



Review of the Methods of Transition from Partial to Ordinary Differential Equations: From Macro- to Nano-structural Dynamics

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Abstract

This review/research paper deals with the reduction of nonlinear partial differential equations governing the dynamic behavior of structural mechanical members with emphasis put on theoretical aspects of the applied methods and signal processing. Owing to the rapid development of technology, materials science and in particular micro/nano mechanical systems, there is a need not only to revise approaches to mathematical modeling of structural nonlinear vibrations, but also to choose/propose novel (extended) theoretically based methods and hence, motivating development of numerical algorithms, to get the authentic, reliable, validated and accurate solutions to complex mathematical models derived (nonlinear PDEs). The review introduces the reader to traditional approaches with a broad spectrum of the Fourier-type methods, Galerkin-type methods, Kantorovich–Vlasov methods, variational methods, variational iteration methods, as well as the methods of Vaindiner and Agranovskii–Baglai–Smirnov. While some of them are well known and applied by computational and engineering-oriented community, attention is paid to important (from our point of view) but not widely known and used classical approaches. In addition, the considerations are supported by the most popular and frequently employed algorithms and direct numerical schemes based on the finite element method (FEM) and finite difference method (FDM) to validate results obtained. In spite of a general aspect of the review paper, the traditional theoretical methods mentioned so far are quantified and compared with respect to applications to the novel branch of mechanics, i.e. vibrational behavior of nanostructures, which includes results of our own research presented throughout the paper. Namely, considerable effort has been devoted to investigate dynamic features of the Germain–Lagrange nanoplate (including physical nonlinearity and inhomogeneity of materials). Modified Germain–Lagrange equations are obtained using Kirchhoff’s hypothesis and relations based on the modified couple stress theory as well as Hamilton’s principle. A comparative analysis is carried out to identify the most effective methods for solving equations of mathematical physics taking as an example the modified Germain–Lagrange equation for a nanoplate. In numerical experiments with reducing the problem of PDEs to ODEs based on Fourier’s ideas (separation of variables), the Bubnov–Galerkin method of static problems and Faedo–Galerkin method of dynamic problems are employed and quantified. An exact solution governing the behavior of nanoplates served to quantify the efficiency of various reduction methods, including the Bubnov–Galerkin method, Kantorovich–Vlasov method, variational iterations and Vaindiner’s method (the last three methods include theorems regarding their numerical convergence). The numerical solutions have been compared with the solutions obtained by various combinations of the mentioned methods and with solutions obtained by FDM of the second order of accuracy and FEM for triangular and quadrangular finite elements. The studied methods of reduction to ordinary differential equations show high accuracy and feasibility to solve numerous problems of mathematical physics and mechanical systems with emphasis put on signal processing.

1 Introduction

In this review, we consider numerous computational approaches derived from mathematical physics and more widely used over the years to find nonlinear PDE solutions that govern the dynamic behavior of structural members. This section will include the description of advantages (sometimes disadvantages) of the classical and extended

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theoretical variants of methods developed by Fourier and Galerkin, variational methods, as well as the variational iteration and Kantorovich–Vlasov methods, and others.

1.1 Fourier Methods (FM)

A number of computational methods aimed at solving a wide variety of problems in mathematical physics and technology were based on the ideas of the great French scientist J. Fourier including the method of separation of variables or the method of eigenfunctions [1]. He formulated the methods which later gave impetus to the use of the variable separation procedure when creating new approaches to solving equations for functions of several variables. William Thomson (Lord Kelvin) called this work “The Great Mathematical Poem”.

The advantage of the Fourier method is that it creates a simple way to obtain an explicit solution. This method allows us to present a series of very wide class of functions, since it requires that one smoothness condition is satisfied inside the area, and others on the boundary. Employment of the apparatus of asymptotic analysis to obtain estimates of the Fourier coefficients improved the convergence and highlighted the features of the series. There are many other benefits while using Fourier’s method, which are specific to each particular problem. For example, it includes the clear physical meaning of Fourier coefficients in vibrations problems and heat dissipation problems.

Nowadays the Fourier-based methods have found various applications while analyzing linear and nonlinear PDEs. Propagation processes governed by PDEs can be solved using the so-called split-step Fourier method. It requires a simple numerical implementation and can cause phenomena which are not detected by other numerical methods. This approach is widely used in the numerical analysis of nonlinear optical fiber channels [2]. The algorithm is a combination of reducing the step-size for the integration along a chosen variable, and then by utilizing a discrete-time representation of the propagating signals to get back and forth between time and frequency domains using FFT (Fast Fourier Transform).

Though the direct Fourier method and the split-step Fourier method are faster than most commercial programs, in general they are computationally time-consuming, and often the finite-difference methods are used instead [3].

When analyzing mechanical systems, machines and mechanisms, the signal processing is based on the time-frequency resolution of Fourier methods and wavelets. The Fourier methods belong to the oldest and have been used to solve static and dynamic problems of structural members, including rods, beams, plates and shells, and the solutions can be obtained either in the form of stresses or displacement functions. Numerous problems of nonlinear discrete

and continuum mechanics are solved with a help of the Fourier methods. In the former case the system of governing ODEs is solved using either single (periodicity) or double (quasi-periodicity) Fourier series, while in the latter case usually the governing PDEs are solved using the Galerkin and Bubnov–Galerkin methods, as well as the separation of variable method matched with the Fourier series. The Fourier methods and the associated harmonic analysis are always involved either explicitly or implicitly in solving a variety of problems of mathematical physics including the Navier–Stokes equations, Littlewood–Paley decompositions, Besov spaces, etc.

Fourier series representation includes both real and complex sinusoids. In the case of signals (noise) often measured in engineering problems, the Fourier representation has a direct physical interpretation. Various waveforms have Fourier series representations, and therefore the Fourier-based analysis is inherently involved in any engineering action. Any periodic/quasi-periodic signal with the given period/periods can be approximated by 1D/2D Fourier series. The Fourier series can approximate odd and even square waves/signals. In addition, regular pulse trains, triangle waveform as well as saw tooth waveforms have their Fourier expressions.

As it has already been mentioned, in engineering Fourier methods serve as a paradigm for solving majority of problems dealing with signal processing, including the Fourier transform, discrete-time Fourier transform, discrete Fourier transform, short-time Fourier transform and different wavelets. Though FFT does not belong strictly to the Fourier methods, it can be understood also as the Fourier idea matched with a computational technique for evaluating the discrete Fourier transforms. It should be noticed that the short-time Fourier transform makes it also possible to trace time evolution of the frequency components, although usually wavelet transforms are used for this purpose.

From the engineering point of view the Fourier methods represent the strength of a signal measured by its amplitudes versus its frequencies, i.e. an arbitrary signal can be presented as a sum of sinusoids (harmonics) of different frequencies. This allows us to carry out the spectral analysis and detect numerous important frequencies, including those occurring in periodic, quasi-periodic and chaotic signals. The discrete-time/discrete Fourier transforms, and the short Fourier transform use the discrete-time complex sinusoids, whereas the Fourier transform employs signals as weighted combinations of continuous-time complex-valued sinusoids.

In mathematical modeling of mechanical/mechatronic systems mainly differential equations are studied. ODEs describe lumped mass mechanical systems and refer to displacements (positions), velocity and accelerations, whereas PDEs describe the behavior (statics and dynamics) of

continuously distributed mass systems and enable the prediction of vibrations, sound, fluid flow, waves, etc.

A property of the Fourier series, i.e. differentiation of a sum of harmonics of different frequencies yields a sum of harmonics of the same frequencies corresponds to the behavior met in real world processes, physics and engineering. It is well known that the engineering approaches based on measuring and control of the mechanical, electric and mechatronic systems involve time delay. This is well fitted by the Fourier series, since a delay of the sum of harmonics of different frequencies results in the sum of harmonics of the same frequencies.

The Fourier methods are well suited to study signal-system interactions. In the case of linear time-invariant systems, the so-called “eigenfunction” property holds, i.e. an input described as an arbitrary sum of complex harmonics of different frequencies yields an output given by the sum of complex harmonics of the same frequencies. The FMs allow also for employing filters to separate certain frequency components of the input signal. In addition, they give efficient computational results of the FFT algorithms based on the properties of complex harmonics. The Fourier methods are extensively described in numerous books [4–6]. Fornberg [7] employed the Fourier method based on FFT to approximate space derivatives for hyperbolic PDEs with emphasis put on its stability and accuracy.

Kreiss and Olinger [8] developed a stability theory for the Fourier pseudo-spectral method with the application to the linear hyperbolic/parabolic PDEs with variable coefficients.

Majda et al. [9] employed the Fourier method to general linear hyperbolic Cauchy problems with nonsmooth initial data. It was proven that the appropriate smoothing techniques give stability, whereas smoothing combined with smoothing of the initial data yielded infinite order accuracy of the obtained results.

Kosloff et al. [10] used the Fourier method to construct a 2D forward modeling algorithm. The solution included discretization in both space and time. Temporal derivatives were approximated by the second order differencing, whereas spatial derivatives were estimated via FFT. The algorithm was validated using the analytic solutions for 2D Lamb problem.

Peyret [11] employed the Fourier method to solve the problems of spatial periodicity in one/two directions matched with the Chebyshev polynomials.

1.2 Galerkin Method (GM)

The Galerkin method can be thought as an extension of the method of Ritz [12] who considered the problem $au = f$ (a stands for a differential operator acting in a Banach space). The approximating solution is $u(x) = U_N(x) = \sum_{i=1}^N \alpha_i w_i(x)$, where N is the natural number and $w_i \in X_N \subset X$ forms a

linearly independent system. It is expected that α_i (weights) can be defined using the condition that the residual (i) $r = au - f \rightarrow \min$. However, a question arises how to define coordinate functions and test functions to find a suitable form of the residual to satisfy the requirement that u_N tends to u , where u is the exact solution. Galerkin proposed a general method and pointed out that the residual should be orthogonal to a system of test functions (linearly independent) where the orthogonality refers to the integral sense [13]. (Nowadays, it is recognized that Galerkin employed not widely known, brilliant idea published by Bubnov (an engineer and marine officer) who applied his method to ship building problems. This is why in many cases the method is referred to as the Bubnov–Galerkin method, while some researchers distinguish the Galerkin and the Bubnov–Galerkin methods).

Therefore, after modifications, the general Galerkin-based idea is to find weight coefficients α_i using the orthogonality relation of the form (ii) $\langle au_N - f, \eta_k \rangle = 0$ for all $\eta_k \in Y_M$, where u_N is given by (i), and $Y_M \subset Y'$ is the set of independent test functions in the space conjugate to Y ; duality pairing of Y and Y' is defined by $\langle \cdot, \cdot \rangle$. The latter general formulation includes various special cases: (a) the Bubnov–Galerkin method when X and Y are Hilbert spaces and spaces X_N and Y_N coincide (see [14]); (b) the Ritz–Galerkin method when (ii) is generated by a quadratic equation and comes from the Euler equation ([15], chapter 4; [16]); (c) the Petrov–Galerkin method where the problem was applied to fit the eigenvalues ([17–19]); (d) the Taylor–Galerkin method where emphasis is laid on the finite element method applied to PDEs [20]; the Faedo–Galerkin method where dynamic properties are concerned [21].

Nowadays the Galerkin methods play a fundamental role in numerical solutions of PDEs including elliptic, parabolic and hyperbolic equations [22].

The method is widely used by the classical and extended discretization algorithms employed explicitly and/or implicitly in the finite element methods (FEMs), finite difference methods (FDMs), and spectral element methods (SEMs).

Nodal (Lagrangian interpolants/polynomials) and modal Fourier basis and Legendre polynomials may serve as basis functions in 1D, and they can be extended to higher space dimensions by using SEM and a concept of tensor products. It should be emphasized that the global modes can be formulated in a mathematical, physical and/or empirical way.

A particular role in finding solutions to a PDE plays the modified FEM known as the Galerkin FEM. In this case the Lagrange and Hermite elements are used as trial functions and this approach can be formulated in both global and local way.

More recently the classical FEM includes extension to fit the discontinuous problems based on completely discontinuous basis functions instead of the piece-wise polynomials

[23]. The review paper [23] addressed the problems of the discontinuous Galerkin methods for solving high-order time-dependent PDEs with emphasis put on the consideration of numerical flux, stability, time discretizations, accuracy of the solution and error estimates.

As already mentioned, Galerkin-type approaches have found great application in both mathematical physics [24–28] and mechanical engineering as well as signal processing [29] (due to the page limit of our review, we have omitted here numerous articles dealing with these issues).

Importance of the Galerkin methods, its sound application and possible further extension can be found in the editorial written by Repin [30].

1.3 Kantorovich–Vlasov Methods (KVM)

Among variety of the numerical methods devoted to the solution of static and dynamic problems of structural members, a key role is played by the methods introduced by Kantorovich [31] and Vlasov [32], further known as the Kantorovich–Vlasov method (KVM). It has the following advantages:

- (i) solution to a nonlinear PDE(s) considered in the Cartesian coordinates is represented by a single series of infinite terms as a combination of functions with separable variables where all components of the approximation are unknown. For example, in the case of uniformly loaded Kirchhoff–Love rectangular plate deflection, $w(x, y)$ is represented as $w(x, y) = \sum_{n=1}^{\infty} f(x)g_n(y)$;
- (ii) the original 2D (3D) problem is reduced to a system of two (three) coupled 1D problems;
- (iii) the iterative procedure of KVM is efficient for computations, since it converges rapidly (usually, it takes a few iterations with a few terms in the approximating function to achieve the validated and stable solution);
- (iv) KVM is useful and shows advantages over other traditional methods of reduction of PDEs to ODEs while dealing with vibrations of beams, plates and shells.

In the beginning, the reduction of PDEs dimension occurring in mathematical physics and mechanics to that of the counterpart 1D problem was carried out by Grigorenko and his disciples [33–36] as well as by Krysko and his co-workers [37–39]. Then, Bepalova successfully employed this method to various problems of nonlinear structural mechanics [40–42].

More recently, Bepalova and Urusova [43] used the advanced Kantorovich–Vlasov approach to reduce the 3D problem of torsion of an anisotropic prism to three

corresponding 1D problems. In addition, they reported new results regarding warping of the cross-section and deformation of the prism axis for different types of anisotropy.

Bepalova [44] proposed also to match the inverse-iteration method of successive approximations and the advanced KVM to calculate natural frequencies of an elastic parallelepiped with different boundary conditions. The dependence of lower frequencies of a 3D cantilever beam was reported.

Surianinov and Chaban [45] used the KVM to reduce the 2D problem of calculating the anisotropic plate based on the proposed analytical-numerical method. This approach yielded the solution of a basis differential equation of bending an anisotropic plate with any boundary conditions and without any restrictions given to the external load.

Nwoji et al. [46] applied the KVM to the flexural investigation of simply supported Kirchhoff plates subjected to transverse uniformly distributed load. The Vlasov approach was applied to construct coordinate functions in the x direction, whereas the Kantorovich method was used to consider the displacement field. They reported that the obtained solutions were rapidly convergent for both deflection and bending moments. The results obtained were validated by the Levi-Nadai solutions.

Then, Kerr [47] extended the Kantorovich method to get the exact solution in both directions by using the iterative procedure. Kapuria and Kumari [48] applied KVM to study a 3D problem of elasticity solution of laminated composite structures in cylindrical bending. Aghdam et al. [49] used successfully the Kantorovich method in the static analysis of moderately thick, functionally graded sector plates. Wu et al. [50] found the Kantorovich-based solution when studying a simply supported beam under linearly distributed loads. The multi-term extended KVM was applied to find analytical solution for the bending problem of moderately thick composite annular sector plates with general boundary conditions and loadings [51].

The free vibration analysis based on KVM of symmetrically laminated fully clamped skew plates and of moderately thick, functionally graded plates was carried out in the works [52, 53]. Vibrations of laminated plates with various boundary conditions based on the extended Kantorovich method were analyzed [54].

Kapur et al. [55] studied the 3D elasticity solution of laminated plates with a help of multi-term extended Kantorovich method, whereas Kumari et al. [56] used the 3D extended Kantorovich solution for Levy-type rectangular laminated plates with edge effects.

Huang et al. [57] used KVM to estimate local stresses in composite laminates upon polynomial stress function.

Kumari and Behera [58] obtained a closed form 3D solution for a rectangular laminated plate with Levy type support based on the extended Kantorovich method. Natural frequencies were estimated for composite and sandwich

laminates. In addition, the effect of the span to thickness ratio and in-plane modulus ratio on the natural frequencies was also investigated. Ike and Mama [59] carried out the flexural analysis of Kirchhoff–Love plates using the variational Kantorovich method.

