved when including the RW regression model. As expected, the significance decreases with increasing strategies that maximize the expected return. In relation to these facts, it can be noted that the use of RW regression shows generally higher SR results. The most striking peculiarity shown in Table 1 is that TDR1 and TDR2 often contradict the results observed for the other indicators due to the noticeably higher values for most of RW strategies, except for those around 30. This is mainly caused by the lower rate of the trend-dependent risk measure. In addition, the ex-post returns of all strategies with OLS and especially with the RW approximation are negatively skewed and strongly leptokurtic.

However, to obtain deeper insights into these findings, Figs. 3 and 4 illustrate the Log-Wealth paths of all strategies considering both the parametric OLS-Pearson regression approach and the nonparametric RW-Pearson correlation approach, respectively.

It is clearly observable that the application of the nonparametric regression technique generates a similar final Log-Wealth for the less risky strategy and outperforms the OLS in the middle strategies, as concluded from Table 1. However, from the surface plot of wealth paths for all strategies, we can better see the difference in riskiness between these regressions. Especially in the part of the figures monitoring riskier strategies, the impact of the financial crisis in 2008–2009 can be seen, as well as the ability of the strategy to adapt. These figures confirm the findings that using the RW approximation smooths the wealth paths compared to their corresponding strategies with OLS, which concurrently means a reduction of the portfolio risk. Recall that these findings are based on a strategy with a mean–variance optimization.

According to the previous discussion and the existing literature that evaluates the accuracy of the OLS and RW approximation by testing the concave dominance between the estimated error parts Ortobelli et al. (2019), the RW method is preferred for further analysis.

Furthermore, in Tables 2 and 3, we illustrate the same ex-post statistics of all portfolio strategies, while applying nonparametric RW-Ruttiens and RW-modRuttiens

Footnote 5 (continued)

the portfolio and r_b is the benchmark return, see Rachev et al. (2008). Following Ortobelli et al. (2017), we define new types of trend-dependent ratios TDR1 and TDR2. These two measures indicate the value of excess wealth per unit of various kinds of risk, static or trend-dependent. Specifically, the formulations of TDR1 and TDR2 are given by $TDR1 = \frac{W_T - 1}{ARV_{\mathrm{mod}}(x'r)}$ and $TDR2 = \frac{W_T - 1}{\sigma_{x'r} + ARV_{\mathrm{mod}}(x'r)}$.

Table 1 Statistics computed for daily ex-post returns obtained for particular mean–variance strategies with parametric OLS-Pearson and nonparametric RW-Pearson ex-ante return approximations

Strategy	Mean	SD	Skew	Kurt	VaR5%	CVaR5%	SR	TDR1	TDR2	Final W
OLS-Pearson										
1	0.0323	0.9508	−0.4774	12.1262	1.4248	2.2728	3.3038	41.3731	37.3870	4.6895
2	0.0320	0.9531	−0.4726	11.9761	1.4175	2.2838	3.2691	40.9149	36.9485	4.6325
3	0.0318	0.9564	−0.4635	11.8450	1.4390	2.2897	3.2301	43.3375	38.8344	4.5745
4	0.0312	0.9612	−0.4472	11.7609	1.4611	2.2935	3.1527	46.8754	41.4570	4.4473
5	0.0308	0.9678	−0.4263	11.5823	1.4666	2.3024	3.0939	47.3463	41.6810	4.3713
6	0.0306	0.9760	−0.4010	11.3729	1.4845	2.3148	3.0480	45.4332	40.0953	4.3309
7	0.0292	0.9855	−0.3880	11.2043	1.4940	2.3341	2.8724	52.7164	45.0267	4.0420
8	0.0283	0.9964	−0.3748	11.0591	1.5109	2.3577	2.7520	64.2157	52.5236	3.8744
9	0.0277	1.0091	−0.3542	10.9379	1.5571	2.3858	2.6602	70.8484	56.3093	3.7690
10	0.0274	1.0217	−0.3513	10.7531	1.5739	2.4166	2.5964	55.7920	46.1020	3.7119
11	0.0268	1.0360	−0.3486	10.5452	1.6124	2.4557	2.5051	38.3824	33.3101	3.6114
12	0.0265	1.0511	−0.3489	10.3216	1.6466	2.4942	2.4401	33.6046	29.5289	3.5592
13	0.0266	1.0688	−0.3493	10.0493	1.6789	2.5393	2.4019	31.7359	28.0269	3.5631
14	0.0258	1.0881	−0.3456	9.7604	1.7187	2.5915	2.2885	25.6657	23.0242	3.4342
15	0.0251	1.1095	−0.3461	9.5022	1.7553	2.6474	2.1799	23.2372	20.9120	3.3186
16	0.0248	1.1324	−0.3551	9.3578	1.8044	2.7049	2.1078	21.8878	19.7321	3.2687
17	0.0246	1.1573	−0.3610	9.2633	1.8602	2.7641	2.0514	19.6085	17.8110	3.2485
18	0.0249	1.1824	−0.3717	9.1817	1.9001	2.8208	2.0326	18.3474	16.7626	3.2945
19	0.0252	1.2087	−0.3839	9.1521	1.9439	2.8859	2.0115	16.3823	15.1036	3.3390
20	0.0262	1.2369	−0.3953	9.1445	1.9848	2.9594	2.0466	15.2186	14.1541	3.5028
21	0.0277	1.2660	−0.3975	9.1067	2.0322	3.0307	2.1148	14.6683	13.7424	3.7560
22	0.0293	1.2954	−0.3924	9.0745	2.0592	3.0975	2.1949	13.5736	12.8375	4.0667
23	0.0317	1.3281	−0.3857	9.0275	2.1128	3.1723	2.3215	13.6370	12.9770	4.5610
24	0.0340	1.3632	−0.3772	8.9984	2.1727	3.2513	2.4259	13.6372	13.0424	5.0765

Table 1 (continued)

