



Investigation and Implementation of Model Order Reduction Technique for Large Scale Dynamical Systems

Santosh Kumar Suman¹ · Awadhesh Kumar¹

Received: 8 April 2021 / Accepted: 26 November 2021 / Published online: 14 January 2022

© The Author(s) under exclusive licence to International Center for Numerical Methods in Engineering (CIMNE) 2021

Abstract

Model Order Reduction (MOR) has demonstrated its robustness and wide use in engineering and science for the simulation of large-scale mathematical systems over the last few decades. MOR is currently being intensively optimized for dynamic systems that are becoming increasingly complex. MOR broad applications have been identified not only in the modeling but also for optimization and control engineering applications. In the present article, various methods related to MOR for large-scale linear and nonlinear dynamic systems have been analyzed, mainly pertaining to electrical power systems, control engineering, computational system theory, and design. The paper focuses on a detailed theoretic Perspective for MOR of the large-scale dynamical system to address the key challenges to the approximation along with their application. Firstly, a complete description of the literature search for various approximation techniques has been presented and the various inferences have been mentioned as the outcome. One of the drawbacks that we have found out of the investigation, has been taken as a sample problem. The key demerit of the balanced truncation approach is that the ROM steady-state values do not correspond with the higher-order system (HOS). This drawback has been eliminated in the proposed approach, which leads to the hybridization of balanced truncation and singular perturbation approximation (SPA) into a novel reduction method without the loss of retaining its dynamic behavior. The reduced system has been so designed to preserve the complete parameters of the original system with reasonable accuracy. The approach is based on the retention of the dominant states or modes of the system and is comparatively less important once. The reduced system evolves from the preservation of the dominant states or modes of the original system and thus retains stability intact. The methodology presented has been tested on two typical numerical examples taken from the literature review, to examine the performance, precision, and comparison with other available order reduction methods.

1 Introduction

The approximation of linear time-invariant large scale dynamical (LSD) systems is important in many engineering problems, particularly in the design of control systems, where the engineer is confronted with the control of a physical system whose analytical model is described as a high order linear time-invariant (LTI) system. LSD systems can be used to model a wide range of practical systems. It is critical to investigate the control problem

of LSD systems to enhance the control performance of such systems [1, 2]. LSD systems, which have been made up of a series of connected subsystems are widely used in practical applications [3, 4]. The development of a mathematical model begins with the investigation of all physical systems, such as aircraft, chemical plants, refineries, electrical grids, traffic congestion networks, digital communication, and control systems, etc. In many practical situations, a complex, HOS is derived from theoretical considerations [5, 6]. Realistic systems have often been observed to consist of many interactive subsystems with their characteristics, and the resulting system size may be too large to be conveniently handled when such subsystems are interconnected. A typical example of this situation is the dynamic stability studies of modern interconnected power systems. System equations are linear under dynamic conditions, but the total number of differential equations describing system performance increases rapidly

✉ Santosh Kumar Suman
sksumanee@gmail.com
Awadhesh Kumar
awadhesg26@gmail.com

¹ Department of Electrical Engineering, Madan Mohan Malaviya University of Technology, Gorakhpur, Uttar Pradesh, India

as the number of interconnected machines increases. Special numerical techniques for calculations are required for a system of a high order. The analysis of such a HOS is not only time-consuming but also not cost-effective for online implementation.

The complexity of the system also makes it difficult to gain a thorough understanding of its behavior. An uncomfortably HOS may present difficulties in its analysis, synthesis, or identification. Preliminary design and optimization of such systems can often be accomplished more easily if a ROM is derived that provides a good approximation of the system. It is thus preferable to replace a high order system with a low order system while retaining the main qualitative features of the original system, such as the constant time, damping ratio, natural frequency, and stability ratio. Thus, the MOR method contributes to a better understanding of the system. The model must also be mathematically simple so that it can be easily analyzed, and the results studied. In short, the main goals of obtaining ROMs are to develop a better understanding of the higher order of the original system, to reduce computational complexity, to reduce hardware complexity, to make designs feasible, and to obtain simpler control legislation.

This paper is split into six parts. The first part provides an overview and detailed summary of the literature review for large-scale systems. Section 2 defines the Problem Statement. Theoretical Perspective for MOR is described in Sect. 3. Section 4 describes the application of the MOR method for LTI large-scale dynamical systems to reduce, followed by numerical experiments and results that are compared to methods from the literature to determine the validity of the proposed method in Sect. 5. Finally, Sect. 6 discusses the premise and future scope of the work.

2 Problem Statement

Let us consider an *n*th order SISO linear time-invariant higher-order system represented by the following transfer state-space representation by Eq. (1):

$$\Sigma := \begin{cases} \frac{dx(t)}{dt} = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \Leftrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ with ‘p’ inputs and ‘q’ outputs and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{q \times n}$ and $D \in \mathbb{R}^{q \times p}$ are constant matrices of appropriate size. $p=q=1$, the original system is referred to as the SISO system, otherwise, it will be called the multi-dimensional system. The MOR problem consists of finding an approximate system of order ‘r’ ($r \ll n$) described by

$$\Sigma_r := \begin{cases} \frac{dx_r(t)}{dt} = A_r x_r(t) + B_r u(t) \\ y_r(t) = C_r x_r(t) + D_r u(t) \end{cases} \Leftrightarrow \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} \quad (2)$$

Such a dimensional SISO Dynamic system and order model are similar in the important aspects of their characteristics and $x_r \in \mathbb{R}^r$, $u \in \mathbb{R}^p$, $y_r \in \mathbb{R}^q$ and $A \in \mathbb{R}^{r \times r}$, $B \in \mathbb{R}^{r \times p}$, $C \in \mathbb{R}^{q \times r}$ and $D \in \mathbb{R}^{q \times p}$ are constant matrices of reduced-order model and reduced output y_r should be a close approximation of y for a given set of inputs [7, 8]. The transfer function associated with original system Eq. (1) representations may be expressed by

$$G(s) = D + C[sI_n - A]^{-1}B \quad (3)$$

$$G(s) = \frac{N(s)}{D(s)} = \frac{n_0 + n_1s + n_2s^2 + \dots + n_{m-1}s^{m-1}}{d_0 + d_1s + d_2s^2 + \dots + d_ns^n} \quad (4)$$

or

$$G(s) = \frac{\sum_{i=0}^{m-1} n_i s^i}{\sum_{i=0}^n d_i s^i} \quad (5)$$

where $G(s)$ is expressed as and n_i, d_i are scalar constants of the HOS system.

The challenge is to design a ROM (5) that retains the important properties of the original and approximates the output as closely as possible [9, 10], The ROM is expressed as $G_r(s)$ and obtained is described as follows:where

$$G_r(s) = R_r(s) = D_r + C_r(sI_r - A_r)^{-1}B_r \quad (6)$$

$$G_r(s) = \frac{N_r(s)}{D_r(s)} = \frac{\hat{n}_0 + \hat{n}_1s + \hat{n}_2s^2 + \dots + \hat{n}_{r-1}s^{r-1}}{\hat{d}_0 + \hat{d}_1s + \hat{d}_2s^2 + \dots + \hat{d}_r s^r} \quad (7)$$

or

$$R_r(s) = \frac{\sum_{j=0}^{r-1} \hat{n}_j s^j}{\sum_{j=0}^r \hat{d}_j s^j} \quad (8)$$

where suffix ‘r’ is denoted and m_j, n_j are a scalar constant of the ROM [11].

3 Theoretical Perspective for MOR

Simplicity while preserving the features of interest will be one of the most desirable features of such models. Since the models can be developed with different goals and objectives

or with different points of view, a given system can have more than one model, each satisfying some predefined goals. Dissimilar people progress different models for the same system; if their perception or point of view changes over time, the same individual may develop dissimilar models for the same system. The model's usability is a crucial aspect, especially in online controls. Simpler models commonly give a better feel of the original system [12–14]. The approximation method theoretical survey is provided in Tables 1 and 2. Several researchers have proposed a wide variety of MOR methods over the past few decades. The goal of MOR can be summarised as follows:

(i) The prominent points that should always be taken care of while deriving the ROM, are as follows [15]

- Stability should be preserved.
- Passivity should also be maintained or preserved.
- The method should be reliable and effective in computation.
- The method should be reliable and effective in computation.
- The methods must adopt some specific requirements for error tolerance.
- High fidelity representation of the original large-scale system.
- A considerable difference between the size of the ROM and the original system.
- Lesser approximation error and existence of global error bound.
- Numerically stable and efficient procedure.
- Economize in terms of hardware while synthesizing the system.

(ii) Some of the reasons for MOR are as follows

- Fast and ease in the understanding of the system
- Reduced computational burden.
- Reduced computational burden
- Reduced Controller synthesis
- Controller procedure making reasonable designs.
- Improving the computer-aided system design approach.
- All of the above results in the best cost-effective solution.

3.1 A Systematic Literature Review of Inference Strategies for MOR

- Researchers are either developing new algorithms or improving the existing algorithms for MOR.

- Control design can be done after the approximation of a system or even approximation-based control design can be achieved.
- Mixed methods (a combination of two existing methods) are widely available in the literature.
- MOR for real-time systems such as the power system model is the focus of many researchers.
- Much attention has been given to stability preserving ROM.
- Structure and passivity preserving ROM is a new shift in this research area.
- Work on MOR for parametric systems, interval systems, and hybrid systems can be found and still work on it is going on.
- Researchers are also developing nonlinear balancing methods, but so far these can only be used for systems of very limited size (< 100).

3.2 MOR From Mathematics to Innovative Applications

MOR was initially being developed in system theory and control engineering, which studies the properties of dynamic systems in application to reduce their complexity while preserving as much of their input–output behavior as possible. Numerical Mathematicians have also taken up the field, particularly after publishing methods such as PVL (Padé via Lanczos). MOR is a multidisciplinary topic of great interest over the past 40 years. This is because engineering problems often involve large-scale systems or very complex processes that have to be controlled using low-order controllers. The ROMs are required to minimize the computational effort during analysis, simulation and controller design of high order practical systems [16–18]. ROMs are valuable for the following reasons: ROM are neither robust concerning parameter changes nor cheap to generate. A scheme based on a ROM database dramatically reduces the cost of computing aeroelastic forecasts while maintaining good accuracy.

- Analysis and synthesis of the system
- Predicting transient response sensitivities of high order systems using ROMs
- Design of controllers and observers
- Advanced applications in the area of integrated circuits, microfluidic devices, innovative materials, etc.
- Innovative methods to reliable MOR in networks.
- Development of online simulators of system
- They are predicting transient response sensitivities of high-order systems using ROMs.
- The area of economic and financial systems.
- Control system design
- New mathematical tools for ROM

Table 1 A summary of investigation for MOR

Method & References	How does it work	Advantage	Disadvantage
<i>Frequency-domain analysis</i>			
Eigen-mode analysis [16, 63]	<p>For linear systems only Pole-Residue Form: $G(s) = \sum_{i=1}^n \frac{R_i}{s-p_i}$</p> <p>Pole-Zero Form: $G(s) = \frac{\prod_{i=1}^{n-1} (s-\zeta_i)}{\prod_{i=1}^n (s-\zeta_i)}$</p> <p>Rational form (point-matching): $G(s) = \frac{\sum_{i=0}^{n-1} n_i s^i}{\sum_{i=0}^n d_i s^i}$</p> <p>where n_0 represents the numerator coefficients, d_0 represents the denominator coefficients</p> <p>A finite continued fraction is a representation $G(s) = \frac{1}{H_1 s^4 + \frac{1}{H_2 s^4 + \frac{1}{H_3 s^4 + \frac{1}{H_4 s^4 + \dots}}}}$</p>	Simple physical interpretation	Computationally inefficient To achieve models of reduced order, only linear algebraic equations need to be solved
Continued fraction expansion [64]	<p>We obtain the “fractional” expansion with retaining dominance terms</p>	It results in a good approximation thus maintaining stability and minimum-phase characteristics	This method is that, although the original high-order system is stable, the ROM sometimes becomes unstable and the reduced models also show poor approximation in the transient part of the response
Time moment matching category [65]	<p>Let the Impulse response of $G(s) = \int_0^\infty g(t)e^{-st} dt$</p> <p>The expansion of e^{-st} about $s=0$ gives: $G(s) = \int_0^\infty g(t) \left\{ 1 - e^{-st} + \frac{\omega^2 t^2}{2!} - \frac{\omega^4 t^4}{4!} + \dots \right\} dt$</p>	<p>Taylor series expansion is utilized Time moments are matched for deriving the ROM</p>	<p>Very Low accuracy Does not guarantee for the preservation of passivity or stability</p>
Padé approximants [66–71]	<p>Padé approximations $G(s) = \frac{\sum_{i=0}^{m-1} n_i s^i}{\sum_{i=0}^n d_i s^i}$</p> <p>$G(s) = q_0 + q_1 s + q_2 s^2 + q_3 s^3 + \dots$</p>	<p>Rational function fitting via moment matching Power series Expansion Time moments and Markov parameters both are matched to derive the ROM</p>	<p>Instability problem This method has also a serious disadvantage of producing an unstable reduced model for an original stable system in which case the ROM is worthless</p>

