Arch Computat Methods Eng (2018) 25:785–815
DOI 10.1007/s11831-017-9214-7

ORIGINAL PAPER

A Review and Study on Ritz Method Admissible Functions with Emphasis on Buckling and Free Vibration of Isotropic and Anisotropic Beams and Plates

Pablo Moreno-García[1] · José V. Araújo dos Santos[2] · Hernani Lopes[3]

Received: 26 July 2016 / Accepted: 7 February 2017 / Published online: 24 March 2017

Abstract The first goal of this work is to present a literature review regarding the use of several sets of admissible functions in the Ritz method. The papers reviewed deal mainly with the analysis of buckling and free vibration of isotropic and anisotropic beams and plates. Theoretically, in order to obtain a correct solution, the set of admissible functions must not violate the essential or geometric boundary conditions and should also be linearly independent and complete. However, in practice, some of the sets of functions proposed in the literature present a bad numerical behavior, namely in terms of convergence, computational time and stability. Thus, a second goal of the present work is to compare the performance of several sets of functions in terms of these three features. To achieve this objective, the free vibration analysis of a fully clamped rectangular plate is carried out using six different sets of functions, along with the study of the convergence of natural frequencies and mode shapes, the computational time and the numerical stability.

✉ Pablo Moreno-García
pablo.morenogarcia@uca.es

José V. Araújo dos Santos
viriato@tecnico.ulisboa.pt

Hernani Lopes
hml@isep.ipp.pt

1 Departamento de Ingeniería Mecánica y Diseño Industrial, Escuela Superior de Ingeniería, Universidad de Cádiz, Cádiz, Spain

2 IDMEC, Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal

3 DEM, ISEP, Instituto Politécnico do Porto, Porto, Portugal

1 Introduction

The Ritz method has been used for several decades in the solution of static, buckling and free vibration problems of beams, plates, and shells. It was formulated in 1909 by Ritz [142] and applied to the transverse vibration of a square plate with free edges by this same researcher [141]. Although over the years the name of Rayleigh has also been attached to this method, according to Leissa [100], the Rayleigh method is formally different from the method proposed by Ritz. Regarding this issue, it is also interesting to cite here the words of Timoshenko [153], who considered the Ritz method an improvement upon the Rayleigh method: "Rayleigh used the method only for an approximate calculation of frequency of the gravest mode of vibration of complicated systems, and was doubtful regarding its application to the investigation of higher modes of vibration." It should be pointed out, nevertheless, that this issue is not consensual and, for instance, Ilanko [81] disagree with the conclusion of Leissa about the name of the method. The option of the authors of the present paper is to name it Ritz method mainly because of Rayleigh's own words [109]: "I wish to call attention to a remarkable memoir by W. Ritz in which, somewhat on the above lines, is developed with great skill what may be regarded as a practically complete solution of the problem of Chladni's figures on square plates." We may also bear in mind, as stated by Gaul [68], that Rayleigh's exposition does not resembles the modern version.

Besides the efficiency and technical interest that this method exhibits in describing and predicting the behavior of important structural members, such as beams, plates, and shells, the Ritz method can be used to motivate students for the study of partial differential equation eigenvalue problems [63]. The teaching of this method to students show several benefits [145]. According to Gander and Wanner [64], it is also possible to trace back the development of the finite element method to the work of Trefftz, which used local basis functions in the Ritz method. A relation between the finite differences for problems with derivatives not higher than the first order and the Ritz method is also described by Courant [49]. Courant suggested the use of polyhedral functions such that in the end the minimum conditions become difference equations. According to Williamson [157] one can see here the essentials of the finite element method. A more comprehensive description of the links, as well as the differences and similarities, between the Ritz method and the finite element method can be found in reference [68].

Over the years, several admissible functions that satisfy the essential boundary conditions, or as Ritz wrote: "our method [is] applicable, as long as the essential boundary conditions are not violated" [100], have been used. Some of these functions are based on trigonometric and hyperbolic functions, and others are based on polynomial functions. However, these different admissible functions present different convergencies to the solution, different computational burdens as well as different numerical stabilities. The present work consists in a comparison of the performance of six sets of admissible functions, which are commonly found in the literature, in terms of these three numerical features. The purpose of this analysis is to find the best type of admissible functions to compute a large number of natural frequencies and mode shapes with sufficient and adequate accuracy. With this objective in mind, the free vibration problem of an isotropic fully clamped rectangular plate is solved. The selection of this type of plate and boundary condition is justified by the fact that a rectangular plate, which is usually riveted to a rigid frame along its edges, is one of the most popular structural members [116]. Furthermore, due to the mathematical simplicity of this boundary condition, the fully clamped plate is frequently used as a test for analytical methods [99]. The fully clamped plate also shows signs of earlier numerical instability for lower degree polynomials than other boundary conditions, namely simply supported on all edges [147], thus presenting a challenge to a commonly used set of admissible functions in the Ritz method. The six sets of admissible functions, studied in the present paper, are: (1) Characteristic Functions [160], (2) Modified Characteristic Functions [67], (3) Orthogonal Polynomials [22], (4) Non-orthogonal polynomials [86], (5) Product of Trigonometric Functions [32], and (6) Static Beam Functions [166].

Young [160] wrote what is one of the first papers with applications of the Ritz method in English language. The author used as admissible functions the characteristic functions of the normal modes of a uniform beam. Besides a clamped plate with six terms in the series, which defines the mode shapes, a cantilever plate and a plate with two adjacent edges clamped and the other two edges free were also analyzed. Due to the numerical problems presented by the characteristic functions of a beam, Gartner and Olgac [67] proposed a modified set of these functions so that the magnitude of the terms is in the range $[-1, 1]$. This is accomplished by replacing the hyperbolic terms by negative exponential terms. A set of orthogonal polynomials was proposed as admissible functions in the Ritz method by Bhat [22]. Simply supported and clamped plates were studied, as well as plates with two adjacent edges simply supported and the other two free and plates with two adjacent edges clamped and the other two free. It was found that this kind of functions presents superior results for lower modes. A different set of polynomials was proposed by Kim et al. [86]. This set is generated by multiplying the previous polynomial by the corresponding co-ordinate, being the first polynomial one that obey the essential boundary conditions. This set is non-orthogonal, and thus present some computational disadvantages. On the other hand, the procedure to generate it is simpler than for the orthogonal set. Frequency parameters of different isotropic plates are reported for different cases, i.e. various boundary conditions, springs at points and concentrated masses. Simply supported and clamped plates under shear load and a clamped orthotropic plate under hydrostatic in-plane loads are also studied. Also, the transition from fully simply supported to fully clamped plate, from fully free to fully simply supported plate, and from fully free to fully clamped plate is studied. Another set of functions, based on trigonometric functions, was used by Chai [32]. Six different types of support conditions were considered, among them the clamped case. For these conditions, the mode shape is approximated by the product of sine functions in each direction. The analysis showed that for lower modes the natural frequencies are consistently higher than those obtained with other sets. For higher modes, the natural frequencies are lower than previous results. The last set of functions studied in the present paper is based on static beam functions, and was proposed by Zhou [166], who studied the first nine eigenvalues of isotropic rectangular plates with various aspect ratios and boundary conditions.

2 Review on Admissible Functions

As discussed above, the results of an analysis with the Ritz method are largely dependent on the type of admissible functions we use. Therefore, it is of paramount importance to select a set of functions which allows, not only physically sound results, but also presents a good numerical behavior. In the next sections, we review the works reporting the results of the Ritz method with the six different sets of admissible functions mentioned above.

2.1 Characteristic Functions

Characteristic functions or eigenfunctions, which define the normal modes of vibration of a beam, were used by Young for the free vibration analysis of a clamped plate, a cantilever plate, and a plate with two adjacent edges clamped and the other two edges free [160]. A more complete set of data related with this work can be found in the technical report by Young and Felgar published in 1949 [161], which presents values of the characteristic functions and its first three derivatives of clamped-clamped, clamped-free and clamped-supported beams. The functions are tabulated for the first five modes to five decimal places and increments of the argument of $0.02L$, where L is the length of the beam. Since the Ritz method requires the computation of integrals involving the admissible functions and their derivatives, over the years several results of such computations have been presented [29, 60, 102, 103, 118, 146]. This type of functions were also used by Warburton [155] to obtain approximate frequency expressions using the Rayleigh method. Leissa [99] presented a comprehensive work on free vibration of rectangular plates, considering, for each edge, the three possible combinations of boundary conditions: clamped, simply supported and free. These boundary conditions originate 21 different cases. Initially, the six cases having two opposite sides simply supported are analytically studied and their frequencies are obtained. The effect of the Poisson's ratio for the case of a plate with two adjacent edges simply supported and the other two edges free is also studied. The other 15 cases are studied using the Ritz method to obtain the frequencies and the results are compared with those obtained with Warburton's approximate formulas [155]. Bassily and Dickinson [16] used the Ritz method with characteristic functions to obtain frequency parameters and buckling loads of fully clamped plates under a combination of uniform shear and direct in-plane loads, plates clamped in two parallel edges and simply supported on the other two edges under uniform shear in-plane loads, and plates clamped in two parallel edges and free on the other two edges under in-plane stress fields involving linearly varying direct stress and parabolically varying shear stress. The convergence of the solution with the number of terms in the series is also reported in a tabular form in this work. The frequency parameters and buckling loads, as well as nodal and contour lines at buckling, are also extensively presented in graphical format. Due to an error made in evaluating one of the integrals, some of the numerical and graphical results reported in reference [16] for plates with free edges were corrected by the authors in a subsequent publication [17].