Strategy	Mean	SD	Skew	Kurt	VaR5%	CVaR5%	SR	TDR1	TDR2	Final W
25	0.0361	1.4003	−0.3647	8.9320	2.2671	3.3347	2.5118	13.1951	12.6871	5.6139
26	0.0383	1.4405	−0.3521	8.8097	2.3366	3.4217	2.5991	11.9452	11.5666	6.2578
27	0.0406	1.4844	−0.3345	8.6917	2.4260	3.5152	2.6761	10.7791	10.4982	6.9795
28	0.0425	1.5314	−0.3139	8.5974	2.4867	3.6219	2.7147	9.2445	9.0511	7.6255
29	0.0453	1.5808	−0.2910	8.5478	2.5658	3.7379	2.8077	8.6129	8.4637	8.7234
30	0.0481	1.6332	−0.2686	8.4996	2.6550	3.8603	2.8910	8.0777	7.9609	9.9896
31	0.0509	1.6890	−0.2426	8.4590	2.7342	3.9871	2.9619	7.2483	7.1641	11.4274
32	0.0535	1.7471	−0.2154	8.4390	2.8123	4.1252	3.0143	6.9835	6.9130	12.9628
33	0.0566	1.8045	−0.2070	8.2912	2.9258	4.2718	3.0859	7.6039	7.5300	14.9808
34	0.0592	1.8670	−0.1998	8.1453	3.0423	4.4340	3.1250	7.8317	7.7608	17.0132
35	0.0616	1.9348	−0.1963	7.9997	3.1572	4.6051	3.1372	9.2854	9.1939	19.0433
36	0.0622	2.0142	−0.1961	7.9165	3.2925	4.8006	3.0466	10.5581	10.4391	19.6599
37	0.0619	2.1065	−0.1941	7.9641	3.4078	5.0316	2.8957	12.4383	12.2629	19.3178
38	0.0622	2.2132	−0.1942	8.2740	3.5703	5.2981	2.7695	15.6266	15.3413	19.5938
39	0.0637	2.3446	−0.1902	8.9275	3.7974	5.6074	2.6813	16.7518	16.4311	21.1231
40	0.0673	2.5107	−0.1680	10.071	4.0304	5.9675	2.6436	15.1543	14.9177	24.9819
RW-Pearson										
1	0.0303	0.7718	−0.6341	20.0788	1.1249	1.8500	3.8056	11.6324	11.3200	4.2534
2	0.0285	0.7766	−0.6897	19.8412	1.1149	1.8743	3.5612	10.3252	10.0490	3.9183
3	0.0277	0.7821	−0.7085	19.3800	1.1191	1.8901	3.4332	9.0110	8.7874	3.7699
4	0.0270	0.7898	−0.7549	18.9993	1.1259	1.9137	3.3053	8.0577	7.8678	3.6377
5	0.0258	0.8006	−0.8139	19.0378	1.1360	1.9438	3.1135	7.5406	7.3584	3.4381
6	0.0250	0.8145	−0.8939	19.5813	1.1502	1.9818	2.9582	6.9946	6.8258	3.3041
7	0.0240	0.8308	−0.9793	20.5779	1.1713	2.0207	2.7813	6.7017	6.5327	3.1518

Table 1 (continued)

Strategy	Mean	SD	Skew	Kurt	VaR5%	CVaR5%	SR	TDR1	TDR2	Final W
8	0.0239	0.8445	−1.0277	21.3724	1.1811	2.0487	2.7209	6.5743	6.4074	3.1323
9	0.0244	0.8607	−1.0800	22.2841	1.2116	2.0829	2.7287	6.4087	6.2526	3.2094
10	0.0253	0.8800	−1.1395	23.2228	1.2320	2.1236	2.7769	6.1476	6.0100	3.3587
11	0.0264	0.9017	−1.2035	24.0191	1.2609	2.1715	2.8327	5.8740	5.7541	3.5410
12	0.0276	0.9259	−1.2392	24.9237	1.2825	2.2219	2.8897	5.6847	5.5780	3.7528
13	0.0287	0.9534	−1.2482	26.1189	1.3132	2.2805	2.9180	5.6042	5.5045	3.9490
14	0.0291	0.9790	−1.3102	27.1846	1.3428	2.3419	2.8871	5.6999	5.5970	4.0338
15	0.0295	1.0069	−1.3671	28.3020	1.3905	2.4081	2.8438	5.8635	5.7542	4.1055
16	0.0297	1.0374	−1.4273	29.6526	1.4428	2.4822	2.7800	6.0274	5.9100	4.1465
17	0.0300	1.0695	−1.4893	31.1782	1.4771	2.5565	2.7188	6.3212	6.1902	4.1939
18	0.0303	1.1033	−1.5503	32.7167	1.5129	2.6365	2.6659	6.6845	6.5367	4.2617
19	0.0310	1.1384	−1.5771	33.5255	1.5680	2.7199	2.6432	7.0987	6.9340	4.4022
20	0.0317	1.1759	−1.5820	33.9001	1.6390	2.8100	2.6166	7.4542	7.2744	4.5475
21	0.0324	1.2158	−1.6074	34.6494	1.6843	2.9068	2.5928	7.7803	7.5871	4.7144
22	0.0326	1.2575	−1.6255	35.0342	1.7237	3.0146	2.5231	8.2870	8.0635	4.7606
23	0.0328	1.3004	−1.6298	35.1924	1.7854	3.1265	2.4549	8.8595	8.5992	4.8058
24	0.0326	1.3442	−1.6265	35.2356	1.8545	3.2434	2.3614	9.5032	9.1913	4.7639
25	0.0328	1.3909	−1.6168	35.0948	1.9381	3.3678	2.2931	10.0744	9.7160	4.7988
26	0.0328	1.4427	−1.5980	34.6779	2.0220	3.5041	2.2150	10.2713	9.8871	4.8132
27	0.0337	1.4933	−1.4061	30.7625	2.1024	3.6405	2.2001	10.7392	10.3276	5.0244
28	0.0348	1.5438	−1.1752	26.4713	2.1723	3.7785	2.1946	11.3192	10.8749	5.2776
29	0.0363	1.5953	−0.9347	22.5628	2.2629	3.9169	2.2200	12.0115	11.5390	5.6794
30	0.0377	1.6514	−0.7603	20.1953	2.3302	4.0673	2.2299	12.1059	11.6471	6.0757
31	0.0390	1.7124	−0.6460	18.9477	2.4272	4.2311	2.2286	11.4866	11.0884	6.4769

Table 1 (continued)

Strategy	Mean	SD	Skew	Kurt	VaR5%	CVaR5%	SR	TDR1	TDR2	Final W
32	0.0418	1.7744	−0.5543	18.2907	2.5102	4.3953	2.3083	10.4327	10.1397	7.4045
33	0.0456	1.8391	−0.4738	17.8982	2.6006	4.5509	2.4319	8.8091	8.6314	8.8676
34	0.0495	1.9061	−0.3988	17.5333	2.7085	4.7027	2.5519	7.4293	7.3224	10.6954
35	0.0535	1.9798	−0.3389	17.2986	2.8080	4.8643	2.6552	6.1563	6.0939	12.9056
36	0.0565	2.0628	−0.2824	17.0743	2.9488	5.0507	2.6983	4.5809	4.5501	14.9639
37	0.0588	2.1540	−0.2161	16.8714	3.0923	5.2529	2.6908	3.3457	3.3304	16.7046
38	0.0610	2.2430	−0.1530	16.4795	3.1862	5.4430	2.6823	2.4270	2.4195	18.5604
39	0.0644	2.3538	−0.0954	16.1396	3.3157	5.6837	2.6999	1.5657	1.5630	21.8271
40	0.0623	2.5233	−0.0772	16.6392	3.5239	6.0534	2.4325	0.8460	0.8450	19.6743