1.4 Variational Methods (VM)

The variational methods belong to the fundamental tools for solving variety of linear/nonlinear problems not only in mechanics but in mathematical physics due to the rigorous mathematical bases. In the classical mechanics it concerns the mechanics of solids and fluids, materials modeling and properties (creeps, plasticity and smart materials fabrication, materials with memories, etc.), the statics and dynamics of structural members dealing with real engineering constructions, heat transfer, and others. It allows us to model complex behavior of both discrete systems and continuum.

Theoretical achievements, counted from the 18th century, introduced by seminal works of Euler, Lagrange, Bubnov, Galerkin, Kantorovich, Nowacki, Vlasov, Vaindiner, Reissner and Ritz along with Hamilton's principle have recently attracted great interest of the researches, and have brought back the classical approaches based on the variational methods and energy principles. This is motivated by growing attractiveness of numerical/computational methods in applied sciences, and in particular in engineering including the finite element method, finite difference method, Ritz method, weighted-residual methods, and more.

The variational methods require good mathematical background, as it is shown by the authors of the outstanding monographs [60–65], but it seems that the pressure of rapidly developing technology and engineering approaches shifted and somehow shallowed the fundamental achievements of variational classical theories to engineering and applied sciences with direct application to the finite element methods.

However, the classical variational approaches can be extended to mathematical branches of science (geometry, topology, analysis), mathematical physics, optimization problems and mathematical modeling and analysis of nonlinear behavior in biology, medicine, and economics.

There are numerous publications devoted to the theory and application of variational methods. Since the latter have been closely matched with the computational methods, it is almost impossible and perhaps useless to cite all or majority of them. Therefore, only few of them will be mentioned.

Courant [66] pointed out importance of the calculus of variations and its equivalence with the boundary value problems of PDEs. He considered variational problems with reference to quadratic functionals, rigid and natural boundary conditions, the Rayleigh–Ritz method, the method of finite differences, and the method of gradients.

The rigorous treatment and extension of the classical variational methods is offered by Marsden et al. [67] who proposed a variational and multisymplectic formulation of compressible/incompressible models of continuum mechanics on general Riemannian manifolds. An interplay of compressible/incompressible continuum mechanics, constrained multisymplectic field theories, symmetries, momentum maps and Noether's theorem was offered.

Direct applications of variational methods for engineers can be found in the book [68]. It gives friendly written basic material devoted to variational methods for algebraic equations, and to differential equations, the treatment of Hilbert and functional spaces feasible for engineers, as well as functionals and calculus of variations.

Finally, it should be emphasized that the classical variational methods still attract interest of the scientists which is documented by the original treatment and extension to the thematic issues presented in the monographs/books [69–74].

1.5 Variational Iteration Method (VIM)

The next method is the variational iteration method (VIM), which relieves a researcher from the need to construct a system of approximating functions like in the Galerkin method. Arbitrarily defined functions (satisfying only the smoothness conditions) are taken and then they are refined in the process of calculation by VIM based on the solutions of the original system of differential equations.

This method was first proposed and applied in 1933 by Shunk [75] to calculate the bending of cylindrical panels. But the work went unnoticed and the method was rediscovered again in 1964 by Zhukov [76], who used it to calculate a thin rectangular plate. Further, VIM was widely used by many researchers to solve the problems of shell and plate theory [37]. The substantiation of this method for the class of equations described by positive definite operators was given in [38].

It is worth noting here that in the literature there is a large number of different names for the method of variational iterations. This method is often called by Kerr [77–79] the extended Kantorovich method and it was published 38 years after Shunk and 5 years after Zhukov. Given this, we can say that the method has been rediscovered.

The variational iteration method or extended Kantorovich method has been used over the past half century to solve the problems of statics, stability, natural frequencies and dynamics.

The VIM was extended by Ji-Huan He [80, 81] who used the old concept of Lagrange multiplier method, with emphasis put on analytical solutions. The approach proposed by He found many followers and his modification to the classical VIM was employed in a wide range of issues of mathematical physics dealing with ODEs and PDEs. On the other hand,

the convergence of VIM was discussed in many papers, including [82–86].

Solodov and Svaiter [87] proposed a novel projection algorithm to solve variational inequality. First, an appropriate hyperplane separating the current iterate from the solution was employed. It required a single projection onto the feasible set and application of the Amijo-type line search along a feasible direction. The next iteration was a projection of the current iterate onto the intersection of the feasible set with the halfspace including the solution.

Matinfar et al. [88] developed a so called modified variational iteration method (MVIM) by merging the classical VIM with He's polynomials and applied it to solve heat transfer problems governed by PDEs with variable coefficients. The proposed algorithms matched the variational iterations with the homotopy, perturbation methods and exhibited all advantages of the coupled technique. The authors showed that MVIM was employed without any discretization procedures and strong assumptions and was free from errors.

Noor and Mohyud-Din [89] developed an analytical version of VIM called the variational decomposition method based on the example of the eighth-order boundary value problem. Analytical solutions were found as the conventional technique confirmed the reliability, validity and high accuracy of the proposed method.

Ji-Huan He [90] again considered VIM with emphasis laid on recent results and new interpretations. Procedures illustrating the solution were based on nonlinear oscillators governed by ODEs. Then, He and Wu [91] discussed the application of VIM to nonlinear wave equations and nonlinear fractional differential equations occurring in various engineering problems. They noticed that VIM may serve as a candidate to solve problems of strongly nonlinear equations which cannot be attacked by the classical approximate methods.

Barari et al. [92] combined the homotopy perturbation method and VIM to solve nonlinear ODEs with an oscillating solution. They stressed that the method did not require either linearization or small perturbation which is needed in the classical approaches.

Saadatmandi and Dehghan [93] employed He's variational method to solve the generalized pantograph equation. They demonstrated that only a few iterations were sufficient to guarantee a highly accurate solution, and it was not affected by computation round of errors.

Abbasbandy and Shivanian [94] used VIM successfully to solve nonlinear Volterra's integro-differential PDEs. They observed that VIM yielded more accurate results as compared to the homotopy perturbation method and the Adomian decomposition method.

Salehooor and Jafari [95] employed the VIM extended by He to find a solution to nonlinear gas dynamics equation and Stefan equation. The method involved the Lagrange

multipliers to identify the optimal values of parameters in a functional.

Heydari et al. [96] combined the spectral method and VIM for heat transfer problems with high order of nonlinearity. The method was implemented to solve an unsteady nonlinear convective-radiative equation with two small independent parameters.

Wazwaz [97] applied VIM for solving linear/nonlinear ODEs with variable coefficients. He tested a variety of models including the hybrid selection model, Thomas-Fermi equations, as well as the Kidder and Riccati equations.

Zhang et al. [98] used a new scheme while adopting VIM to solve the problem of infrared radiative transfer in a scattered medium. Both upward and downward intensities were calculated separately by VIM and it was found that VIM was more accurate comparing to the discrete-ordinates method.

Narayanamoorthy and Mathankumar [99] used VIM to solve problems governed by the linear Volterra fuzzy integro-differential equations. Two test problems were considered and the approximate versus exact solutions were validated.

El-Sayed and El-Mongy [100] employed the modified VIM to solve the free vibration problem of a tapered Euler-Bernoulli beams mounted on 2DoF mass-spring-damper subsystem. Natural frequencies and mode shapes for uniform/tapered beam were studied. It was demonstrated how highly accurate results were obtained by adjusting only a few iterations of VIM.

1.6 Vaindiner's Method (VaM)

This method, while rigorous and powerful, did not attract, however, special attention particularly among the western-oriented researchers. The method can be viewed as an extension and modification of the Kantorovich and Bubnov–Galerkin methods. An approximate solution to a problem governed by PDEs was searched in the form of the sum of products of the coordinate and sought functions with applications to the Fourier general polynomials and series [101, 102]. In the first product, a given function depends on x , whereas in the second product it depends on y . The approach was then further extended and developed by studying many problems of the convergence of the methods used, the decreasing order of the studied systems, interpolation of functions with many variables, lattice collocation, boundary value problems and singular integral equations of elasticity in 3D [101–107].

Vaindiner's method was widely used to study stationary geometrically nonlinear problems of the shell theory [108]. The system of nonlinear ordinary differential equations was reduced to a nonlinear system of algebraic equations by the finite difference method. And then, the problem was solved by the general iteration method. A solution for

square plates with three types of boundary conditions was obtained including the analysis of a simple support, sliding clamping of the contour, a simple support on two opposite edges and sliding clamping on the other two. More recently, Vaindiner's method was used to study the static bending of plates taking into account physical nonlinearity and different moduli of material [109, 110], and also to consider the problems of one-sided corrosion of shells [111].

A brief review of the literature reveals that there is no available comparative analysis of the methods so far described and schematically shown in Fig. 1 (description of the Agranovskii–Baglai–Smirnov method (ABSM) is omitted here, because it is presented in Sect. 3.3). Many of the above methods have not been applied to problems in the theory of plates and shells. This is why we have employed and quantified the effectiveness of the methods based on the example of the Germain–Lagrange PDE.

1.7 Closing Remarks

It is well recognized that in solving multidimensional problems of mathematical physics, the reduction of dimension plays a crucial role, and next an attempt to achieve a solution in a compact way is highly required. In this respect, an original multidimensional problem is substituted by its counterpart sequence of one-dimensional problems. This makes it possible to obtain solutions without time-consuming numerical computations, which is important for engineering applications. Since in most cases it is impossible to obtain an exact match between the original multidimensional problem and the sequence of one-dimensional problems, appropriate functionals are proposed and used to solve the problem in an approximate way.

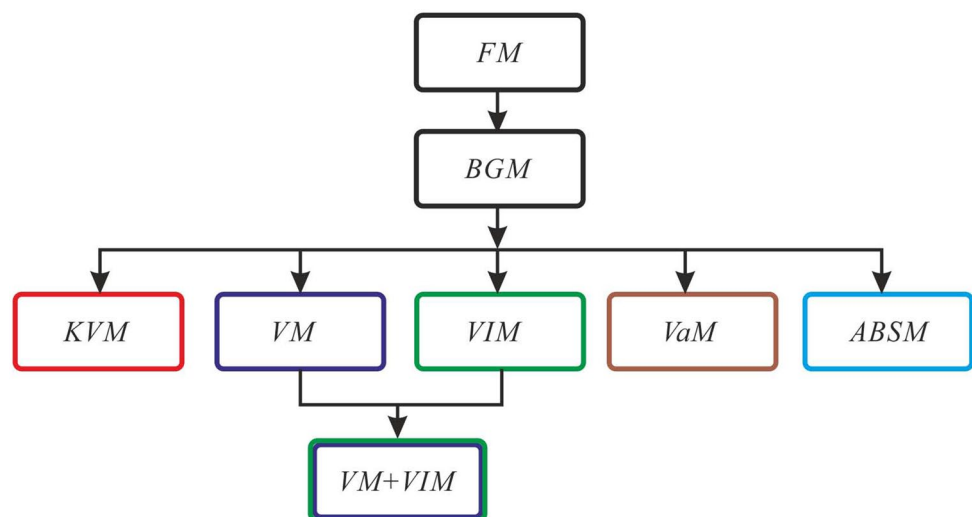
Nonlinear dynamics of the one-dimensional macro-structural-mechanical issues has been recently deeply revisited

with regard to chaotic and bifurcational features (see books [112–114] and review paper [115]). Thus, with the appropriate methodology suitable from the engineering point of view and reliable and accurate approaches to the transition from nonlinear multidimensional problems to a set of nonlinear but one-dimensional problems, progress can be made in the computational aspects of modern mechanics, aeronautics and civil engineering.

Worth noting is the interest of engineering community in the so called reduced order models (ROMs), where the transition from multidimensional infinite problems governed by nonlinear PDEs is reduced to a set of nonlinear ODEs, where the problem is highly truncated to the single-mode approximation. Although this approach allows us to find a relationship between the dynamics of structural members with infinite degrees of freedom and the dynamics of lumped massed systems with one degree of freedom governed by the second order ODE, in many cases it does not allow us to obtain reliable results. Therefore, recognizing several advantages of this engineering-oriented approach, such as direct fit to the well-known classical 1 DoFs oscillators, including Duffing, Duffing-Helmholtz and van der Pol archetypal systems, and capturing the basic dynamic phenomena of structural members under study, we prefer to highlight more rigorous mathematically and reliable methods to be used by engineering community, which is why ROMs are not covered by our review article.

The rest of the paper is organized in the following way. Problem statement is given in Sect. 2, whereas solutions based on the methods described in Sect. 1 are reported in Sect. 3. Section 4 presents solutions of the problem using FEM and FDM, and concluding remarks are given in Sect. 6.

Fig. 1 The scheme of connections of the Fourier method (FM) with the Bubnov–Galerkin method (BGM), Kantorovich–Vlasov method (KVM) and their modifications



2 Problem Statement

To construct a mathematical model of a Germain–Lagrange nanoplate based on the modified couple stress theory, an elastic isotropic rectangular plate is considered. The rectangular coordinate system associated with the nanoplate is introduced in the following way: $\Omega = \{x, y, z / (x, y, z) \in [0, a] \times [0, b] \times [-h/2, h/2]\}$. The nanoplate has dimensions a, b, h along axes x, y, z , respectively. Its middle surface at $z = 0$ is denoted by $B = \{x, y / (x, y) \in [0, a] \times [0, b]\}$ (see Fig. 2).

The displacements along the x, y axes refer to u, v , respectively. Deflection (normal plate movement) along the z axis is denoted by $w = w(x, y)$. In order to construct the mathematical model, we introduce the following assumptions and hypotheses: (i) Kirchhoff hypothesis holds; (ii) rotation inertia of the plate elements are not taken into account; (iii) plate material is homogeneous and isotropic, i.e. it is assumed that for the same stress the same deformations occur in all points and elastic properties in each point of the plate are the same in all directions; (iv) all displacement components are considered to be significantly smaller than the characteristic size of the considered rectangular plate; (v) the deformations of middle surface $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy}$ are assumed to be negligible compared to unity (this does not mean that the relationship between movements and deformations must be linear).

Owing to the modified couple stress theory [116], the accumulated energy of deformation U of the plate, taking its size-dependent behavior into account, is governed by the following formula

$$U = \frac{1}{2} \int_{\Omega} \left(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{xy}\epsilon_{xy} + m_{xx}\chi_{xx} + m_{yy}\chi_{yy} + m_{xy}\chi_{xy} \right) d\Omega \tag{1}$$

where $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy}, m_{xx}, m_{yy}, m_{xy}, \chi_{xx}, \chi_{yy}, \chi_{xy}$ denote the components of the symmetric part of stress tensor σ ,

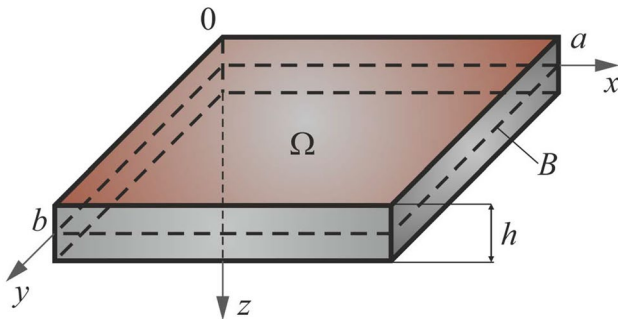


Fig. 2 General view of the nanoplate

components of deformation tensor ϵ , components of the deviatoric part of the symmetric tensor of the higher order m , and the components of the symmetric part of curvature tensor χ .

Linear deformations and displacements are coupled via the following relations

$$\epsilon_{xx} = -z \frac{\partial^2 w}{\partial x^2}, \quad \epsilon_{yy} = -z \frac{\partial^2 w}{\partial y^2}, \quad \epsilon_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y} \tag{2}$$

The following relation for the non-zeroth component of the symmetric curvature tensor part holds

$$\chi_{xx} = \frac{\partial^2 w}{\partial x \partial y}, \quad \chi_{yy} = -\frac{\partial^2 w}{\partial x \partial y}, \quad \chi_{xy} = \frac{1}{2} \left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} \right). \tag{3}$$

For non-zero components of the symmetric part of stress tensor σ and for the deviatoric part of the tensor of the higher order m , the following relations take place [116]:

$$\sigma_{xx} = (\lambda + 2\mu)\epsilon_{xx}, \quad \sigma_{yy} = (\lambda + 2\mu)\epsilon_{yy}, \quad \sigma_{xy} = 2\mu\epsilon_{xy}, \tag{4}$$

$$m_{xx} = 2\mu l^2 \chi_{xx}, \quad m_{yy} = 2\mu l^2 \chi_{yy}, \quad m_{xy} = 2\mu l^2 \chi_{xy},$$

where: $\lambda(x, y, z, e_i) = \frac{E\nu}{(1+\nu)(1-2\nu)}$, $\mu(x, y, z, e_i) = \frac{E}{2(1+\nu)}$ are the Lamé parameters; $e_i = \frac{\sqrt{2}}{3} \sqrt{(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{yy}^2 + \epsilon_{xx}^2 + \frac{3}{2}\epsilon_{xy}^2}$ describes the intensity of deformation; E - Young modulus; ν - Poisson's coefficient; l - material length parameter associated with the symmetric tensor of the relation gradient.