Table 1 (continued)

Method & References	How does it work	Advantage	Disadvantage
Routh–Hurwitz array method [72, 73]	$G(s) = \frac{N(s)}{D(s)} = \frac{n_0+n_1s+n_2s^2+\dots+n_{m-1}s^{m-1}}{d_0+d_1s+d_2s^2+\dots+d_{l-1}s^{l-1}}$ <p>where</p> $n_{i,j} = n_{i-2,j+1} - \frac{(n_{i-2,1}n_{i-1,j+1})}{(n_{i-1,1})}$ $d_{i,j} = d_{i-2,j+1} - \frac{(d_{i-2,1}d_{i-1,j+1})}{(d_{i-1,1})}$ $i \geq 3; 1 \leq j \leq \left\lfloor \frac{(m-i+3)}{2} \right\rfloor$ <p>or</p> $\left\lfloor \frac{(l-i+3)}{2} \right\rfloor$	<p>This method gives the guarantee that a stable is reduced only to a stable one but by this method several full models can have the same ROM</p> <p>This method also extended to the reduction of the discrete-time system</p>	<p>Only limited order reduction</p> <p>Computationally inefficient some-times</p>
Stability equation method [71, 74–77]	$= \frac{n_0+n_1s+n_2s^2+\dots+n_{m-1}s^{m-1}}{d_0+d_1s+d_2s^2+\dots+d_{l-1}s^{l-1}}$ <p>Separation for numerator polynomial</p> $N(s) = N_e(s) + N_o(s)$ $N(s) = \sum_{i=0,2,4,\dots}^m m_i s^i + \sum_{i=1,3,5,\dots}^m m_i s^i$ $N_e(s) = \sum_{i=0,2,4,\dots}^m m_i s^i \text{ (Even - Part)}$ $N_o(s) = \sum_{i=1,3,5,\dots}^m m_i s^i \text{ (Odd - Part)}$ <p>where</p> $N_e(s) = m_0 + m_2s^2 + m_4s^4 + \dots$ $N_o(s) = m_1 + m_3s^3 + m_5s^5 + \dots$ <p>The even and odd parts in the form of stability equation as</p> $N_e(s) = m_0 \prod_{i=1}^{k_1} \left(1 + \frac{s^2}{z_i^2} \right)$ $N_o(s) = m_1 \prod_{i=1}^{k_2} \left(1 + \frac{s^2}{p_i^2} \right)$	<p>These stability equations are factorized, and only roots close to the origin are kept</p> <p>Based on the separation base in the form of even and odd terms</p> <p>Based on even and odd parts in the form of stability equation</p>	<p>This approach ensures stability in the ROM if the original system is stable and preserves the system's first two-time moments, maintaining a steady-state response that fits the impulse, step, Ramp input between LSD system and ROM</p> <p>Only LTI system</p> <p>Computationally complex</p>

Table 1 (continued)

Method & References	How does it work	Advantage	Disadvantage
<i>Time-domain methods</i>			
Krylov subspaces by means of either the Arnoldi or Lanczos process (implicit moment matching) [25, 78–82]	Krylov based subspace-scheme (explicit): (Arnoldi, lanczos, interpolation, PVL, PRIMA) $H(s) = \sum_{i \geq 0} h_i s^{-i}$ The coefficients h_i are known as Markov parameters and $H(s)$ called the transfer function of the system	Improve the ill-conditioning Numerical efficient Applicability: $n \gg 1000$ Computationally very useful, can be extended to dynamic systems of very large dimensions	Cannot handle system problems with MIMO Preservation of stability is still open for investigation No global error bound Arnoldi/Lanczos methods lose orthogonality It does not have a global error estimate
SVD based Approximation (BT, SPA) [80]	The proper orthogonal decomposition (POD) methods $A = U \sum V^*$	Improve the ill-conditioning e can use Nonlinear systems Preservation of stability Global error bound Applicability up to $n \approx 1000$ depending on computer specification The calculation can be used entirely correctly in case of a global error	Computationally inefficient So it is used for low order systems The computational complexity is conventional implementations is $O(N_3)$ which can only be used for systems with orders of less than a few thousand unknowns
Hankel-norm approximation [65, 83–90]	The Hankel norm of state rational transfer function: $\ G(s)\ _H = \lambda_{\max}^{\frac{1}{2}}(P_c P_o)$ Where controllability Grammian (P_c) and observability Grammian (P_o)	Priori additive approximation error for upper and lower bound Extended to include a certain class of frequency weights The theory is that the norm of the all-pass scheme defines the order of the optimal and sub-optimal approximations	It is important to note that the optimal Hankel norm approximation does not have to provide a better approximation than a balanced approximation in terms of the infinity norm of the error system
Low-rank Gramian approximations [91–96]	A hybrid method utilizing BT and Krylov	Have an estimate of global error and computational complexity as appropriate	It is under development at present

Table 2 A summary of general comparisons of approximation techniques

S. No.	Techniques	References	Computational efforts	Stability/passivity preservation	Applicability to multi-variables systems	Special attribute
<i>Frequency-domain</i>						
1	Continued fraction expansion (CFE) method	[97]	Low	No	Yes (but no. of inputs must be equal to no. of output)	Gives good matching in the low and high-frequency ranges
2	Moment matching method (MMM)	[98]	Low	No	No	ROM depends on the choice of the no of moments to be matched
3	Padé approximation (PA)	[68, 69, 99–101]	Low	No	No	Taylor series expansion
4	Routh Approximation (RA)	[102]	Medium	Yes	Yes	Can handle full models' expansions belong to Padé expansion belong to Padé expansions about one or more points
5	Routh Hurwitz array (RHA), other mixed	[66, 103, 104]	Low	Yes	Yes	A matching Taylor series $s = 0$ can handle unstable full models
6	Error minimization	[105–107]	High	Yes	yes	Commonly involves extended computations and often leads to iterations solution of nonlinear equations
7	Stability equation method (SEM)	[75, 76]	Low	Yes	Yes	The proposed method and the form of stability equation produce identical results for low-frequency systems
8	Differentiation method	[108]	Low	No	Yes	
9	Truncation	[109–111]	High	Yes	Yes	
10	Dominant pole retention	[112]	Medium	Yes	No	
11	Factor division	[113–115]	Low	Yes	Yes	
12	Mihailov stability criterion	[116–118]	Low	Yes	No	
13	Induced norm minimization	[119, 120]	Low	Yes	No	Effective for non-dominant or all dominant full models when other methods fail
14	Mixed Padé-Routh	[66, 121]	Low	Yes	Yes	Routh expansion may fail for the case of pole-zero cancellation straight Padé' would not fail
15	Mixed stability equation-Padé, other mixed	[74, 122, 123]	Low	Yes	yes	Used the benefit of each method to avoid the drawback of the method
<i>Time-domain</i>						
17	Modal analysis	[124, 125]	High	Yes	Yes	Computation of eigenvalues and eigenvectors is necessary
18	Minimal realization	[126–128]	Medium	Yes	No	Requires evaluation of system moments and Hankel values matrix formulations

Table 2 (continued)

S. No.	Techniques	References	Computational efforts	Stability/passivity preservation	Applicability to multi-variables systems	Special attribute
19	Aggregation	[129–131]	Medium	Yes	Yes	Also deriving state feedback suboptimal control, preserved internal structural properties (eigenvalues)
20	Optimal reduction	[132]	Low	Yes	No	
21	Balanced realization	[101, 134]	High	Yes	Yes	
	[133]					
22	Hankel norm approximation (HNA)	[85, 90, 135, 136]	High	Yes	Yes	Priori upper and lower bound approximation error, Guaranteed error bound
23	Singular perturbation approximation	[137–139]	High	Yes	Yes	If the calculation has a global error, it can be used entirely automatically
24	Krylov	[25, 26, 81, 140–142]	High	No	Yes	Improve the ill-conditioning

- Adaptive control design with the help of low order models.
- Suboptimal control derived by simplified models.
- Power system stability
- Providing that a guideline for online interactive modeling

3.3 Key Challenges of MOR

Rethinking about the linear/nonlinear MOR problem, a few prominent and significant challenges in MOR has been pointed out which are as follows:

- Modeling uncertain mechanical systems is challenging and necessitates the careful analysis of an enormous amount of data- Uncertain mechanical systems [19, 20].
- Multi-disciplinary optimization in MOR [14, 21]
- Stability preservation for large scale dynamical systems [22–30]
- Passivity Preservation [31–36]
- MOR for nonlinear complex systems & dimensionality reduction [12, 37–41]
- Challenges in MOR for industrial problems [21, 42–44]
- VLSI devices and layout optimization [27, 45, 46]
- Optimization technology and device modeling in micro and nano-electronics [47–52]
- Optimization of the electrical power system and smart city [14, 53]
- A Posteriori error estimation [54–57]
- Projection-based ROM [58–60]
- Aeroelasticity loads analysis [61]
- The financial and economic system in MOR [62]

4 MOR of LTI Large-Scale Dynamical System

The most important problem in the appearance of complex activities of a higher dimension system is that it occurs in many areas, including complicated transport, ecological systems, electrical equipment, aeronautics, hydraulics, etc. [5, 143–145].

All these complex and large systems with conventional techniques are difficult to model. The combination of these is also considered to be big (large) if it wishes to be detached for each numerical measurement to many structured machineries or small structures for practical purposes [16, 146, 147]. Then perhaps a system is complex and wide enough to fail to generate the proper solutions with realistic computational efforts by conventional modeling, analysis, device design, and approximation strategies [148, 149]. Studying this physical system [17] starts with structuring the model, which can be considered as an enthusiastic example of this kind of structure, which is motivated by a task of control in preparing and evaluating a model [149–151]. We are presenting a high stage of negotiation on computing in this first segment, which is important for detailed incident model observations in perspective and industry implementation [16, 152].

Several MOR solutions were mainly provided in two ways, namely frequency and time domain [155]. Researchers' reduction techniques have both benefits and inconveniences. One common weakness in the methods is that even if the HOS is stable, the reduced-order system is unstable [5, 149, 153] and steady-state matching. The other drawbacks are the low precision in average ranges as well as high

frequency and the non-minimum phase characteristics [136, 154]. Based upon the dominant poles method, numerous mixed methods have been suggested by [155, 156], the continued method and time matching fraction expansion can produce stable systems models. In the literature search, there are numerous approaches for reducing models of LSD system, such as a ROM algorithm, which was presented with a Padé approximation [63, 157, 158] and MOR of state linear time-invariant system based on the theory of balanced realization was initially firstly suggested by [159] in which the realization term balanced is selected for the system state configuration and partitions of the modes [160]. If the steady-state matches the balanced truncation of the steady-state value and the steady-state error of the LSD system is not kept, the BT reduction models obtained after truncation would have a less controllable and less measurable status. Addressed that the weak subsystem removed can be used to maintain the steady-state BT gain using the SPA approach [137, 161–166]. Preserving the ratio of steady-state output to steady-state input of the BT model for the minimal system using the SPA approach, which can be used to reduce the system to stable, minimal, and internal balancing [137].

The contribution of the work is that, with the traditional BT method, it is easy to derive a ROM that may fit well with the original system, but in most cases, it may be possible to observe a steady-state value mismatch. Although various researchers have used SPA to prevent such demerits from occurring. We, too, have successfully applied this concept in one of our works. The interesting part of the present work is that, with simple algorithmic modifications, it leads to a new modified algorithm based on a hybrid approach with the BT method and the SPA approach. It is referred to as a balanced singular perturbation approximation (BSPA) method. The advantage of the methodology lies not only in its steady-state matching, but also in its applicability to LSD systems, as shown by some of the examples derived from the published work, and also in comparison with the existing methods available in the literature to validate the effectiveness and superiority of the proposed method.