The plates analyzed in the works cited above were uniform and isotropic. The beginning of the application and study of composite materials in the 1950s and 1960s led to the use of the Ritz method for the analysis of bending, buckling, and free vibration of anisotropic plates. A paper by Hearmon [70] reports the use of characteristic functions for the vibration analysis of rectangular orthotropic plates with clamped and supported edges. Ashton and Waddoups [12] presented numerical solutions obtained with the Ritz method using the characteristic functions, as well as a comparison with experimental results. This work was extended to non-uniform plate properties and loadings [8]. Reference [8] also presents a comparison of experimental buckling results for a tapered beam with the results obtained with the Ritz method for a linearly tapered beam. A comparison of experimental mode shapes, obtained by photographing node lines described by aluminum granules, of fully clamped square boron-epoxy composite plates and those obtained with the Ritz method is presented by Ashton and Anderson [11]. An excellent agreement between the computed and experimental nodal lines is observed. Analytical mode shapes of free rectangular anisotropic plates, obtained with the Ritz method with free-free characteristic functions, are reported in reference [9]. This reference also presents a study on the effects of fiber orientation, relation between Young's modulus and stacking sequence. According to Ashton, the application of the Ritz method for bending, buckling, and dynamics of anisotropic plates, described in [8, 12], originates acceptable solutions, but they may not be acceptable if the derivatives of the deflection are required [10]. The natural frequencies of unsymmetrically laminated anisotropic plates with fully clamped edges were obtained by Bert and Mayberry [19], using the Ritz method and the characteristic functions tabulated in reference [161]. By relying on the work of Warburton [155], Dickinson [53] analyzed orthotropic plates and extended the formulation to include the effect of uniform and direct in-plane forces. The approach allows the computation of natural frequencies and/or buckling loads of plates with any combination of free, simply supported or clamped edges. The Ritz method with one term defined by the clamped or simply supported edges characteristic functions was used by Kollar and Veres [88] to obtain the buckling loads of rectangular orthotropic plates subjected to biaxial normal forces.

A combination of characteristic functions has also been used in the study of vibration of rectangular plates with elastically restrained edges by Warburton and Edney [156]. The variation of the fundamental frequency with in-plane forces and the computation of critical loads with these restrained conditions are also reported. The authors of reference [156] concluded that by using characteristic functions, instead of polynomials, one obtains better accuracy. Plunkett [135] compared experimental and computational results of flat (uniform) and variable thickness (non-uniform) cantilever plates. It was concluded that this last type of plates need a great number of terms in the series of the Ritz method in order to correctly represent the complex behavior of the higher order mode shapes. Aref et al. [6] computed the static deflection of a fiber-reinforced plastic skew bridge superstructure with the Ritz method and characteristic functions. The formulation of a transformed plate, with two opposite edges simply supported and the other two edges free was presented. The results show a good agreement with finite element results. Ciancio et al. [48] presented a study on cantilevered rectangular anisotropic plates with a concentrated mass rigidly attached to the center. The authors of reference [48] applied the Ritz method with characteristic functions as admissible functions and reported the natural frequencies and mode shapes for different mass magnitudes. The dynamic analysis of a cantilever plate with attached spring-mass system on an arbitrary point was presented by Chiba and Sugimoto [47]. The displacement of the plate was defined by clamped-free and free-free characteristic functions, whereas the displacement of the attached mass was defined by a constant. Extensive results were reported for both uncoupled and coupled vibrations of plate and spring-mass system. The free vibration analysis of thin rectangular plates with holes or orthotropic patches and an elastically attached mass was carried out by Bambill et al. [15]. Simply supported and clamped isotropic and orthotropic plates were studied using characteristic functions. Very recently, Afshari and Inman [1] studied the vibration of a piezoelectrically driven beam with a single growing crack. The mode shapes and natural frequencies of cracked simply supported and clamped beams were approximated by the Ritz method with characteristic functions. The effect of the crack, which is considered as a massless rotational spring, was modeled by a loss of energy. These approximate results are compared with the ones from the reference model of the cracked beam. Numerical results are also reported for various design parameters of crack position and depth and location of the piezoceramic patch. The analysis of thick plates was carried out by Lim et al. [108] using this type of functions and a simplification of Reddy's displacement functions [137]. The results are compared with 3D results and the ones obtained with Mindlin's theory for simply supported, fully clamped plates, and plates with adjacent edges simply supported and clamped. Kim [87] analyzed the vibration of fully clamped functionally graded rectangular plates made up of metal and ceramic. In this work, the material properties are temperature dependent and vary through the thickness according to a power law distribution in terms of the volume fractions of the constituents. The Ritz method is applied with characteristic functions of clamped beams and a third order shear deformation theory is used to account for rotary inertia and transverse shear strains. The numerical results reported show that the vibration characteristics are significantly influenced by the materials composition, the plate geometry and the temperature rise.

A method for the study of the boundary conditions effects on the free vibration characteristics of multi-layered cylindrical shells, based on Love's theory, was proposed by Lam and Loy [95]. The Ritz method is used with the characteristic functions as axial modal functions. According to the authors of reference [95], the proposed method is less computational demanding since it is a non-iterative method, unlike other methods. A multi-layered cylindrical shell made of three homogeneous isotropic layers, being the outer and inner layers of the same material and the middle layer of a different material, is studied. Extensive results are presented in reference [95], namely fundamental frequencies and mode shapes of the nine boundary conditions in the three directions. An extension of this approach for the frequency analysis of multi-layered cylindrical shells under lateral pressure and with asymmetric boundary conditions was presented by Isvandzibaei et al. [83]. In order to study the vibration of cylindrical shells with a ring support arbitrarily placed along the shell, the characteristic functions are also chosen as axial functions by Loy and Lam [114], along with Sanders' shell theory. Studies on the frequency parameter, the ring support position and the boundary conditions are reported. It was found out that the ring support significantly influences the frequencies and this influence depends on the position of the ring support and the boundary conditions. The vibration analysis of functionally graded cylindrical shells with an exponential volume fraction law was carried out by Arshad et al. [7]. The thin shell theory of Love is employed and the characteristic functions are used to describe the axial displacements. Several boundary conditions, such as free-free, free-simply supported, simply supported-simply supported, clamped-clamped, clamped-free, clamped-simply supported, and different values of the power law exponent are studied. It was found out that, for all boundary conditions, the natural frequencies of the functionally graded cylindrical shell are in an interval defined by the natural frequencies of shells made of the pure constituent materials. The use of characteristic functions for the vibration analysis of a fluid coupled with a structure, namely cylindrical shells, has also been reported. Such is the case of a work by Kwak et al. [90], who studied the effect of both internal and external fluids coupled with a partially submerged clamped-free cylindrical shell. The authors derived the virtual mass matrix that is added to the matrix equations of motion for a cylindrical shell in vacuo based on the Sanders' shell theory. An extension of the work presented in reference [90] was developed by Bae et al. [14]. The authors analyzed the case of a submerged shell incorporating an external wall, an interior shaft and a bottom. The experimental verification of the theoretical results presented in references [90] and [14] was also carried out. Among other conclusions, it was found out that the natural frequencies decrease substantially even for a small water height. This decrease is observed both in theoretical as well as experimental results. Characteristic functions of clamped free and simply supported boundary conditions are also used by Askari et al. [13] to define the axial mode shape of an isotropic cylindrical shell. The cylindrical shell is a component of a liquid-storage elastic cylindrical container with internal body and partially filled with a sloshing fluid. Reference [13] presents an analytical method for the analysis of the fluid structure interaction which rely on the Rayleigh quotient and the Ritz and Galerkin methods. The analytical results obtained are compared with data available in the literature and finite element results.

Maheri and Adams [115] also used characteristic functions to analyze the vibration damping of anisotropic fiber-reinforced plastic laminates and they show that the results correlate well with experimental data obtained for freely held plates. A similar approach to the study of damping in laminated beams and plates was presented by Berthelot [20], who found that the damping depends on the modes and its evaluation is related with the mode shapes considered. The author used clamped-free and free-free characteristic functions. This work is an extension of a previous study by Berthelot and Sefrani [21], in which the damping properties of unidirectional plates are described. In order to estimate the vibration of a floor, Kato and Honma [85] modeled building floors with the Ritz method and characteristic functions. A square isotropic plate simply supported at the four corner points and restrained against rotation along the four edges with infinite spring constants is defined as the base model. Reinforcements with four beams along each boundary are also included. Besides the base model, other four models are analyzed, which are characterized by adding two parallel beams in the same interval, adding a simply supported point near the center of the plate, adding in-plane forces, and adding a tuned mass damper at the center of the plate to the case with two parallel beams added in the same interval to the base model. The first six natural frequencies and the time history curves for the deflection at the center of the plate are reported. A good agreement between the results of the proposed models and finite element models is observed. The problem posed by free edges is also addressed in reference [85]. An application of the Ritz method with characteristic functions as admissible functions was presented by Deobald and Gibson [52], who determined four elastic constants of orthotropic plates with clamped and free edges. Two Young's moduli, the in-plane shear modulus, and a Poisson's ratio were determined using four natural frequencies. Lai and Ip [91] also applied the Ritz method with characteristic functions of a free-free beam to estimate three elastic moduli and two Poisson's ratios of three orthotropic plates. A statistical Bayesian estimation method was used and seven natural frequencies were considered in the computations. A similar approach to characterize the transverse modulus and the in-plane shear modulus of free-free thin orthotropic shells was developed by Ip et al. [82].

An increase in the development of numerical models for the analysis of structures presenting size effects has been observed in recent years. One of the theories which account for size effects is the modified couple stress proposed by Yang et al. [158]. In this theory, the size effect is accounted for by including a material length scale parameter related with the couple stress. The theory described in reference [158] was applied for the free vibration and buckling analyses of Timoshenko beams by dos Santos and Reddy [144]. The authors used the Ritz method with characteristic functions to compute the natural frequencies and buckling loads of beams with any of the six possible boundary conditions. Kong [89] studied the pull-in behavior of cantilever and clamped microbeams with the modified couple stress, considering the displacement as a one term series of characteristic functions. It was found that the normalized pull-in voltage of the microbeam exhibits remarkably the size effect and that this effect diminishes when the thickness is greater than the material length scale parameter. The pull-in displacement is, on the other hand, constant and thus independent of the material length scale. Another elasticity theory which has been applied to the analysis of nanostructures is the nonlocal theory proposed by Eringen [57, 58]. The buckling and the free vibration of single- and multi-walled carbon nanotubes have been studied with this theory by Ansari and co-workers. Ansari et al. [5] applied the Ritz method with characteristic functions describing the axial functions, which appears in the components of the displacements of the elastic Donnell's shell theory, for the analysis of axial buckling of singe-walled carbon nanotubes. These authors [4] extended this numerical model by including the thermal environment effect and showed that the difference between the thermal axial buckling responses is more prominent for higher values of the nonlocal elasticity constant. Numerical models based on Flügge's shell theory complemented with Eringen's nonlocal theory were also developed by Ansari et al. for the free vibration analysis of single-walled [2] and multi-walled [143] carbon nanotubes and for the analysis of thermal buckling of multi-walled carbon nanotubes [3]. Similarly to the previous works, the Ritz method is applied with characteristic functions as admissible axial functions. With the model presented in reference [3] it is possible to specify the boundary conditions in a layerwise manner.