According to the method of variable parameters of elasticity [117], $E(x, y, z, e_i)$ and $\nu(x, y, z, e_i)$ are not constant but they depend on the coordinates and intensity of deformations.

Equations (2), (3) and (4) are used to derive deformation energy U of the nanoplate, i.e. we have

$$U = \frac{1}{2} \int_0^a \int_0^b \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{ (\lambda + 2\mu) \left(-z \frac{\partial^2 w}{\partial x^2} \right)^2 + (\lambda + 2\mu) \left(-z \frac{\partial^2 w}{\partial y^2} \right)^2 + 2\mu \left(-z \frac{\partial^2 w}{\partial x \partial y} \right)^2 + \frac{1}{2} \mu l^2 \left[-\frac{\partial^2 w}{\partial x^2} \right]^2 + \frac{1}{2} \mu l^2 \left[-\frac{\partial^2 w}{\partial y^2} \right]^2 + \mu l^2 \left[-\frac{\partial^2 w}{\partial x \partial y} \right]^2 \right\} dz dy dx. \tag{5}$$

We introduce the following notation: $\int_{-\frac{h}{2}}^{\frac{h}{2}} (\lambda + 2\mu) z^2 dz = \alpha_1$, $\int_{-\frac{h}{2}}^{\frac{h}{2}} \mu z^2 dz = \alpha_2$, $\int_{-\frac{h}{2}}^{\frac{h}{2}} \mu dz = \alpha_3$. Then, $\alpha_i = \alpha_i(x, y, e_i)$ are the functions of coordinates and intensity of deformations. Equation (5) takes the following form

$$\begin{aligned}
 U &= \frac{1}{2} \int_0^a \int_0^b \left\{ \alpha_1 \left(\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right) + 2\alpha_2 \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right. \\
 &\quad \left. + \frac{1}{2} \alpha_3 l^2 \left(\left[\frac{\partial^2 w}{\partial x^2} \right]^2 + \left[\frac{\partial^2 w}{\partial y^2} \right]^2 + 2 \left[\frac{\partial^2 w}{\partial x \partial y} \right]^2 \right) \right\} dx dy, \tag{6}
 \end{aligned}$$

$$\begin{aligned}
 \delta W &= \iint_B q(x, y, t) \delta w dB, \\
 \delta K &= \int_0^b \int_0^a I \frac{\partial^2 w(x, y, t)}{\partial t^2} \delta w dx dy,
 \end{aligned}$$

where: $I = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho dz$.

where the underlined members include higher order terms.

External work related to the distributed forces takes the following form

Employing Hamilton’s principle, we get

$$W = \iint_B q(x, y) w(x, y) dB. \tag{7}$$

$$\int_{t_0}^{t_1} \delta(K - U + W) dt = 0, \tag{11}$$

Kinetic plate energy is as follows

and we obtain the following equation of dynamics of the nanoplate based on the modified couple stress theory with variable elastic parameters

$$K = \frac{1}{2} \int_B \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(x, y, z) \left(\frac{\partial w}{\partial t} \right)^2 dz dB. \tag{8}$$

The associated variations of U, W, K are as follows

$$\begin{aligned}
 \delta U &= \iint_B \left\{ \alpha_1 \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial \alpha_1}{\partial x} \frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 \alpha_1}{\partial x^2} \frac{\partial^2 w}{\partial x^2} + \right. \\
 &\quad + \alpha_1 \frac{\partial^4 w}{\partial y^4} + 2 \frac{\partial \alpha_1}{\partial y} \frac{\partial^3 w}{\partial y^3} + \frac{\partial^2 \alpha_1}{\partial y^2} \frac{\partial^2 w}{\partial y^2} \\
 &\quad + 2 \left(\frac{\partial^2 \alpha_2}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial \alpha_2}{\partial x} \frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial \alpha_2}{\partial y} \frac{\partial^3 w}{\partial x^2 \partial y} + \alpha_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} \right) \\
 &\quad + \frac{1}{2} l^2 \left[\alpha_3 \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial \alpha_3}{\partial x} \frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 \alpha_3}{\partial x^2} \frac{\partial^2 w}{\partial x^2} \right. \\
 &\quad + \alpha_3 \frac{\partial^4 w}{\partial y^4} + 2 \frac{\partial \alpha_3}{\partial y} \frac{\partial^3 w}{\partial y^3} + \frac{\partial^2 \alpha_3}{\partial y^2} \frac{\partial^2 w}{\partial y^2} \\
 &\quad \left. + 2 \left(\frac{\partial^2 \alpha_3}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial \alpha_3}{\partial x} \frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial \alpha_3}{\partial y} \frac{\partial^3 w}{\partial x^2 \partial y} + \alpha_3 \frac{\partial^2 w}{\partial x \partial y} \right) \right] \delta w dx dy \tag{9} \\
 &\quad + \int_0^a \alpha_1 \frac{\partial^2 w}{\partial y^2} \delta \left(\frac{\partial w}{\partial y} \right) \Big|_0^b - \left(\frac{\partial \alpha_1}{\partial y} \frac{\partial^2 w}{\partial y^2} + \alpha_1 \frac{\partial^3 w}{\partial y^3} \right) \delta(w) \Big|_0^b \\
 &\quad - 2 \left(\frac{\partial \alpha_2}{\partial x} \frac{\partial^2 w}{\partial x \partial y} + \alpha_2 \frac{\partial^3 w}{\partial x^2 \partial y} \right) \delta(w) \Big|_0^b dx \\
 &\quad + \int_0^b \alpha_1 \frac{\partial^2 w}{\partial x^2} \delta \left(\frac{\partial w}{\partial x} \right) \Big|_0^a - \left(\frac{\partial \alpha_1}{\partial x} \frac{\partial^2 w}{\partial x^2} + \alpha_1 \frac{\partial^3 w}{\partial x^3} \right) \delta(w) \Big|_0^a \\
 &\quad - 2 \left(\frac{\partial \alpha_2}{\partial y} \frac{\partial^2 w}{\partial x \partial y} + \alpha_2 \frac{\partial^3 w}{\partial x \partial y^2} \right) \delta(w) \Big|_0^a dy + 2\alpha_2 \frac{\partial^2 w}{\partial x \partial y} \delta(w) \Big|_{00}^a b,
 \end{aligned}$$

$$\begin{aligned}
 & \left(\alpha_1 + \frac{1}{2}l^2\alpha_3\right) \frac{\partial^4 w}{\partial x^4} + 2\left(\alpha_2 + \frac{1}{2}l^2\alpha_3\right) \frac{\partial^4 w}{\partial x^2 \partial y^2} \\
 & + \left(\alpha_1 + \frac{1}{2}l^2\alpha_3\right) \frac{\partial^4 w}{\partial y^4} + 2\left(\frac{\partial\alpha_1}{\partial x} + \frac{1}{2}l^2 \frac{\partial\alpha_3}{\partial x}\right) \frac{\partial^3 w}{\partial x^3} \\
 & + \left(\frac{\partial^2\alpha_1}{\partial x^2} + \frac{1}{2}l^2 \frac{\partial^2\alpha_3}{\partial x^2}\right) \frac{\partial^2 w}{\partial x^2} + 2\left(\frac{\partial\alpha_1}{\partial y} + \frac{1}{2}l^2 \frac{\partial\alpha_3}{\partial y}\right) \frac{\partial^3 w}{\partial y^3} \\
 & + \left(\frac{\partial^2\alpha_1}{\partial y^2} + \frac{1}{2}l^2 \frac{\partial^2\alpha_3}{\partial y^2}\right) \frac{\partial^2 w}{\partial y^2} + 2\left(\frac{\partial^2\alpha_2}{\partial x \partial y} + \frac{1}{2}l^2 \frac{\partial^2\alpha_3}{\partial x \partial y}\right) \frac{\partial^2 w}{\partial x \partial y} \\
 & + 2\left(\frac{\partial\alpha_2}{\partial x} + \frac{1}{2}l^2 \frac{\partial\alpha_3}{\partial x}\right) \frac{\partial^3 w}{\partial x \partial y^2} + 2\left(\frac{\partial\alpha_2}{\partial y} + \frac{1}{2}l^2 \frac{\partial\alpha_3}{\partial y}\right) \frac{\partial^3 w}{\partial x^2 \partial y} \\
 & = q(x, y) + I \frac{\partial^2 w}{\partial t^2}.
 \end{aligned} \tag{12}$$

The corresponding choice of the boundary conditions is given with regard to the plate edge [0; b]:

$$\begin{aligned}
 & \alpha_1 \frac{\partial^2 w}{\partial y^2} \Big|_0^b = 0 \text{ or } \delta \left(\frac{\partial w}{\partial y} \right) \Big|_0^b = 0, \\
 & \left(\frac{\partial\alpha_1}{\partial y} \frac{\partial^2 w}{\partial y^2} + \alpha_1 \frac{\partial^3 w}{\partial y^3} \right) - 2 \left(\frac{\partial\alpha_2}{\partial x} \frac{\partial^2 w}{\partial x \partial y} + \alpha_2 \frac{\partial^3 w}{\partial x^2 \partial y} \right) \\
 & + 2\alpha_2 \frac{\partial^2 w}{\partial x \partial y} \Big|_0^b = 0 \\
 \text{or } & \delta(w) \Big|_0^b = 0.
 \end{aligned} \tag{13}$$

For the case, when elasticity parameters are independent of both coordinates and time, the modified couple stress theory, the Germain–Lagrange equations of the following form can be used

$$D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = q, \tag{14}$$

where $D = \frac{Eh^3}{12(1-\nu^2)} + \frac{Ehl^2}{2(1+\nu)}$, and it will be further referred to as the modified Germain–Lagrange equation.

In the future, we will use two types of boundary conditions and their combinations:

$$w|_B = 0, \quad \frac{\partial^2 w}{\partial n^2} \Big|_B = 0 \quad (\text{simple support}), \tag{15}$$

$$w|_B = 0, \quad \frac{\partial w}{\partial n} \Big|_B = 0 \quad (\text{clamping}), \tag{16}$$

where n stands for a normal to the plate boundary.

For the numerical study, we use the non-dimensional form of Eq. (14). We introduce the following non-dimensional variables

$$\begin{aligned}
 \bar{w} &= \frac{w}{h}, \quad \bar{x} = \frac{x}{a}, \quad \bar{y} = \frac{y}{b}, \quad \bar{q} = \frac{a^2 b^2}{Eh^4} q, \quad \gamma = \frac{l}{h^2}, \\
 \lambda &= \frac{a}{b}.
 \end{aligned} \tag{17}$$

Then Eq. (14) can be rewritten as follows (bars over the dimensionless parameters are omitted)

$$D \left(\frac{1}{\lambda^2} \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \lambda^2 \frac{\partial^4 w}{\partial y^4} \right) = q, \tag{18}$$

where: $D = \frac{1}{12(1-\nu^2)} + \frac{\gamma^2}{2(1+\nu)}$, γ - non-dimensional material length parameter equal to 0 for the classical mechanics case, and it is in the interval (0; 0.5) for higher order mechanics.

Boundary conditions (15), (16) should be attached to Eq. (18) in the same dimensionless form. The derived PDEs will be solved both analytically and numerically by several methods: the Kantorovich–Vlasov method, variational iteration method, Vaindiner’s method, Bubnov–Galerkin method, by the use of double trigonometric series, the finite element method and the finite difference method. In order to keep accuracy of the solution, the Agranovskii–Baglai–Smirnov method will be used [118, 119].

Equation (18) can be presented in its counterpart operator form

$$L(w) = q, \tag{19}$$

where L is the operator (in general nonlinear) acting in a Hilbert space H . Boundary conditions (14) and (15) have the following operator representations

$$l(w) = f. \tag{20}$$

Remark 1 It should be noted that Eq. (12) with the corresponding boundary conditions stands for the generalization of the modified Germain–Lagrange equation (MGLE). It allows one to take into account the heterogeneous structure of the nanoplates. The properties of nanoplates can be heterogeneous prior to deformation and can be changed during loading. This theory is based on the deformation theory of plasticity and makes it possible to recalculate elastic moduli and Poisson’s ratios, depending on the deformation diagram and the loading process.

Remark 2 Equation (12) is used to consider physical non-linearity for nanostructures using the iterative scheme of the method of variable elastic parameters. The method of variable parameters of elasticity [117], used to solve a physically nonlinear problem, employs the fact that $E_i(x, y, z, e_i)$, $\nu_i(x, y, z, e_i)$ are not constant. They depend on the deformed state at a structure point and are defined by the following formulas

$$E_i = \frac{9K_{1i}G_i}{3K_{1i} + G_i}, \quad \nu_i = \frac{1}{2} \frac{3K_{1i} - 2G_i}{3K_{1i} + G_i}.$$

Here we consider that volumetric deformation $K_{1i} = const$. In the deformation theory of plasticity [120], the shear modulus is defined in the following form:

$$G_i = \frac{1}{3} \frac{\sigma_i(e_i)}{e_i}$$

where σ_i, e_i stand for the plate stress intensity and plate deformations, respectively. The relationship $\sigma_i(e_i)$ is determined experimentally for a given nanomaterial.

3 Analytical and Numerical Methods for Solving MGLE

3.1 Exact Solution of MGLE Based on Navier’s Double Trigonometric Series

We consider first a possibility to get analytical solution to Eq. (18) with boundary conditions (15). External transverse load $q(x, y)$ is taken into account in the following form

$$q(x, y) = \sum_{m=1}^N \sum_{n=1}^N B_{mn} \sin m\pi x \sin n\pi y, \tag{21}$$

where

$$B_{mn} = \int_0^1 \int_0^1 q(x, y) \sin m\pi x \sin n\pi y \, dx dy. \tag{22}$$

Deflection function $w(x, y)$ is taken in the form of the double trigonometric series

$$w(x, y) = \sum_{m=1}^N \sum_{n=1}^N A_{mn} \sin m\pi x \sin n\pi y, \tag{23}$$

and each term of solution (23) satisfies all boundary conditions (15).

Equations (21), (23) and (18) allow us to derive the following algebraic equation

$$\sum_{m=1}^N \sum_{n=1}^N \left\{ A_{mn} D [(m\pi)^2 + (n\pi)^2]^2 - B_{mn} \right\} \sin m\pi x \sin n\pi y = 0. \tag{24}$$

Taking into account the linear independence of the terms in Eq. (24), we obtain

$$A_{mn} = \frac{B_{mn}}{D [(m\pi)^2 + (n\pi)^2]^2}. \tag{25}$$

Then the counterpart solution in a general form can be written as follows

$$w = \frac{1}{D\pi^4} \sum_{m=1}^N \sum_{n=1}^N B_{mn} \frac{\sin m\pi x \sin n\pi y}{[(m\pi)^2 + (n\pi)^2]^2}. \tag{26}$$

For uniformly distributed constant load $q = q_0$, coefficients B_{mn} take the following form

$$B_{mn} = \frac{16q_0}{mn\pi^2}; \quad m = 1, 3, 5, \dots; \quad n = 1, 3, 5, \dots \tag{27}$$

Therefore, we obtain

$$w = \frac{16q_0}{D\pi^6} \sum_{m=1}^N \sum_{n=1}^N \frac{\sin m\pi x \sin n\pi y}{(m^2 + n^2 \lambda^2)^2 mn}, \tag{28}$$

where

$$\lambda = \frac{a}{b}, \quad \xi = \frac{x}{a}, \quad \eta = \frac{y}{b}. \tag{29}$$

For Eq. (18), convergence of the solution is studied depending on the number of terms of series N in (28), for the following fixed values $\gamma = 0, \gamma = 0.3, \gamma = 0.5$ (see Table 1).

Analysis of the results given in Table 1 shows that for $\gamma = 0$, five members of the series should be considered, whereas for $\gamma \neq 0$ three members of the series are sufficient to get a reliable solution. In what follows, we will consider this as an exact solution and compare it with the solutions obtained numerically using other methods.

3.2 Bubnov–Galerkin Method

Further, we will formulate an application scheme of the Bubnov–Galerkin method using the example of operator equation (19) and taking into account boundary conditions (20).