4.1 Balanced Truncation Method

A systems realization is balanced if its observability and controllability gramians are equal, meaning each state is controllable and observable. When this is done, one finds a reduced-order model by deleting those states that are least controllable and observable (as measured by the size of HSVs) provides a measure of energy for each state in a system structure in control theory. They are the basis for a balanced reduction of the system, which retains high energy states while discarding low energy states. The ROM obtained via this method has significant characteristics of the original systems [92, 167, 168].

The main idea is that the singular values of the controllability gramians correspond to the amount of energy required to move the corresponding states in the system. This balanced truncation process is a very interesting and powerful generalization of minimal realization theory, which only eliminates the completely unobservable and uncontrollable states from a given system model to furnish a minimal realization. This paper aims to construct a new model order reduction strategy to simplify a large-scale linear dynamical (LSD) system.

In Table 3, the Balanced Realization (BR) Algorithm to derive ROM from higher-dimensional systems has been presented. In the BR process, a higher dimensional stable structure may be controllable and observable at once. However, as reported in the literature, a transformation still needs to be established in many situations [159]. Then, it is transformed into a unique form of controllability and observability gramians, which are further equal. This leads to a diagonal matrix Σ , consisting of Hankel singular values at its diagonal, ultimately ordered in their dominance. Such kind of realization is called BR or internally balanced realization. This balancing of a given system is the first step into a category of methods for MOR, referred to as the BT method [159, 169].

4.2 Proposed Hybrid Method for Approximation

The proposed algorithm is the result of the hybridization of the Balanced truncation and Singular perturbation approximation approach. It consists of two steps as follows:

Step 1 The ROM obtained using the balanced truncation method [141, 148] algorithm has been discussed in Table 3 of Sect. 4.1.

The Steps of the order reduction algorithm using the Balanced Truncation Method may be described as followed.

To understand the balanced truncation method, we need to introduce two characteristics of a state: controllability and observability.

The controllability gramian (G_c) and observability gramian of the system is defined as follows:

$$G_c = \int_0^{\infty} e^{A\tau} B B^T e^{A^T \tau} d\tau \quad (9)$$

$$G_o = \int_0^{\infty} e^{A\tau} C^T C e^{A\tau} d\tau \quad (10)$$

The matrix G_c, G_o is a symmetric positive-semidefinite matrix called controllability and observability gramian, respectively. It is a solution of the following Lyapunov equation

Table 3 Balanced realization algorithm

Input System: Original System (A, B, C, D)

Output: Reduced Order Model (A_r, B_r, C_r, D_r)

1. Solve $AP_C + P_C A^T = -BB^T$ for P_C ;
2. Solve $A^T P_O + P_O A = -CC^T$ for P_O ;
3. Compute Cholesky factors $P_C = L_C L_C^T$ and $P_O = L_O L_O^T$;
4. Compute **SVD** of Cholesky factors $U \Sigma V^T = L_O^T L = [U_1, U_2] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} [V_1, V_2]^T$

where Σ is diagonal positive and U, V has orthonormal columns matrix; with $\Sigma_1 = \text{diag}(\xi_1, \dots, \xi_i)$, $\Sigma_2 = \text{diag}(\xi_{i+1}, \dots, \xi_n)$.

5. Compute the balancing transformation matrices

$$W^{-1} = \Sigma^{-\frac{1}{2}} U^T L_O^T, \quad W = L_C V \Sigma^{-\frac{1}{2}};$$

6. From the balancing realization as: $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} = D) = (WAW^{-1}, WB, CW^{-1}, D)$
7. Select number of dominant Hankel singular value (HSV) will be the order of reduction r and the eigen values of the original system and partition $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ conformally;
8. Truncate $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ to form the reduced realization (A_r, B_r, C_r, D_r) .

Properties

- (A_r, B_r, C_r, D_r) is asymptotically stable
- Error bound: $\|G_r(s) - G(s)\|_{H_\infty} \leq 2(\xi_{r+1} + \dots + \xi_n)$

$$\left. \begin{aligned} AG_C + G_C A^T &= -BB^T \\ A^T G_O + G_O A &= -CC^T \end{aligned} \right\} \quad (11)$$

Assumption: The nth-order dimensional system is an asymptotically stable system and also minimal. Moreover, the state-space equation of the original system or the pair (A_0, B_0) states controllable if and only if the $n \times nm$ state controllability matrix and pair (A_0, C_0) is observable if the $np \times n$ observability matrix [159].

By assumption, both gramians G_C and G_O are a positive definite and unique symmetric matrix explanation to the couple of gramians. Since their implementation is minimal.

Both gramians satisfy the following linear Lyapunov equations [170, 171].

In control philosophy, eigenvalues express system stability, although HSV describes the “energy” of each state in the system.

Again, G, eigenvectors (as well as eigenvalues) are completely dependent on the choice of basis. Therefore, one may

speak of dominant controllable states only relative to a certain basis.

Numerically we express as a stable state-space system Eq. (1), its HSV are well-defined as the square roots of the eigenvalues of P Q, ordered non increasingly, are called Hankel Singular Values: $\sigma_i = \sqrt{\lambda_i(G_C G_O)}$, respectively. For simplicity, such singular values (SV) are generally ordered downward to truncate states that match smaller Hankel singular values as follows.

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \sigma_4 \geq \sigma_r \geq \sigma_{r+1} \geq \dots \sigma_n > 0 \quad (12)$$

The Hankel singular values are also the singular values of the (infinite-dimensional, but finite rank) Hankel operator, which maps past inputs to future system outputs.

This is also a significant action of the minimality of realization of the original system is the diminishing positive number such that

$$P_c = P_0 = \Sigma = \text{Diagonal}\{\sigma_1, \sigma_2, \sigma_3, \sigma_4 \dots \sigma_n\} \quad (13)$$

The diagonal matrix (Σ) if such a matrix realization exists [170, 172, 173].

$$\sigma(\omega) = \|G_o(j\omega)\|^2 \tag{14}$$

Any symmetric positive definite matrix may be decomposed into a product.

Compute (Cholesky) factors (CF) of the gramians are often obtained by this factorization according to [146, 174]. The lower triangular matrix (CF) Q_c and Q_o of both gramians P_c and P_o is obtained as [146, 161].

$$P_c = Q_c Q_c^T \tag{15}$$

$$P_o = Q_o Q_o^T \tag{16}$$

Compute SVD, the $Q_o Q_c^T$ is singular value decomposition of gramians, also known as SVD of the system, found as follows:

$$SVD(Q_o^T Q_c) = U \Sigma V^* \tag{17}$$

where U and V are a vector, define as left and right singular. Also, unitary matrices (orthogonal).

This system matrix may be transformed into the balanced model by a similarity transformation matrices W , which may be achieved as follows [146, 171, 175].

ROM is (WAW^{-1}, WB, CW^{-1}) . where W is a transformation matrix

$$W = Q_c V \Sigma^{-\frac{1}{2}} \tag{18}$$

The original system has been completely balanced, which is partitioned as:

$$G_{Bal}(s) = \left[\begin{array}{c|c} WAW^{-1} & WB \\ \hline CW^{-1} & D \end{array} \right] : \text{Balanced System} \tag{19}$$

$$= \underbrace{\left[\begin{array}{c|c} A_B & B_B \\ \hline C_B & D_B \end{array} \right]}_{\text{Balanced System}} \Leftrightarrow \Sigma : \left[\begin{array}{cc} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{array} \right] \tag{20}$$

where the singular value $\Sigma_1 = \text{diag}(\sigma_1, \dots, \sigma_r)$ and $\Sigma_2 = \text{diag}(\sigma_{r+1}, \dots, \sigma_n)$. It is seen that Σ_1 corresponds to the ‘‘strong’’ sub-systems to be retained and Σ_2 the ‘‘weak’’ sub-systems to be deleted.

$$\left. \begin{array}{l} A_B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B_B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ C_B = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D_B = D \end{array} \right\} : \underbrace{\hspace{10em}}_{\text{(to be retained)}} + \underbrace{\hspace{10em}}_{\text{(to be retained)}} \Leftrightarrow \Sigma \tag{21}$$

Hence, the reduced-order model is defined as

$$A_r = A_{11}, B_r = B_1, C_r = C_1$$

where A_{11} is part of a strong subsystem and Σ_1 are $r \times r (r < n)$ matrixes. We call this ROM a balanced system approximation of direct-truncation (DT). There are some well-known results on approximation. There are some well-known results on the approximation [137, 176].

Lemma 1 (Pernebo et al. 1982) *The subsystem matrix A_{ii}, B_i, C_i is the minimal and internally balanced realization through Grammian $\Sigma_i (i = 1, 2) (i = 1, 2)$.*

Lemma 2 (Pernebo et al. 1982) *The subsystem matrix $A_{ii} (i = 1, 2)$ is asymptotically stable if Σ_1 and Σ_2 has no common diagonal component. Furthermore, the subsystem $(A_{11}, B_1, C_1) (i = 1, 2)$ is both completely controllable and observable [177].*

Step 2 Now, let us focus on applying the SPA derived from the ROM of an LTI system [177, 178].

In numerous engineering, the system’s steady-state gain, usually referred to as DC gain value (the system gains at an infinitive time, equivalent to $G(0)$, plays an essential role in evaluating system performance. It is, therefore, restored to preserve the DC gain value in the ROM, i.e., $G_r(0) = G(0)$. The balanced truncation approach introduced in the preceding subsection does not retain the DC gain value unchanged [179].

Suppose that (A, B, C, D) is compatible with minimal and balanced truncation of the stable system and the partitioned system as in the previous subsection. Then, it can be demonstrated that stable is A_{22} .

In this section, we address the order reducing procedure for higher-dimensional systems resulting in a hybrid approach using BT and balanced SPA. In the BT method, all balanced systems are separated into two parts as a slow and fast mode by defining the lower Hankel singular values (HSV) as fast mode, with the others defined as a slow mode. First, the derivative of all states equal to zero in fast mode may be obtained by defining a reduced system. The main aim of structure preservation in the ROM is to preserve the dominant frequencies of the original system. Hence, to preserve dominant dynamic modes in the reduced system. This work introduces a new MOR algorithm applied for a linear large-scale dynamical system, based on the idea of preserving the dominant poles of the original system during the order reduction. The notion

of Hankel singular values is a superior criterion for deciding the order of ROM verified and validated with various test problems. The approach is based on retaining the dominant eigenvalues or modes of the system and truncating the less significant eigenvalues comparatively.

Equation (22) has been attained as a minimal realized model containing strong and weak subsystems. Thus, SPA may be effortlessly applied on subsystems of Eq. (3.39). In the BT model, reduced (r) balanced states are retained, which are completely controllable and observable, so balanced states are preserved and remaining weakly controllable and observable states are truncated. SPA is used to maintain the DC gain of the original system in the model [139, 180]. The concerned researcher may refer to [177, 178] for more indications of the method.

The portioned form as above may be used to construct a singular perturbation approximation. As the balanced realized system determine, it can be re-written in the form of given as

$$\left. \begin{aligned} \left[\begin{array}{c} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{array} \right] &= \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{122} \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] + \left[\begin{array}{c} B_1 \\ B_2 \end{array} \right] u(t) \\ y &= \left[\begin{array}{cc} C_1 & C_2 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] + Du(t) \end{aligned} \right\} : \text{Balanced Model} \tag{22}$$

Again, re-write is equation form

$$\frac{dx_1}{dt} = A_{11}x_1 + A_{12}x_2 + B_1u \text{ (Slower)} \tag{23}$$

$$\mu \frac{dx_2}{dt} = A_{21}x_1 + A_{22}x_2 + B_2u \text{ (Faster)} \tag{24}$$

where, μ is a positive small perturbational parameter of singular perturbation approximation approach [181, 182].

By comparing the derivative of the weakly subsystem to zero below, the BSPA model may be achieved [138, 139].

Now the final system $(\hat{A}_{BSPA}, \hat{B}_{BSPA}, \hat{C}_{BSPA}, \hat{D}_{BSPA})$ conformally as in (25).

$$G(s)_{BSPA} = \left[\begin{array}{c|c} \hat{A}_{BSPA} & \hat{B}_{BSPA} \\ \hline \hat{C}_{BSPA} & \hat{D}_{BSPA} \end{array} \right] \Leftrightarrow \underbrace{\left[\begin{array}{cc|cc} A_{11} - A_{12} A_{22}^{-1} A_{21} & B_1 - A_{12} A_{22}^{-1} B_2 \\ \hline C_1 - C_2 A_{22}^{-1} A_{21} & D - C_2 A_{22}^{-1} B_2 \end{array} \right]}_{\text{Balanced SPA}} \tag{25}$$

In the preceding section, the technique will be verified, and the proposed method will be successfully validated.