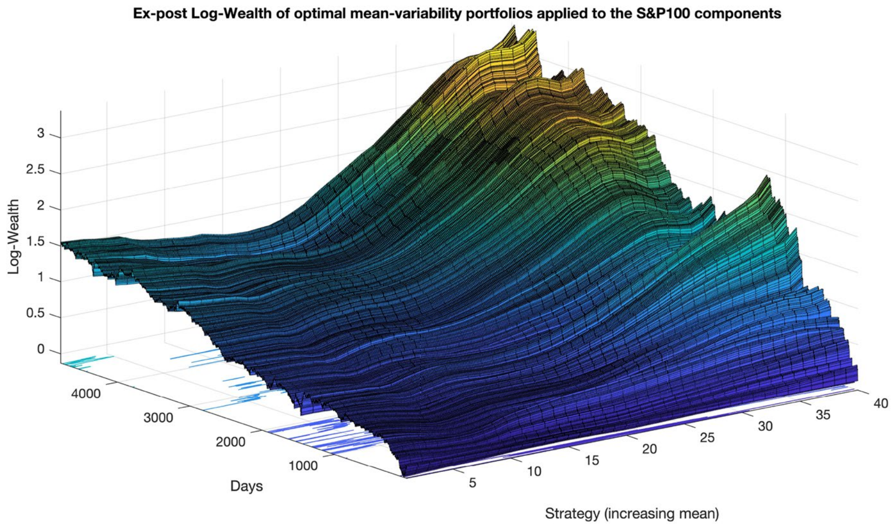

Fig. 3 Ex-post Log-Wealth of mean–variability strategies with parametric OLS return approximation scenario

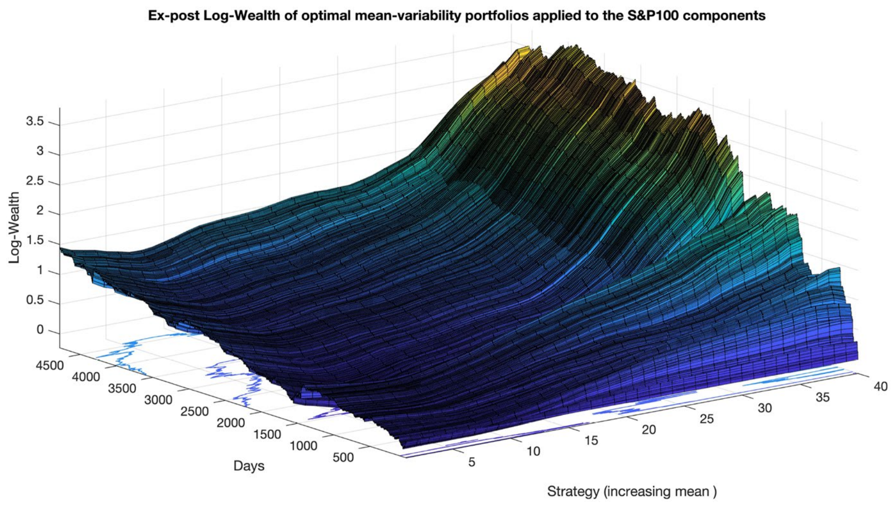

Fig. 4 Ex-post Log-Wealth of mean–variability strategies with nonparametric RW-Pearson return approximation scenario

return approximation scenario simultaneously with a double optimization strategy for a monthly re-calibration.

The results give a broad overview of the profitability and benefits of incorporating the proposed trend measures into the double optimization portfolio strategy. In particular, we observe that in most strategies, the proposed approach using RW

Table 2 Statistics computed for daily ex-post returns obtained for particular mean–trend–risk strategies with nonparametric RW-Ruttiens ex-ante return approximation

Strategy	Mean	SD	Skew	Kurt	VaR5%	CVaR5%	SR	TDR1	TDR2	Final W
1	0.0248	1.4976	−0.1274	12.0370	2.3002	3.5323	1.5951	22.1006	19.2900	3.2716
2	0.0266	1.4904	−0.1328	11.9950	2.2501	3.5062	1.7269	20.7030	18.4873	3.5746
3	0.0289	1.4817	−0.1448	11.9152	2.2410	3.4887	1.8881	19.1217	17.4608	3.9787
4	0.0300	1.4775	−0.1529	11.9984	2.2329	3.4863	1.9740	18.9593	17.4382	4.2113
5	0.0317	1.4726	−0.1759	11.8270	2.2384	3.4850	2.0905	15.9643	14.9730	4.5507
6	0.0329	1.4683	−0.1938	11.5760	2.2331	3.4748	2.1822	13.8031	13.1099	4.8327
7	0.0326	1.4709	−0.2522	11.7137	2.2073	3.4845	2.1576	13.9106	13.1931	4.7626
8	0.0322	1.4771	−0.3211	12.0791	2.1968	3.4941	2.1223	15.2052	14.3296	4.6753
9	0.0329	1.4884	−0.3977	12.9958	2.2021	3.5088	2.1481	15.1351	14.2916	4.8170
10	0.0333	1.4945	−0.4262	13.7542	2.1862	3.5150	2.1672	13.0901	12.4673	4.9138
11	0.0333	1.4958	−0.4569	14.4156	2.1885	3.5117	2.1648	12.0881	11.5541	4.9119
12	0.0348	1.4996	−0.4946	15.2474	2.2057	3.5143	2.2621	10.1604	9.8117	5.2878
13	0.0363	1.5077	−0.5045	16.6059	2.2143	3.5189	2.3478	10.0201	9.7064	5.6748
14	0.0379	1.5126	−0.4918	18.0950	2.2277	3.5153	2.4487	10.2034	9.9058	6.1383
15	0.0385	1.5192	−0.4896	19.7392	2.2076	3.5170	2.4768	9.9432	9.6684	6.3137
16	0.0386	1.5365	−0.5171	22.1777	2.2549	3.5557	2.4548	9.2697	9.0290	6.3409
17	0.0404	1.4817	−0.3002	9.8633	2.2532	3.4852	2.6695	8.7821	8.5933	6.9236
18	0.0410	1.4925	−0.3252	9.9658	2.2800	3.5116	2.6897	8.6576	8.4786	7.1220
19	0.0423	1.5060	−0.3364	10.0698	2.2952	3.5432	2.7487	8.4765	8.3148	7.5615
20	0.0427	1.5225	−0.3623	10.2610	2.2965	3.5998	2.7486	8.6693	8.5024	7.7262
21	0.0432	1.5408	−0.3984	10.4561	2.3135	3.6634	2.7453	8.8633	8.6412	7.8961
22	0.0431	1.5630	−0.4274	10.6882	2.3525	3.7406	2.7019	8.9712	8.7917	7.8692
23	0.0413	1.6367	−0.5910	22.0047	2.3919	3.8736	2.4694	9.6409	9.4022	7.2149
24	0.0413	1.6577	−0.5860	21.6620	2.3952	3.9358	2.4376	11.1167	10.7964	7.2125
25	0.0437	1.6823	−0.5770	21.2301	2.4086	4.0122	2.5455	11.6157	11.3044	8.0957
26	0.0449	1.7132	−0.5641	20.6179	2.4881	4.0978	2.5714	14.6838	14.2126	8.5863

Table 2 (continued)