Let an algebraic base $\{\phi_j\} \subset D(L)$ be chosen in H (Hilbert space) and approximate solution (19) be sought in the following form

Table 1 Quantifying reliable solutions of the nanoplate deflection versus series terms N for different parameter γ

Values N	$w_N(0.5;0.5), \gamma = 0$	$w_N(0.5;0.5), \gamma = 0.3$	$w_N(0.5;0.5), \gamma = 0.5$
1	0.318	0.231	0.155
2	0.318	0.231	0.155
3	0.310	0.225	0.151
5	0.311	0.225	0.151
10	0.311	0.225	0.151

$$w = w_0(x, y) + \sum_{j=1}^N a_j \phi_j(x, y), \tag{30}$$

where: a_j are the unknown numerical coefficients, $\phi_j(x, y)$ are the known analytical functions (approximation), and $w_0(x, y)$ is the function satisfying boundary conditions (20).

Substitution of (30) into (19) yields the following residual

$$\begin{aligned} R(a_0, a_1, a_2, \dots, a_N, x, y) &= L(w) - q \\ &= L(w_0) + \sum_{j=1}^N a_j L(\phi_j) - q. \end{aligned} \tag{31}$$

In order to determine unknown coefficients $a_j, j = 1, \dots, N$, according to the Bubnov–Galerkin method [121, 122], we require that the basis functions $\{\phi_j\}_{k=1}^N$ were also orthogonal to residual (31), i.e. so that the following condition

$$\begin{aligned} w(x, y) &= \sum_{i,j=1}^N \frac{64\lambda^2 q \sin\left(\frac{\pi i}{2}\right)^2 \sin\left(\frac{\pi j}{2}\right)^2}{\pi^4 D(\sin(2\pi i) - 2\pi i)(\sin(2\pi j) - 2\pi j)(\lambda^2 j^2 + i^2)} \sin(i\pi x) \sin(j\pi y). \end{aligned} \tag{36}$$

$$(R, \phi_k) = \iint_{\Omega} R \phi_k \, dx dy \tag{32}$$

is satisfied.

For the Bubnov–Galerkin method the unknown coefficients a_j included in (31) must be determined from the solution of the following system of linear algebraic equations

$$(R, \phi_k) = 0, \quad k = 1, 2, \dots, N \tag{33}$$

where R is the residual equation (19), and ϕ_k are the same analytic functions (weights) that appear in (30). Since the linear problem is considered in (19), Eq. (33) can be written in its counterpart matrix form

$$\sum_{j=1}^N a_j (L(\phi_j) - q, \phi_k) = -L(w_0, \phi_k). \tag{34}$$

Solving the system of algebraic equations by any method aimed at estimating a_j , and substituting the obtained solution into (31), we obtain an approximate solution w .

Applying the Bubnov–Galerkin method when choosing weights $\phi_k(x, y)$ and approximating functions $\phi_j(x, y)$, the following conditions are hypothesized:

1. $\phi_j(x, y) \in H, \phi_k(x, y) \in H$ where H is the Hilbert space;
2. $\forall j, k$ function $\phi_j(x, y), \phi_k(x, y)$ are linearly independent and continuous in space Ω ;

3. $\phi_j(x, y), \phi_k(x, y)$ satisfy rigorously the main boundary conditions (and arbitrary initial conditions);
4. $\phi_j(x, y)$ and $\phi_k(x, y)$ should represent the first N elements of the complete system of functions;
5. $\phi_j(x, y)$ and $\phi_k(x, y)$ satisfy a completeness property in H .

Now, let us illustrate the advantages of BGM. We illustrate the process of finding solutions of the modified Germain–Lagrange equation for boundary conditions (15).

Displacements w can be expanded by using the following series:

$$w(x, y) = \sum_{i=1}^N \sum_{j=1}^N A_{ij}(\sin(i\pi x) \sin(j\pi y)). \tag{35}$$

After applying the Bubnov–Galerkin method for $q(x, y) = q = const$, we find

Table 2 presents solutions of the deflection in the center of the plate $w_m(0.5;0.5)$ depending on the number of terms in the series (36) for three values of γ and fixed parameters $q = 7, \lambda = 1$.

Solutions, which coincide completely with the solution obtained by Navier, are highlighted in Table 2. The given example illustrates the fact that inclusion of γ in the modified Germain–Lagrange equation improves the convergence of the Bubnov–Galerkin method.

3.3 Agranovskii–Baglai–Smirnov Method (ABSM)

The idea is to construct an iterative procedure for solving the original equation (19) with the right-hand side as a residual from the solution at the previous step [118, 119]. We consider the algorithm for applying the

Table 2 Quantification of reliable solutions for nanoplate deflection versus a series of N terms for different γ ($q = 7, \lambda = 1$)

Values N	$w_N(0.5;0.5), \gamma = 0$	$w_N(0.5;0.5), \gamma = 0.5$	$w_N(0.5;0.5), \gamma = 0.5$
1	0.318	0.231	0.155
2	0.318	0.231	0.155
3	0.310	0.225	0.151
5	0.311	0.225	0.151
10	0.311	0.225	0.151

Agranovskii–Baglai–Smirnov method (ABSM) using an example of the operator equation (19).

A new differential equation is constructed in the following way

$$L(w_1(x, y)) = q(x, y), \tag{37}$$

where we changed the right-hand side in Eq. (19); now it plays the role of the residual of the solution to the original equation. This process is repeated n times. Finally, the following series is used as the input solution

$$w(x, y) = \sum_{i=1}^n w_i(x, y). \tag{38}$$

The proof of the convergence of the ABSM method was given in references [118, 119]. In the following sections, the method of combinations of the Kantorovich–Vlasov method, the variational iteration method, and the Vaindiner method will be considered.

3.4 Solving MGLE by Kantorovich–Vlasov Method

Owing to the Kantorovich–Vlasov method (KVM) [31, 32], a solution to Eq. (19) can be searched in the following form

$$w_N(x, y) = \sum_{j=1}^N X_j(x)Y_j(y). \tag{39}$$

Weight functions $X_j(x)$ satisfy boundary conditions (15) and (16). Functions $Y_j(y)$ are the searched functions defined by the following system of projected equations (we substitute (39) into (19) and employ the BGM with respect to co-ordinate x):

$$(L_j(w) - q), X_k(x) = 0, \quad k = 1, 2, \dots, N. \tag{40}$$

Procedure (40) reduces the problem of PDEs to ODEs by employing the Bubnov–Galerkin method regarding one of the coordinates. Solving the system of ODEs with the corresponding boundary conditions, we obtain a set of functions $Y_j(y)$, and next substituting them into (39), we obtain the desired solution.

3.5 Proof of the Convergence of KVM for Some Class of Problems

In this section we will prove that the KVM method for a certain class of problems (including geometrically nonlinear problems of the theory of plates and shells [39]) coincides with the projection methods. A distinctive feature of the method relies on the way of constructing an extremely dense system of subspaces [123].

Let G be a bounded space R^2 with boundary Γ . Consider the first boundary problem for a quasi-linear elliptic equation type of order $2m$ of the following form

$$Lu \equiv \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u) = h(x) \quad (|\gamma| \leq m), \tag{41}$$

$$D^\omega u|_\Gamma = 0, \quad |\omega| \leq m - 1, \tag{42}$$

where: $\alpha = (\alpha_1 + \alpha_2)$ - integer multi-index of differentiation; $|\alpha| = \alpha_1 + \alpha_2$; $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2}$; $D_i = \partial/\partial x_i$; the same holds for γ and ω . Function $A_\alpha(x, \xi_\gamma)$ where $\xi_\gamma = (\xi_{\gamma_1}, \xi_{\gamma_2})$ is nonlinear and depends on all possible indices γ at $|\gamma| \leq m$, $x = (x_1, x_2)$. Finally, let $h(x) \in L_2(G)$. Function $u(x)$ is called the generalized solution of problem (31) and (32), if $u(x) \in W_2^m(G)$ and if for any function $v(x) \in W_2^m(G)$ the following integral identity holds

$$\sum_{|\alpha| \leq m} (A_\alpha(x, D^\gamma u), D^\alpha v) = (h, v). \tag{43}$$

We introduce a few assumptions, which will be implemented in the next steps of the article.

1. The problem of solving Eq. (41) under condition (42) is equivalent to solving the operator equation

$$u = Tu, \quad T : \overset{\circ}{W}_2^m(G) \rightarrow \overset{\circ}{W}_2^m(G), \tag{44}$$

2. Area $G = E_1 \times E_2$, where E_1 and E_2 are the bounded sets of space R^1 .
3. Each element of the complete system of functions $\{\theta_i(x)\}$ of space $W_2^m(E_1 \times E_2)$ takes the form

$$\theta_i(x_1, x_2) = \phi_i(x_1)\psi_i(x_2), \quad x_1 \in E_1, \quad x_2 \in E_2, \tag{45}$$

and moreover, the system of functions $\{\psi_i(x_2)\}$ is complete in space $W_2^m(E_2)$, and $\{\phi_i(x_1)\}$ in $W_2^m(E_1)$. (In the future, we will employ the following lemma).

Lemma 1 *If sequence $\{v_n(x_1, x_2)\}$ of the elements of space $W_2^m(G)$ converges in norm to a certain element $v^* \in W_2^m(G)$, i.e.*

$$\|v_n - v^*\|_{W_2^m}^2 = \int_{E_1 \times E_2} \sum_{k=0}^m \sum_{(k)} \left| \frac{\partial^k (v_n - v^*)}{\partial^{k_1} x_1 \partial^{k_2} x_2} \right|^2 dx_1 dx_2 \rightarrow 0, \quad n \rightarrow \infty, \tag{46}$$

and for $\forall n$, function v_n can be taken as

$$v_n(x_1, x_2) = \sum_{j=1}^N \omega_j^n(x_x)g_i(x_2),$$

where $g_i(x_2)$ are the given elements of the complete system of functions $\{\phi_i(x_2)\}$, and N is the fixed number, then function $v^*(x_1, x_2)$ can also be represented in a similar way, i.e.

$$v^*(x_1, x_2) = \sum_{j=1}^N \omega_j^n(x_x)g_i(x_2). \tag{47}$$

In order to prove the Lemma, it suffices to consider only one term in expression (47). Indeed, $\int_{E_1 \times E_2} |v_n - v^*|^2 dx_1 dx_2 \rightarrow 0, n \rightarrow \infty$, or, according to Fubini's theorem, we have

$$\int_{E_1} F^n dx_1 \rightarrow 0, n \rightarrow \infty, \tag{48}$$

where $0 \leq F^n(x_1) = \int_{E_2} |v_n - v^*|^2 dx_2$.

It follows from (48) that there is a sequence $\{F^{n_k}(x_1)\}$ such that

$$\int_{E_2} \left| \sum_{j=1}^N \omega_j^{n_k}(x_1)g_j(x_2) - v^*(x_1, x_2) \right|^2 dx_2 \rightarrow 0, k \rightarrow \infty, \tag{49}$$

for almost all x_1 .

Formula (49) shows that a certain sequence of elements from a finite-dimensional subspace of dimension N (which linearly spans N given elements $g_j(x_2) \in W_2^m(E_2)$) converges for almost all x_1 to the element $v^*(x_1, x_2)$. Since every finite-dimensional subspace is closed, it is obvious that $v^*(x_1, x_2)$ has the representation (47). The Lemma is proved. ■

Now let us pay attention to the proof of convergence of the Kantorovich–Vlasov method.

Definition 1 We will say that an approximate solution to problem (41) and (42) is sought by the Kantorovich–Vlasov method, if the following assumptions hold:

- (1) the solution to problem (41) and (42) is sought in the form

$$u_N(x_1, x_2) = \sum_{j=1}^N \omega_j(x_1)g_j(x_2), \tag{50}$$

where: $\forall j \omega_j(x_1) \in W_2^m(E_1), g_j(x_2) N$ - known elements from the complete system of functions $\{\psi_i(x_2)\}, N$ - fixed number;

- (2) N unknown functions $\omega_j(x_1)$ are estimated from the following system of equations:

$$\sum_{|\alpha| \leq m} \int_{E_1} \left[\int_{E_2} F(x_1, x_2) D_2^{\alpha_2} g_l(x_2) dx_2 \right] D_1^{\alpha_1} \phi(x_1) dx_1 = 0,$$

$$F(x_1, x_2) = A_\alpha \left(x, D^\gamma \sum_{j=1}^N \omega_j(x_1)g_j(x_2) \right) - h(x_1, x_2),$$

$$\forall \phi(x_1) \in W_2^m(E_1), l = 1, \dots, N. \tag{51}$$

The formulated method with the defined assumptions satisfies the following convergence theorem.

Theorem 1 Let T be a completely continuous operator on a certain non-empty open set $\Omega \subset W_2^m(E_1 \times E_2)$, and let Eq. (44) have an isolated solution $u_0 \in \Omega$ with non-zero index, then there is N_0 and δ_0 such that when $N \geq N_0$, the system of equations (51) has in the area $\|u - u_0\|_{W_2^m} \leq \delta_0$ at least one solution u_N and all solutions u_N when $N \rightarrow \infty$ tend to u_0 by the norm.

Indeed, we see that the system of equations (51) is equivalent to the system obtained from projection of Eq. (44) onto the set

$$M_N = \left\{ v(x_1, x_2) \left| v(x_1, x_2) = \sum_{j=1}^N \omega_j(x_1)g_j(x_2) \right. \right\}, \tag{52}$$

where: $g_j(x_2) N$ - fixed elements from the system $\{\psi_i(x_2)\}; \forall j \omega_j(x_1)$ go over the whole space $W_2^m(E_1)$, and N is the given number.

It is enough to observe that we can substitute $v \rightarrow \phi(x_1)g_j(x_2)$ to identity (43), where $\phi(x_1)$ is the arbitrary function from $W_2^m(E_1)$. Obviously, set M_N is linear and, as follows from the Lemma, it is a closed set. Thus, M_N is the subspace of space $W_2^m(E_1 \times E_2)$ (in general, of infinite dimension). Taking into account the latter statement and (51), we see that Eq. (44) is projected onto subspace M_N orthogonally, and therefore projection operator P_N is bounded. Further, from the representation of the elements of the basic system $\{\theta_i(x_1, x_2)\}$ of space $W_2^m(E_1 \times E_2)$ in the form (45) it follows that the sequence of subspaces M_N is extremely dense in $W_2^m(E_1 \times E_2)$, i.e. $\forall y \in W_2^m \|P_N y - y\|_{W_2^m} \rightarrow 0, N \rightarrow \infty$.

The theorem is proved. This proof of the theorem is similar to the proof of the convergence of the Galerkin method for nonlinear problems (see [119]).

Comment The obtained results can be easily extended to fit the nanoplate behavior, and the authors refer to the modified Germain–Lagrange equation.

3.5.1 Analytical Solution of MGLE by KVM in the First Approximation

The approximate solution $w(x, y)$ can be written in the form

$$w(x, y) = X_0(x)Y_1(y). \tag{53}$$

As an initial approximation for boundary conditions (15), we take function $X_0(x) = \sin(\pi x)$ and use notation $Q = \frac{q}{D}$. Substituting (53) into (18), we obtain

$$\frac{1}{\lambda^2}X_0^{(4)}Y_1 + 2X_0^{(2)}Y_1^{(2)} + \lambda^2X_0Y_1^{(4)} = Q, \tag{54}$$

where $X_0^{(4)} = \frac{d^4X_0(x)}{dx^4}$.

Using the Bubnov–Galerkin method, we get

$$\int_0^1 \left(\frac{1}{\lambda^2}X_0^{(4)}Y_1 + 2X_0^{(2)}Y_1^{(2)} + \lambda^2X_0Y_1^{(4)} - Q \right) X_0 dx = 0, \tag{55}$$

and we obtain the following differential equation:

$$A_1Y_1^{(4)} + B_1Y_1^{(2)} + C_1Y_1 = D_1, \tag{56}$$

where

$$A_1 = \lambda^2 \int_0^1 X_0^2 dx = \frac{\lambda^2}{2}, \quad B_1 = \int_0^1 2X_0^{(2)}X_1 dx = -\pi^2,$$

$$C_1 = \frac{1}{\lambda^2} \int_0^1 X_0^{(4)}X_0 dx = \frac{\pi^4}{\lambda^2 2}, \quad D_1 = Q \int_0^1 X_0 dx = Q \frac{2}{\pi}. \tag{57}$$

The following function is used to solve Eq. (56):

Table 3 Numerical solution of Eq. (25) using FDM

Number of partition points n	$w_n(0.5;0.5)$
5	0.296
9	0.310
15	0.313
19	0.313
25	0.313

Table 4 Computation of the nanoplate centre deflection for various methods and difference length scale parameter γ

γ (length parameter of material)	$w(0.5; 0.5)$ Analytical solution	$w_{Ag}(0.5;0.5)$ Numerical solution	$w(0.5; 0.5)$ Exact Navier’s solution
	KVM	KVM + ABSM	
0	0.313	0.308	0.311
0.3	0.228	0.228	0.225
0.5	0.153	0.153	0.151

$$Y_1(y) = c_1e^{S_1y} + c_2ye^{S_1y} + c_3e^{-S_1y} + c_4ye^{-S_1y} + k_1, \tag{58}$$

where

$$c_1 = -\frac{k_1(2e^{-2S_1} - 2e^{-S_1} - 2 - 2S_1 + 2e^{S_1} + S_1e^{-S_1} + S_1e^{S_1})}{2e^{-2S_1} + 2e^{2S_1} - 4},$$

$$c_2 = \frac{k_1(S_1 - S_1e^{-S_1})}{2e^{S_1} - 2e^{-S_1}}, \quad S_1 = \lambda\pi,$$

$$c_3 = -\frac{k_1(2 - 2S_1e^{-S_1} - 2e^{2S_1} - 2S_1 + 2e^{S_1} + S_1e^{-S_1} + S_1e^{S_1})}{2e^{-2S_1} + 2e^{2S_1} - 4},$$

$$c_4 = \frac{k_1(S_1 - S_1e^{S_1})}{2e^{S_1} - 2e^{-S_1}}, \quad k_1 = \frac{D_1}{C_1}. \tag{59}$$

Finally, substituting (58) into (53) we find the desired solution

$$w(x, y) = \sin(\pi x)[c_1e^{S_1y} + c_2ye^{S_1y} + c_3e^{-S_1y} + c_4ye^{-S_1y} + k_1]. \tag{60}$$

Equation (56) can be solved numerically, for example, by the finite difference method (FDM) with approximation $O(h^2)$. A very important point plays a number of partitions of variable y . Results of the study on convergence of the solution depending on the number of partitions n , are reported in Table 3 for $\gamma = 0$.