To compare the effectiveness and performance of the proposed methodology with other existing reduced models available in the literature review. The Accuracy and performance of the proposed method are also validated by calculating performance indices such as an integral square error (ISE), integral absolute error (IAE), relative integral

square error (RISE), integral time-weighted absolute error (ITAE) [8, 145, 183, 184], in between the original system and its reduced-order model will be calculated and these are defined as

$$ISE = \int_0^\infty [y_1(t) - y_2(t)]^2 dt \tag{26}$$

$$IAE = \int_0^\infty |y_1(t) - y_2(t)| dt \tag{27}$$

$$ITAE = \int_0^\infty t|y_1(t) - y_2(t)| dt \tag{28}$$

$$RISE = \int_0^\infty (y_1(t) - y_2(t))^2 dt / \int_0^\infty (g(t))^2 dt \tag{29}$$

where $y_1(t)$ and $y_2(t)$ are the outputs of the original system and ROM [63].

The reduced system's RISE values should be nearby (close) the original system, and ISE should be as small as possible. Respectively, $y_1(t)$ and $y_2(t)$ are system under the consideration and the ROM step responses obtained respectively from the proposed method. $g(t)$ is the impulse response of the system [5, 76, 185–187] To obtain a lower-order model from the more complex model is an issue in control systems like stability, realizability, and large-order capability. Thus, there is significant interest in investigating new algorithms that work faster and with greater precision. To find out which results come from the proposed method, which results are used, which ones are given, and which ones are used in place of these, the ISE, IAE, and ITAE and RISE known methods are different, are evaluated for accuracy.

5 Numerical Experiments and Results

Example 1 Consider the following 4th order system [63].

$$G_4(s) = \frac{s^3 + 7.00s^2 + 24.00s + 24.00}{s^4 + 10.00s^3 + 35.00s^2 + 50.00s + 24.00} \tag{30}$$

The HSVs of the original system are calculated as σ is given by

$$\sigma = [0.5179, 0.0309, 0.0124, 0.0006] \tag{31}$$

The first–second singular values are important here $\sigma_2 \gg \sigma_3$, and the third singular values quickly decay, as can be seen from the matrix, Eq. (31). As a result, the second-order reduction order has been chosen.

The ROM matrices obtained by using BSPA, are given as

$$\left. \begin{aligned} A_{BSPA} &= \begin{bmatrix} -0.7417 & 0.7286 \\ -0.7286 & -2.656 \end{bmatrix}, B_{BSPA} = \begin{bmatrix} -0.8765 \\ -0.4049 \end{bmatrix}, \\ C_{BSPA} &= [-0.8765 \ 0.4049], D_{BSPA} = 0.02597 \end{aligned} \right\} \tag{32}$$

Finally, the ROM of representation in the form of the transfer function is expressed as $R_2(s)$ of test system 2.

$$R_2(s) = \frac{0.02597s^2 + 0.6925s + 2.501}{s^2 + 3.98s + 2.501} \tag{33}$$

The results of the simulation are shown in Fig. 1. The second-order ROM obtained by the suggested method is very close to the original system to the other methods available in the literature review for the same step unit, respectively. Also, the performance indices error values are calculated to check the modeling error and the closeness of the original system, as shown in Table 4. The proposed method shows that the ISE value is much lower than the value obtained from other literature. It has been seen that the ISE values are calculated to be, for Example 2 is that 4.427e–05. whereas the least value of ISE, using Moore (1981), Suman et al. (2019), Pal (1980), and Pati et al. (2014), and others are shown in the literature. Furthermore, in Table 5, a comparison analysis of the time-domain specifications between the various second-order model and the original system with

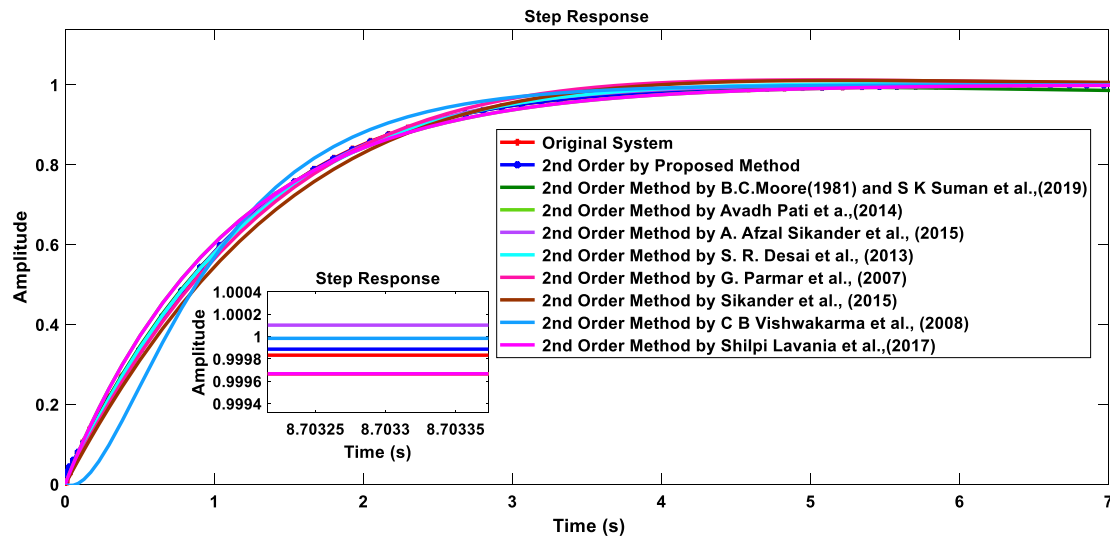


Fig. 1 Qualitative comparison of the proposed method and original system with other ROM methods in terms of step response for Example 1

Table 4 Performance analysis of the proposed method and other existing ROM methods for Example 1

MOR method & References	ROM	ISE	IAE	ITAE	RISE
Proposed method	$\frac{0.02597 s^2 + 0.6925 s + 2.501}{s^2 + 3.98 s + 2.501}$	4.427e–05	0.009887	0.01892	1.107e–05
Moore (1981) [159] and Suman (2019) [149]	$\frac{0.8216 s + 0.4542}{s^2 + 1.268 s + 0.4663}$	0.001448	0.09609	0.6479	0.000362
Jayanata pal [188] and Avadh Pati et al. (2014) [189]	$\frac{s^2 + 27.01 s + 24.01}{s + 24.01}$	0.001112	0.06246	0.1317	0.000278
Afzal Sikander et al. (2015) [190] and Narwal et al. (2016) [185]	$\frac{0.7751 s + 1.258}{s^2 + 2.12 s + 1.258}$	0.0001324	0.02732	0.09423	3.311e–05
Desai et al. (2013) [191]	$\frac{0.8058 s + 0.7944}{s^2 + 1.65 s + 0.7944}$	0.0002839	0.0002839	0.1738	7.098e–05
Parmar et al. (2007) [192]	$\frac{0.7442575 s + 0.6991576}{s^2 + 1.45771 s + 0.69997}$	0.001698	0.1051	0.3954	0.0004245
Sikander et al. (2015) [8]	$\frac{0.6997 s + 0.6997}{s^2 + 1.45771 s + 0.6997}$	0.00278	0.1315	0.4172	0.0006953
C B. Vishwakarma et al. (2008) [193]	$\frac{-0.189762 s + 4.5713}{s^2 + 4.76187 s + 4.5713}$	0.008021	0.1455	0.24	0.002006
Shilpi Lavania et al. (2017) [194]	$\frac{0.992 s + 24.096}{s^2 + 27.096 s + 24.096}$	0.001105	0.6223	0.1313	0.0002763

help of literature has been presented. The response is to demonstrate the exact representation and effectiveness of the proposed method. The results of the proposed method were compared with existing ROM methods which show an improvement in performance error indices, time response characteristics, and time domain specifications with the same DC gain as the original system. This response allows an accurate approximation and confirms the efficacy of the technique.

Example 2 Let us consider the eighth order system represented by the following transfer function, which many researchers have previously considered [73, 195]

$$G_8(s) = \frac{18s^7 + 514s^6 + 5982s^5 + 36380s^4 + 12264s^3 + 222088s^2 + 185760s + 40320}{s^8 + 36s^7 + 546s^6 + 4536s^5 + 22449s^4 + 67284s^3 + 118124s^2 + 109584s + 40320} \tag{34}$$

The HSVs of the original system is calculated as σ is given by

$$\sigma = [1.216652465935050, 0.746403486833304, 0.027915998321307, 0.001940648946887, 0.000107069466391, 0.000001588964686, 0.000000145817410, 0.00000000050828] \tag{35}$$

From the matrix σ , it can be proved that $\sigma_2 \gg \sigma_3$. The first and second singular values are extremely important, and the third singular values quickly disappear away. As a result, the reduction order has been chosen as the second order.

The ROM matrices obtained by using BSPA, are given as

$$\left. \begin{aligned} A_{BSPA} &= \begin{bmatrix} -6.644 & -2.201 \\ 2.201 & -0.0444 \end{bmatrix}, B_{BSPA} = \begin{bmatrix} -4.021 \\ 0.2575 \end{bmatrix}, \\ C_{BSPA} &= \begin{bmatrix} -4.021 & -0.2575 \end{bmatrix}, D_{BSPA} = 0.0595 \end{aligned} \right\} \tag{36}$$

Finally, the ROM of representation in the form of the transfer function is expressed as $R_2(s)$ of Example 2.

Table 5 A Comparison of time-domain specification with Other available ROM Methods according to literature for Example 1

The method used by Author	Rise time (s)	Settling time (s)	Settling minimum (s)	Settling maximum (s)	Overshoot (s)	Undershoot (s)	Peak (s)	Peak time (s)
Four order original System	2.2603	3.9308	0.9019	0.9990	0	0	0.9990	6.8847
Second Order Model by Proposed Method	2.2583	3.9178	0.9034	0.9990	0	0	0.9990	6.7109
Second Order Model by Moore (1981) and Suman et al. (2019)	2.0892	5.3643	0.8819	0.9941	2.0631	0	0.9941	4.9396
Second Order Model by Jayanata pal (1980) and Avadh Pati et al. (2014)	2.3872	4.2476	0.9003	1.0000	0	0	1.0000	11.4593
Second Order Model by Afzal Sikander et al. (2015) and Narwal et al. (2016)	2.2409	3.6722	0.9017	1.0002	0.0248	0	1.0002	6.9078
Second Order Model by Desai et al. (2013)	2.2785	3.6199	0.9054	1.0027	0.2738	0	1.0027	5.9728
Second Order Model by Parmar et al. (2007)	2.1870	3.2168	0.9005	1.0119	1.3061	0	1.0119	4.9915
Second Order Model by Sikander et al. (2015)	2.3011	3.4104	0.9019	1.0107	1.0722	0	1.0107	5.2442
Second Order Model by Vishwakarma et al. (2008)	1.8368	3.3413	0.9021	0.9999	0	0.3202	0.9999	7.2263
Second-Order Model by Shilpi Lavania et al. (2017)	2.3865	4.2467	0.9003	1.0000	0	0	1.0000	11.4559

$$R_2(s) = \frac{0.0595s^2 + 16.5s + 5.141}{s^2 + 6.688s + 5.141} \tag{37}$$

It is shown in Fig. 2 that The ROM obtained by the proposed method is very close to the original system, compared with other methods also available in the literature. As well, the comparative analyses of ROMs in terms of ISE, IAE, ITAE, and RISE, are given in Table 6 for illustration. From this comparison, it can be observed that the proposed method obtained a good result and gives the closest approximation to the original system with less error than other ROM models. It has been seen that the ISE values are calculated to be, for example, 2, 0.0005563, whereas the least value of ISE, by other methods and recently published work by Moore (1981), Suman et al. (2019), Afzal Sikander et al.

(2015) and Narwal et al. (2016) and others are depicted in Table 6. Furthermore, in Table 7, the comparison of time-domain specifications between different 2nd order models and the original system has been presented. The response is to illustrate the exact representation and effectiveness of the proposed method. The results of the proposed method have been compared with existing ROM methods which show an improvement in performance error indices, time response characteristics, and time-domain specifications with the same DC gain or steady-state value as the original system. This response allows for an accurate approximation and confirms the efficacy of the technique. Hence, it is clear that the proposed method is much better than the other well-known methods available in the literature review.