2.2 Modified Characteristic Functions

Due to the numerical problems presented by the characteristic functions of a beam, Gartner and Olgac [67] proposed a modified set of these functions so that the magnitude of the terms is in the range $[-1, 1]$. This is accomplished by replacing the hyperbolic terms by negative exponential terms. A similar idea was proposed by Dowell [56] studying the asymptotic approximations of the characteristic functions, with similar results. The modified characteristic functions were used by Gartner and Cobb [66] in a computational procedure to describe the biplanar dynamic and static responses of rotating spindle systems.

The set of modified characteristic functions was used in the Ritz method by Pao and Peterson [132] to plot the contour of free vibration and buckling mode shapes of plates. The plates analyzed were square, fully clamped, and made of isotropic and composite materials. Besides showing the peaks and valleys of the first six mode shapes, the contour plots also reveal the fiber direction of the single-layer laminate plate analyzed. Dasgupta and Huang [51] proposed a layer-wise model for the free vibration analysis of thick, arbitrarily laminated spherical panels with boundary conditions at four edges being any combination of simply-supported, free, clamped and guided in each lamina. The model relies on a displacement field described by finite element interpolation shape functions in the thickness direction and the modified characteristic functions in the in-plane and latitudinal directions. The authors state that, for a given accuracy, the combination of the Ritz and the finite element methods allows significant saving in computational resources relatively to a pure three-dimensional finite element analysis. More recently, van Hulzen et al. [80] obtained the mode shapes needed to compute the axial and radial deformations of piezoelectric tube actuators using the Ritz method with the modified characteristic functions of fixed-free and fixed-fixed beams. Moreno-García et al. [121] proposed a damage localization method where higher order derivatives of displacements of composite laminated plates are obtained by direct differentiation of the series defined by the Ritz method with modified characteristic functions. Taking advantage of the direct differentiation described in this paper, Moreno-García et al. [122, 123] proposed a technique to define an optimal spatial sampling for damage localization in laminated composite plates. The modified characteristic functions have also been used as admissible functions in the Ritz method for the analysis of delaminated carbon fiber-reinforced polymer plates by Gallego et al. [62], carbon fiber-reinforced polymer plates with Young's moduli reduction by Moreno-García et al. [120] and Moreno-García [119]. In the aforementioned three works was proposed a damage localization method based on the Ritz method and wavelet analysis. Moreover, in [119] one can find more applications of these functions: plates with local density and stiffness changes, plates with cut-outs and stepped plates. A convergence study of natural frequencies and mode shapes is also included for this last type of plates. Besides the use of the modified characteristic functions in the Ritz method mentioned above, they were also used in the formulation of hierarchical beam finite elements by Ganesan and Engels [65]. This kind of finite elements were applied to a simply supported beam and a four-bay frame. An excellent convergence of frequencies was observed.

2.3 Orthogonal Polynomials

A set of orthogonal polynomials was proposed as admissible functions in the Ritz method by Bhat [22]. The orthogonal polynomials are obtained with the Gram–Schmidt process and the paper contains an appendix showing how to obtain the first polynomial in order to start this process. This first polynomial is such that it satisfies the essential (geometric) as well as the natural boundary conditions. Simply supported and clamped plates, plates with two adjacent edges simply supported and the other two free, and plates with two adjacent edges clamped and the other two free were studied. The natural frequencies of simply supported plates were compared with exact theoretical results, whereas the natural frequencies of the other three cases were compared with results reported in Leissa [99] and Dickinson and Li [55], where characteristic beam functions and simply supported plate functions were used, respectively. It was found that the use of orthogonal polynomials leads to superior results for lower modes. Bhat [23] also used orthogonal polynomials to obtain the deflections of a fully clamped plate and a plate with three edges clamped and the other free. Both plates were subjected to uniform loading and hydrostatic loading. It was found that the results correlate well with those presented by Timoshenko and Woinowsky-Krieger [154]. A graphical comparison of the first six orthogonal polynomials and corresponding characteristic functions, as defined in [160], for clamped-clamped and clamped-free beams is also presented in [23]. This comparison shows a very close agreement between the results of these two types of functions. The orthogonal polynomials also provided good results of computations of natural frequencies and mode shapes of a rotating uniform cantilever beam, mounted on a hub, with a tip mass [24]. A parameter study of natural frequencies and mode shapes as a function of different rotational speeds and different combinations of tip mass, hub radius, and setting angles is also presented. Descriptions and applications of other type of orthogonal polynomials can be found in the extensive review written by Chakraverty et al. [38], namely orthogonal polynomials in two variables, which allow the analysis of polygonal plates [26]. It is also worth mentioning the work of Bhat [25] in which the construction of the higher order orthogonal polynomials is made with fractional power increments. The author mentions that this construction is more cumbersome than the definition of simple orthogonal polynomials.

Due to the efficiency and accuracy of the results obtained with Bhat's orthogonal polynomials for isotropic plates, this set of admissible functions has also been used in the analysis with the Ritz method of plates made of anisotropic materials. For instance, besides isotropic plates, Dickinson and Di Blasio [54] studied orthotropic plates using orthogonal polynomials and the Ritz method. For clamped-free and simply supported-free boundary conditions, they defined a simpler first polynomial that obey the essential (geometric) boundary conditions, but not the natural boundary conditions. A convergence study of the natural frequencies is performed and the results, presented in tabular and graphical format, are very satisfactory. Mode shapes, shear forces and bending moments are also computed for isotropic plates, simply supported along two parallel edges and free on the other two or clamped-free on the other two. The results are compared with those obtained using characteristic functions, degenerated functions [18], and exact results, and it is found that the use of the proposed functions yields better results than the ones in reference [18]. Frederiksen [61] applied the Ritz method with orthogonal polynomials for the free vibration analysis of completely free thick laminates using two different higher order plate theories. The first theory contains six unknown functions and the second only three. The author concluded that the second theory shows a good compromise between accuracy and computational efficiency. Nevertheless, since both theories are single layer theories they present some limitations on the analysis of single-ply and cross-ply laminates. Another application of the Ritz method with orthogonal polynomials was presented by Cupial [50]. Although the author restricted the analysis to symmetrically laminated plates and made use of the classical plate theory of composite plates, results for seven different cases of boundary conditions are reported in the paper. Nallim and Grossi [126] computed fundamental frequencies, maximum deflections and center moments of rectangular anisotropic plates with orthogonal polynomials and characteristic functions. The convergence study performed in a simply supported plate and a plate with three edges clamped and one free showed that the orthogonal polynomials are faster and the results do not present oscillations. A general approach for the study of static and dynamic responses of arbitrary quadrilateral anisotropic plates with various boundary conditions was presented by Nallim et al. [128]. The procedure relies on orthogonal polynomials and natural coordinates. The numerical results presented in the paper include trapezoidal, skew, rhomboidal, and general quadrilateral symmetrically laminated composite plates with several stacking sequences and different fiber angles. Nallim and Oller [127] extended the work reported in reference [128] to unsymmetrically laminated plates. The orthogonal polynomials are used to approximate the three field displacement components, i.e. the transverse deflection and the two in-plane stretching deformations. Very recently, Rango et al. [136] extended and generalized the method presented in references [128] and [127] for thick quadrilateral laminated plates, based on the trigonometric shear deformation theory (TSDT). Sets of orthogonal polynomials are also selected by Hu et al. [73] to define the three linear displacement functions and the two angular displacement functions of angle-ply laminated plates with twist. Frequency parameters and mode shapes are reported and the effects of fiber angle, twist angle, thickness ratio and stacking sequence is also presented in reference [73].

The orthogonal polynomials also allow the analysis of beams and plates with non-uniform properties [23]. Indeed, Bhat et al. [28] presented a comparative study of four methods in the computation of natural frequencies of plates with linearly varying thickness. One of the methods is the Ritz method with orthogonal polynomials as admissible functions and the other three are the Ritz method with functions including two exponents, the optimized Kantorovich method and the finite element method. Natural frequencies of stepped plates using the Ritz method with orthogonal polynomials were also computed by Lam and Amrutharaj [92]. The natural frequencies of plates with a linear variation of the thickness in one direction and subjected to rotational and translational elastic restrains at the edges were computed by Grossi and Bhat [69] using orthogonal polynomials in the Ritz method. The orthogonal polynomials are also used in a method developed by Muthukumaran et al. [124] to study the effect of boundary conditioning on vibrations of a rectangular plate. According to the authors, in order to achieve the required results, and thus the tuning of the structure, the boundary conditioning procedure implies the modification of translational and rotational stiffness distribution on the edges. The proposed method was later applied in reference [125] to structural tuning of a square plate, along with a fuzzy logic approach. With this approach, it is possible to obtain fuzzy sets of boundary stiffnesses and eigenvalue ratios. Liew et al. [107] analyzed moderately thick plates with the first order shear deformation theory of Mindlin. The transverse deflection and the two cross-sectional rotations are defined by three sets of orthogonal polynomials. Convergence studies for simply supported and fully clamped plates are also reported. Comparison of the results for simply supported thin and thick plates are in excellent agreement with the ones using trigonometric admissible functions. There is also a close agreement of the solutions with the three-dimensional elasticity solution results for moderately thick plates. It was also found that there are significant discrepancies between the results reported and the results of the classical plate theory for moderately thick plates, in particular for the higher modes.

A method relying on the division of a structure with complex geometry into an assembly of simpler geometries was proposed by Bhat [27]. The displacement in each one of these simpler geometries is based on orthogonal functions and the relation among them is defined by continuity conditions. An analysis of a beam fixed at both ends and with an intermediate support is reported in the paper. Two orthogonal functions in each one of the two segments are used. According to Bhat [27], the method can be viewed as a combination of the Ritz method and the finite element method and is named domain decomposition method by some authors. The extension of this method to two dimensional structures, such as isotropic plates, single layer anisotropic plates, and symmetric laminated plates was reported by Liew et al. in [105], [104], and [106], respectively. These papers present frequency parameters and mode shapes of plates with a large variety of mixed edge boundary conditions. Using Bhat's orthogonal polynomials, piecewise integration and an algorithm taking into account the continuity, Lam et al. [96] obtained the fundamental frequency of a rectangular plate with one or two rectangular cutouts, a plate with abrupt change in thickness, and a plate with a rectangular non-homogeneity. Both isotropic and orthotropic materials are considered. The method described in [96] was also applied to the analysis of simply supported and fully clamped isotropic plates with stiffened openings by Lam and Hung [94]. The authors report parametric studies of the fundamental frequency as a function of the geometry of the stiffened opening.