On the other hand, Table 4 shows the results achieved using the Kantorovich–Vlasov method (KVM) in the first approximation, obtained analytically (54) and numerically (w_{Ag}) applying the Agranovskii–Baglai–Smirnov method (ABSM). The analytical solution of MGLE by the Kantorovich–Vlasov method in the first approximation taking into account the Agranovskii–Baglai–Smirnov approach is difficult to find. The numerical solution obtained by matching KVM + ABSM is presented in Table 4.

Let us consider the solution of Eq. (18) found via the Kantorovich–Vlasov method in the first approximation for the clamped boundary conditions (16).

As an initial approximation, we take function $X_1(x) = \sin^2(\pi x)$. Similarly, for the boundary condition (15) we obtain the following solution to the original equation:

Table 5 Nanoplate center deflection obtained using the FDM for different number of partitions n ($\gamma = 0$)

Number of points n	$w_n(0.5; 0.5)$
5	0.337
9	0.308
15	0.298
19	0.296
25	0.294
35	0.293
55	0.292
75	0.292

Table 6 Nanoplate center deflection using the analytical (KVM) and numerical (KVM+ABSM) solutions for different γ

γ (parameter length of material)	$w(0.5; 0.5)$	
	Analytical solution KVM	Numerical solution KVM+ABSM
0	0.292	0.290
0.3	0.212	0.210
0.5	0.142	0.142

$$w(x, y) = \sin^2(\pi x) [e^{N_1 y} (c_1 \cos E_1 y + c_2 \sin E_1 y) + e^{-N_1 y} (c_1 \cos E_1 y + c_2 \sin E_1 y) + k_1], \tag{61}$$

where

$$K_{1,2} = N_1 \pm E_1 i, \quad K_{3,4} = -N_1 \pm E_1 i,$$

$$N_1 = \sqrt{|t|} \cos\left(\frac{\varphi + 4\pi}{2}\right),$$

$$E_1 = \sqrt{|t|} \sin\left(\frac{\varphi + 4\pi}{2}\right), \quad t_1 = \frac{-B_1}{2A_1} + \frac{\sqrt{-(B_1^2 - 4A_1 C_1)}}{2A_1} i,$$

$$\varphi_1 = \arctan\left(\left(\frac{\sqrt{-(B_1^2 - 4A_1 C_1)}}{2A_1}\right) / \left(\frac{-B_1}{2A_1}\right)\right), \quad k_1 = \frac{D_1}{C_1}.$$

Coefficients c_1, c_2, c_3, c_4 are determined from the boundary conditions as the solution of the following system of algebraic equations

$$\begin{aligned} c_1 + c_3 + k_1 &= 0, \\ k_1 + e^{-N_1} (c_3 \cos(E_1) + c_4 \sin(E_1)) + e^{N_1} (c_1 \cos(E_1) + c_2 \sin(E_1)) &= 0, \\ E_1 c_2 + E_{14} + N_1 c_1 - N_1 c_3 &= 0, \\ e^{N_1} (c_2 E_1 \cos(E_1) - c_1 E_1 \sin(E_1)) + e^{N_1} (c_4 E_1 \cos(E_1) - c_3 E_1 \sin(E_1)) &= 0. \end{aligned} \tag{62}$$

The equations are solved using the finite difference method. Table 5 shows the deflection values depending on the number of partitions for $\gamma = 0$. The results reported in Table 5 show that $n = 55$ partitions are required to achieve convergence of the results. However, for boundary condition (15) $n = 15$ partitions are sufficient. The increase in the number of partitions is directly related to the edge effect caused by boundary condition (16).

Table 6 gives analytical results obtained by the KVM method in the first approximation and numerical results for (w_{Ag}) taking into account the ABSM method for boundary condition (16).

3.5.2 Analytical Solution of MGLE by KVM in the Second Approximation

Now, we consider a similar approach but for the second approximation of the KVM. Deflection function is represented by $w(x, y) = \sin(\pi x)Y_1(y) + \sin(3\pi x)Y_2(y)$. After applying the Bubnov–Galerkin method, the following system of differential equations is obtained

$$\begin{aligned} \int_0^1 \left(\frac{1}{\lambda^2} X_1^{(4)} Y_1 + 2X_1^{(2)} Y_1^{(2)} + \lambda^2 X_1^{(2)} Y_1^{(4)} - Q \right) X_1 dx &= 0, \\ \int_0^1 \left(\frac{1}{\lambda^2} X_2^{(4)} Y_2 + 2X_2^{(2)} Y_2^{(2)} + \lambda^2 X_2^{(2)} Y_2^{(4)} - Q \right) X_2 dx &= 0, \end{aligned} \tag{63}$$

or equivalently

$$\begin{aligned} A_1 Y_1^{(4)} + B_1 Y_1^{(2)} + C_1 Y_1 &= D_1, \\ A_2 Y_2^{(4)} + B_2 Y_2^{(2)} + C_2 Y_2 &= D_2, \end{aligned} \tag{64}$$

where coefficients $A_1, C_1, B_1, D_1, A_2, C_2, B_2, D_2$ are defined in the following way:

$$A_1 = \lambda^2 \int_0^1 (X_1 + X_2)X_1 dx = \frac{\lambda^2}{2},$$

$$B_1 = \int_0^1 2(X_1^{(2)} + X_2^{(2)})X_1 dx = -\pi^2,$$

$$C_1 = \frac{1}{\lambda^2} \int_0^1 (X_1^{(4)} + X_2^{(4)})X_1 dx = \frac{\pi^4}{\lambda^2 2},$$

$$D_1 = Q \int_0^1 X_1 dx = Q \frac{2}{\pi}.$$

$$A_2 = \lambda^2 \int_0^1 (X_1 + X_2)X_2 dx = \frac{\lambda^2}{2},$$

$$B_2 = \int_0^1 2(X_1^{(2)} + X_2^{(2)})X_2 dx = -9\pi^2,$$

$$C_2 = \frac{1}{\lambda^2} \int_0^1 (X_1^{(4)} + X_2^{(4)})X_2 dx = \frac{\pi^4}{\lambda^2 2},$$

$$D_2 = Q \int_0^1 X_2 dx = Q \frac{2}{\pi}.$$

Solving the system of equations (64) we get

$$w(x, y) = \sin(\pi x) [c_1 e^{S_1 y} + c_2 y e^{S_1 y} + c_3 e^{-S_1 y} + c_4 y e^{-S_1 y} + k_1] + \sin(3\pi x) [c_5 e^{S_2 y} + c_6 y e^{S_2 y} + c_7 e^{-S_2 y} + c_8 y e^{-S_2 y} + k_2], \tag{66}$$

where

$$\begin{aligned} S_1 &= \lambda\pi, \quad S_2 = 3\lambda\pi, \quad k_1 = \frac{D_1}{C_1}, \quad k_2 = \frac{D_2}{C_2}, \\ c_1 &= -\frac{k_1(2e^{-2S_1} - 2e^{-S_1} - 2 - 2S_1 + 2e^{S_1} + S_1 e^{-S_1} + S_1 e^{S_1})}{2e^{-2S_1} + 2e^{2S_1} - 4}, \\ c_2 &= \frac{k_1(S_1 - S_1 e^{-S_1})}{2e^{S_1} - 2e^{-S_1}}, \\ c_3 &= -\frac{k_1(2 - 2S_1 e^{-S_1} - 2e^{2S_1} - 2S_1 + 2e^{S_1} + S_1 e^{-S_1} + S_1 e^{S_1})}{2e^{-2S_1} + 2e^{2S_1} - 4}, \\ c_4 &= \frac{k_1(S_1 - S_1 e^{S_1})}{2e^{S_1} - 2e^{-S_1}}, \\ c_5 &= -\frac{k_2(2e^{-2S_2} - 2e^{-S_2} - 2 - 2S_2 + 2e^{S_2} + S_2 e^{-S_2} + S_2 e^{S_2})}{2e^{-2S_2} + 2e^{2S_2} - 4}, \\ c_6 &= \frac{k_2(S_2 - S_2 e^{-S_2})}{2e^{S_2} - 2e^{-S_2}}, \\ c_7 &= -\frac{k_2(2 - 2S_2 e^{-S_2} - 2e^{2S_2} - 2S_2 + 2e^{S_2} + S_2 e^{-S_2} + S_2 e^{S_2})}{2e^{-2S_2} + 2e^{2S_2} - 4}, \\ c_8 &= \frac{k_2(S_2 - S_2 e^{S_2})}{2e^{S_2} - 2e^{-S_2}}. \end{aligned} \tag{65}$$

Table 7 shows the results obtained using the Kantorovich–Vlasov method in the second approximation w_{2m} for boundary conditions (15), while w_{1m} is used in the first approximation.

Analysis of the results given in Table 7 leads to the conclusion that the analytical solution of the first approximation according to the Kantorovich–Vlasov method gives results close to those obtained by Navier, the difference is 0.9% for $\gamma = 0$, and 1.3% for $\gamma = 0.3$ and $\gamma = 0.5$. Obtaining an analytical solution using the Agranovskii–Baglai–Smirnov method matched with the Kantorovich–Vlasov method is problematic due to the large amount of calculations. Comparison with the results obtained by the above combinations of methods in comparison with the Navier method gives

Table 7 Comparison of the nanoplate centre deflection using the analytical solutions of the KVM (the first and second approximation) and the numerical solution of the KVM+ABSM versus the Navier solution for different γ

γ (parameter length of material)	$w_1(0.5;0.5)$ Analytical solution of KVM in the first approximation	$w(0.5;0.5)$ Numerical solution in the first approximation KVM + ABSM	$w_2(0.5;0.5)$ Analytical solution of KVM in the second approximation	$w(0.5; 0.5)$ Exact Navier's solution
0	0.314	0.308	0.310	0.311
0.3	0.228	0.223	0.227	0.225
0.5	0.153	0.150	0.152	0.151

the same result, and the comparison of results obtained by the Kantorovich–Vlasov method in the second approximation with the Navier results yields the following errors: 0.3% for $\gamma = 0$, and 0.7% for $\gamma = 0.3$ and $\gamma = 0.5$. Though the Kantorovich–Vlasov method in the second approximation slightly improves the solution, it does not coincide with the exact one. Apparently, the reason for this lies in the idea of separating variables.

3.6 Variational Iteration Method (VIM)

This method presents a modification of the Kantorovich–Vlasov method (KVM). Owing to VIM, an n term trial solution for a two-dimensional problem is taken to have the following form

$$w_N(x, y) = \sum_{j=1}^N X_j^{(0)}(x)Y_j(y). \tag{67}$$

Here, $X_j^{(0)}(x)$ is a priori specified, and in general may not satisfy the given boundary conditions $Y_j^{(1)}(y)$. We employ the Bubnov–Galerkin method with regard to coordinate x and we get a set of n ordinary differential equations (ODEs) with regard to coordinate y . Its solution with an account of the boundary conditions yields the system of functions where $Y_j^{(1)}(y)$ is substituted into (67):

$$w_N^{(1)}(x, y) = \sum_{j=1}^N X_j^{(0)}(x)Y_j^{(1)}(y). \tag{68}$$

Therefore, the solutions by the Kantorovich–Vlasov method coincides exactly with the first iteration in the VIM. Further, we use $Y_j^{(1)}(y)$ obtained through KVM as the test functions $X_j(x)$ are unknown to be re-determined after applying the Bubnov–Galerkin method and with regard to y we obtain a set of n ordinary differential equations (ODEs) with the respect to x . They yield a set of functions $X_j^{(1)}$, which allows us to present the solution in the following form

$$w_N(x, y) = \sum_{j=1}^N X_j^{(1)}(x)Y_j^{(2)}(y). \tag{69}$$

The so far described procedure can be repeated m -times until the result converges to a desired degree:

$$w_N^{(m)}(x, y) = \sum_{j=1}^N X_j^{(m-1)}(x)Y_j^{(m)}(y), \quad m = 1, 2 \dots n. \tag{70}$$

This method eventually removes all assumptions on both sets of $X_j(x)$ and $Y_j(y)$, but the accuracy of the obtained approximate solution depends on the number of trial solution terms.

3.7 Proof of the Convergence of VIM While Solving the Operator Equation

Let us present a proof of the convergence of the variational iteration method (VIM) for the operator equation reported in [38].

The scheme of VIM can be formally described in the following way. It is necessary to find a solution to the equation

$$L[w(x, y)] = q(x, y), \quad x, y \in \Omega(x, y), \tag{71}$$

where $L[w]$ is the operator defined on the manifold $D(L)$ of the Hilbert space $L_2(\Omega)$; $q(x, y)$ stands for a given function of two variables x and y ; $w(x, y)$ is the searched function of the same variables; and $\Omega(x, y)$ is the space associated with variations of x and y .

If $\Omega(x, y) = T \times P$ (T - a certain bounded set of variables x , P bounded set of y), then the solution of Eq. (71) takes the form

$$w_N(x, y) = \sum_{i=1}^N X_i(x)Y_i(y), \tag{72}$$

where functions $X_i(x)$ and $Y_i(y)$ are defined by the system of equations

$$\begin{aligned} \int_T (L[w_N] - q)X_1(x)dx &= 0, & \int_P (L[w_N] - q)Y_1(y)dy &= 0, \\ \vdots & & \vdots & \\ \int_T (L[w_N] - q)X_N(x)dx &= 0, & \int_P (L[w_N] - q)Y_N(y)dy &= 0, \end{aligned} \tag{73}$$

in the following way. A certain system composed of N functions with respect to one of the variables, for instance, $X_1^0(x), X_2^0(x), \dots, X_N^0(x)$ is given. Then, the first N equations of system (73) is determined by the system of N functions $Y_1^1(x), Y_2^1(x), \dots, Y_N^1(x)$. Next, the obtained functions are employed to create a new set of functions x , i.e. $X_1^2(x), X_2^2(x), \dots, X_N^2(x)$, which is further used to construct a set of new functions with respect to variable y , i.e. $Y_1^3(x), Y_2^3(x), \dots, Y_N^3(x)$, and so on.

Definition 2 The process of calculating a given system of functions by another one will be called the VIM step. The number of steps taken to determine any set of functions corresponds to the superscript (number) of functions from the set under consideration.

Breaking the process of finding functions $X_i(x)$ and $Y_i(y)$ on the k -th step (which, for example, corresponds to getting a set of functions $(Y_1^k(y), Y_2^k(y), \dots, Y_N^k(y))$, allows us to construct function $w_N^k = \sum_{i=1}^N X_i^{k-1}(x)Y_i^k(y)$, which is taken as an approximate solution to Eq. (71) obtained by the VIM.

Remark 1 Here and further on, by operator $L[w]$ we mean a certain differential operator, which is defined on set $D(L)$ of Hilbert space $L_2(\Omega)$. Thus, at each step, system (73) will disintegrate into the corresponding system of ordinary differential equations, with resolved the solvability problem.

Remark 2 Function $w_N(x, y)$ is called the N -th approximation of the solution for Eq. (71), if the number of terms of the series in representation (70) is equal to N .