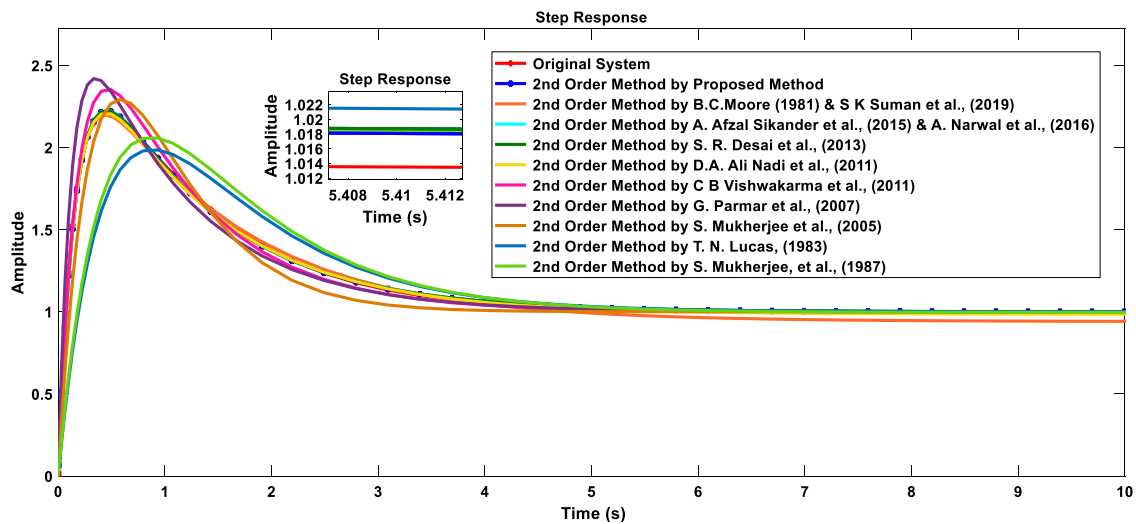


Fig. 2 Qualitative comparison of the proposed method and original system with other ROM methods in terms of step response for Example 2

Table 6 Performance analysis of the proposed method and other existing ROM methods for Example 2

MOR method & References	ROM	ISE	IAE	ITAE	RISE
Proposed method	$\frac{0.0595s^2+16.5s+5.141}{s^2+6.688s+5.141}$	0.0005563	0.05262	0.1476	0.000139
Moore (1981) and Suman et al. (2019)	$\frac{17.77s+4.546}{s^2+7.365s+4.834}$	0.01398	0.3164	2.112	0.003492
Afzal Sikander et al. (2015)	$\frac{16.97s^2+5.262}{s^2+6.893s+5.262}$	0.006967	0.6077	0.1769	0.01741
Desai et al. (2013) [196]	$\frac{16.91s^2+5.255}{s^2+6.87s+5.26}$	0.006923	0.05965	0.17	0.0001729
Ali Nadi et al. (2011) [197]	$\frac{17s^2+5.074}{s^2+6.972s+5.151}$	0.001223	0.09536	0.5199	0.0003056
Vishwakarma et al. (2011) [198]	$\frac{16.51137s^2+5.45971}{s^2+6.19642s+5.45971}$	0.01406	0.1972	0.3471	0.003512
Parmar et al. (2007) [63]	$\frac{24.11429s+8}{s^2+9s+8}$	0.04809	0.3007	0.3891	0.01201
Mukherjee et al. (2005) [199]	$\frac{11.3909s+4.4357}{s^2+4.2122s+4.4357}$	0.05692	0.4575	0.9475	0.01422
Lucas (1983) [113]	$\frac{6.7786s+2}{s^2+3s+2}$	0.2792	0.7628	1.02	0.06976
Mukherjee, et al. (1987) [157]	$\frac{7.0903s+1.9907}{s^2+3s+2}$	0.269	0.8098	1.209	0.06719

Table 7 A Comparison of time-domain specification with Other available ROM Methods according to literature for Example 2

The method used by Author	Rise time (s)	Settling time (s)	Settling minimum (s)	Settling maximum (s)	Overshoot (s)	Undershoot (s)	Peak (s)	Peak time (s)
8th order original System	0.0569	4.8201	0.9712	2.2035	120.3496	0	2.2035	0.4490
Second Order Model by Proposed Method	0.0581	5.0580	1.0017	2.2250	122.4975	0	2.2250	0.4603
Second Order Model by Moore (1981) and Suman et al. (2019)	0.0528	5.9713	0.9436	2.2010	134.0255	0	2.2010	0.4441
Second Order Model by Afzal Sikander et al. (2015) and Narwal et al. (2016)	0.0594	5.1041	1.0004	2.2277	122.7736	0	2.2277	0.4438
Second Order Model by Desai et al. (2013)	0.0596	5.0928	1.0003	2.2267	122.6722	0	2.2267	0.4457
Second Order Model by Ali Nadi et al. (2011)	0.0583	5.2816	0.9859	2.2106	124.4157	0	2.2106	0.4506
Second Order Model by Vishwakarma et al. (2011)	0.0597	4.3581	0.9644	2.3562	135.6204	0	2.3562	0.4666
Second-Order Model by Parmar et al. (2007)	0.0409	4.3942	0.9140	2.4214	142.1376	0	2.4214	0.3454
Second-Order Model by Mukherjee et al. (2005)	0.0856	3.3527	1.0003	2.2925	129.2490	0	2.2925	0.5904
Second-Order Model by Lucas (1983) Lucas (1983)	0.1507	5.4712	0.9769	1.9878	98.7846	0	1.9878	0.8750
Second-Order Model by Mukherjee et al. (1987)	0.1416	5.4717	0.9954	2.0620	107.1669		2.0620	0.8750

6 Conclusion and Future Scope

In this contribution, the various MOR methods for the LSD system have been thoroughly and comprehensively reviewed. It has been focused on methods with their detailed theoretical background as applied to the system. We have also discussed similarities and differences between several approaches along with their merits and demerits which may be useful to the research community. Two numerical comparisons show the advantages and disadvantages of these approaches. There are a few new viewpoints to this area which has been elaborated. We have also, highlighted the impressive improvements made over the last few years with respect to MOR applied to linear systems, although several key challenges remain to be investigated and further developments in numerical methods are, yet to be addressed. In addition, an application to my previous work for reducing the order of the large-scale dynamic LTI system has been further investigated in this paper. The hybrid technique applied was found to be superior to the conventional method (BT) or other existing methods. This hybridization approach using BT and SPA Approach has been found to effectively compensate for the demerits of each other. Furthermore, the same technique has been illustrated with a couple of very promising examples of a continuous LTI system. The step response comparison shows that the ROM obtained by the

method applied provides a close approximation to the HOS. In addition, the accuracy, validation, and superior performance of the presented method have been demonstrated by comparing the performance indices with various similar outcomes existing in the literature. Applicability to large-scale systems may increase the benefits of the method however it which is a matter of further investigation. Some of them are currently ongoing at the present. This procedure can be extended to the design of the state feedback controller, optimum, H-infinity controller, etc.

Data Availability My manuscript has no associated data.

References

1. Li S, Xiang Z (2020) Sampled-data decentralized output feedback control for a class of switched large-scale stochastic nonlinear systems. *IEEE Syst J* 14(2):1602–1610. <https://doi.org/10.1109/JSYST.2019.2934512>
2. Patalano S, Mango Furnari A, Vitolo F, Dion JL, Plateaux R, Renaud F (2021) A critical exposition of model order reduction techniques: application to a slewing flexible beam. *Arch Comput Methods Eng*. <https://doi.org/10.1007/s11831-019-09369-1>
3. He W, Li S, Ahn CK, Guo J, Xiang Z (2020) Global decentralized control of p-normal large-scale nonlinear systems based on

- sampled-data output feedback. *IEEE Syst J*. <https://doi.org/10.1109/jsyst.2020.2997029>
4. Chan SC, Wu HC, Ho CH, Zhang L (2019) An augmented Lagrangian approach for distributed robust estimation in large-scale systems. *IEEE Syst J* 13(3):2986–2997. <https://doi.org/10.1109/JSYST.2019.2897788>
 5. Sikander A, Prasad R (2015) Soft computing approach for model order reduction of linear time invariant systems. *Circuits Syst Signal Process* 34(11):3471–3487. <https://doi.org/10.1007/s00034-015-0018-4>
 6. Kumar J, Sikander A, Mehrotra M, Parmar G (2020) A new soft computing approach for order diminution of interval system. *Int J Syst Assur Eng Manag* 11:366–373. <https://doi.org/10.1007/s13198-019-00865-y>
 7. Antoulas AC, Benner P, Feng L (2018) Model reduction by iterative error system approximation. *Math Comput Model Dyn Syst* 42(2):103–118. <https://doi.org/10.1080/13873954.2018.1427116>
 8. Sikander A, Prasad R (2015) Linear time-invariant system reduction using a mixed methods approach. *Appl Math Model* 39(16):4848–4858. <https://doi.org/10.1016/j.apm.2015.04.014>
 9. Sikander A, Prasad R (2019) Reduced order modelling based control of two wheeled mobile robot. *J Intell Manuf* 30(3):1057–1067. <https://doi.org/10.1007/s10845-017-1309-3>
 10. Uniyal I, Sikander A (2018) A comparative analysis of PID controller design for AVR based on optimization techniques. In: *Advances in intelligent systems and computing*, pp 1315–1323. https://doi.org/10.1007/978-981-10-5903-2_138
 11. Prajapati AK, Prasad R (2019) Reduced-order modelling of LTI systems by using Routh approximation and factor division methods. *Circuits Syst Signal Process* 38(7):3340–3355. <https://doi.org/10.1007/s00034-018-1010-6>
 12. Baur U, Benner P, Feng L (2014) Model order reduction for linear and nonlinear systems: a system-theoretic perspective. *Arch Comput Methods Eng* 21(4):331–358. <https://doi.org/10.1007/s11831-014-9111-2>
 13. Rowley CW (2005) Model reduction for fluids, using balanced proper orthogonal decomposition. *Int J Bifurcat Chaos* 15(3):997–1013. <https://doi.org/10.1142/S0218127405012429>
 14. Schilders WHA, Van Der Vorst HA, Rommes J (2008) *Model order reduction: theory, research aspects and applications*, vol 13. Springer, Berlin
 15. Sundström D (1985) Mathematics in industry. *Int J Math Educ Sci Technol*. <https://doi.org/10.1080/0020739850160226>
 16. Mohamed KS (2018) *Machine learning for model order reduction*. Springer, Berlin
 17. Schilders WHA, van der Vorst HA, Rommes J (2008) *Model order reduction: theory, research aspects and applications*. Springer, Berlin
 18. Chaturvedi DK (2017) Model order reduction. In: *Modeling and simulation of systems using MATLAB® and Simulink®*. CRC Press, Boca Raton
 19. Pivovarov D et al (2019) Challenges of order reduction techniques for problems involving polymorphic uncertainty. *GAMM Mitteilungen*. <https://doi.org/10.1002/gamm.201900011>
 20. Bui-Thanh T, Willcox K, Ghattas O (2008) Parametric reduced-order models for probabilistic analysis of unsteady aerodynamic applications. *AIAA J* 46(10):2520–2529. <https://doi.org/10.2514/1.35850>
 21. Mendonça G, Afonso F, Lau F (2019) Model order reduction in aerodynamics: review and applications. *Proc Inst Mech Eng Part G J Aerosp Eng* 233(15):5816–5836. <https://doi.org/10.1177/0954410019853472>
 22. Baur U, Benner P, Greiner A, Korvink JG, Lienemann J, Moosmann C (2011) Parameter preserving model order reduction for MEMS applications. *Math Comput Model Dyn Syst* 17(4):297–317. <https://doi.org/10.1080/13873954.2011.547658>
 23. Anand S, Fernandes BG (2013) Reduced-order model and stability analysis of low-voltage dc microgrid. *IEEE Trans Ind Electron*. <https://doi.org/10.1109/TIE.2012.2227902>
 24. Mariani V, Vasca F, Vásquez JC, Guerrero JM (2015) Model order reductions for stability analysis of islanded microgrids with droop control. *IEEE Trans Ind Electron* 62(7):4344–4354. <https://doi.org/10.1109/TIE.2014.2381151>
 25. Bai Z (2002) Krylov subspace techniques for reduced-order modeling of large-scale dynamical systems. *Appl Numer Math* 43(1–2):9–44. [https://doi.org/10.1016/S0168-9274\(02\)00116-2](https://doi.org/10.1016/S0168-9274(02)00116-2)
 26. Freund RW (2000) Krylov-subspace methods for reduced-order modeling in circuit simulation. *J Comput Appl Math* 123(1–2):395–421. [https://doi.org/10.1016/S0377-0427\(00\)00396-4](https://doi.org/10.1016/S0377-0427(00)00396-4)
 27. Freund RW (2004) SPRIM: structure-preserving reduced-order interconnect macromodeling. In: *IEEE/ACM international conference on computer-aided design, Digest of Technical Papers, ICCAD*, pp 80–87. <https://doi.org/10.1109/iccad.2004.1382547>
 28. Wang X, Yu M, Wang C (2018) Structure-preserving-based model-order reduction of parameterized interconnect systems. *Circuits Syst Signal Process* 37(1):19–48. <https://doi.org/10.1007/s00034-017-0561-2>
 29. Freund RW (2011) The SPRIM algorithm for structure-preserving order reduction of general RCL circuits. In: *Lecture notes in electrical engineering*, pp 25–52. https://doi.org/10.1007/978-94-007-0089-5_2
 30. Beattie C, Gugercin S (2009) Interpolatory projection methods for structure-preserving model reduction. *Syst Control Lett* 58(2):225–232. <https://doi.org/10.1016/j.sysconle.2008.10.016>
 31. Odabasioglu A, Celik M, Pileggi LT (1997) PRIMA: Passive reduced-order interconnect macromodeling algorithm. In: *IEEE/ACM international conference on computer-aided design, Digest of Technical Papers*, pp 433–450. https://doi.org/10.1007/978-1-4615-0292-0_34
 32. Odabasioglu A, Celik M, Pileggi LT (1998) PRIMA: Passive reduced-order interconnect macromodeling algorithm. *IEEE Trans Comput Des Integr Circuits Syst* 17(8):645–654. <https://doi.org/10.1109/43.712097>
 33. Ionutiu R, Rommes J, Antoulas AC (2008) Passivity-preserving model reduction using dominant spectral-zero interpolation. *IEEE Trans Comput Des Integr Circuits Syst* 27(12):2250–2263. <https://doi.org/10.1109/TCAD.2008.2006160>
 34. Fanizza G, Karlsson J, Lindquist A, Nagamune R (2006) A global analysis approach to passivity preserving model reduction. In: *Proceedings of the IEEE conference on decision and control*, pp 3399–3404. <https://doi.org/10.1109/cdc.2006.376706>
 35. Antoulas AC (2005) A new result on passivity preserving model reduction. *Syst Control Lett* 54(4):361–374. <https://doi.org/10.1016/j.sysconle.2004.07.007>
 36. Farle O, Burgard S, Dyczij-Edlinger R (2011) Passivity preserving parametric model-order reduction for non-affine parameters. *Math Comput Model Dyn Syst* 17(3):279–294. <https://doi.org/10.1080/13873954.2011.562901>
 37. Fuchs A (2013) *Nonlinear dynamics in complex systems*. Springer, Berlin, p 2008
 38. Burkardt J, Du Q, Gunzburger M, Lee H-C (2003) Reduced order modeling of complex systems. NA03 Dundee
 39. Sargsyan S, Brunton SL, Kutz JN (2015) Nonlinear model reduction for dynamical systems using sparse sensor locations from learned libraries. *Phys Rev E Stat Nonlinear Soft Matter Phys*. <https://doi.org/10.1103/PhysRevE.92.033304>
 40. Rafiq D, Bazaz MA (2021) Nonlinear model order reduction via nonlinear moment matching with dynamic mode decomposition.