The buckling analysis of rectangular anisotropic plates under stress gradient and general boundary conditions was presented by Pandey and Sherbourne [130]. The authors used the results of the Ritz method with orthogonal polynomials as benchmark solutions for a comparison with results obtained with the differential quadrature method. Although the analysis is limited to uniformly distributed edge load, an extensive discussion on the buckling behavior of anisotropic composite plates with different boundary conditions is presented in the paper. Pandey and Sherbourne [131] also proposed a method for the inhomogeneity design by controlling the spatial fiber distribution in a lamina of a composite plate. The aim of the authors of reference [131] is to improve the uniaxial and shear buckling behavior of rectangular, uni-directional and cross-ply laminates under a variety of boundary conditions. The uniaxial and shear buckling analysis of rectangular, inhomogeneous, orthotropic, laminated composite plates under a variety of combinations of simple and clamped edges is performed by the Ritz method with orthogonal polynomials.

The use of orthogonal polynomials in the Ritz method for the free vibration analysis of Mindlin plates with side and internal cracks was reported by Huang et al. [76]. Since these functions are continuous and are not singular anywhere in the domain, supplementary special functions are needed to describe the existence of the crack. According to the authors of reference [76], the asymptotic solutions at the neighborhood of the crack tip are usually good candidates for these admissible functions. Thus, for a complete description of the cracked Mindlin plate, the transverse displacement of the mid-plane, and the two rotations of the mid-plane normal to each one of the two directions are defined by two sets of functions. Extensive convergence studies for simply supported and cantilevered rectangular plates with side cracks and internal cracks, as well as comparison with published results are reported in reference [76]. The approach is also used to compute natural frequencies and nodal patterns as function of the length, location, and orientation of the cracks. An extension of the approach used in reference [76] to the free vibration of functionally graded material plates with side cracks [77] and internal cracks [78] was carried out by Huang et al. In the first case, the theory used was Reddy's third-order plate theory and in the second case a three-dimensional elasticity theory was employed. Three-dimensional vibrations of rectangular parallelepipeds of functionally graded material having side cracks were also object of study by Huang et al. [79]. Very recently, Bose and Mohanty [30] used the Ritz method for the analysis of forced vibration of simply supported and clamped plates with a side crack. Like in similar works regarding cracked plates, the deflection function is defined by a first part describing the displacement of the uncracked plate and a second part is used to generate the presence of the crack. The first part is composed of orthogonal polynomials, whereas the second part are the corner functions which are expanded by finite terms of polynomials, as proposed by Hung and Leissa [74]. Although the Ritz method was also used in reference [74], the functions relative to the uncraked plate are not orthogonal polynomials, like in reference [75], but algebraic. The corner functions satisfy the natural boundary conditions of zero moment and shear force along the crack. There is a good agreement between the first five modes obtained with the Ritz method and the finite element method. The normalized mobility curves of a square plate with a side crack of arbitrary length, orientation, and position are also reported in reference [30]. The variation of the mobility is also parametrically studied for changes in crack length, angle, and position.

The free vibration characteristics of transverse shear deformable cross-ply laminated circular cylindrical shells was studied by Soldatos and Messina [148] using the Ritz method with orthogonal polynomials. The analysis relies on several shear deformable Love-type shell theories. The results were compared with available experimental data and relevant analytical results. The comparisons showed a fast convergence of the method, independently of the type of shell theory. Later, these authors extended the applicability of the method described in [148] to the advance study of the influence of edge boundary conditions on the vibration characteristics of complete cross-ply laminated cylindrical shells [117] and angle-ply laminated plates, circular cylinders and cylindrical panels [149]. More recently, similar approaches were developed by Sun et al. [152] and Song et al. [150], who studied the vibration of rotating cylindrical shells with arbitrary edges and symmetrically laminated composite cylindrical shells with arbitrary boundary conditions, respectively. The previous models based on the Ritz method are extended for the vibration analysis of the rotating laminated composite cylindrical shells with elastic edges supports by Song et al. [151].

The Ritz method with orthogonal polynomials is also used in the modeling and theoretical analysis of micro-electro-mechanical systems (MEMS) by Rinaldi et al. [138–140]. In these works the authors applied the Ritz method to obtain the natural frequencies and mode shapes, along with a quantitative experimental approach for the characterization of non-classical boundary conditions, of cantilever probes for atomic force microscopy (AFM) by electro-thermal–mechanical testing. The microcantilever end support is modeled with artificial rotational and translational springs. In more recent years, the Ritz method with orthogonal polynomials has also been applied to other coupled problems. One example is the work of Jeong and Kang [84], who developed a theoretical method to compute the natural frequencies and mode shapes of multiple rectangular plates fully in contact with a laterally bounded liquid. An approximation of the wet dynamic displacement of the plates was given by a combination of the orthogonal polynomials. A derivation of the liquid displacement potential satisfying the liquid boundary conditions is formulated and, for a compatibility requirement along the contacting surface between the plates and the liquid, the wet dynamic modal functions of the plates were expanded by the finite Fourier transform. An excellent agreement is observed between the results of the proposed method and those from a three-dimensional finite element analysis. Typical wet mode shapes of three and four rectangular plates are reported as well as studies of the effects of the number of plates and the liquid gap size on the natural frequencies. An electromechanically-coupled analytical model of piezoelectric energy harvesting skin based on Kirchhoff plate theory was proposed by Yoon et al. [159]. The Ritz method with orthogonal polynomials takes into account the inertia and stiffness effects of the surface-bonded piezoelectric patch in order to enhance the predictive capability of the electromechanically-coupled analytical model. Parashar and Kumar [133] developed a model for the calculation of natural frequencies and modes of piezoceramic cylindrical shells. The results obtained with the Ritz method were compared with the ones from shell theory, experiment and finite element analysis and a close agreement is observed. Bose and Mohanty [31] developed a sound radiation model of a cracked plate using the Ritz method. In addition to the orthogonal polynomials, corners functions are introduced to generate the crack tip singularity. It is found a good agreement between the natural frequencies obtained from the Ritz method and those obtained from the finite element method. The radiation efficiency and sound power obtained from the Ritz method are also close to those obtained from the boundary element method. A study on the variations of normalized sound power, which are shown to be due to a change in the crack parameters, is also reported.

2.4 Non-orthogonal Polynomials

Kim et al. [86] proposed a set of simpler polynomials generated by incrementing the power of the multiplying co-ordinate. This set does not form an orthogonal set and thus one losses some computational advantages. Nevertheless, according to the authors, the main advantage of these functions is that there is no need for complicated generating procedures similar to the one applied in generating the orthogonal polynomials. The evaluation of the integrals is also much simpler. This set of non-orthogonal polynomials is equivalent to the one obtained using the orthogonally generated polynomials if, in the process of generating this polynomials, a constant weight function and a specific starting function are used. The authors demonstrate this equivalence in an appendix presented in reference [86]. The frequency parameters of different isotropic plates, with various boundary conditions, springs at points and concentrated masses are also investigated. Square isotropic simply supported and clamped plates under shear load and a square clamped orthotropic plate under hydrostatic in-plane loads are also studied. A study of transitions from fully simply supported to fully clamped plate, from fully free to fully simply supported plate, and from fully free to fully clamped plate is also reported. The transitions are accomplished by applying increasingly rotational and/or translational springs of equal value along the four edges. Several of the results reported are compared with those available in the literature.

The non-orthogonal polynomials proposed in [86] are also used by Young and Dickinson [163] for the analysis of free vibration of rectangular plates with straight or curved internal line supports. Natural frequencies and mode shapes are reported for simply supported plates with oblique straight line internal supports, with curved line supports extending between two diagonally opposite corners, and with central internal circular supports. The natural frequencies for the two first cases are compared with available results found in the literature. In order to study the dynamic response of laminated angle-ply plates with clamped conditions subjected to explosive blast loading, Lam and Chun [93] computed the mode shapes using the Ritz method with the non-orthogonal polynomials. They applied the mode superposition method to obtain the displacement, at different times, of symmetrically, anti-symmetrically, and non-symmetrically square plates with four layers and found out that the symmetrically stacked plate has the least central deflection. The set of non-orthogonal polynomials proposed by Kim et al. [86] was also used in the Ritz method by Fasana and Marchesiello [59] for the vibration analysis of sandwich beams. Both the transverse and in-plane displacements are considered in the analysis. The loss factor of the beam is computed by substituting the shear modulus of the core with the complex form. In such a case, the ratio of the imaginary part to the real part of each eigenvalue gives the loss factor of the beam at each vibration frequency. Only simply supported and clamped-free beams are analyzed, but the paper presents a large number of comparisons with values found in the literature. The authors state that the results have a good agreement with the values reported in literature. Young [162] expressed the three displacements of thick shells arbitrarily deep in one direction with the non-orthogonal polynomials and the Ritz method was used to obtain the frequency parameters of simply supported and cantilevered cylindrical panels, cylindrical shells with shear diaphragm conditions at both ends, and closed barrel shells clamped at both ends. The results obtained are compared with published results or with results from the finite element method.

2.5 Product of Trigonometric Functions

Functions defined by the product of trigonometric functions can also be used in the Ritz method. The first application of this type of functions was carried out by Chai [32]. This researcher studied the free vibration of plates with and without a concentrated mass. The study relies on the Rayleigh method and a single term of the product of trigonometric functions is used. Comparative results for a square plate and the effect on the first natural frequency of the concentrated mass placed at the center and along the length of the plate are reported in [32]. It was found out that, although the first natural frequency for the concentrated mass placed at the center of a plate with two opposite simply supported edges and two opposite clamped edges is well predicted, the same is not true when the mass is placed away from the center. A related and complementary work to the one reported in [32] was published by Low et al. [110]. Later, the results for off-center concentrated mass were improved by Chai [35]. The author used the Ritz method to compute the first natural frequency and observed that the percentage difference between experimental results and the computed results with 100 terms is in the range $\pm 15\%$, whereas this percentage can be as high as 43% with just one term. The case of a plate clamped in two opposite edges and simply supported in the other two is extensively studied in reference [34]. Besides this type of plate, Low et al. [112] analyzed fully clamped plates with an attached mass in a large number of positions and found out that the series with 100 terms can generally predict well the first three experimental natural frequencies of the plates regardless of the mass position, whereas if only one term is used the estimation is good only for the first natural frequency. Plots of frequency ratio as function of length ratio describing the position of the mass are reported. It was found that these plots show similar trends when the mass attached to the plates is different. The authors also describe a relation between the mode shape of the plates and the plots of the frequency ratio versus the length ratios in the two directions. Thus, it is possible to predict the mode shape using the iso-frequency curves. A complementary work to the one described in [112] was presented by Low et al. [113], who used a spectrum analyzer and a TV-holographic system to obtain mode shapes of plates with mounted masses. The experimental data thus obtained was used to validate the results from the Ritz method with product of trigonometric functions. It is suggested in this work that more terms in the series are needed to analyze the higher modes and heavier loads.