Consider the case of the first approximation, i.e. the solution of Eq. (73) will be sought in the form $w_1(x, y) = X(x)Y(y)$, where functions $X(x)$ and $Y(y)$ are determined in the above described way from the system of equations

$$\int_x (L[X(x) \cdot Y(y)] - q)X(x)dx = 0, \tag{74}$$

$$\int_y (L[X(x) \cdot Y(y)] - q)X(y)dy = 0.$$

Let operator L in (71) be positive definite. Let us introduce the notation: $H_L(T \times P)$ is the energy space of operator L [123], $[\bullet, \bullet]$ is the scalar product of elements in H_L ; and w_0 is the exact solution to Eq. (71).

Theorem 2 *If L is a positive definite operator with its action space $D(L) \subset H_L$, then the sequence of elements of $\alpha_k = w_1^k(x, y) - w_{0H_L}$ is monotonically decreasing, that is, for arbitrary i and j and if $i \geq j$, the following inequality holds*

$$\|w_1^i - w_0\|_{H_T} \leq \|w_1^j - w_0\|_{H_T}. \tag{75}$$

Proof of Theorem 2 Consider subset M_1^1 of space H_L , which takes the form

$$M_1^1 = \{w(x, y) | w(x, y) = X^0(x)Y(y), X^0(x) \in H_L(T), Y(y) \in H_L(P)\}.$$

Obviously, set M_1^1 is the subspace of space $H_L(T \times P)$ (in the general case the problem has infinite dimension). Accordingly, it is possible to define the projection of element w_0 onto space M_1^1 . It is known that to satisfy the condition for element $X^0(x)Y^*(y) \in M_1^1$ as the projection of w_0 onto M_1^1 it is required that

$$[X^0(x)Y^*(y) - w_0, X^0(x)Y(y)]_{H_L} = 0, \tag{76}$$

for arbitrary elements $X_0(x)Y(y) \in M_1^1$. Hence, if $X^0(x)Y^*(y) \in M_1^1$, then Eq. (76) coincides with the first equation of system (64).

Consequently, element $X^0(x)Y^1(y)$, which is a result of the first step of the variational iteration method, stands as the projection of element w_0 onto subspace M_1^1 . Therefore, the inequality

$$w_1^1(x, y) - w_{0H_L} \leq X^0(x)Y(y) - w_{0H_L} \tag{77}$$

for elements $X^0(x)Y(y) \in M_1^1$, holds. By a similar construction, we obtain the inequality for subspace $M_2^1 = \{w(x, y) | w(x, y) = X^0(x)Y^1(y); X(x) \in H_L(T), Y^1(y) \in H_L(P)\}$ which corresponds to the second step of the VIM, i.e.

$$X^2(x)Y^1(y) - w_{0H_L} \leq X(x)Y^1(y) - w_{0H_L} \tag{78}$$

for arbitrary elements $X(x)Y^1(y) \in M_2^1$. From (77) and (79) it follows that $\|X^2(x)Y^1(y) - w_0\|_{H_T} \leq \|X^0(x)Y^1(y) - w_0\|_{H_L}$. Arguments similar to those given above are valid for the functions obtained at the k -step of the VIM. Finally, the mathematical induction method proves the theorem and at the same time inequality (75). \square

Proof of the Theorem (the case of higher approximations) The result of Theorem 1 is extended to the case of higher approximations. Namely, inequality (75) takes the following form

$$w_N^m(x, y) - w_{0H_L} \leq w_N^n(x, y) - w_{0H_L}, m \geq n.$$

Further investigation requires proof of the following lemma. \square

Lemma 2 *Let each element of the basis system of the space H_L satisfy the following conditions*

$$\theta_i(x, y) = \phi_i(x)\psi_i(y), \quad \forall^i \phi_i(x) \in H_L(T), \quad \psi_i(y) \in H_L(P).$$

If the components of some basis function θ_i serve as the initial approximation of VIM, i.e. $X^0(x) \equiv \phi_i(x)$, then for any

number k of steps taken by VIM, the following inequality holds $w_1^k(x, y) - w_{0H_L} \leq c\phi_i(x)\psi_i(y) - w_{0H_L}$, where c is the arbitrary real number.

Proof of the Lemma Since $X^0(x)Y^1(y) \equiv \phi_i(x)Y^1(y)$, then by Theorem 2 we have

$$w_1^k(x, y) - w_{0H_L} \leq \phi_i(x)Y^1(y) - w_{0H_L} \leq c\phi_i(x)\psi_i(y) - w_{0H_L}.$$

On the basis of the above Lemma, one criterion for VIM convergence is formulated. First, we identify the space H_L with the space $W_2^m(\Omega)$, which is obtained by enclosing the norm

$$\|w\|_{W_2^m} = \left\{ \int_{\Omega} \sum_{k=0}^m \sum_{(k)} \left| \frac{\partial^k w}{\partial^{k_1} x \partial^{k_2} y} \right|^2 dx dy \right\}^{1/2}$$

of the set of infinitely differentiable functions $C^{\infty}(\Omega)$ with compact carrier in Ω . □

Theorem 3 *Let each element of the basis system of the space $W_2^m(T \times P)$ take the following form*

$$\theta_i(x, y) = \phi_i(x)\psi_i(y), \tag{79}$$

where $\{\phi_i(x)\}$ is the basis system in space $W_2^m(T)$, similarly $\{\psi_i(y)\}$ in space $W_2^m(P)$ and to obtain an arbitrary N -th approximation by VIM, the base elements of the system are taken $\{\theta_i(x, y)\}$ as the first approximation. Then, for a large value of N , VIM gives a unique approximate solution w_N and the sequence $\{w_N\}$ converges by the norm of the space $W_2^m(T \times P)$ to the exact solution w_0 regardless of the number of steps k , which can be performed for every N -th approximation, i.e.

$$\|w_N^k - w_0\|_{W_2^m} \rightarrow 0, \quad N \rightarrow \infty.$$

Proof of Theorem 3 If the theorem is proved for the approximations obtained in the first step of VIM, then based on the results of the Lemma it will be valid for any chosen k -th step. Therefore, we consider the N -th approximation of the solution to problem (71), which was obtained in the first step. Similarly to the statement used in Theorem 2, it can be shown that each w_N^1 is the projection of the element w_0 into the subspace

$$M_N^1 = \left\{ w(x, y) | w(x, y) = \sum_{i=1}^N X^0(x)Y^1(y) \right\},$$

where $X_i^0(x)$ consists of N fixed elements from the system $\{\phi_i(x)\}$ for arbitrary i , $Y_i(y)$ takes all values from space $W_2^m(P)$. Thus $w_N^1 = \Pi_N w_0$, where Π_N is the operator of an orthogonal projection on the subspace M_N^1 , which is bounded. Since the elements of the basis system $\theta_i(x, y)$ have the form (73), it is extremely dense in $W_2^m(T \times P)$ [123]. The proof can be completed similarly to the proof of Theorem 16.2 [123]. □

Remark 3 The results of Theorem 1 and the Lemma show that using VIM, one can obtain an approximate solution to

Eq. (71) not worse than using the Ritz method in the corresponding subspace.

VIM can be extended to cover more variables. For example, if the desired solution to Eq. (71) is a function of three variables x, y, z , then an approximate solution by VIM will be searched for in the form $Q(x, y, z) = X(x)Y(y)Z(z)$, and the VIM procedure can be applied to each variable.

Remark 4 When implementing VIM, it is not necessary to construct an initial approximation that satisfies the boundary conditions of the problem. Suppose that operator $L[w]$ defines a boundary problem. As an initial approximation we take any function from the domain of definition of the differential operator of this problem. Then, in the first step, we obtain a system of functions satisfying the boundary conditions in one of the variables, and in the second step - in both variables.

Remark 5 These results can be extended to the equations of the nanoplate, which the authors called the modified Germain–Lagrange equation.

3.8 Variational Iteration Method for Solving MGLE

Our goal is to apply the variational iteration method to solve the modified Germain–Lagrange equation with boundary conditions (15).

The Kantorovich–Vlasov method in the first approximation yielded the solution $Y_1(y) = c_1 e^{s_1 y} + c_2 y e^{s_1 y} + c_3 e^{-s_1 y} + c_4 y e^{-s_1 y} + k_1$ to Eq. (56).

As mentioned above, further the approximate solution can serve as an initial approximation to obtain the solution with respect to x . The obtained $Y_1(y)$ as the specified functions and $X_2(x)$ as unknown are to be re-determined after applying the Bubnov–Galerkin method. For this, we follow a similar procedure and, as a result, we find the function

Table 8 Estimation of deflection of the nanoplate centre using VIM and ABSM versus the exact solutions for different γ

γ (parameter length of material)	$w(0.5; 0.5)$ VIM	$w_{Ag}(0.5; 0.5)$ ABSM	$w(0.5; 0.5)$ Exact solution
0	0.307	0.308	0.311
0.3	0.223	0.224	0.225
0.5	0.150	0.150	0.151

$$X_2(x) = c_5 e^{S_2 x} + c_6 x e^{S_2 x} + c_7 e^{-S_2 x} + c_8 x e^{-S_2 x} + k_2. \tag{80}$$

Then, using the variational iteration procedure, we find functions X_i, Y_i that allow us to obtain a solution with a given accuracy ϵ . The solution takes the following form:

$$w_i(x, y) = [c_{4i+1} e^{S_i x} + c_{4i+2} x e^{S_i x} + c_{4i+3} e^{-S_i x} + c_{4i+4} x e^{-S_i x} + k_i] \times [c_{4i+1} e^{S_i y} + c_{4i+2} x e^{S_i y} + c_{4i+3} e^{-S_i y} + c_{4i+4} x e^{-S_i y} + k_i], \tag{81}$$

where

$$S_i = \sqrt{\frac{B_i}{2C_i}}, \quad k_i = \frac{D_i}{B_i},$$

$$c_{4i+1} = -\frac{k_i(2e^{-2S_i} - 2e^{-S_i} - 2 - 2S_i + 2e^{S_i} + S_2 e^{-S_i} + S_i e^{S_i})}{2e^{-2S_2} + 2e^{2S_2} - 4},$$

$$c_{4i+2} = \frac{k_i(S_2 - S_2 e^{-S_2})}{2e^{S_2} - 2e^{-S_2}},$$

$$c_{4i+3} = -\frac{k_i(2 - 2S_2 e^{-S_2} - 2e^{2S_2} - 2S_2 + 2e^{S_2} + S_2 e^{-S_2} + S_2 e^{S_2})}{2e^{-2S_2} + 2e^{2S_2} - 4},$$

$$c_{4i+4} = \frac{k_i(S_2 - S_2 e^{S_2})}{2e^{S_2} - 2e^{-S_2}}. \tag{82}$$

Next, the Agranovskii–Baglai–Smirnov procedure is used in combination with the variational iteration method in the first term approximation. It should be mentioned that the analytical solution of the problem is not an easy task. This is why in this case, we use finite difference method of the second-order accuracy to solve the obtained ODEs. The convergence of the solution is investigated as a function of the number of partition points along the appropriate coordinate. It was found that the optimal partitioning in spatial coordinates for a hinged support is 15 points, and for clamping it is 55 points. The same partition is equal for all values of the dimensionless parameter length of material γ .

Table 8 shows results of the analytical procedure using the variational iteration method with respect to the first term approximation for boundary conditions (15), and numerical results for w_{Ag} using the Agranovskii–Baglai–Smirnov method versus the exact solution.

Table 9 Analytical versus numerical (MABS) solutions for different γ

γ (parameter length of material)	w Analytical solution	w_{Ag} Numerical solution (ABSM)
0	0.292	0.292
0.3	0.212	0.212
0.5	0.142	0.142

The solution for boundary conditions (16) will be the same as for boundary conditions (15). Following the above reasoning, we obtain function Y_i, X_i in the following form:

$$w_i(x, y) = [e^{N_i x} (c_{4i+1} \cos(E_i x) + c_{4i+2} \sin(E_i x)) + e^{-N_i x} (c_{4i+1} \cos(E_i x) + c_{4i+2} \sin(E_i x)) + k_i] \times [e^{N_i y} (c_{4i+1} \cos(E_i y) + c_{4i+2} \sin(E_i y)) + e^{-N_i y} (c_{4i+1} \cos(E_i y) + c_{4i+2} \sin(E_i y)) + k_i], \tag{83}$$

where

$$c_{4i+1} + c_{4i+3} + k_1 = 0,$$

$$k_1 + e^{-N_1} (c_{4i+3} \cos(E_1) + c_{4i+4} \sin(E_1)) + e^{N_1} (c_{4i+1} \cos(E_1) + c_{4i+2} \sin(E_1)) = 0, \tag{84}$$

$$E_1 c_{4i+2} + E_1 c_{4i+4} + N_1 c_{4i+1} - N_1 c_{4i+3} = 0,$$

$$e^{N_1} (c_{4i+2} E_1 \cos(E_1) - c_{4i+1} E_1 \sin(E_1)) + e^{-N_1} (c_{4i+4} E_1 \cos(E_1) - c_{4i+3} E_1 \sin(E_1)) = 0.$$

Table 9 shows the values of displacement w obtained analytically and displacement w_{Ag} obtained using the Agranovskii–Baglai–Smirnov method, for boundary conditions (16).

A few comments on the application of the variational iteration method in the analytical and numerical solution using the Agranovskii–Baglai–Smirnov method in the first approximation are given below. The numerical results obtained by the Kantorovich–Vlasov method in the first and second term approximations, by the Kantorovich–Vlasov method using the variational iteration method and by the Agranovskii–Baglai–Smirnov method give practically the same results. The use of the variational iteration method offers the following advantage: the system of approximating functions does not need to be known, while in the Bubnov–Galerkin method two coordinates should be defined, whereas in the Kantorovich–Vlasov method it is required to know one coordinate. The system of approximating functions is obtained from the solutions of ODEs. The variational iteration procedure converges even when the set of approximating functions does not satisfy boundary conditions of the problem.

3.9 Vaindiner’s Method

Vaindiner’s method (VaM) can be considered as an extension of the Kantorovich–Vlasov method. The proof of the convergence of this method is given in [103] for a linear inhomogeneous second order equation with variable coefficients.

Consider the application of this method to the operator equation (19). This equation can be both linear and non-linear and up to the eighth order. In our case, operator equation (19) is an inhomogeneous fourth-order equation, i.e.

the modified Germain–Lagrange equation. An n term trial solution is assumed in the following form

$$w_{ij}(x, y) = \sum_{j=1}^N X_{1j}(x)Y_{1j}(y) + X_{2j}(x)Y_{2j}(y), \quad i = 1, 2, \quad (85)$$

where weight functions $Y_{1j}(y)$ and $X_{2j}(x)$, and functions $X_{1j}(x)$ and $Y_{2j}(y)$ are unknown. They are determined using the Bubnov–Galerkin procedure, i.e. we have

$$\begin{aligned} (L(w_{ij}) - q, Y_{1k}(y)) &= 0, \quad k = 1, 2, \dots, N, \quad i = 1, 2, \\ (L(w_{ij}) - q, X_{2k}(x)) &= 0, \quad k = 1, 2, \dots, N, \quad i = 1, 2, \end{aligned} \quad (86)$$

In other words, a system of $2N$ ordinary differential equations is obtained. This allows us to find the $\{X_{1j}(x)\}_N$ and $\{Y_{2j}(y)\}_N$ functions, and then substituting them in (85) we get a trial solution of the problem.