Strategy	Mean	SD	Skew	Kurt	VaR5%	CVaR5%	SR	TDR1	TDR2	Final W
27	0.0453	1.7465	−0.5429	19.9996	2.5424	4.1930	2.5454	19.4392	18.6237	8.7532
28	0.0478	1.7792	−0.5069	19.4526	2.5792	4.2745	2.6364	22.9580	21.9443	9.8424
29	0.0496	1.8159	−0.4735	18.9225	2.6014	4.3756	2.6851	23.2207	22.2585	10.7548
30	0.0506	1.8637	−0.4298	18.4341	2.6590	4.5071	2.6701	22.0615	21.2134	11.2843
31	0.0506	1.9040	−0.4212	18.0779	2.7439	4.6251	2.6134	19.1799	18.5220	11.2823
32	0.0532	1.9502	−0.4002	17.6796	2.7978	4.7432	2.6816	16.1711	15.7481	12.7391
33	0.0546	1.9944	−0.3706	17.3014	2.8688	4.8419	2.6938	12.8843	12.6276	13.6392
34	0.0552	2.0444	−0.3252	16.9197	2.9482	4.9530	2.6564	7.9014	7.8046	14.0252
35	0.0577	2.0916	−0.2846	16.6803	2.9911	5.0670	2.7157	5.8617	5.8135	15.8041
36	0.0611	2.1521	−0.2286	16.3776	3.0814	5.2078	2.7993	5.2987	5.2647	18.6335
37	0.0637	2.2293	−0.1603	16.0372	3.2198	5.3996	2.8163	5.1504	5.1211	21.0387
38	0.0689	2.2981	−0.1043	15.8507	3.2823	5.5282	2.9610	4.0086	3.9945	27.0636
39	0.0712	2.4030	−0.0460	15.5730	3.4125	5.7592	2.9255	2.3450	2.3405	30.1504
40	0.0655	2.5486	−0.0553	16.4383	3.5693	6.0915	2.5346	1.0691	1.0678	22.9471

Table 3 Statistics computed for daily ex-post returns obtained for particular mean–trend–risk strategies with nonparametric RW-modRuttiens ex-ante return approximation

Strategy	Mean	SD	Skew	Kurt	VaR5%	CVaR5%	SR	TDR1	TDR2	Final W
1	0.0276	1.5224	0.3042	15.9323	2.2293	3.4867	1.7560	54.0686	41.6090	3.7488
2	0.0263	1.5169	0.2106	15.1448	2.2201	3.4899	1.6771	44.7456	35.2622	3.5238
3	0.0259	1.5069	0.1455	14.2330	2.2138	3.4729	1.6629	24.7883	21.5205	3.4599
4	0.0263	1.4988	0.0709	13.4050	2.1929	3.4650	1.6967	15.0828	13.8422	3.5223
5	0.0283	1.4935	−0.0094	12.7081	2.2028	3.4629	1.8347	11.0896	10.4847	3.8706
6	0.0290	1.4911	−0.0759	12.1905	2.2129	3.4553	1.8835	8.9150	8.5366	3.9992
7	0.0289	1.4978	−0.1898	12.3134	2.2395	3.4761	1.8702	8.1805	7.8580	3.9853
8	0.0310	1.5066	−0.3151	13.0662	2.2408	3.5034	1.9977	8.0077	7.7336	4.4035
9	0.0320	1.5010	−0.3464	12.7538	2.1943	3.5015	2.0746	8.0180	7.7606	4.6286
10	0.0337	1.5094	−0.3741	13.7362	2.1892	3.5247	2.1769	7.8600	7.6350	5.0255
11	0.0332	1.5226	−0.4407	17.3635	2.1943	3.5394	2.1216	7.3428	7.1379	4.8937
12	0.0326	1.5216	−0.4834	18.5636	2.2111	3.5382	2.0831	7.7387	7.5033	4.7535
13	0.0320	1.5221	−0.5140	19.9205	2.1934	3.5375	2.0453	8.2800	8.0020	4.6271
14	0.0326	1.5253	−0.5358	20.9739	2.1978	3.5544	2.0818	8.7432	8.4442	4.7668
15	0.0341	1.5278	−0.5432	21.6797	2.1962	3.5561	2.1712	9.2465	8.9387	5.1016
16	0.0360	1.5359	−0.5507	22.1940	2.1947	3.5851	2.2852	8.4908	8.2564	5.5938
17	0.0402	1.4795	−0.3211	9.5422	2.1910	3.5150	2.6595	6.8508	6.7342	6.8557
18	0.0418	1.4908	−0.3196	9.6449	2.2142	3.5404	2.7464	6.0893	6.0042	7.3993
19	0.0432	1.5033	−0.3179	9.7507	2.2192	3.5760	2.8128	5.8594	5.7855	7.8899
20	0.0447	1.5165	−0.3173	9.9494	2.2598	3.6187	2.8860	6.0283	5.9554	8.4700
21	0.0457	1.5336	−0.3220	10.1859	2.2673	3.6660	2.9252	6.1139	6.0424	8.9246
22	0.0452	1.5554	−0.3530	10.4483	2.2688	3.7281	2.8498	6.3986	6.3170	8.6998
23	0.0435	1.6285	−0.5205	22.0544	2.2975	3.8492	2.6150	7.3389	7.2158	8.0038
24	0.0443	1.6489	−0.5077	21.6869	2.3535	3.8988	2.6304	8.1546	8.0073	8.3108
25	0.0462	1.6736	−0.5039	21.1996	2.4054	3.9709	2.7094	8.3690	8.2273	9.1343
26	0.0476	1.7029	−0.4997	20.6251	2.4603	4.0542	2.7418	10.1614	9.9642	9.7415

Table 3 (continued)

Strategy	Mean	SD	Skew	Kurt	VaR5%	CVaR5%	SR	TDR1	TDR2	Final W
27	0.0488	1.7309	−0.4933	20.1180	2.5130	4.1360	2.7709	13.4715	13.1438	10.3523
28	0.0522	1.7611	−0.4786	19.5553	2.5220	4.2190	2.9115	16.8614	16.4233	12.1314
29	0.0533	1.7952	−0.4698	19.0989	2.5442	4.3274	2.9200	19.6028	19.0357	12.8131
30	0.0531	1.8425	−0.4397	18.6509	2.6268	4.4691	2.8318	19.0568	18.5000	12.6659
31	0.0528	1.8836	−0.4417	18.3342	2.7080	4.5892	2.7559	15.6900	15.2971	12.5060
32	0.0543	1.9338	−0.4190	17.9132	2.7895	4.7135	2.7649	11.6458	11.4392	13.4723
33	0.0559	1.9814	−0.4005	17.5968	2.8408	4.8203	2.7774	8.8364	8.7234	14.5201
34	0.0555	2.0377	−0.3754	17.2677	2.8614	4.9539	2.6798	5.1765	5.1356	14.2269
35	0.0574	2.0972	−0.3567	17.1236	2.9229	5.1001	2.6962	3.7959	3.7754	15.6104
36	0.0613	2.1610	−0.2822	16.6463	3.0627	5.2506	2.7951	3.3729	3.3591	18.7737
37	0.0647	2.2366	−0.1751	16.0576	3.2307	5.4302	2.8549	3.8525	3.8369	22.1449
38	0.0701	2.3016	−0.1007	15.8406	3.2662	5.5398	3.0056	3.9292	3.9164	28.5729
39	0.0722	2.4043	−0.0415	15.5677	3.3949	5.7601	2.9682	2.4224	2.4178	31.7241
40	0.0659	2.5496	−0.0542	16.4350	3.5693	6.0942	2.5521	1.0705	1.0892	23.4688