Chai et al. [37] developed a numerical model based on the product of trigonometric functions and the Ritz method for the study of tension effects on natural frequencies of clamped beams with a mass at mid-span. The main objective of including the tension effects was to justify the poor correlation of theory and experiment results for thin beams, as reported previously in reference [111]. Four beam specimens with different dimensions and properties were studied and a good correlation between experimental data and the proposed numerical model results is observed when the tension at the extremities of the beams are accounted for. A single term of the product of trigonometric functions was also used by Chai and Khong [36], along with the strip method for the buckling analysis of laminated composite plates. The results are compared with those available in the literature and a parametric study is also carried out. The authors found out that the finite strip method may not be able to predict the buckling loads of laminated plates with significant magnitudes of the bending/twisting stiffnesses. A more complete study of buckling analysis of laminated composite plates is presented by Chai [33]. In this work, a series of product of trigonometric functions is used in the Ritz method. The results with a series of 144 terms are compared with the ones in reference [36]. It was observed that the results correlate reasonably well with experimental data of five different plates with two opposite clamped edges and the other two edges simply supported, although in a case a different of +19% is observed. Another two sets of experimental data of eleven fully clamped plates and eleven plates with two opposite clamped edges and the other two edges simply supported are compared to the results and, in general, a good agreement is verified. The convergence studies show that 144 terms are necessary and sufficient to provide a converged solution. Very recently, Pirnat et al. [134] developed a structural-acoustic model of a rectangular plate-cavity system with an attached distributed mass and internal sound source. The authors used the Ritz method because the finite element was found unsuitable since it gives a solution in discrete points and not as a continuous function. The natural frequencies and mode shapes of simply supported and clamped plates with attached mass were computed with different numbers of terms in the series defined by sine functions and products of trigonometric functions, respectively. The model was validated by comparing its results with those obtained experimentally.

2.6 Static Beam Functions

The use of static beam functions as admissible functions in the Ritz method was first proposed by Zhou [164]. These functions are based on the general solution of the differential equation of a Euler–Bernoulli beam under a point load. By varying the location of the point load applied to the beam a set of third order polynomial functions can be defined. The coefficients of the polynomials are determined from the boundary conditions. Besides the frequency parameters of a clamped beam, the eigenvalues of five plates with different boundary conditions are also computed. A close agreement with exact solution is observed, as well as a good accuracy relatively to results with characteristic functions. However, according to Zhou [164], the computations of the matrices are simpler with static beam functions than with characteristic functions. It was also found that plates with clamped and/or simply supported edges present eigenvalues more accurate than those of a plate with free edges. The accuracy of the eigenvalues of a plate with no free corners is better than the accuracy of the eigenvalues of a plate with free corners. Lee and Lee [98] also applied the static beam functions to the free vibration analysis of rectangular plates on elastic point supports. The authors report the first three modes of a square plate with all four edges simply supported and with a central elastic point support. A study on the influence of the stiffness and the location of the elastic point support on the frequency parameter is also presented in reference [98]. The static beam functions based on a point load have been recently used by Lee et al. [97] in the free vibration analysis of simply supported and clamped thin isotropic plates with voids. The position of the voids is arbitrary and they are defined continuously by using the extended Dirac function. The numerical results are in agreement with alternative ones, but, according to the authors of reference [97], the model presented is less computational expensive.

A different set of static beam functions was proposed by Zhou [166]. In this case, the functions are the solution of the differential equation of a beam where the loads are expanded into a sine series. These static beam functions are a combination of a sine series and a third order polynomial. The coefficients of the polynomial are defined by the boundary conditions. The accuracy and convergence is illustrated by numerical results of free vibration of isotropic rectangular plates with various aspect ratios and boundary conditions. This type of admissible functions was also used by Zhou [165] to determine natural frequencies of rectangular plates with elastic translational and/or rotational edges. The coefficients of the third order polynomial depend on the translational and rotational stiffnesses at the edges. In reference [165] several numerical results are reported for square plates with various symmetrically distributed elastic stiffnesses. Although the convergence is very good, the accuracy worsens for small stiffnesses of the elastic restraints. This is because these admissible functions cannot be applied to fully free plates.

The free vibration analysis of laminated composite plates has also been carried out with static beam functions. Cheung and Zhou [41] established relations between point loaded beams and point supported plates, thus defining a new set of admissible functions which are composed of the static solutions of a beam under sine loads and under point loads. Unlike other functions, the proposed functions directly satisfy both the geometric boundary conditions and the zero displacements at the point supports. Besides results for isotropic square plates with point supports and symmetrically laminated square plates, which are compared with results available in the literature, results for point supported plates with different angles of fiber orientations, material properties, numbers of layers, locations of point supports and aspect ratios are also reported in reference [41]. After the studies described in reference [41], Cheung and Zhou published two papers where static beam functions were used for the free vibration analysis of rectangular symmetrically [45] and unsymmetrically [44] laminated composite plates with intermediate line supports. These authors also reported in reference [43] the use of static beam functions for the analysis of orthotropic rectangular plates with elastic intermediate line supports and edge constraints. In this work, the elastic rotational and the elastic translational constraints along the edges of the plate are considered simultaneously.

Tapered plates have also been analyzed by Cheung and Zhou [39]. They developed a set of admissible functions which are the solutions of a tapered beam under an arbitrary load expanded into a Taylor series. This beam can be viewed as a strip with unit breadth taken from the rectangular plate in the longitudinal or transverse directions. Extensive results for plates with different taper factors and boundary conditions are reported in reference [39]. The free vibration of tapered beams has also been object of study with static beam functions, as proposed by Zhou and Cheung [169]. Cheung and Zhou demonstrate that it is also possible to analyze tapered plates with intermediate line supports by using functions which are the solutions of a tapered beam with intermediate supports [40]. In a similar fashion, point supported plates with variable thickness can also be studied using a set of static beam functions, as reported by Zhou [168]. Mindlin plates were also studied using static Timoshenko beam functions for different cases, like arbitrary boundary conditions and thickness [42], elastically restrained edges [167], internal line supports [171], and variable thickness [46]. Zhou and Cheung [170] also studied the vibrations of tapered Timoshenko beams using the functions of Timoshenko beams with continuously varying cross section. Sets of static Timoshenko beam functions have also been used more recently in the free vibration and buckling analysis of vertical and horizontal Mindlin plates resting on a Pasternak elastic foundation and in contact with a fluid by Hosseini-Hashemi et al. [71, 72].

3 Ritz Method for the Analysis of Free Vibrations

In the analysis of free vibrations, the Ritz method relies on the Hamilton's principle, which in turn is based on strain and kinetic energies. The strain energy stored in an elastic body of volume V is given by

$$U = \frac{1}{2}\int_V (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \sigma_{xy}\epsilon_{xy} + \sigma_{xz}\epsilon_{xz} + \sigma_{yz}\epsilon_{yz})dV, \tag{1}$$

where σ_x, σ_y, σ_z, σ_{xy}, σ_{xz}, σ_{yz} are the stress components, and ϵ_x, ϵ_y, ϵ_{xy}, ϵ_{xz}, ϵ_{yz} are the strain components. Considering a rectangular plate of length a, width b and thickness h with a Cartesian system of coordinates, as shown in Fig. 1, and bearing in mind the Kirchhoff assumptions, the Equation above simplifies to

$$U = \frac{1}{2}\int_V (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_{xy}\epsilon_{xy})dV. \tag{2}$$

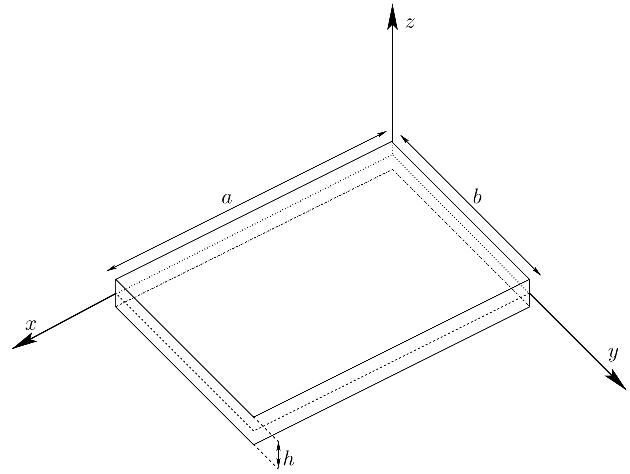

Fig. 1 Geometry of a rectangular plate

Taking into account the constitutive relations, the kinematic assumptions, and integrating in the z direction, Eq. (2) can be written as a function of the out-of-plane displacement $w(x, y, t)$ and the bending stiffness $D = Eh^3/\left[12(1-\nu^2)\right]$, being ν the Poisson's ratio:

$$U = \frac{1}{2}D\int_A \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] dA, \tag{3}$$

where A is the in-plane surface area of the plate. The kinetic energy of a body with density ρ is given by

$$T = \frac{1}{2}\rho\int_V \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dV, \tag{4}$$

where $u(x, y, t)$ and $v(x, y, t)$ are the in-plane displacements, and $w(x, y, t)$ is the out-of-plane displacement. For a rectangular plate subjected to out-of-plane vibrations, such that $u(x, y, t) = v(x, y, t) = 0$, and after integrating in the z direction, Eq. (4) reduces to

$$T = \frac{1}{2}\rho h\int_A \left(\frac{\partial w}{\partial t}\right)^2 dA. \tag{5}$$

Considering the free vibration of a plate with harmonic motion of angular frequency ω, the kinetic energy becomes

$$T = \frac{1}{2}\rho h\omega^2\int_A w^2 dA. \tag{6}$$

The Ritz method is based on the minimization of the energy functional defined by the difference between the kinetic energy T and the strain energy U:

$$\frac{\partial (T-U)}{\partial W_{kl}} = 0, \tag{7}$$

with $k = 1, \dots, M$; $l = 1, \dots, N$; and where W_{kl} are parameters or coefficients of a series describing the out-of-plane displacement:

$$w(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{N} W_{mn} X_m(x) Y_n(y), \tag{8}$$

with $X_m(x)$ and $Y_n(y)$ being admissible functions that must satisfy the essential boundary conditions at the edges $(x = 0, y)$, $(x = a, y)$ and $(x, y = 0)$, $(x, y = b)$, respectively. As seen in Eq. (8), there are M admissible functions in the x direction and N admissible functions in the y direction. Inasmuch as W_{mn} are parameters without dependence of the spatial variables, the derivative of order p of the out-of-plane displacement with respect to x can be easily computed as:

$$\frac{\partial^p w(x, y)}{\partial x^p} = \sum_{m=1}^{M} \sum_{n=1}^{N} W_{mn} \frac{d^p X_m(x)}{dx^p} Y_n(y). \tag{9}$$

The derivatives in the y direction are computed in a similar way.