- Int J Non Linear Mech. <https://doi.org/10.1016/j.jnonlinmec.2020.103625>
41. Venna J, Kaski S, Aidos H, Nybo K, Peltonen J (2010) Information retrieval perspective to nonlinear dimensionality reduction for data visualization. *J Mach Learn Res* 11(2):451–490
 42. Lassila T, Manzoni A, Quarteroni A, Rozza G (2014) Model order reduction in fluid dynamics: challenges and perspectives. In: *Reduced order methods for modeling and computational reduction*
 43. Preisner T, Mathis W (2010) *Scientific computing in electrical engineering SCEE 2008*. Springer, Berlin
 44. Rudnyi E, Korvink J (2006) *Modern model order reduction for industrial applications*. Report, Germany
 45. Gray PR, Meyer RG (2009) *Analysis and design of analog integrated circuits*. Wiley, Hoboken
 46. Tan S, He L (2007) *Advanced model order reduction techniques in VLSI design*. Cambridge University Press, New York
 47. Kerschen G, Golinval JC, Vakakis AF, Bergman LA (2005) The method of proper orthogonal decomposition for dynamical characterization and order reduction of mechanical systems: an overview. *Nonlinear Dyn* 41(1):147–169. <https://doi.org/10.1007/s11071-005-2803-2>
 48. Bond B, Daniel L (2005) Parameterized model order reduction of nonlinear dynamical systems. In: *IEEE/ACM international conference on computer-aided design, Digest of Technical Papers, ICCAD*, pp 487–494. <https://doi.org/10.1109/ICCAD.2005.1560117>
 49. Qu Z-Q (2004) *Model order reduction techniques with applications in finite element analysis*. Springer, London
 50. Brozek T, Iniewski KK (2017) *Micro-and nanoelectronics: emerging device challenges and solutions*. CRC Press, Boca Raton
 51. Lienemann J, Billger D, Rudnyi EB, Greiner A, Korvink JG (2004) MEMS compact modeling meets model order reduction: examples of the application of Arnoldi methods to microsystem devices
 52. Rudnyi EB, Korvink JG (2006) Model order reduction for large scale engineering models developed in ANSYS. In: *International workshop on applied parallel computing*. Springer, Berlin, Heidelberg, pp 349–356. https://doi.org/10.1007/11558958_41
 53. Djukic S, Saric A (2012) Dynamic model reduction: an overview of available techniques with application to power systems. *Serbian J Electr Eng* 9(2):131–169. <https://doi.org/10.2298/sjee1202131d>
 54. Veroy K, Prud'Homme C, Rovas DV, Patera AT (2003) A posteriori error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations. In: *16th AIAA computational fluid dynamics conference*, p. 3847. <https://doi.org/10.2514/6.2003-3847>
 55. Rozza G, Huynh DBP, Patera AT (2008) Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations: application to transport and continuum mechanics. *Arch Comput Methods Eng* 15(3):229–275. <https://doi.org/10.1007/s11831-008-9019-9>
 56. Jung N, Patera AT, Haasdonk B, Lohmann B (2011) Model order reduction and error estimation with an application to the parameter-dependent eddy current equation. *Math Comput Model Dyn Syst*. <https://doi.org/10.1080/13873954.2011.582120>
 57. Haasdonk B, Ohlberger M (2011) Efficient reduced models and a posteriori error estimation for parametrized dynamical systems by offline/online decomposition. *Math Comput Model Dyn Syst*. <https://doi.org/10.1080/13873954.2010.514703>
 58. Lorenzi S, Cammi A, Luzzi L, Rozza G (2016) POD-Galerkin method for finite volume approximation of Navier–Stokes and RANS equations. *Comput Methods Appl Mech Eng* 311:151–179. <https://doi.org/10.1016/j.cma.2016.08.006>
 59. Star K, Stabile G, Georgaka S, Belloni F, Rozza G, Degroote J (2019) Pod-Galerkin reduced order model of the Boussinesq approximation for buoyancy-driven enclosed flows. In: *International conference on mathematics and computational methods applied to nuclear science and engineering, M and C 2019*, pp. 2452–2461
 60. Bui-Thanh T, Damodaran M, Willcox K (2003) Proper orthogonal decomposition extensions for parametric applications in compressible aerodynamics. In: *21st AIAA applied aerodynamics conference*, p. 4213. <https://doi.org/10.2514/6.2003-4213>
 61. Thomas PV, ElSayed MSA, Walch D (2019) Review of model order reduction methods and their applications in aeroelasticity loads analysis for design optimization of complex airframes. *J Aerosp Eng*. [https://doi.org/10.1061/\(asce\)as.1943-5525.0000972](https://doi.org/10.1061/(asce)as.1943-5525.0000972)
 62. Binder A, Jadhav O, Mehrmann V (2020) Model order reduction for parametric high dimensional interest rate models in the analysis of financial risk. *arXiv*
 63. Parmar G, Mukherjee S, Prasad R (2007) System reduction using factor division algorithm and Eigen spectrum analysis. *Appl Math Model* 31(11):2542–2552. <https://doi.org/10.1016/j.apm.2006.10.004>
 64. Hwang C, Chen MY (1987) Stable linear system reduction via a multipoint tangent phase continued-fraction expansion. *J Chin Inst Eng Trans Chin Inst Eng A/Chung-kuo K Ch'eng Hsueh K'an*. <https://doi.org/10.1080/02533839.1987.9676990>
 65. Antoulas AC (2005) An overview of approximation methods for large-scale dynamical systems. *Annu Rev Control*. <https://doi.org/10.1016/j.arcontrol.2005.08.002>
 66. Pal J (1979) Stable reduced-order Padé approximants using the Routh–Hurwitz array. *Electron Lett* 15(8):225–226. <https://doi.org/10.1049/el:19790159>
 67. Pindor M (2006) Padé approximants. *Lect Notes Control Inf Sci*. https://doi.org/10.1007/11601609_4
 68. Guillaume P, Huard A (2000) Multivariate Padé approximation. *J Comput Appl Math*. [https://doi.org/10.1016/S0377-0427\(00\)00337-X](https://doi.org/10.1016/S0377-0427(00)00337-X)
 69. Shamash Y (1975) Linear system reduction using Padé approximation to allow retention of dominant modes. *Int J Control* 21(2):257–272. <https://doi.org/10.1080/00207177508921985>
 70. Shamash Y (1975) Multivariable system reduction via modal methods and Padé approximation. *IEEE Trans Autom Control* 20(6):815–817. <https://doi.org/10.1109/TAC.1975.1101090>
 71. Verma P, Juneja PK, Chaturvedi M (2017) Various mixed approaches of model order reduction. In: *Proceedings - 2016 8th international conference on computational intelligence and communication networks, CICN 2016*, pp 673–676. <https://doi.org/10.1109/CICN.2016.138>
 72. Adamou-Mitiche ABH, Mitiche L, Larbi S (2013) Time and frequency approaches in the approximation problem: a comparative study. <https://doi.org/10.1109/ICoSC.2013.6750916>
 73. Shamash Y (1975) Model reduction using the Routh stability criterion and the Padé approximation technique. *Int J Control* 21(3):475–484. <https://doi.org/10.1080/00207177508922004>
 74. Chen TC, Chang CY, Han KW (1980) Model reduction using the stability-equation method and the Padé approximation method. *J Frankl Inst* 309(6):473–490. [https://doi.org/10.1016/0016-0032\(80\)90096-4](https://doi.org/10.1016/0016-0032(80)90096-4)
 75. Chen TC, Chang CY, Han KW (1979) Reduction of transfer functions by the stability-equation method. *J Frankl Inst* 308(4):389–404. [https://doi.org/10.1016/0016-0032\(79\)90066-8](https://doi.org/10.1016/0016-0032(79)90066-8)
 76. Sikander A, Thakur P, Uniyal I (2017) Hybrid method of reduced order modelling for LTI system using evolutionary algorithm. In: *Proceedings on 2016 2nd international*