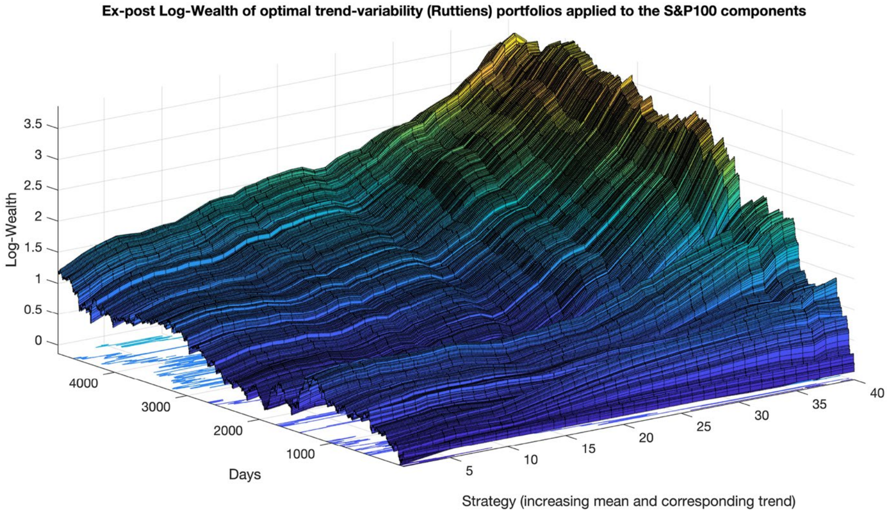

Fig. 5 Ex-post Log-Wealth of mean–variability strategies with nonparametric RW-Ruttiens return approximation scenario

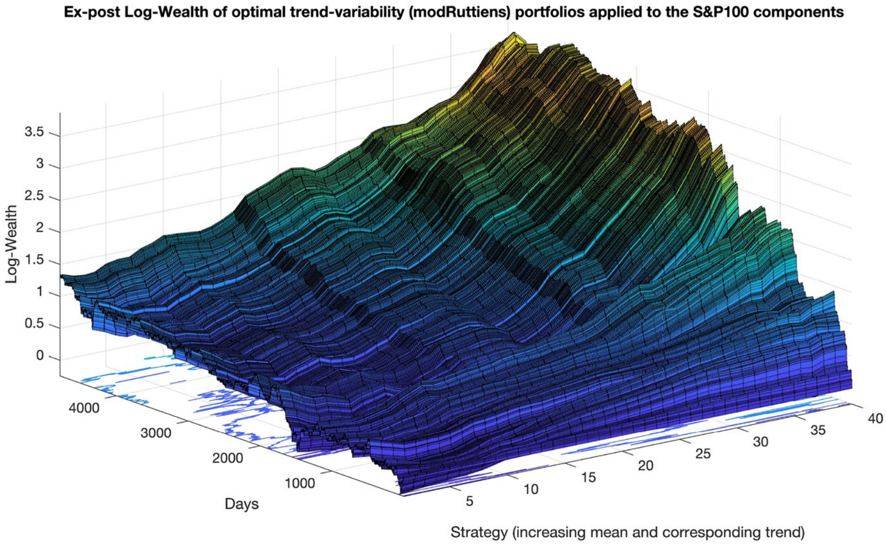

Fig. 6 Ex-post Log-Wealth of mean–variability strategies with nonparametric RW-modRuttiens return approximation scenario

approximated returns with Ruttiens and modRuttiens correlations outperforms the RW-Pearson approximation with a mean–variance optimization in terms of profitability, except for the first risk-minimizing strategies that achieve higher wealth using the RW-Pearson approach. Furthermore, using the RW-Pearson scenario leads to a lower level of riskiness (standard deviation, VaR5%, CVaR5%) than using the other two approaches, where the high risk-averse strategies show approximately one-half the RW-Ruttiens results, but with higher risk tolerance, the differences narrow almost to zero. Probably the reason for this is that the second optimization focussing on minimizes the deviations from the trend allow smoothing the wealth path, but the weights vary significantly from the optimum of the global minimum variance strategy. However, we can observe a trend that as the rate of return increases, portfolio performance expressed by SR increases. It is also evident that more risky mean–trend–risk strategies (above the average) generally outperform the mean–variance ones. Nevertheless, RW-modRuttiens generates higher trend-dependent performance ratios only for the least risky and the most profitable strategies. Furthermore, it is evident that the ex-post return series of all RW-Pearson strategies are more asymmetric and leptokurtic than RW-Ruttiens or RW-modRuttiens. Generally, according to the findings above, the RW-modRuttiens approach is more appropriate for less risk-averse investors than RW-Ruttiens.

In Figs. 5 and 6, we show the Log-Wealth paths of all strategies based on the nonparametric RW-Ruttiens and RW-modRuttiens scenarios. They essentially confirm that trend–risk strategies have a fundamental impact on ex-post performance in portfolio selection strategies even if they have a slightly higher variability in their ex-post wealth paths.

To show and easily compare the results of the different strategies, Fig. 7 with selected portfolio indicators is presented. In particular, we also present the results of trend–risk measures and performance ratios. For all partial panels a–h, the x axis represents an individual portfolio strategy (1–40), where 1 means the GMV strategy and 40 is the MER strategy. On the y axes, we show Final Wealth (panel a), Mean (panel b), Standard Deviation (panel c), Sharpe Ratio (panel d), ARV_{mod} (panel e), ARV (panel f), TDR1 (panel g), and TDR2 (panel h). The discussion of the classical portfolio statistics presented in Tables 1, 2, and 3 was provided above. However, when we focus on both trend-dependent risk measures ARV_{mod} and ARV, it can be surprising that this kind of risk of whole portfolio paths with the double optimization strategy is habitually higher than the simple mean–variance model. Only for the less and relatively high risky portfolios are the obtained results of these scenarios lower than the other ones. We can observe that the results of ARV_{mod} for the most risky strategies are not depicted in the figure due to the scale used (around a value of 20) in order to better show the differences.

4.2 Ex-ante analysis of diversification impact

Having shown the comparison of portfolios between the three strategies in terms of ex-post portfolio performance, we proceed to analyse the diversification effect of individual strategies. To do so, we select four basic diversification measures.

The first simple indicator is the number of assets ($DM1$) to which the investment funds are allocated, i.e. the number of assets with the non-zero weight after the kth re-calibration. The results of $DM1$ are in the interval $[0, z]$, where 0 indicates that the investor interrupted the investment at the re-calibration time k and z means that all funds are split into all available assets.

The second diversification measure ($DM2$) is the simple concentration index, also well known as the Herfindahl–Hirschman index, which is generally used to measure market concentration (Hirschman 1964). However, because of its explanatory power, it can be applied to portfolio analysis. Its formulation is as follows:

$$DM2 = \sum_{i=1}^{z} x_i^2, \tag{18}$$

where $DM2 \in [\frac{1}{z}, 1]$ due to the cases where naive and single asset portfolios are considered. If the value of $DM2$ approaches zero (one), then the investment is divided into a huge (small) number of assets and vice-versa.