Since the kinetic and strain energies are quadratic in $w(x, y)$, and the vibration is harmonic, Eq. (7) defines an eigenvalue problem of size $M \times N$:

$$\mathbf{KW} = \mathbf{MW}\lambda, \tag{10}$$

where the elements of matrices $\mathbf{K}$ and $\mathbf{M}$ for a plate with bending stiffness D, Poisson's ratio ν, material density ρ, in-plane area A and thickness h are given by

$$\begin{aligned} K_{kl} = \textstyle\sum_{m=1}^{M} \sum_{n=1}^{N} D \int_A \Bigg[& \frac{d^2 X_k}{dx^2} \frac{d^2 X_m}{dx^2} Y_l Y_n \\ & + \nu \left(\frac{d^2 X_k}{dx^2} X_m Y_l \frac{d^2 Y_n}{dy^2} + X_k \frac{d^2 X_m}{dx^2} \frac{d^2 Y_l}{dy^2} Y_n \right) \\ & + X_k X_m \frac{d^2 Y_l}{dy^2} \frac{d^2 Y_n}{dy^2} + 2(1 - \nu) \frac{dX_k}{dx} \frac{dX_m}{dx} \frac{dY_l}{dy} \frac{dY_n}{dy} \Bigg] dA, \end{aligned} \tag{11}$$

$$M_{kl} = \rho h \sum_{m=1}^{M} \sum_{n=1}^{N} \int_A X_k X_m Y_l Y_n dA. \tag{12}$$

The matrices $\mathbf{K}$ and $\mathbf{M}$ are real positive definite if one considers a motion without rigid modes, i.e. if the plate has isostatic or hyperstatic conditions. In this case the eigenvalues and eigenvectors are real numbers greater than zero. The solution of the eigenvalue problem consists in the full matrix $\mathbf{W}$ and the diagonal matrix λ, containing the parameters W_{mn} and the angular natural frequencies $\omega_n = \sqrt{\lambda_n}$, respectively. Each row of $\mathbf{W}$ contains a set of parameters W_{mn}, which after being introduced in Eq. (8), allow the computation of the mode shape corresponding to the respective eigenvalue. It should be pointed out that in the present work all the integrals in Eqs. (11) and (12) were computed analytically, since this type of computation is much more efficient than numerical integration. For instance, the computation of matrices $\mathbf{K}$ and $\mathbf{M}$ using the Simpson's rule is about forty times slower than the direct computation with analytical integrations [121].

The six sets of admissible functions for the direction x, $X_m(x)$, of a clamped rectangular plate are defined as follows:

(1) Characteristic Functions (CF) [160]:

$$\begin{aligned} X_m(x) = {} & A_m \cos\left(\frac{\gamma_m x}{a}\right) + B_m \sin\left(\frac{\gamma_m x}{a}\right) \\ & + C_m \cosh\left(\frac{\gamma_m x}{a}\right) + D_m \sinh\left(\frac{\gamma_m x}{a}\right), \end{aligned} \tag{13}$$

where the parameters γ_m are the solutions of the non-linear equation

$$\cos(\gamma_m) - \frac{2e^{-\gamma_m}}{1 + e^{-2\gamma_m}} = 0, \tag{14}$$

and with

$$A_m = -1, \quad B_m = -\frac{\cosh(\gamma_m) - \cos(\gamma_m)}{\sinh(\gamma_m) - \sin(\gamma_m)},$$
$$C_m = 1, \quad D_m = \frac{\cosh(\gamma_m) - \cos(\gamma_m)}{\sinh(\gamma_m) - \sin(\gamma_m)}.$$

The functions in Eq. (13) are, therefore, the same as the ones that define the mode shapes of a vibrating uniform clamped beam.

(2) Modified Characteristic Functions (MCF) [67]:

$$\begin{aligned} X_m(x) = {} & A_m \cos\left(\frac{\gamma_m x}{a}\right) + B_m \sin\left(\frac{\gamma_m x}{a}\right) \\ & + C_m e^{\frac{-\gamma_m x}{a}} + D_m e^{\frac{-\gamma_m (a-x)}{a}}, \end{aligned} \tag{15}$$

with

$$A_m = 1, \quad B_m = -\frac{1 + (-1)^m e^{-\gamma_m}}{1 - (-1)^m e^{-\gamma_m}},$$
$$C_m = -\frac{1}{1 - (-1)^m e^{-\gamma_m}}, \quad D_m = \frac{(-1)^m}{1 - (-1)^m e^{-\gamma_m}},$$

Table 1 Number of functions for which the computation fails and type of failure for each set of admissible function

Set of admissible functions	Number of functions M	Type of failure
CF	8	Results with complex eigenvalues
MCF	97	Huge increase of computational time
OP	9	Results with complex eigenvalues
NOP	8	Results with complex eigenvalues
PTF	96	Huge increase of computational time
SBF	73	Results with complex eigenvalues

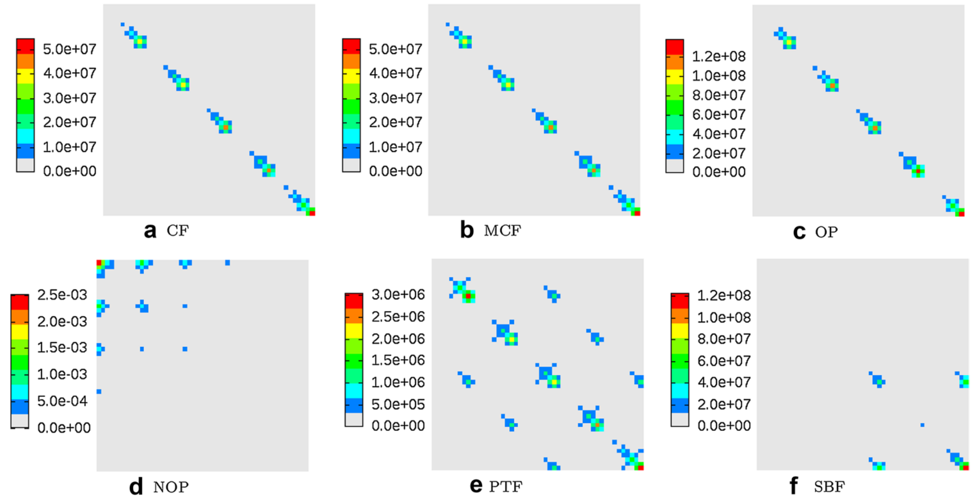

Fig. 2 Graphical representation of the matrix **K** using $M = 5$ and the six sets of admissible functions. (Color figure online)

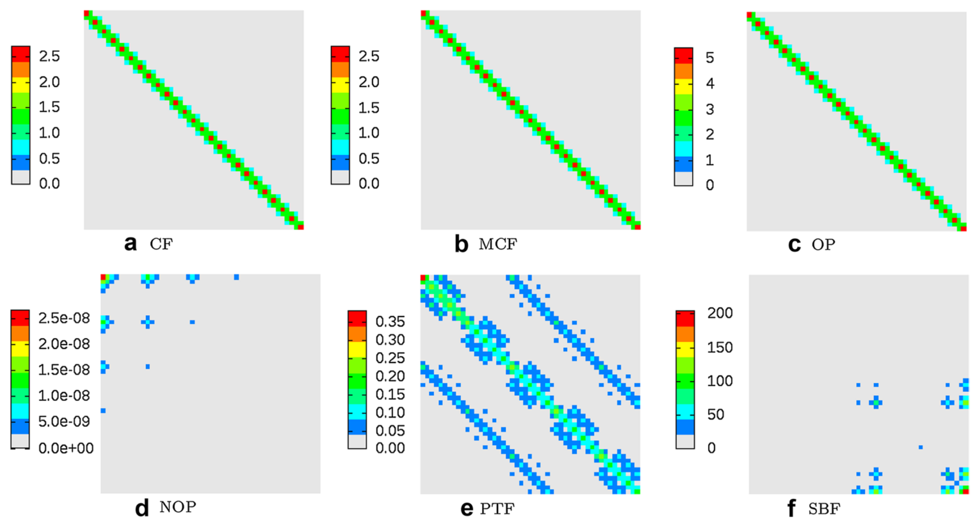

Fig. 3 Graphical representation of the matrix **M** using $M = 5$ and the six sets of admissible functions. (Color figure online)

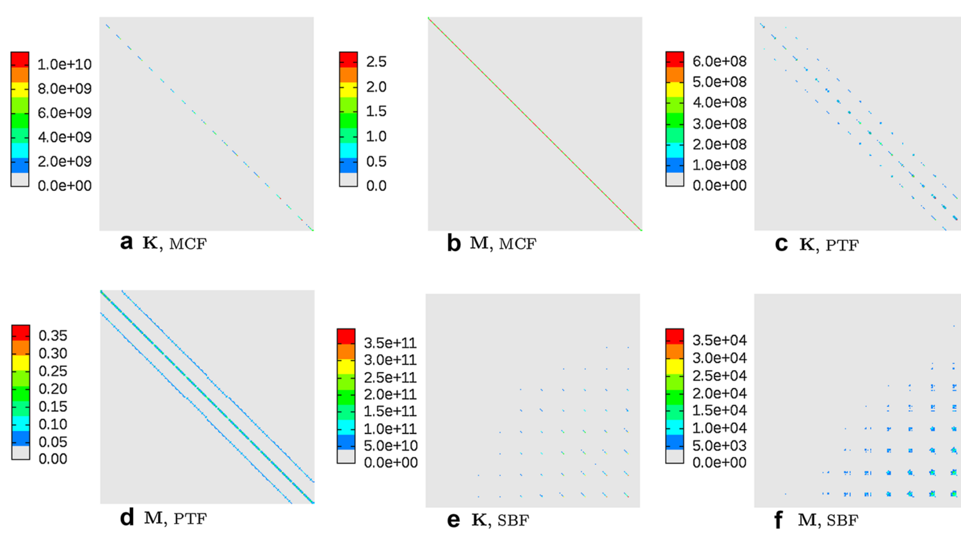

Fig. 4 Graphical representation of the matrices **K** and **M** using $M = 20$ and three sets of admissible functions. (Color figure online)

and where the parameters γ_m are the same ones as the ones needed to define the characteristic functions [Eq. (14)].