- conference on next generation computing technologies, NGCT 2016, pp 84–88. <https://doi.org/10.1109/NGCT.2016.7877394>
77. Lucas TN (1992) A tabular approach to the stability equation method. *J Frankl Inst* 329(1):171–180. [https://doi.org/10.1016/0016-0032\(92\)90106-Q](https://doi.org/10.1016/0016-0032(92)90106-Q)
 78. Mohammadpour J, Grigoriadis KM (2010) Efficient modeling and control of large-scale systems. Springer, Berlin
 79. Grimme EJ (1997) Krylov projection methods for model reduction. University of Illinois at Urbana-Champaign
 80. Antoulas AC (2005) Approximation of large-scale dynamical systems. Society for Industrial and Applied Mathematics, Philadelphia
 81. Hochbruck M, Lubich C (1997) On Krylov subspace approximations to the matrix exponential operator. *SIAM J Numer Anal.* <https://doi.org/10.1137/S0036142995280572>
 82. Bruaset AM (2019) Krylov subspace methods. In: A survey of preconditioned iterative methods. Routledge, Boca Raton
 83. Kumar D, Nagar SK (2014) Model reduction by extended minimal degree optimal Hankel norm approximation. *Appl Math Model* 38(11–12):2922–2933. <https://doi.org/10.1016/j.apm.2013.11.012>
 84. Adamjan VM, Arov DZ, Krein MG (1971) Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur–Takagi problem. *Math USSR Sb* 15(1):31. <https://doi.org/10.1070/SM1971v015n01ABEH001531>
 85. Kung SY, Lin DW (1981) Optimal Hankel-norm model reductions: multivariable systems. *IEEE Trans Autom Control* 26(4):832–852. <https://doi.org/10.1109/TAC.1981.1102736>
 86. Latham GA, Anderson BDO (1985) Frequency-weighted optimal Hankel-norm approximation of stable transfer functions. *Syst Control Lett* 5(4):229–236
 87. Hung YS, Glover K (1986) Optimal Hankel-norm approximation of stable systems with first-order stable weighting functions. *Syst Control Lett* 7(3):165–172. [https://doi.org/10.1016/0167-6911\(86\)90110-6](https://doi.org/10.1016/0167-6911(86)90110-6)
 88. Antoulas AC, Beattie CA, Gugercin S (2010) Interpolatory model reduction of large-scale dynamical systems. In: Efficient modeling and control of large-scale systems
 89. Korvink JG, Rudnyi EB, Greiner A, Liu Z (2006) MEMS and NEMS simulation. In: MEMS: a practical guide of design, analysis, and applications
 90. Zhou K (1995) Frequency-weighted L_∞ norm and optimal Hankel norm model reduction. *IEEE Trans Autom Control.* <https://doi.org/10.1109/9.467681>
 91. Benner P, Quintana-Orti ES, Quintana-Orti G (2000) Balanced truncation model reduction of large-scale dense systems on parallel computers. *Math Comput Model Dyn Syst.* <https://doi.org/10.1076/mcmd.6.4.383.3658>
 92. Antoulas AC, Sorensen DC, Zhou Y (2002) On the decay rate of Hankel singular values and related issues. *Syst Control Lett* 46(5):323–342. [https://doi.org/10.1016/S0167-6911\(02\)00147-0](https://doi.org/10.1016/S0167-6911(02)00147-0)
 93. Powel ND, Morgansen KA (2015, December) Empirical observability Gramian rank condition for weak observability of nonlinear systems with control. In 2015 54th IEEE conference on decision and control (CDC). IEEE, pp 6342–6348. <https://doi.org/10.1109/CDC.2015.7403218>
 94. Grippo L, Palagi L, Piccialli V (2011) An unconstrained minimization method for solving low-rank SDP relaxations of the maxcut problem. *Math Program.* <https://doi.org/10.1007/s10107-009-0275-8>
 95. Freitas FD, Rommes J, Martins N (2011, March). Low-rank gramian applications in dynamics and control. In 2011 international conference on communications, computing and control applications (CCCA). IEEE, pp 1–6. <https://doi.org/10.1109/CCCA.2011.6031400>
 96. Markovsky I (2008) Structured low-rank approximation and its applications. *Automatica.* <https://doi.org/10.1016/j.automatica.2007.09.011>
 97. Pal J, Prasad R (1986) Stable low order approximants using continued fraction expansions. In MSE international conference on modelling simulation
 98. Baštuš M, Petreczky M, Wisniewski R, Leth J (2014, June) Model reduction by moment matching for linear switched systems. In 2014 american control conference. IEEE, pp 3942–3947. <https://doi.org/10.1109/ACC.2014.6858983>
 99. Bultheel A, Van Barel M (1986) Padé techniques for model reduction in linear system theory: a survey. *J Comput Appl Math* 14(3):401–438. [https://doi.org/10.1016/0377-0427\(86\)90076-2](https://doi.org/10.1016/0377-0427(86)90076-2)
 100. Parmar G, Mukherjee S, Prasad R (2007) System reduction using Eigen spectrum analysis and Padé approximation technique. *Int J Comput Math.* <https://doi.org/10.1080/00207160701345566>
 101. Prajapati AK, Prasad R (2019) Order reduction in linear dynamical systems by using improved balanced realization technique. *Circuits Syst Signal Process* 38(11):5289–5303. <https://doi.org/10.1007/s00034-019-01109-x>
 102. Hutton MF, Friedland B (1975) Routh approximations for reducing order of linear, time-invariant systems. *IEEE Trans Autom Control* 20(3):329–337. <https://doi.org/10.1109/TAC.1975.1100953>
 103. Pal J (1980) System reduction by a mixed method. *IEEE Trans Autom Control* 25(5):973–976. <https://doi.org/10.1109/TAC.1980.1102485>
 104. Rao AS, Lamba SS, Rao SV (1979) Comments on ‘model reduction using the Routh stability criterion.’ *IEEE Trans Autom Control* 24(3):518–518. <https://doi.org/10.1109/TAC.1979.1102069>
 105. Mishra RN, Wilson DA (1980) A new algorithm for optimal reduction of multivariable systems. *Int J Control.* <https://doi.org/10.1080/00207178008961054>
 106. Hwang C, Wang KY (1984) Optimal Routh approximations for continuous-time systems. *Int J Syst Sci* 15(3):249–259. <https://doi.org/10.1080/00207728408926558>
 107. Goyal R, Parmar G, Sikander A (2019) A new approach for simplification and control of linear time invariant systems. *Microsyst Technol.* <https://doi.org/10.1007/s00542-018-4004-1>
 108. Gutman P, Mannerfelt CF, Molander P (1982) Contributions to the model reduction problem. *IEEE Trans Autom Control* 27(2):454–455. <https://doi.org/10.1109/TAC.1982.1102930>
 109. Gustafson RD (1966) A paper and pencil control system design. *J Fluids Eng Trans ASME* 88(2):329–336. <https://doi.org/10.1115/1.3645858>
 110. Shamash Y, Smamash Y (1981) Truncation method of reduction: a viable alternative. *Electron Lett* 17(2):97–99. <https://doi.org/10.1049/el:19810070>
 111. Prajapati AK, Prasad R (2019) Model order reduction by using the balanced truncation and factor division methods. *IETE J Res* 65(6):827–842. <https://doi.org/10.1080/03772063.2018.1464971>
 112. Davison EJ (1966) A method for simplifying linear dynamic systems. *IEEE Trans Autom Control* 11(1):93–101. <https://doi.org/10.1109/TAC.1966.1098264>
 113. Lucas TN (1983) Factor division: a useful algorithm in model reduction. *IEE Proc D Control Theory Appl* 130(6):362–364. <https://doi.org/10.1049/ip-d.1983.0060>
 114. Prajapati AK, Prasad R (2019) Reduced order modelling of linear time invariant systems using the factor division method to allow retention of dominant modes. *IETE Tech Rev (Inst Electron Telecommun Eng, India)* 36(5):449–462. <https://doi.org/10.1080/02564602.2018.1503567>
 115. Sikander A, Prasad R (2017) A new technique for reduced-order modelling of linear time-invariant system. *IETE J Res* 63(3):316–324. <https://doi.org/10.1080/03772063.2016.1272436>

116. Wan BW (1981) Linear model reduction using Mihailov criterion and Padé approximation technique. *Int J Control* 33(6):1073–1089. <https://doi.org/10.1080/00207178108922977>
117. Rana J (2013) Order reduction using Mihailov criterion and Pade approximations. *Int J Innov Eng Technol* 2:19–24
118. Tomar SK, Prasad R (2008) Linear model reduction using Mihailov stability criterion and continued fraction expansions. *J Inst Eng India* 84:7–10
119. El-Attar RA, Vidyasagar M (1978) System order reduction using the induced operator norm and its applications to linear regulators. *J Frankl Inst* 306(6):457–474. [https://doi.org/10.1016/0016-0032\(78\)90053-4](https://doi.org/10.1016/0016-0032(78)90053-4)
120. Khademi G, Mohammadi H, Dehghani M (2013) LMI based model order reduction considering the minimum phase characteristic of the system. In: 2013 9th Asian control conference, ASCC 2013, pp 1–6. <https://doi.org/10.1109/ASCC.2013.6606180>
121. Pal J, Ray LM (1980) Stable Padé approximants to multivariable systems using a mixed method. *Proc IEEE* 68(1):176–178. <https://doi.org/10.1109/PROC.1980.11603>
122. Chen TC, Chang CY, Han KW (1980) Model reduction using the stability-equation method and the continued-fraction method. *Int J Control* 32(1):81–94. <https://doi.org/10.1080/00207178008922845>
123. Parthasarathy R (1982) System reduction using stability-equation method and modified cauer continued fraction. *Proc IEEE* 70(10):1234–1236. <https://doi.org/10.1109/PROC.1982.12453>
124. Wittmuess P, Tarin C, Sawodny O (2015, December) Parametric modal analysis and model order reduction of systems with second order structure and non-vanishing first order term. In 2015 54th IEEE conference on decision and control (CDC). IEEE, pp 5352–5357. <https://doi.org/10.1109/CDC.2015.7403057>
125. Rahrovani S, Vakilizadeh MK, Abrahamsson T (2014) A metric for modal truncation in model reduction problems part 1: performance and error analysis. In: Conference proceedings of the society for experimental mechanics series, pp 781–788. https://doi.org/10.1007/978-1-4614-6585-0_73
126. Rózsa P, Sinha NK (1975) Minimal realization of a transfer function matrix in canonical forms. *Int J Control*. <https://doi.org/10.1080/00207177508921986>
127. De Schutter B, De Moor B (1995) Minimal realization in the max algebra is an extended linear complementarity problem. *Syst Control Lett*. [https://doi.org/10.1016/0167-6911\(94\)00062-Z](https://doi.org/10.1016/0167-6911(94)00062-Z)
128. Senkal D, Efimovskaya A, Shkel AM (2015, March) Minimal realization of dynamically balanced lumped mass WA gyroscope: dual foucault pendulum. In 2015 IEEE international symposium on inertial sensors and systems (ISISS) proceedings. IEEE, pp 1–2. <https://doi.org/10.1109/ISISS.2015.7102394>
129. Aoki M (1968) Control of large-scale dynamic systems by aggregation. *IEEE Trans Autom Control*. <https://doi.org/10.1109/TAC.1968.1098900>
130. Hickin J, Sinha NK (1975) Aggregation matrices for a class of low-order models for large-scale systems. *Electron Lett*. <https://doi.org/10.1049/el:19750142>
131. Hwang C (1984) Aggregation matrix for the reduced-order modified Cauer CFE model. *Electron Lett* 20(4):150–151. <https://doi.org/10.1049/el:19840100>
132. Bandler JW, Markettos ND, Sinha NK (1973) Optimum system modelling using recent gradient methods. *Int J Syst Sci*. <https://doi.org/10.1080/0020727308919993>
133. Maurya MK, Kumar A (2017, April) Dimension reduction and controller design for large scale systems using balanced truncation. In 2017 1st international conference on electronics, materials engineering and nano-technology (IEMENTech). IEEE, pp 1–4. <https://doi.org/10.1109/IEMENTECH.2017.8076972>
134. Zhou K, Salomon G, Wu E (1999) Balanced realization and model reduction for unstable systems. *Int J Robust Nonlinear Control* 9(3):183–198
135. Antoulas AC (2011) 8. Hankel-Norm approximation. In: Approximation of large-scale dynamical systems, society for industrial and applied mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104)
136. Cao X, Saltik MB, Weiland S (2019) Optimal Hankel norm model reduction for discrete-time descriptor systems. *J Frankl Inst* 356(7):4124–4143. <https://doi.org/10.1016/j.jfranklin.2018.11.047>
137. Liu Y, Anderson BDO (1989) Singular perturbation approximation of balanced systems. *Int J Control* 50(4):1379–1405. <https://doi.org/10.1109/cdc.1989.70360>
138. Guiver C (2019) The generalised singular perturbation approximation for bounded real and positive real control systems. *Math Control Relat Fields* 9(2):313–350. <https://doi.org/10.3934/MCRF.2019016>
139. Kumar D, Tiwari JP, Nagar SK (2012) Reducing order of large-scale systems by extended balanced singular perturbation approximation. *Int J Autom Control* 6(1):21–38. <https://doi.org/10.1504/IJAAC.2012.045438>
140. Van Der Vorst HA (2000) Krylov subspace iteration. *Comput Sci Eng*. <https://doi.org/10.1109/5992.814655>
141. Sorensen DC (2002) Numerical methods for large eigenvalue problems. *Acta Numer*. <https://doi.org/10.1017/s0962492902000089>
142. Grimme EJ (1997) Krylov projection methods for model reduction. University of Illinois at Urbana-Champaign
143. Sambariya DK, Sharma O (2016) Routh approximation: an approach of model order reduction in SISO and MIMO systems. *Indones J Electr Eng Comput Sci* 2(3):486–500. <https://doi.org/10.11591/ijeecs.v2.i3.pp486-500>
144. Antoulas AC, Sorensen DC, Gugercin S (2012) A survey of model reduction methods for large-scale systems
145. Suman SK, Kumar A (2020) Higher-order reduction of linear system and design of controller. *Sci J King Faisal Univ* 2020(3):1–16
146. Boley D, Datta BN (1997) Numerical methods for linear control systems. In: *Systems and control in the Twenty-First Century*, Birkhäuser, Boston, MA, pp 51–74
147. Suman SK, Kumar A (2020) Model order reduction of transmission line model. *WSEAS Trans Circuits Syst* 19:62–68. <https://doi.org/10.37394/23201.2020.19.7>
148. Willcox KE, Peraire J (2002) Balanced model reduction via the proper introduction. *AIAA J*. <https://doi.org/10.2514/2.1570>
149. Suman SK, Kumar A (2019) Investigation and reduction of large-scale dynamical systems. *WSEAS Trans Syst* 18:175–180
150. Dax A (2013) From Eigenvalues to singular values: a review. *Adv Pure Math*. <https://doi.org/10.4236/apm.2013.39a2002>
151. Suman SK (2019) Approximation of large-scale dynamical systems for bench-mark collection. *J Mech Continua Math Sci* 14(3):196–215
152. Gugercin S, Antoulas AC (2006) Model reduction of large-scale systems by least squares. *Linear Algebra Appl*. <https://doi.org/10.1016/j.laa.2004.12.022>
153. Gupta AK, Samuel P, Kumar D (2019) A mixed-method for order reduction of linear time invariant systems using big bang-big crunch and Eigen spectrum algorithm. *Int J Autom Control* 13(2):158–175. <https://doi.org/10.1504/ijaac.2019.10018127>