The last measure of diversification $DM3$ is more significant for evaluating portfolio diversification from a risk perspective. For this purpose, $DM3$ is the ratio between the risk of the portfolio and the average variability of all z assets for the kth re-calibration. It is formulated as

$$DM3 = \frac{x^{k\prime} Q^k x^k}{\frac{1}{z}\sum_{i=1}^{z}\left\{\frac{1}{T}\sum_{t=1}^{T}\left[r_{i,t}^k - \mathrm{E}\left(r_i^k\right)\right]^2\right\}}, \tag{19}$$

where $DM3 \geq 0$ and $T = (1, \ldots, 252)$. If $DM3 = 1$ then the risk of the portfolio is equal to the mean risk of the assets, which can substitute for the average market risk. If $DM3 < 1$ ($DM3 > 1$) the portfolio has a lower (higher) risk than the market average.

The last indicator examined is turnover (Moorman 2014). According to Biglova et al. (2014), turnover is defined as the change in the composition of the optimal portfolio after the kth re-calibration. Therefore, it can be expressed as follows:

$$\varrho_k = \sum_{i=1}^{z} \mid x_i^k - x_i^{k-1} \mid, \tag{20}$$

where x_i^k is the proportion of funds invested in the ith portfolio component at the re-calibration time k. Moreover, based on this formulation, the indicator value ϱ_k belongs to [0, 2] interval, where $\varrho_k = 0$ means the unchanged composition of the portfolio after the kth re-calibration, while $\varrho_k = 2$ indicates the situation in which the portfolio composition has completely changed.

Table 4 reports the average results of all selected diversification measures (i.e. DM1, DM2, and DM3) and the turnover indicator ϱ of all optimal portfolio strategies considering different RW approximation scenarios.

From the results in Table 4 we observe that according to the number of non-zero weight assets (i.e. DM1) optimal portfolios are always divided among more than 2 assets and less than 17 assets on average for RW-Pearson and for RW with trend-correlation measures less than 5 assets on average. As might be expected, risk-averse strategies are more diversified than the riskiest strategies. This fact has

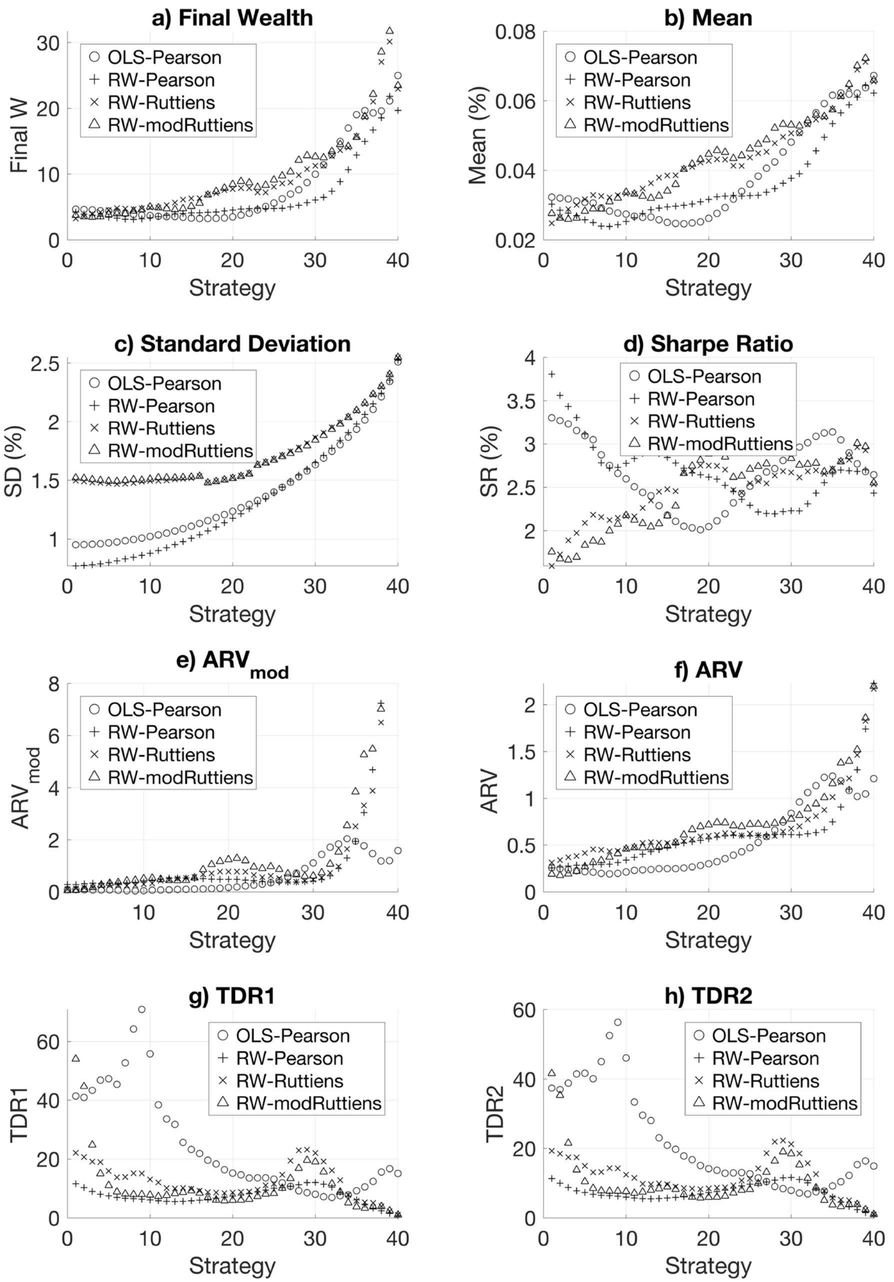

Fig. 7 Comparison of portfolio Final Wealth and selected daily statistics for particular strategies

been confirmed in research on portfolio diversification (Egozcue et al. 2011; Woerheide and Persson 1992, and the references therein). As a result, the values of DM2 behave in inverse relationship depending on the selected strategy, meaning that a higher value of DM1 corresponds to a lower value of DM2 and vice versa. However, the differences between risk-averse strategies are not very pronounced. In addition, when we compare the more sophisticated diversification measure DM3, we can observe that if the RW-Pearson selection strategy is applied, the interval is deeper compared to the RW-Ruttiens and RW-modRuttiens strategies. Furthermore, due to increased diversification, risk-averse strategies have drastically lower variability (approximately 10%) than the average of assets in the market, and risk-seeking strategies are more than 1.5 times riskier than the market average. In contrast, when considering the RW-Ruttiens and RW-modRuttiens strategies, the variability is almost always below the market average, even for the riskiest portfolio strategies (around 40). In summary, these approaches stabilize the portfolio variability.