(3) Orthogonal Polynomials (OP) [22]: Given the polynomial,

$$X_0(x) = \frac{A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4}{\sqrt{\int_0^a \left[A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4\right]^2 dx}}, \quad (16)$$

satisfying the essential (geometric) boundary conditions, the remaining orthogonal polynomials of the set are generated by the Gram–Schmidt process:

$$X_1(x) = \frac{(x-B_1)X_0(x)}{\sqrt{\int_0^a \left[(x-B_1)X_0(x)\right]^2 dx}},$$
$$X_m(x) = \frac{((x-B_m)X_{m-1}(x) - C_mX_{m-2}(x))}{\sqrt{\int_0^a \left[(x-B_m)X_{m-1}(x) - C_mX_{m-2}(x)\right]^2 dx}}, \quad (17)$$

with

$$B_m = \int_0^a xX_{m-1}^2(x)dx \Big/ \int_0^a X_{m-1}^2(x)dx,$$
$$C_m = \int_0^a xX_{m-1}(x)X_{m-2}(x)dx \Big/ \int_0^a X_{m-2}^2(x)dx.$$

For clamped plates, the coefficients of the polynomial are $A_0 = A_1 = 0$, $A_2 = a^2$, $A_3 = -2a$ and $A_4 = 1$.

(4) Non-Orthogonal Polynomials (NOP) [86]:

$$X_m(x) = (a - x)^2 x^{m+1}. \quad (18)$$

(5) Product of Trigonometric Functions (PTF) [32]:

$$X_m(x) = \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right). \quad (19)$$

(6) Static Beam Functions (SBF) [166]:

$$X_m(x) = A_m + B_mx + C_mx^2 + D_mx^3 + \sin\left(\frac{m\pi x}{a}\right), \quad (20)$$

where

$$A_m = 0, \quad B_m = -\frac{m\pi}{a},$$
$$C_m = \frac{m\pi((-1)^m + 2)}{a^2},$$
$$D_m = -\frac{m\pi((-1)^m + 1)}{a^3}.$$

Since both edges in the y direction, $(x, y = 0)$, $(x, y = b)$, are also clamped, the functions $Y_n(y)$ are defined in a similar way. Thus, the admissible functions $Y_n(y)$ can be defined for the y direction by replacing the variable x for y, the length a for the width b, and m for n in the Equations above. To better grasp the differences of these sets of functions, the plots of functions with $m = 1, 2, 10$ and 15 are presented in The Appendix.

4 Results of the Comparative Study

The plate analyzed is a clamped rectangular plate with constant thickness and the standard properties of aluminum ($E = 70$ GPa, $\nu = 0.33$ and $\rho = 2700$ kg/m^3). The length, width and thickness dimensions of the plate are $1 \times 0.5 \times 0.002$ m^3, respectively. The computations presented in this work were carried out in an Intel® Core™ i7 with 8 cores at 3.4 GHz, 8 GB of RAM, and double precision calculations in MATLAB® platform. In all the computations, the number of functions is the same in the two directions and thus $M = N$ in Eqs. (8), (11) and (12). Table 1 shows the values of M for which the computation fails and the type of failure for each set of admissible

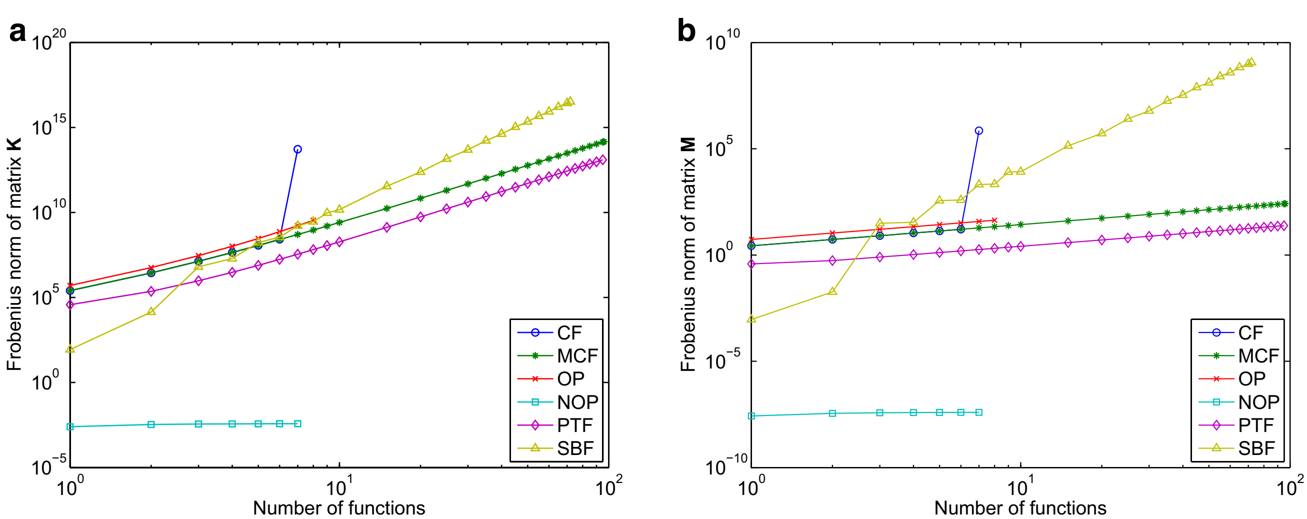

Fig. 5 Frobenius norm of (**a**) matrix **K** and (**b**) matrix **M** versus the number of functions. (Color figure online)

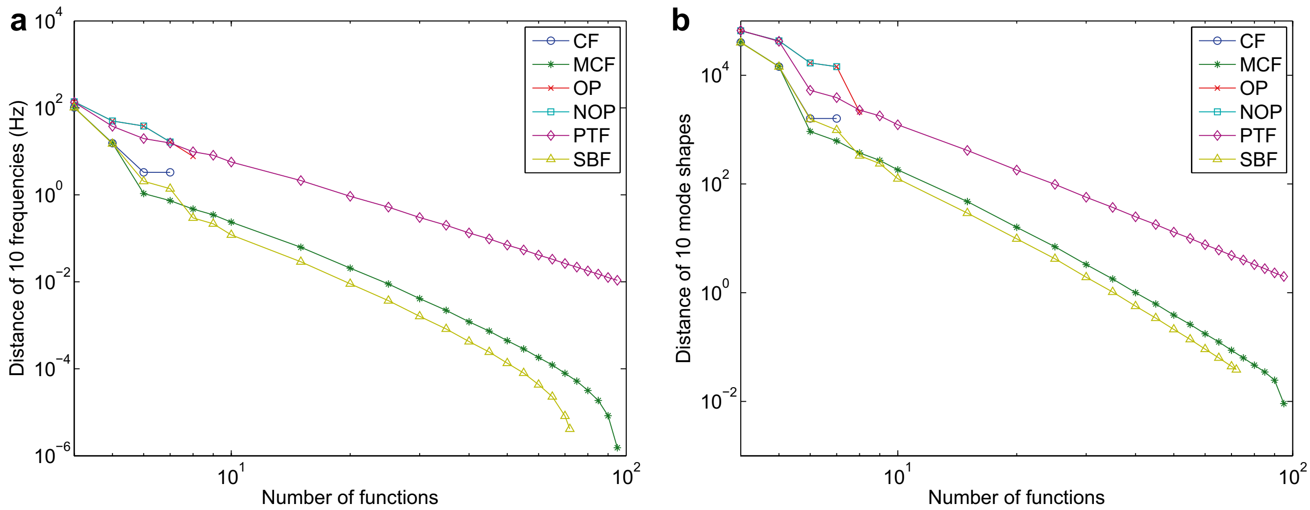

Fig. 6 Distance of (**a**) the frequencies and (**b**) the mode shapes to the references versus the number of functions. (Color figure online)

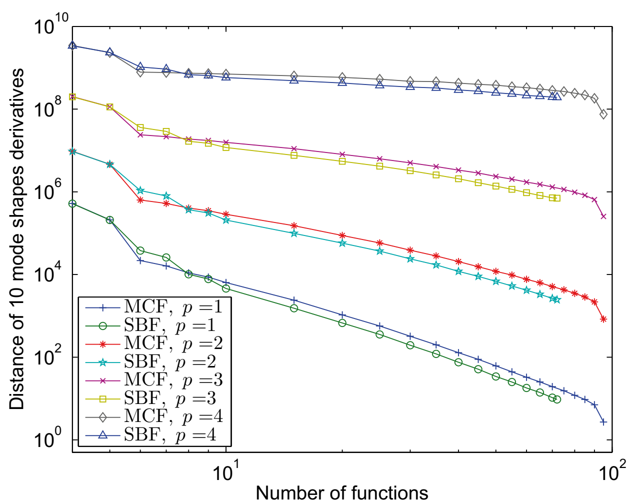

Fig. 7 Distance of the mode shapes derivatives to the references versus the number of functions. (Color figure online)

functions. A first type of failure observed is such that one obtains eigenvalues in the complex domain and, thus, natural frequencies with real and imaginary parts (CF, OP, NOP and SBF functions). It is also observed that some functions (MCF and PTF functions) and some values of M lead to a huge increase in the computational time comparatively to the time in the computation with $M-1$. In these cases the computations were stopped without obtaining a result. This problem is mainly due to the amount of physical memory available and the need to use the swap memory for the computations.