154. Benner P, Gugercin S, Willcox K (2013) A survey of model reduction methods for parametric systems. In: SIAM Review, Max Planck Institute for dynamics of complex technical systems, pp 1–36
155. Singh J, Vishwakarma CB, Chatterjee K (2016) Biased reduction method by combining improved modified pole clustering and improved Pade approximations. *Appl Math Model* 40(2):1418–1426. <https://doi.org/10.1016/j.apm.2015.07.014>
156. Sandberg H, Rantzer A (2004) Balanced truncation of linear time-varying systems. *IEEE Trans Autom Control* 49(2):217–229. <https://doi.org/10.1109/TAC.2003.822862>
157. Mukherjee S, Mishra RN (1987) Order reduction of linear systems using an error minimization technique. *J Frankl Inst*. [https://doi.org/10.1016/0016-0032\(87\)90037-8](https://doi.org/10.1016/0016-0032(87)90037-8)
158. Mittal AK, Prasad R, Sharma SP (2004) Reduction of linear dynamic systems using an error minimization technique. *J Inst Eng Electr Eng Div* 84:201–206
159. Moore BC (1981) Principal component analysis in linear systems: controllability, observability, and model reduction. *IEEE Trans Autom Control* 26(1):17–32. <https://doi.org/10.1109/TAC.1981.1102568>
160. Fernando KV, Nicholson H (1983) On the structure of balanced and other principal representations of SISO systems. *IEEE Trans Autom Control* 28(2):228–231. <https://doi.org/10.1109/TAC.1983.1103195>
161. Al-Saggaf UM, Franklin GF (1988) Model reduction via balanced realizations: an extension and frequency weighting techniques. *IEEE Trans Autom Control* 37(3):687–692. <https://doi.org/10.1109/9.1280>
162. Samar R, Postlethwaite I, Gu DW (1995) Model reduction with balanced realizations. *Int J Control* 62(1):33–64. <https://doi.org/10.1080/00207179508921533>
163. Clapperton B, Crusca F, Aldeen M (1996) Bilinear transformation and generalized singular perturbation model reduction. *IEEE Trans Autom Control*. <https://doi.org/10.1109/9.489281>
164. Glover K (1984) All optimal Hankel-norm approximations of linear multivariable systems and their L_∞ -error bounds†. *Int J Control* 39(6):1115–1193. <https://doi.org/10.1080/00207178408933239>
165. Škatarić D, Kovačević NR (2010) The system order reduction via balancing in view of the method of singular perturbation. *FME Trans* 38(4):181–187
166. Benner P, Schneider A (2010) Balanced truncation model order reduction for LTI systems with many inputs or outputs. In: Proceedings of the 19th international symposium on mathematical theory of networks and systems—MTNS, pp 1971–1974, [Online]. Available: http://www.tu-chemnitz.de/mathematik/syrene/papers/BenS_2010_MTNS.pdf
167. Yasuda M (2004) Spectral characterizations for hermitian centrosymmetric K-matrices and hermitian skew-centrosymmetric K-matrices. *SIAM J Matrix Anal Appl* 25(3):601–605. <https://doi.org/10.1137/S0895479802418835>
168. Prajapati AK, Prasad R (2020) Model reduction using the balanced truncation method and the Padé approximation method. *IETE Tech Rev (Inst Electron Telecommun Eng, India)*. <https://doi.org/10.1080/02564602.2020.1842257>
169. Ferranti F, Deschrijver D, Knockaert L, Dhaene T (2011) Data-driven parameterized model order reduction using z-domain multivariate orthonormal vector fitting technique. In: Lecture notes in electrical engineering, pp 141–148. https://doi.org/10.1007/978-94-007-0089-5_7
170. Gugercin S, Antoulas AC (2004) A survey of model reduction by balanced truncation and some new results. *Int J Control* 77(8):748–766. <https://doi.org/10.1080/00207170410001713448>
171. Imran M, Ghafoor A, Sreeram V (2014) A frequency weighted model order reduction technique and error bounds. *Automatica* 50(12):3304–3309. <https://doi.org/10.1016/j.automatica.2014.10.062>
172. Lall S, Marsden JE, Glavaški S (2002) A subspace approach to balanced truncation for model reduction of nonlinear control systems. *Int J Robust Nonlinear Control* 12(6):519–535. <https://doi.org/10.1002/rnc.657>
173. Segalman DJ (2007) Model reduction of systems with localized nonlinearities. *J Comput Nonlinear Dyn* 2(3):249–266. <https://doi.org/10.1115/1.2727495>
174. Pernebo L, Silverman LM (1982) Model reduction via balanced state space representations. *IEEE Trans Autom Control* 27(2):382–387. <https://doi.org/10.1109/TAC.1982.1102945>
175. Gugercin S (2008) An iterative SVD-Krylov based method for model reduction of large-scale dynamical systems. *Linear Algebra Appl* 428(9):1964–1986. <https://doi.org/10.1016/j.laa.2007.10.041>
176. Fernando KV, Nicholson H (1982) Singular perturbational model reduction of balanced systems. *IEEE Trans Autom Control* 27(2):466–468. <https://doi.org/10.1109/TAC.1982.1102932>
177. Fernando KV, Nicholson H (1982) Singular perturbational model reduction in the frequency domain. *IEEE Trans Autom Control* 27(4):969–970. <https://doi.org/10.1109/TAC.1982.1103037>
178. Kokotovic PV, O'Malley RE, Sannuti P (1976) Singular perturbations and order reduction in control theory—an overview. *Automatica* 12(2):123–132. [https://doi.org/10.1016/0005-1098\(76\)90076-5](https://doi.org/10.1016/0005-1098(76)90076-5)
179. Gu DW, Petkov PH, Konstantinov MM (2013) Lower-order controllers. In: Advanced textbooks in control and signal processing, pp 73–91. Springer, London
180. Safonov MG, Chiang RY (1989) A Schur method for balanced-truncation model reduction. *IEEE Trans Autom Control* 34(7):729–733. <https://doi.org/10.1109/9.29399>
181. Fernando KV, Nicholson H (1983) Singular perturbational approximations for discrete-time balanced systems. *IEEE Trans Autom Control* 28(2):240–242. <https://doi.org/10.1109/TAC.1983.1103202>
182. Gajic Z, Lelic M (2001) Improvement of system order reduction via balancing using the method of singular perturbations. *Automatica* 37(11):1859–1865. [https://doi.org/10.1016/S0005-1098\(01\)00139-X](https://doi.org/10.1016/S0005-1098(01)00139-X)
183. Vishwakarma CB, Prasad R (2009) MIMO system reduction using modified pole clustering and genetic algorithm. *Model Simul Eng* 2009:1–5. <https://doi.org/10.1155/2009/540895>
184. Singh V, Chandra D, Kar H (2004) Improved Routh-Padé approximants: a computer-aided approach. *IEEE Trans Autom Control* 49(2):292–296. <https://doi.org/10.1109/TAC.2003.822878>
185. Narwal A, Prasad BR (2016) A novel order reduction approach for LTI systems using cuckoo search optimization and stability equation. *IETE J Res* 62(2):154–163. <https://doi.org/10.1080/03772063.2015.1075915>
186. Sikander A, Thakur P (2018) Reduced order modelling of linear time-invariant system using modified cuckoo search algorithm. *Soft Comput* 22(10):3449–3459. <https://doi.org/10.1007/s00500-017-2589-4>
187. Sikander A, Thakur P, Bansal RC, Rajasekar S (2018) A novel technique to design cuckoo search based FOPID controller for AVR in power systems. *Comput Electr Eng* 70:261–274
188. Pal J (1980) Suboptimal control using Pade approximation techniques. *IEEE Trans Autom Control* 25(5):1007–1008. <https://doi.org/10.1109/TAC.1980.1102490>

189. Pati A, Kumar A, Chandra D (2014) Suboptimal control using model order reduction. *Chin J Eng* 2014(2):1–5. <https://doi.org/10.1155/2014/797581>
190. Sikander A, Rajendra Prasad B (2015) A novel order reduction method using cuckoo search algorithm. *IETE J Res*. <https://doi.org/10.1080/03772063.2015.1009396>
191. Desai SR, Prasad R (2013) A novel order diminution of LTI systems using big bang big crunch optimization and Routh approximation. *Appl Math Model* 37(16–17):8016–8028. <https://doi.org/10.1016/j.apm.2013.02.052>
192. Parmar G, Ssi LM, Prasad R, Mukherjee S (2007) Order reduction of linear dynamic systems using stability equation method and GA. *Int J Electr Robot Electron Commun Eng* 1(1):26–32
193. Vishwakarma CBCB, Prasad R (2009) Clustering method for reducing order of linear system using Pade approximation. *IETE J Res* 54(5):326–330. <https://doi.org/10.4103/0377-2063.48531>
194. Lavania S (2017) Hybrid techniques for reduction of linear time-invariant systems. *J Simul Syst Sci Technol Int*. <https://doi.org/10.5013/jssst.a.18.04.03>
195. Komarasamy R, Albhonso N, Gurusamy G (2012) Order reduction of linear systems with an improved pole clustering. *J Vib Control* 18(12):1876–1885. <https://doi.org/10.1177/1077546311426592>
196. Desai SR, Prasad R (2013) A new approach to order reduction using stability equation and big bang big crunch optimization. *Syst Sci Control Eng* 1(1):20–27. <https://doi.org/10.1080/21642583.2013.804463>
197. Abu-Al-Nadi D (2011) Reduced order modeling of linear MIMO systems using particle swarm optimization. In: Seventh international conference on autonomous systems
198. Vishwakarma CB (2011) Order reduction using modified pole clustering and Pade approximations. *World Acad Sci Eng Technol* 56:787–791
199. Mukherjee S, Mittal RC (2005) Model order reduction using response-matching technique. *J Frankl Inst*. <https://doi.org/10.1016/j.jfranklin.2005.01.008>

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.