3.9.1 Analytical Solution of MGLE by Vaindiner’s Method

To solve (18) with boundary conditions (15) using VaM, a trial solution takes the form $w(x, y) = \sin(\pi x)Y_{11}(y) + X_{12}(x)\sin(\pi y)$. Applying the Bubnov–Galerkin procedure, the following system of differential equations is obtained

$$\begin{aligned} A_{11}Y_{11}^{(4)} + B_{11}Y_{11}^{(2)} + C_{11}Y_{11} &= D_{11}, \\ A_{12}X_{12}^{(4)} + B_{12}X_{12}^{(2)} + C_{12}X_{12} &= D_{12}, \end{aligned} \quad (87)$$

where coefficients $A_{11}, B_{11}, C_{11}, D_{11}, A_{12}, B_{12}, C_{12}, D_{12}$ are as follows

$$\begin{aligned} A_{11} &= \lambda^2 \int_0^1 X_1^2 dx = \frac{1}{2}, \\ B_{11} &= \int_0^1 2X_1^{(2)}X_1 dx = -\lambda^2 \pi^2, \\ C_{11} &= \frac{1}{\lambda^2} \int_0^1 X_1^{(4)}X_1 dx = \frac{\lambda^4 \pi^4}{2}, \\ D_{11} &= Q_1 \int_0^1 X_1 dx = Q_1 \frac{2}{\pi}, \\ A_{12} &= \frac{1}{\lambda^2} \int_0^1 Y_1^2 dy = \frac{\lambda^4}{2}, \\ B_{12} &= \int_0^1 2Y_1^{(2)}Y_1 dy = -\lambda^2 \pi^2, \\ C_{12} &= \lambda^2 \int_0^1 Y_1^{(4)}Y_1 dy = \frac{\pi^4}{2}, \\ D_{12} &= Q_2 \int_0^1 Y_1 dy = Q_2 \frac{2}{\pi}. \end{aligned} \quad (88)$$

Solving the system of equations (87), we get

$$\begin{aligned} Y_{11}(y) &= c_{11}e^{S_1 y} + c_{21}ye^{S_1 y} + c_{31}e^{-S_1 y} + c_{41}ye^{-S_1 y} + k_{11}, \\ X_{12}(x) &= c_{12}e^{S_1 x} + c_{22}xe^{S_1 x} + c_{32}e^{-S_1 x} + c_{42}xe^{-S_1 x} + k_{12}, \end{aligned} \quad (89)$$

where

$$\begin{aligned} c_{11} &= -\frac{2k_{11}e^{-2S_1} - 2k_{11}e^{-S_1} - 2k_{11} - 2S_1k_{11} + 2k_{11}e^{S_1} + S_1k_{11}e^{-S_1} + S_1k_{11}e^{S_1}}{2e^{-2S_1} + 2e^{2S_1} - 4}, \\ c_{21} &= \frac{S_1k_{11} - S_1k_{11}e^{-S_1}}{2e^{S_1} - 2e^{-S_1}}, \quad c_{41} = \frac{S_1k_{11} - S_1k_{11}e^{S_1}}{2e^{S_1} - 2e^{-S_1}}, \quad c_{22} = \frac{S_1k_{12} - S_1k_{12}e^{-S_1}}{2e^{S_1} - 2e^{-S_1}}, \\ c_{31} &= -\frac{2k_{11} - 2S_1k_{11}e^{-S_1} - 2k_{11}e^{2S_1} - 2S_1k_{11} + 2k_{11}e^{S_1} + S_1k_{11}e^{-S_1} + S_1k_{11}e^{S_1}}{2e^{-2S_1} + 2e^{2S_1} - 4}, \\ c_{12} &= -\frac{2k_{12}e^{-2S_1} - 2k_{12}e^{-S_1} - 2k_{12} - 2S_1k_{12} + 2k_{12}e^{S_1} + S_1k_{12}e^{-S_1} + S_1k_{12}e^{S_1}}{2e^{-2S_1} + 2e^{2S_1} - 4}, \\ c_{32} &= -\frac{2k_{12} - 2S_1k_{12}e^{-S_1} - 2k_{12}e^{2S_1} - 2S_1k_{12} + 2k_{12}e^{S_1} + S_1k_{12}e^{-S_1} + S_1k_{12}e^{S_1}}{2e^{-2S_1} + 2e^{2S_1} - 4}, \\ c_{42} &= \frac{S_1k_{12} - S_1k_{12}e^{S_1}}{2e^{S_1} - 2e^{-S_1}}, \quad k_{11} = \frac{D_{11}}{C_{11}}, \quad k_{12} = \frac{D_{12}}{C_{12}}, \quad S_1 = \lambda\pi. \end{aligned} \quad (90)$$

By substituting functions $Y_{11}(y)$ and $X_{12}(x)$ into the expression for displacement $w(x, y)$, we obtain the desired solution. This iterative procedure can be repeated.

If function $Y_{11}(y)$, $X_{12}(x)$ is known, we can find function $X_{12}(x)$, $Y_{22}(y)$. Then

$$A_{21} = \frac{1}{\lambda^2} \int_0^1 X_1^2 dx, \quad B_{21} = \int_0^1 2X_1^{(2)} X_1 dx,$$

$$C_{21} = \lambda^2 \int_0^1 X_1^{(4)} X_1 dx, \quad D_{21} = Q_1 \int_0^1 X_1 dx,$$

$$A_{22} = \lambda^2 \int_0^1 Y_1^2 dy, \quad B_{22} = \int_0^1 2Y_1^{(2)} Y_1 dy,$$

$$C_{22} = \frac{1}{\lambda^2} \int_0^1 Y_1^{(4)} Y_1 dy, \quad D_{22} = Q_2 \int_0^1 Y_1 dy,$$

$$S_{21} = \sqrt{\frac{B_{21}}{2C_{21}}}, \quad S_{22} = \sqrt{\frac{B_{22}}{2C_{22}}}, \quad k_{21} = \frac{D_{21}}{B_{21}}, \quad k_{22} = \frac{D_{22}}{B_{22}},$$

$$c_{51} = -\frac{2k_{21}e^{-2S_{21}} - 2k_{21}e^{-S_{21}} - 2k_{21} - 2S_{21}k_{21} + 2k_{21}e^{S_{21}} + S_{21}k_{21}e^{-S_{21}} + S_{21}k_{21}e^{S_{21}}}{2e^{-2S_{21}} + 2e^{2S_{21}} - 4}, \tag{91}$$

$$c_{61} = \frac{S_{21}k_{21} - S_1k_{21}e^{-S_{21}}}{2e^{S_{21}} - 2e^{-S_{21}}}, \quad c_{81} = \frac{S_{22}k_{22} - S_{22}k_{22}e^{S_{22}}}{2e^{S_{22}} - 2e^{-S_{22}}},$$

$$c_{62} = \frac{S_{22}k_{22} - S_{22}k_{22}e^{-S_{22}}}{2e^{S_{22}} - 2e^{-S_{22}}},$$

$$c_{71} = -\frac{2k_{21} - 2S_{21}k_{21}e^{-S_1} - 2k_{21}e^{2S_{21}} - 2S_{21}k_{21} + 2k_{21}e^{S_{21}} + S_{21}k_{21}e^{-S_{21}} + S_{21}k_{21}e^{S_1}}{2e^{-2S_{21}} + 2e^{2S_{21}} - 4},$$

$$c_{52} = -\frac{2k_{22}e^{-2S_{22}} - 2k_{22}e^{-S_{22}} - 2k_{22} - 2S_{22}k_{22} + 2k_{22}e^{S_{22}} + S_1k_{22}e^{-S_{22}} + S_1k_{22}e^{S_{22}}}{2e^{-2S_1} + 2e^{2S_1} - 4},$$

$$c_{72} = -\frac{2k_{22} - 2S_{22}k_{22}e^{-S_{22}} - 2k_{22}e^{2S_{22}} - 2S_{22}k_{22} + 2k_{22}e^{S_{22}} + S_{22}k_{22}e^{-S_{22}} + S_{22}k_{22}e^{S_{22}}}{2e^{-2S_{22}} + 2e^{2S_{22}} - 4},$$

$$c_{82} = \frac{S_{22}k_{12} - S_{22}k_{12}e^{S_{22}}}{2e^{S_{22}} - 2e^{-S_{22}}}, \quad k_{21} = \frac{D_{21}}{C_{21}}, \quad k_{22} = \frac{D_{22}}{C_{22}}.$$

We obtain the functions

$$X_{21}(x) = c_{51}e^{S_{21}x} + c_{61}xe^{S_{21}x} + c_{71}e^{-S_{21}x} + c_{81}xe^{-S_{21}x} + k_{21},$$

$$Y_{22}(y) = c_{52}e^{S_{22}y} + c_{62}ye^{S_{22}y} + c_{72}e^{-S_{22}y} + c_{82}ye^{-S_{22}y} + k_{22}. \tag{92}$$

Table 10 Estimation of the nanoplate centre deflection using VaM and VaM+ABSM versus the Navier solution for different γ

γ (parameter length of material)	$w(0.5; 0.5)$ Analytical solution by VaM	$w_{Ag}(0.5; 0.5)$ Numerical solution (VaM+ABSM)	$w(0.5; 0.5)$ Exact Navier's solution
0	0.307	0.308	0.311
0.3	0.223	0.223	0.225
0.5	0.150	0.150	0.151

As a result, we have the following displacement function

$$w(x, y) = X_{21}(x)Y_{21}(y) + X_{22}(x)Y_{22}(y). \tag{93}$$

In order to solve the modified Germain–Lagrange equation, we will use combined Vaindiner’s and the Agranovskii–Baglai–Smirnov methods (VaM+ABSM). In this case each ordinary differential equation is solved by the finite difference method with the error $O(h^2)$. The convergence of the solution of ordinary differential equations depending

on the number of partitions along the spatial coordinate is investigated. As a result of a numerical experiment, it is found that the required number of partitions in the finite difference method is 9 points for boundary condition (15) and 55 points for boundary condition (16).

Table 10 gives displacement $w_m(0.5;0.5)$ obtained by the analytical Vaindiner’s method and displacement $w(0.5;0.5)$ obtained by Vaindiner’s method combined with the Agranovskii–Baglai–Smirnov method and exact Navier’s solution.

Solution for boundary conditions (16) is obtained like that for boundary conditions (15). In this case, a trial function of displacement $w(x, y)$ takes the following form

$$w(x, y) = \sin^2(\pi x)Y_{11}(y) + X_{12}(x)\sin^2(\pi y).$$

Next, we obtain the system of equations

$$\begin{aligned} A_{11}Y_{11}^{(4)} + B_{11}Y_{11}^{(2)} + C_{11}Y_{11} &= D_{11}, \\ A_{12}X_{12}^{(4)} + B_{12}X_{12}^{(2)} + C_{12}X_{12} &= D_{12}, \end{aligned} \tag{94}$$

where

$$\begin{aligned} A_{11} &= \lambda^2 \int_0^1 X_1^2 dx = \lambda^2 \frac{3}{16}, \quad B_{11} = \int_0^1 2X_1^{(2)}X_1 dx = -\frac{\pi^2}{2}, \\ C_{11} &= \frac{1}{\lambda^2} \int_0^1 X_1^{(4)}X_1 dx = \frac{\pi^4}{\lambda^2}, \quad D_{11} = Q_1 \int_0^1 X_1 dx = Q_1 \frac{1}{4}, \\ A_{12} &= \frac{1}{\lambda^2} \int_0^1 Y_1^2 dy = \frac{1}{\lambda^2} \frac{3}{16}, \quad B_{12} = \int_0^1 2Y_1^{(2)}Y_1 dy = -\frac{\pi^2}{2}, \\ C_{12} &= \lambda^2 \int_0^1 Y_1^{(4)}Y_1 dy = \frac{\lambda^2 \pi^4}{2}, \quad D_{12} = Q_2 \int_0^1 Y_1 dy = Q_2 \frac{1}{4}. \end{aligned} \tag{95}$$

Solving the system of equations (94), we obtain

$$\begin{aligned} Y_{11}(y) &= e^{N_{11}y}(c_{11} \cos(E_{11}y) + c_{21} \sin(E_{11}y)) \\ &\quad + e^{-N_{11}y}(c_{31} \cos(E_{11}y) + c_{41} \sin(E_{11}y)) + k_{11}, \\ X_{12}(x) &= e^{N_{12}x}(c_{12} \cos(E_{12}x) + c_{22} \sin(E_{12}x)) \\ &\quad + e^{-N_{12}x}(c_{32} \cos(E_{12}x) + c_{42} \sin(E_{12}x)) + k_{12}, \end{aligned} \tag{96}$$

Table 11 Estimation of the nanoplate centre deflections using VaM and VaM+ABSM for different γ

γ (parameter length of material)	$w(0.5; 0.5)$ Analytical solution by VaM	$w_{Ag}(0.5;0.5)$ Numerical solution by VaM+ABSM combination
0	0.289	0.290
0.3	0.210	0.211
0.5	0.142	0.142

where

$$\begin{aligned} N_{11} &= \sqrt{|t_{11}|} \cos\left(\frac{\varphi_{11} + 4\pi}{2}\right), \\ E_{11} &= \sqrt{|t_{11}|} \sin\left(\frac{\varphi_{11} + 4\pi}{2}\right), \\ N_{12} &= \sqrt{|t_{12}|} \cos\left(\frac{\varphi_{12} + 4\pi}{2}\right), \\ E_{12} &= \sqrt{|t_{12}|} \sin\left(\frac{\varphi_{12} + 4\pi}{2}\right), \\ t_{11} &= \frac{-B_{11}}{2A_{11}} + \frac{\sqrt{-(B_{11}^2 - 4A_{11}C_{11})}}{2A_{11}}i, \\ \varphi_{11} &= \arctan\left(\left(\frac{\sqrt{-(B_{11}^2 - 4A_{11}C_{11})}}{2A_{11}}\right) / \left(\frac{-B_{11}}{2A_{11}}\right)\right), \\ k_{11} &= \frac{D_{11}}{C_{11}}, \\ t_{12} &= \frac{-B_{12}}{2A_{12}} + \frac{\sqrt{-(B_{12}^2 - 4A_{12}C_{12})}}{2A_{12}}i, \\ \varphi_{12} &= \arctan\left(\left(\frac{\sqrt{-(B_{12}^2 - 4A_{12}C_{12})}}{2A_{12}}\right) / \left(\frac{-B_{12}}{2A_{12}}\right)\right), \\ k_{12} &= \frac{D_{12}}{C_{12}}. \end{aligned}$$

Substituting the obtained values into the form for the displacement, we obtain the following solution to the original equation

Table 12 Square plate centre deflection for different n and γ yielded by FDM (load equal 7, to boundary conditions (102))

Number of partition points for one coordinate	$w_n(0.5;0.5)$		
	$\gamma = 0$	$\gamma = 0.3$	$\gamma = 0.5$
$(n = 5)$	0.307	0.225	0.150
$(n = 11)$	0.310	0.225	0.151
$(n = 15)$	0.310	0.225	0.151
$(n = 21)$	0.310	0.225	0.151
$(n = 25)$	0.310	0.225	0.151
Exact Navier's solution	0.311	0.225	0.151

Table 13 Square plate centre deflection for different n and γ yielded by FDM (load equal 21, boundary conditions (103))

Number of partition points for one coordinate	$w_n(0.5;0.5)$		
	$\gamma = 0$	$\gamma = 0.3$	$\gamma = 0.5$
$(n = 5)$	0.412	0.299	0.201
$(n = 11)$	0.314	0.228	0.153
$(n = 15)$	0.302	0.220	0.148
$(n = 21)$	0.296	0.215	0.145
$(n = 25)$	0.294	0.214	0.144
$(n = 31)$	0.293	0.213	0.143
$(n = 35)$	0.292	0.212	0.142
$(n = 41)$	0.292	0.212	0.142

Table 14 Solution to MGLE for boundary conditions (15) for $q = 7$ and boundary condition (16) for $q = 21$

Number of Triangular Finite Elements	Clamping $w_n(0.5;0.5)$	Simple Support $w_n(0.5;0.5)$	Number of Triangular Finite Elements	Clamping $w_n(0.5;0.5)$	Simple Support $w_n(0.5;0.5)$
5	0.291	0.309	5	0.289	0.309
11	0.290	0.310	11	0.289	0.310
21	0.290	0.310	21	0.290	0.310
31	0.290	0.310	31	0.290	0.310
37	0.290	0.310	37	0.290	0.310
Exact Navier's solution		0.311	Exact Navier's solution		0.311

$$w(x, y) = \sin^2(\pi x) [e^{N_{11}y} (c_{11} \cos(E_{11}y) + c_{21} \sin(E_{11}y)) + e^{-N_{11}y} (c_{31} \cos(E_{11}y) + c_{41} \sin(E_{11}y)) + k_{11}] + [e^{N_{12}x} (c_{12} \cos(E_{12}x) + c_{22} \sin(E_{12}x)) + e^{-N_{12}x} (c_{32} \cos(E_{12}x) + c_{42} \sin(E_{12}x)) + k_{12}] \sin^2(\pi y), \tag{97}$$

where

$$\begin{aligned} c_{11} + c_{31} + k_{11} &= 0, \\ k_{11} + e^{-N_{11}}(c_{31} \cos(E_{11}) + c_{41} \sin(E_{11})) + e^{N_{11}}(c_{11} \cos(E_{11}) + c_{21} \sin(E_{11})) &= 0, \\ E_{11}c_{21} + E_{11}c_{41} + N_{11}c_{11} - N_{11}c_{31} &= 0, \\ e^{N_{11}}(c_{21}E_{11} \cos(E_{11}) - c_{11}E_{11} \sin(E_{11})) + e^{N_{11}}(c_{41}E_{11} \cos(E_{11}) - c_{31}E_{11} \sin(E_{11})) &= 0, \\ c_{12} + c_{32} + k_{12} &= 0, \\ k_{12} + e^{-N_{12}}(c_{32} \cos(E_{12}) + c_{42} \sin(E_{12})) + e^{N_{12}}(c_{12} \cos(E_{12}) + c_{22} \sin(E_{12})) &= 0, \\ E_{12}c_{22} + E_{12}c_{42} + N_{12}c_{12} - N_{12}c_{32} &= 0, \\ e^{N_{12}}(c_{22}E_{12} \cos(E_{12}) - c_{12}E_{12} \sin(E_{12})) + e^{N_{12}}(c_{42}E_{12} \cos(E_{12}) - c_{32}E_{12} \sin(E_{12})) &= 0. \end{aligned} \tag{98}$$

Formula (97) forms a classical solution obtained by Vaindiner's method. However, this solution can be refined. Since we know functions $Y_{11}(y)$, $X_{12}(x)$, we can find functions $X_{12}(x)$, $Y_{22}(y)$. To do this, the iterative procedure can be repeated, and hence

$$\begin{aligned}
 Y_{21}(y) &= e^{N_{21}y} (c_{11} \cos (E_{21}y) + c_{21} \sin (E_{21}y)) + \\
 &+ e^{-N_{21}y} (c_{31} \cos (E_{21}y) + c_{41} \sin (E_{21}y)) + k_{21}, \\
 X_{22}(x) &= e^{N_{22}x} (c_{52} \cos (E_{22}x) + c_{62} \sin (E_{22}x)) + \\
 &+ e^{-N_{12}x} (c_{72} \cos (E_{22}x) + c_{82} \sin (E_{22}x)) + k_{22}.
 \end{aligned}
 \tag{99}$$

As a result, we have the following displacement function

$$w(x, y) = X_{21}(x)Y_{21}(y) + X_{22}(x)Y_{22}(y).
 \tag{100}$$

Table 11 shows the displacement obtained analytically by Vaindiner’s method for boundary conditions (16), and the numerical solution obtained by the combined methods of Vaindiner and Agranovskii–Baglai–Smirnov.