In addition, an examination comparing types of investor reveals that for classical mean–variance optimization in the RW-Pearson scenario, risk-averse investors exhibit lower portfolio turnover than risk-takers. However, using the proposed double optimisation in RW-Ruttiens or RW-modRuttiens, lower turnover is present in the strategies around the middle. Considering the average values of the turnover, which are around 1.4, we can deduce that RW-Ruttiens or RW-modRuttiens (for almost all strategies) reflect a significant change in portfolio composition. Recall that proportional transaction costs are strongly influenced by turnover.

We have also tested the second order stochastic dominance among the ex-post log-returns of the different portfolio selection strategies. We observed that about 23 per cent of trend–risk portfolio strategies second order stochastically dominates the corresponding mean–variance strategies. However, among the trend–risk type strategies, we do not observed the second order stochastic dominance relationship.

4.3 Discussion

According to the previous ex-post and ex-ante empirical analysis, we have demonstrated that:

- The integration of time-dependent or trend–risk measures as an alternative in the optimization process expands our insight into the issue of portfolio selection strategy;
- A trend–risk double optimization portfolio strategy outperforms the profitability of the simple mean–variance selection strategy;
- In general, these risk measures appear to be more useful in the finance sense as well as attractive to risk-avoiding investors.

In addition, we also used exponential and other variations of the trend line as a substitute for the equally accrued return (required trend) in risk measurement equations, which generate similar results and benefits. Nevertheless, we did not cover them in this analysis, but this adjustment could merit further research. In other words, we can integrate different types of function (e.g. the exponential) to capture trend preferences or replace it with the market trend. Furthermore, we could consider replacing the mean–variance model for the first optimization with a max-ratio model.

Table 4 Diversification statistics obtained for all mean–trend–risk strategies with nonparametric RW returns approximation

Strategy	RW-Pearson				RW-Ruttiens				RW-modRuttiens			
	DM1	DM2	DM3	Turnover ϱ	DM1	DM2	DM3	Turnover ϱ	DM1	DM2	DM3	Turnover ϱ
1	16.4626	0.1709	0.0955	0.8210	4.2026	0.3796	0.2404	1.6026	3.8546	0.3941	0.5649	1.4769
2	16.5815	0.1701	0.0959	0.8352	4.2291	0.3793	0.2344	1.5852	3.8767	0.3897	0.5419	1.4609
3	16.5374	0.1680	0.0970	0.8601	4.3040	0.3800	0.2290	1.5663	3.9251	0.3883	0.5201	1.4476
4	16.4009	0.1649	0.0988	0.8864	4.3656	0.3781	0.2239	1.5440	3.9515	0.3896	0.4995	1.4467
5	16.3216	0.1626	0.1012	0.9206	4.4053	0.3759	0.2195	1.5197	4.0088	0.3880	0.4803	1.4468
6	16.2379	0.1608	0.1044	0.9591	4.4053	0.3764	0.2155	1.5022	4.0793	0.3870	0.4626	1.4425
7	16.0441	0.1596	0.1084	1.0015	4.4273	0.3759	0.2121	1.4849	4.1013	0.3874	0.4461	1.4393
8	15.8238	0.1583	0.1131	1.0440	4.4493	0.3763	0.2091	1.4630	4.5110	0.3886	0.4308	1.4368
9	15.6035	0.1575	0.1187	1.0835	4.3965	0.3769	0.2066	1.4432	4.5903	0.3873	0.4167	1.4278
10	15.1630	0.1571	0.1253	1.1234	4.4405	0.3772	0.2045	1.4225	4.6256	0.3851	0.4037	1.4142
11	14.8811	0.1571	0.1330	1.1620	4.3965	0.3774	0.2028	1.4045	4.2599	0.3831	0.3918	1.4018
12	14.4934	0.1580	0.1417	1.1968	4.4317	0.3766	0.2016	1.3853	4.2511	0.3821	0.3808	1.3977
13	14.1278	0.1589	0.1517	1.2293	4.3965	0.3763	0.2008	1.3672	4.2511	0.3830	0.3707	1.3961
14	13.6828	0.1598	0.1631	1.2580	4.3833	0.3742	0.2005	1.3493	4.2335	0.3820	0.3616	1.3853
15	13.2731	0.1618	0.1759	1.2871	4.3700	0.3700	0.2006	1.3312	4.1674	0.3832	0.3534	1.3777
16	12.8943	0.1648	0.1905	1.3168	4.3568	0.3717	0.2011	1.3215	4.1586	0.3846	0.3461	1.3675
17	12.3877	0.1689	0.2068	1.3496	4.3260	0.3711	0.2023	1.3098	4.1322	0.3832	0.3403	1.3561
18	12.0044	0.1735	0.2251	1.3841	4.2599	0.3742	0.2038	1.3002	4.1145	0.3849	0.3355	1.3499
19	11.4405	0.1790	0.2456	1.4151	4.3304	0.3751	0.2061	1.2847	4.0837	0.3854	0.3322	1.3438
20	10.9251	0.1851	0.2683	1.4393	4.1806	0.3764	0.2091	1.2649	4.0749	0.3840	0.3307	1.3312
21	10.4053	0.1922	0.2936	1.4588	4.1850	0.3793	0.2129	1.2520	4.0088	0.3860	0.3308	1.3181
22	9.9559	0.2005	0.3218	1.4769	4.1278	0.3850	0.2170	1.2451	3.9780	0.3918	0.3323	1.3132
23	9.5639	0.2092	0.3531	1.4917	4.0573	0.3910	0.2219	1.2478	3.9559	0.3979	0.3354	1.3135
24	9.0749	0.2177	0.3879	1.5054	4.0617	0.3928	0.2278	1.2575	3.9559	0.4002	0.3413	1.3189

Table 4 (continued)

Strategy	RW-Pearson				RW-Ruttiens				RW-modRuttiens			
	DM1	DM2	DM3	Turnover ϱ	DM1	DM2	DM3	Turnover ϱ	DM1	DM2	DM3	Turnover ϱ
25	8.7533	0.2274	0.4268	1.5185	3.9956	0.3969	0.2338	1.2600	3.9207	0.4025	0.3481	1.3190
26	8.2907	0.2382	0.4700	1.5283	3.9736	0.4050	0.2414	1.2729	3.8811	0.4121	0.3575	1.3218
27	7.8106	0.2495	0.5183	1.5330	3.9648	0.4122	0.2531	1.2841	3.8634	0.4181	0.3868	1.3213
28	7.3656	0.2611	0.5721	1.5351	3.9075	0.4150	0.2632	1.2871	3.8194	0.4239	0.4057	1.3204
29	6.9604	0.2739	0.6321	1.5363	3.8502	0.4186	0.2759	1.2916	3.7665	0.4239	0.4288	1.3224
30	6.5903	0.2888	0.6990	1.5359	3.7445	0.4251	0.2897	1.3056	3.7004	0.4298	0.4544	1.3327
31	6.1850	0.3069	0.7733	1.5367	3.7004	0.4342	0.3034	1.3202	3.6432	0.4394	0.4812	1.3465
32	5.7577	0.3274	0.8562	1.5343	3.6564	0.4490	0.3201	1.3306	3.5683	0.4541	0.5136	1.3586
33	5.3216	0.3505	0.9491	1.5305	3.5683	0.4656	0.3387	1.3542	3.8987	0.4683	0.5475	1.3758
34	4.9780	0.3759	1.0542	1.5260	3.4890	0.4836	0.3605	1.3909	3.5198	0.4865	0.5934	1.4084
35	4.5551	0.4083	1.1736	1.5272	3.4053	0.5044	0.3834	1.4200	3.4714	0.5077	0.6398	1.4319
36	4.1145	0.4485	1.3099	1.5305	3.3304	0.5318	0.4092	1.4423	3.3040	0.5358	0.6917	1.4488
37	3.7269	0.4945	1.4699	1.5328	3.2379	0.5704	0.4411	1.4654	3.2291	0.5738	0.7545	1.4687
38	3.2379	0.5509	1.6648	1.5307	3.0881	0.6149	0.4758	1.4756	3.4802	0.6167	0.8241	1.4745
39	2.8370	0.6281	1.9086	1.5246	3.0176	0.6801	0.5210	1.4800	3.0000	0.6819	0.9098	1.4780
40	2.4317	0.7435	2.2396	1.5024	2.8458	0.7783	0.5751	1.4867	2.8590	0.7798	1.0098	1.4853