The following conclusions can be formulated. The accuracy of the solution for two types of boundary conditions obtained by the Vaindiner method is comparable to the accuracy of the solution obtained by the Kantorovich–Vlasov method and variational iteration method. It is worth noting that the solution by Vaindiner’s method and the Kantorovich–Vlasov method depends on a correct choice of the initial approximation. This limitation can be removed by using the procedure of the variational iteration method in each equation of the system obtained by the Vaindiner method (100). Like in the case of the variational iteration method, this will allow us to use any smooth function for the initial approximation.

4 Numerical Solution of MGLE by FDM and FEM

We will use the well-known methods to obtain numerical solutions for numerous PDEs of mathematical physics. This is the finite difference method (FDM) with approximation $O(h^2)$ and the finite element method (FEM). Recently, these methods have been widely used in various fields of continuum mechanics. It seems interesting to compare the solutions obtained by the methods of reducing partial differential equations to ordinary differential equations with the solutions obtained by the above methods. Next, we compare the numerical solutions of the modified Germain–Lagrange equation obtained by the methods of reducing partial differential equations to ordinary differential equations with the solutions attained by the finite difference method and finite element method for some types of boundary conditions.

4.1 Finite Difference Method

Let us apply the second-order finite difference method to equation (18). We get

$$\begin{aligned}
 D[w_{i+2,j} + w_{i-2,j} + w_{i,j+2} + w_{i,j-2} + \\
 + 2(w_{i+1,j+1} + w_{i+1,j-1} + w_{i-1,j+1} + w_{i-1,j-1}) \\
 + 20w_{i,j} - 8(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1})] = qh^4.
 \end{aligned}
 \tag{101}$$

Table 15 Analytical and numerical solutions obtained by various methods for estimation of the nanoplate centre deflection (b.c. (15), $q = 7$) for different γ

Type of solution	Methods	$\gamma = 0$ $w(0.5; 0.5)$	$\gamma = 0.3$ $w(0.5; 0.5)$	$\gamma = 0.5$ $w(0.5; 0.5)$
Analytical solutions	Navier method	0.311	0.225	0.151
	BGM	0.311	0.228	0.153
	KVM			
	(in first term approximation)	0.314	0.228	0.153
	KVM			
	(in second term approximation)	0.312	0.225	0.152
	VIM	0.307	0.223	0.150
Numerical solution	VaM	0.307	0.223	0.150
	FDM	0.310	0.225	0.151
	FEM - (Triangular Elements (350 Elements))	0.310	0.225	0.151
	FEM - (Quadrangular Elements (165 Elements))	0.310	0.225	0.151
	KVM in first term + ABSM	0.308	0.223	0.150
	KVM in second term + ABSM	0.310	0.225	0.151
	VIM + ABSM	0.308	0.224	0.150
	VaM + ABSM	0.308	0.223	0.150
	VaM + VIM	0.308	0.223	0.150

Table 16 Analytical and numerical solutions obtained by various methods for the estimation of the nanoplate centre deflection (b.c. (16), $q = 21$) for different γ

Type of solution	Methods	$\gamma = 0$ $w(0.5; 0.5)$	$\gamma = 0.3$ $w(0.5; 0.5)$	$\gamma = 0.5$ $w(0.5; 0.5)$
Analytical solutions	KVM (in first term approximation)	0.289	0.210	0.142
	KVM (in second term approximation)	0.291	0.212	0.143
	VIM	0.292	0.212	0.142
	VaM	0.289	0.210	0.142
	Numerical solution	FDM	0.294	0.214
Numerical solution	FEM - (Triangular Elements (350 Elements))	0.297	0.217	0.146
	FEM - (Quadrangular Elements (165 Elements))	0.297	0.217	0.146
	KVM in first term + ABSM	0.290	0.210	0.142
	KVM in second term + ABSM	0.291	0.211	0.143
	VIM + ABSM	0.292	0.212	0.142
	VaM + ABSM	0.290	0.211	0.142
	VaM + VIM	0.290	0.210	0.142

Table 17 Various methods of solution versus MGLE and the number of equations required to obtain a reliable solution (b.c. (15), $\gamma = 0$)

Solution method	The number of equations required to obtain a reliable solution		
	n	N	$n * N$
	Kantorovich–Vlasov method in first term	15	1
Kantorovich–Vlasov method in second term	30	1	30
Variational iteration method in first term	15	3	45
Vaindiner’s method in first term	30	3	90
Finite difference method $O(h^2)$	121	1	121
Finite element method, triangular elements	1498	1	1498
Finite element method, quadrangular elements	1242	1	1242

Table 18 Various methods of solution versus MGLE and the number of equations required to obtain a reliable solution (b.c. (16), $\gamma = 0.5$)

Solution method	The number of equations required to obtain a reliable solution		
	n	N	$n * N$
	Kantorovich–Vlasov method in first term	15	1
Kantorovich–Vlasov method in second term	30	1	30
Variational iteration method in first term	15	2	45
Vaindiner’s method in first term	30	1	30
Finite difference method $O(h^2)$	121	1	121
Finite element method, triangular elements	1498	1	1498
Finite element method, quadrangular elements	1242	1	1242

We obtain a system of linear equations n^2 for each grid node, where n is the number of grid nodes, each on the x and y axes. We solve this system by the Gauss method. In order to determine the values of the extreme grid nodes, we use the following boundary conditions.

For boundary conditions (15), we have

$$\begin{aligned}
 w_{0,0} = w_{n,0} = w_{0,n} = w_{n,n} = 0, \\
 w_{-1,i} = w_{1,i}, w_{n+1,i} = w_{n-1,i}, w_{i,-1} = w_{i,1}, w_{i,n+1} = w_{i,n-1}
 \end{aligned}
 \tag{102}$$

For boundary conditions (16), we have

$$\begin{aligned}
 w_{0,0} = w_{n,0} = w_{0,n} = w_{n,n} = 0, \\
 w_{-1,i} = -w_{1,i}, w_{n+1,i} = -w_{n-1,i}, \\
 w_{i,-1} = -w_{i,1}, w_{i,n+1} = -w_{i,n-1}.
 \end{aligned}
 \tag{103}$$

Tables 12 and 13 show the results obtained for boundary conditions (102) and (103), using the finite difference method for a square plate under the action of a constant transverse load.

Analysis of the results shows that the material length parameter γ can significantly influence the results obtained. The solution for the material length parameter

Table 19 Various methods of solution versus MGLE and the number of equations required to obtain a reliable solution (b.c. (16), $\gamma = 0$)

Solution method	The number of equations required to obtain a reliable solution		
	<i>n</i>	<i>N</i>	<i>n * N</i>
	Kantorovich–Vlasov method in first term	55	1
Kantorovich–Vlasov method in second term	110	1	110
Variational iteration method in first term	55	3	165
Vaindiner’s method in first term	110	3	330
Finite difference method $O(h^2)$	1225	1	1225
Finite element method, triangular elements	1498	1	1498
Finite element method, quadrangular elements	1242	1	1242

Table 20 Various methods of solution versus MGLE and the number of equations required to obtain a reliable solution (b.c. (16), $\gamma = 0.5$)

Solution method	The number of equations required to obtain a reliable solution		
	<i>n</i>	<i>N</i>	<i>n * N</i>
	Kantorovich–Vlasov method in first term	55	1
Kantorovich–Vlasov method in second term	110	1	110
Variational iteration method in first term	55	2	110
Vaindiner’s method in first term	110	1	110
Finite difference method $O(h^2)$	1225	1	1225
Finite element method, triangular elements	1498	1	1498
Finite element method, quadrangular elements	1242	1	1242

$\gamma = 0.3$; $\gamma = 0.5$ coincides with Navier’s exact solution. The convergence of the solution obtained by the finite difference method depends essentially on the type of boundary conditions. Namely, for boundary conditions (103) three times more partition points are required than in the case of boundary condition (102).

4.2 Finite Element Method

In this case we will use linear triangular and quadrangular elements while employing FEM. According to the algorithm for solving the problem, the area is automatically divided into a given number of elements. The solution is obtained by the system of linear algebraic equations using the Gauss method. A convergence study is provided depending on the number of the used elements.

Table 14 shows the results of the study on convergence for the finite element method when solving the modified Germain–Lagrange equation for boundary conditions (15) at a constant transverse load $q = 7$ and for boundary conditions (16) for fixed $q = 21$, and $\gamma = 0$.

Thus, the best convergence is achieved for 11 triangular finite elements, both for boundary conditions (15) and (16).

Table 14 presents the analysis of results of the study on the convergence of the solution of the modified Germain–Lagrange equation by the finite element method. Thus, the best convergence is achieved for 11 triangular finite elements, for both boundary conditions studied. For quadrangular finite elements for both types of boundary conditions, convergence is achieved for 21 finite elements. To reduce the number of elements when solving the problem, it is possible to use symmetry of the solution and reduce the number of elements by 4 times, but in this case it was not done.

5 Comparison of Results Obtained by the Methods of Reducing PDEs to ODEs and by Numerical Methods

Previously, the equation governing bending of a nanoplate taking into account the modified couple stress theory and the method of variable parameters of elasticity has been derived. Now, it is solved by some of the already mentioned methods: the Kantorovich–Vlasov method with various modifications; variational iteration method, Vaindiner’s method, the Agranovskii–Baglai–Smirnov method, Bubnov–Galerkin method, the method of double trigonometric series approximation, the finite element method and the finite difference method.

Table 15 presents the values of the maximum displacement calculated by various methods for solving the equation of bending of a nanoplate under the action of a uniformly distributed load ($q = 7$) for boundary conditions (15).

The analysis of the numerical results and final data reported in Table 15 shows that the solutions obtained by various numerical methods including the finite element method, finite differences method, Kantorovich–Vlasov method in *N* terms and the variational iteration method combined with the Agranovskii–Baglai–Smirnov method are very close to the exact solution.

The maximum difference between the exact solution and solutions obtained by other methods does not exceed 1.2%, so these methods are competitive. Methods for reducing PDEs to ODEs allow us to obtain an analytical solution with extremely high accuracy. *The most important conclusion based on the obtained results is as follows. The cost of computer time when using these methods is several times cheaper than when using the numerical methods based on FEM and FDM.*

In addition, Table 16 presents the values of the maximum nanoplate displacement calculated by various methods for solving the equation of bending of a nanoplate under the action of uniformly distributed load ($q = 21$) for boundary conditions (16).

In this case, the comparative analysis of the numerical results based on the data reported in Table 16 for boundary condition (16) for square nanoplate leads to the conclusion that if $\gamma = 0.3$, $\gamma = 0.5$ the results of all methods are quite close. The largest differences in the solutions arise when $\gamma = 0$. It should be emphasized that the values of solutions by the mentioned methods devoted to the reduction of PDEs to ODEs are close to each other and their difference is 0.6%.

In addition, we have compared the number of algebraic equations while employing the Kantorovich–Vlasov method, variational iteration method, Vaindiner’s method, FEM and FDM. Table 17 shows a comparison of the number of equations required to solve Eq. (18) at $\gamma = 0$ for boundary conditions (15). The following notation is used: n is the number of equations, N - is the number of iterations (if necessary), $n * N$ is the total number of equations to obtain a solution.

Table 17 presents the results of analysis for different number of equations for $\gamma = 0$, and for various methods.

The same is done in Table 18 but for $\gamma = 0.5$.

We have also analyzed the same problem regarding equation (14) for $\gamma = 0$ for boundary conditions (16) (see Table 19).

The same analysis has been extended for the case when $\gamma = 0.5$ (see Table 20).

Based on the analysis of Tables 17, 18, 19 and 20, it can be concluded that in terms of computer time the most expensive are the finite difference and finite element methods.

The analysis of the numerical results given in Tables 17, 18, 19 and 20 leads to the conclusion that in terms of computer time the most expensive are the finite difference and finite element methods. The methods that reduce the original problem for a partial differential equation to solving an ordinary differential equation use a one-dimensional grid. This leads to a decrease in the number of algebraic equations and an increase in the speed of obtaining a solution.

6 Concluding Remarks

This article presents a comprehensive review of the literature on methods of reducing nonlinear partial differential equations to a set of nonlinear differential equations with emphasis put on their reliability, validity, accuracy and computational efficiency. Since the nonlinear PDEs govern dynamic and static behavior of numerous systems in mechanics, aeronautics and civil engineering, a proper choice of the feasible computational method to study mass distributed mechanical systems based on the signal processing plays an important

role in the system modeling, identification, monitoring and control.

The classical methods aimed at solving the problem have their origin in mathematical physics. However, the problem has not yet been fully resolved and, as our review has shown, it requires revision and more attention to the known methods of finite elements and finite differences widely used in engineering community but not well recognized as competitive approaches. The methodology is well known: a given multidimensional problem should be substituted by its counterpart sequence of one-dimensional problems. However, the obtained results strongly depend on the used methods, which answer numerous questions and require checking of the authenticity, reliability, validity, stability of results, etc.

The following methods have been reviewed and examined due to their state-of-the art levels of performance and computational advantages: the Fourier methods, the Galerkin-type methods, the variational methods, the variational iterations, the Kantorovich–Vlasov methods, Vaindiner’s method, the Agranov- skii-Baglai-Smirnov method with respect to the archetypal finite element and finite difference methods.

The quantification and comparison of the tested (above mentioned) methods was carried out on the basis of the modified Germain–Lagrange PDE governing the dynamics of a nanoplate and with comparison to the exact results obtained on the basis of Navier’s method.

Although the paper has a review character, it includes also our original results. The whole work can be briefly summarized based on the following major conclusions.

- A mathematical model of Germain–Lagrange nanoplates, taking into account the physical nonlinearity and inhomogeneity of the material, has been derived.
- The comparative analysis of numerous methods, including the Kantorovich–Vlasov method, the variational iteration method, and Vaindiner’s method aiming at the reduction of PDEs to ODEs was carried out for the first time.
- For each method described above, theorems have been both formulated and proved and thus their mathematical justification has been given.
- The proven convergence theorems for solutions obtained by the Kantorovich–Vlasov methods and variational iteration method can be applied to solve the modified Germain–Lagrange equations.
- The solutions obtained by the mentioned methods of reducing the equations of mathematical physics to ordinary differential equations have been compared with the exact Navier solution obtained by the authors for the modified Germain–Lagrange equation.
- These methods appeared to be effective. The computation time for obtaining a solution by these methods is much

shorter than the computation with the use of either the finite difference or finite element methods.

- A numerical experiment has shown that the investigated methods of reducing partial differential equations to ordinary differential equations are highly accurate. They provide a solution with an error of less than 1 percent.
- The convergence to the exact solution for the investigated methods has been significantly improved for nanostructures.
- For the first time, we have proposed to use the procedure for refining solutions based on the Agranovskii–Baglai–Smirnov method while reducing PDEs to ODEs for problems in the theory of nanoplates.
- When solving ODEs numerically, it has been found that the accuracy of solution depends on the type of boundary conditions and the number of partition intervals.
- Analysis of the methods for reducing elliptic PDEs of mathematical physics to ODEs allows drawing conclusions about their effectiveness in solving problems of the theory of plates, taking into account physical nonlinearity and material heterogeneity.
- The presented review and analysis of the methods imply that they can be applied to hyperbolic equations of mathematical physics, including construction of an iterative procedure within the algorithms of the Faedo–Galerkin method.

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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