Recall that in this paper we expanded the possibility of using trend analysis to measure risk and dependence in the portfolio selection problem. The results should not only compare the modified indicators with the original ones provided by Ruttiens, but rather present the advantage of the proposed portfolio selection strategies.

5 Conclusion

In this paper, we have proposed a new modification of the trend-dependent risk measure based on Ruttiens' ARV, which evaluates the mean of the squared deviations between the returns and their linear trend. In this context, we are able to include the trend in the optimization process more accurately than Ruttiens supposed. Moreover, we have also defined new trend-dependency measures, i.e. covariance and correlation, that are appropriate to use for the PCA.

To empirically apply the proposed trend-dependent measures in the portfolio selection problem, we have considered the mean–variance portfolio model. In this context, we applied a PCA to several correlation matrices (linear or trend-dependent) to reduce the dimensionality of the portfolio and approximate the returns using both parametric and nonparametric regression models. Additionally, we proposed a new double optimization portfolio selection strategy, which consists of the classical Markowitz mean–variance model followed by a minimization of the deviations from the trend alternative obtained from the previous optimization generating at least identical mean return and final wealth. We evaluated the impact on the ex-post portfolio statistics for 40 strategies with different preferences and risk attitudes of investors. The empirical results showed that using the nonparametric RW approximation, the wealth is smoother during the investment and return series are less variable, with insignificant differences in the profitability. Furthermore, double optimization strategies with trend correlation PCA and RW regression model outperforms the final wealth of strategies based on the Pearson correlation PCA. Finally, the average results of ex-ante diversification analysis show that the suggested strategies reduce the number of assets in the portfolio and their allocation compared to the RW-Pearson strategy.

Note that this type of trend-dependent risk measure is only one alternative, which can subsequently be adapted to the requirements of analysts or researchers.

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Data Availability Not applicable.

Declarations

Conflict of interest Not applicable.

References

Altman EI, Caouette JB, Narayanan P (1998) Credit-risk measurement and management: the ironic challenge in the next decade. Financ Anal J 54:7–11
Artzner P, Delbaen F, Eber JM et al (1999) Coherent measures of risk. Math Financ 9:203–228
Biglova A, Ortobelli S, Rachev ST et al (2004) Different approaches to risk estimation in portfolio theory. J Portf Manag 31(1):103–112
Biglova A, Ortobelli S, Fabozzi F (2014) Portfolio selection in the presence of systemic risk. J Asset Manag 15:285–299
Bowman A, Azzalini A (1997) Applied smoothing techniques for data analysis. Oxford University Press, London
Delbaen F, Eber J, Heath D (1998) Coherent measures of risk. Math Financ 4:203–228
Duffie D, Pan J (1997) An overview of value at risk. J Deriv 4:7–49
Egozcue M, Fuentes García L, Wong W et al (2011) Do investors like to diversify? A study of markowitz preferences. Eur J Oper Res 215:188–193
Fama EF (1965) The behavior of stock-market prices. J Bus 38:34–105
Fan J, Fan Y, Lv J (2008) High dimensional covariance matrix estimation using a factor model. J Econom 147:186–197
Fastrich B, Paterlini S, Winker P (2015) Constructing optimal sparse portfolios using regularization methods. Comput Manag Sci 12:417–434
Hallerbach WG, Spronk J (2002) The relevance of mcdm for financial decisions. J Multi-Crit Decis Anal 11:187–195
Hirschman AO (1964) The paternity of an index. Am Econ Rev 54:761–762
Jorion P (2000) VaR: the new benchmark for managing financial risk. McGraw Hill, New York
Kouaissah N, Hocine A (2021) Forecasting systemic risk in portfolio selection: the role of technical trading rules. J Forecast 40:708–729
Malavasi M, Ortobelli S, Trueck S (2020) Second order of stochastic dominance efficiency vs mean variance efficiency. Eur J Oper Res 290:1192–1206
Mandelbrot B (1963) The variation of certain speculative prices. J Bus 36:394–419
Markowitz HM (1952) Portfolio selection. J Financ 7:77–91
Markowitz HM (1959) Portfolio selection: efficient diversification of investment. Wiley, New York
Miller N, Ruszczyński A (2008) Risk-adjusted probability measures in portfolio optimization with coherent measures of risk. Eur J Oper Res 191:193–206
Moorman TC (2014) An empirical investigation of methods to reduce transaction costs. J Empir Financ 29:230–246
Nolan JP, Ojeda-Revah D (2013) Linear and nonlinear regression with stable errors. J Econom 172:186–194
Ortobelli S, Tichý T (2015) On the impact of semidefinite positive correlation measures in portfolio theory. Ann Oper Res 235:625–652
Ortobelli S, Petronio F, Lando T (2017) A portfolio return definition coherent with the investors' preferences. IMA J Manag Math 28:451–466
Ortobelli S, Kouaissah N, Tichý T (2019) On the use of conditional expectation in portfolio selection problems. Ann Oper Res 274:501–530
Rachev S, Mittnik S (2000) Stable paretian models in finance. Wiley, Chichester
Rachev S, Ortobelli S, Stoyanov S et al (2008) Desirable properties of an ideal risk measure in portfolio theory. Int J Theor Appl Financ 11:447–469
Rockafellar R, Uryasev S (2002) Conditional value-at-risk for general loss distributions. J Bank Financ 26:1443–1471
Ruppert D, Wand M (1994) Multivariate locally weighted least squares regression. Ann Stat 22:1346–1370
Ruttiens A (2013) Portfolio risk measures: the time's arrow matters. Comput Econ 41:407–424
Scott D (2015) Multivariate density estimation: theory, practice, and visualization. Wiley, New York
Sharpe WF (1994) The sharpe ratio. J Portf Manag 21(1):49–58
Szegö G (2002) Measures of risk. J Bank Financ 26:1253–1272
Tobin J (1958) Estimation of relationships for limited dependent variables. Econometrica 26:24–36
Woerheide W, Persson D (1992) An index of portfolio diversification. Financ Serv Rev 2:73–85

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