4.1 Matrices Topology

In this section the topology of the matrices **K** and **M** generated by the Ritz method [see Eqs. (10), (11) and (12)] will be analyzed. Figures 2 and 3 present a graphical representation of the matrices **K** and **M**, respectively, for the six sets of admissible functions and $M = 5$. It can be seen that the CF, MCF and OP functions lead to matrices where the higher values are close to the diagonal, being more scatter the matrices obtained when the NOP, PTF and SBF functions are used. The NOP functions present the lowest values of K_{kl}, being this a remarkable behavior of these functions. This type of functions show a decrease of the K_{kl} values as k and l increase. An inverse and also remarkable behavior is observed for the SBF functions. To analyze the behavior with a higher number of terms in the series defining the displacement [see Eq. (8)], Fig. 4 shows a graphical representation of the matrices **K** and **M** using $M = 20$ for the MCF, PTF and SBF functions. The results are similar to the ones in Figs. 2 and 3b, e, f, respectively.

To study quantitatively the variation of K_{kl} and M_{kl} with the number of functions, the Frobenius norms of the matrices were computed. The Frobenius norms of matrices **K** and **M** are defined, respectively, by:

$$
\begin{aligned}
||\mathbf{K}||_F &= \sqrt{\sum_{k=1}^{M\times N}\sum_{l=1}^{M\times N} |K_{kl}|^2} \\
||\mathbf{M}||_F &= \sqrt{\sum_{k=1}^{M\times N}\sum_{l=1}^{M\times N} |M_{kl}|^2}
\end{aligned}
\tag{21}
$$

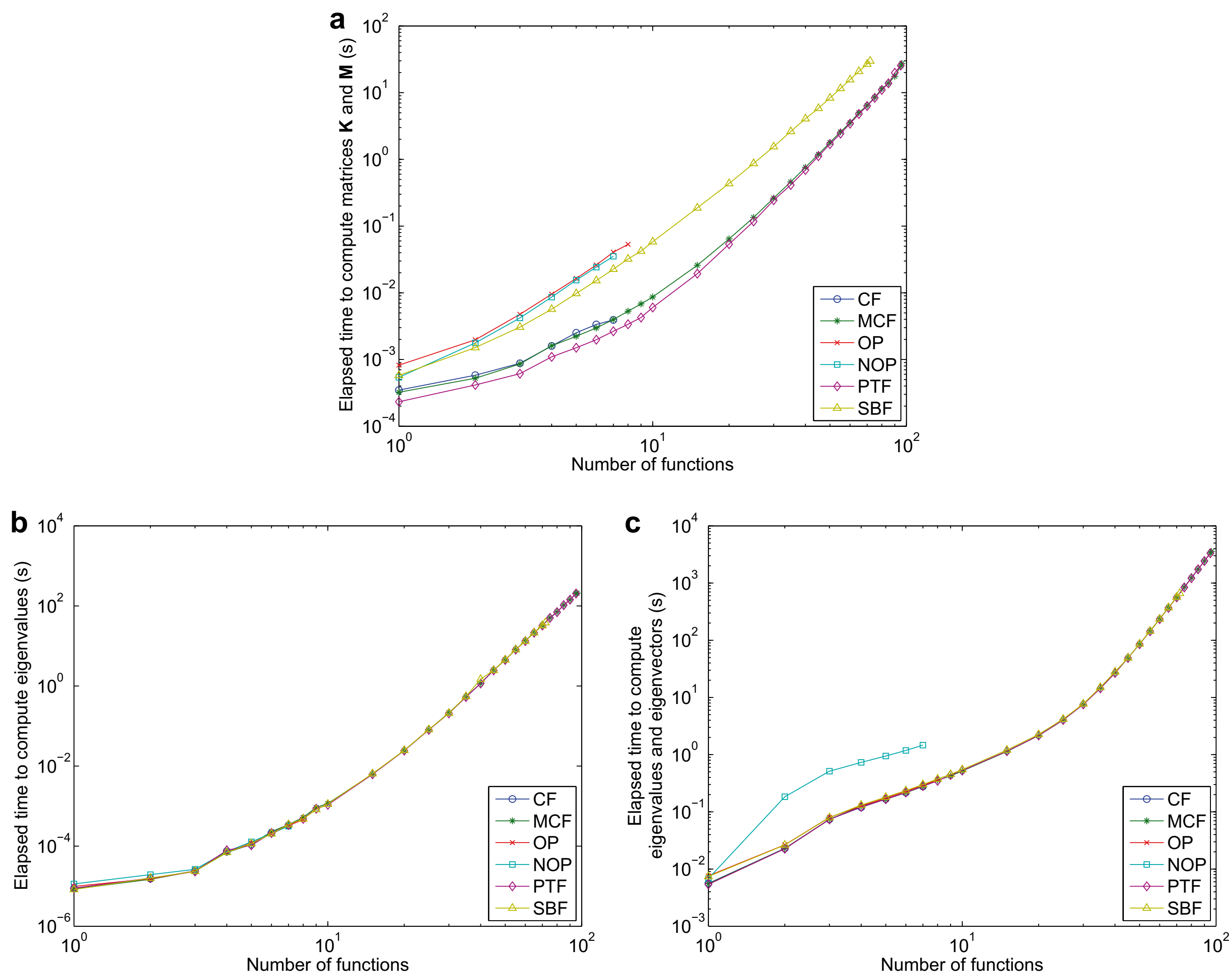

Fig. 8 Elapsed time to compute (**a**) the matrices **K** and **M**, (**b**) the eigenvalues and (**c**) the eigenvalues and eigenvectors versus the number of functions. (Color figure online)

Figure 5 shows the values of the Frobenius norm of matrices **K** and **M** versus the number of functions, using different sets of admissible functions. It is observed that the NOP functions present very low values, in agreement with what is observed in Figs. 2 and 3. The OP, PTF and MCF functions present moderate values of the Frobenius norm, being the SBF functions the ones with higher values. The CF functions have a very large increase from $M = 6$ to $M = 7$.

4.2 Convergence

To study the convergence in respect to the number of functions used, the distance to a reference was defined, both for frequencies and mode shapes. This reference was chosen as the solution obtained using MCF with $M = 96$, which corresponds to the maximum number of functions used in this work. Therefore, the distance d_f relative to frequencies was defined as:

$$d_f = \sum_{i=1}^{nm} \left(f_i - f_i^{(\mathrm{ref})} \right), \tag{22}$$

where nm is the number of mode shapes taken into account, f_i the computed frequencies and $f_i^{(\mathrm{ref})}$ the frequencies of the reference. Also, to compare the mode shapes and its derivatives, the corresponding values were computed in a square mesh of 200 points in the x-direction and 100 points in the y-direction, and the distance for the transversal displacement was defined as:

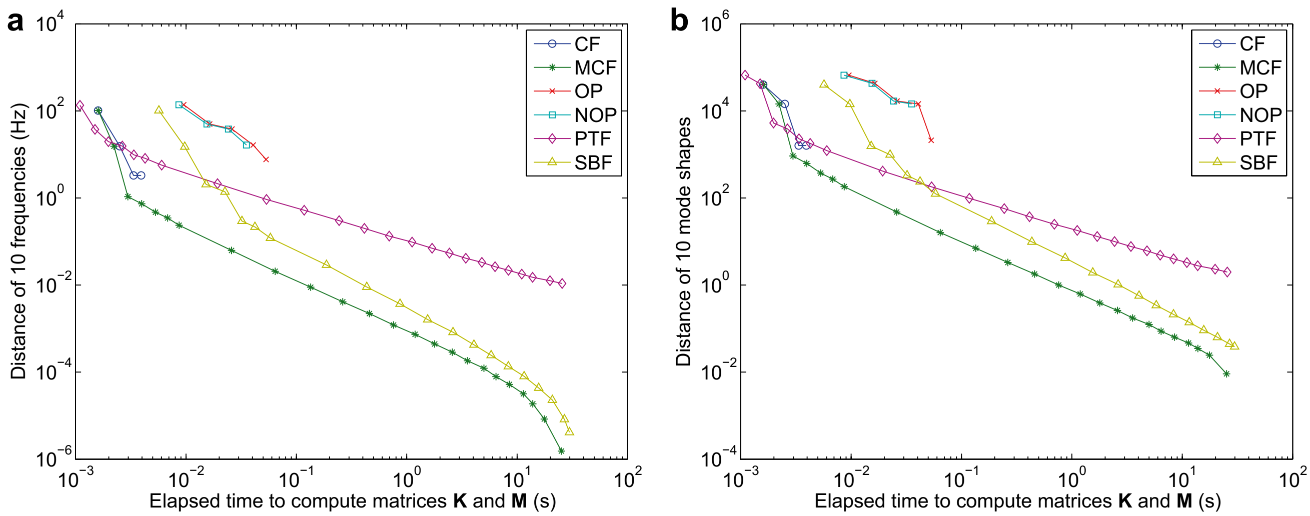

Fig. 9 Distance of (**a**) the frequencies and (**b**) the mode shapes to the references versus elapsed time to compute the matrices **K** and **M**. (Color figure online)

$$d_m = \frac{\sum_{i=1}^{nm}\sum_{j=1}^{np}\left|w_{i,j}(x,y) - w_{i,j}^{(\mathrm{ref})}(x,y)\right|}{\dfrac{\sum_{i=1}^{nm}\sum_{j=1}^{np}\left(w_{i,j}^{(\mathrm{ref})}(x,y)\right)}{nm \times np}}, \tag{23}$$

where np is the number of points in the mesh ($np = 200 \times 100$), nm is the number of mode shapes considered ($nm = 10$), $w_{i,j}(x, y)$ is the value of the transversal displacement of the computed mode shape i in point j and $w_{i,j}^{(\mathrm{ref})}(x, y)$ is the corresponding value of the transversal displacement of the reference mode shape in the same point.

To study the convergence of the mode shape derivatives the following distance was defined and used:

$$d_m^{(p)} = \frac{\sum_{i=1}^{nm}\sum_{j=1}^{np}\left|\dfrac{\partial^p w_{i,j}(x,y)}{\partial x^p} - \dfrac{\partial^p w_{i,j}^{(\mathrm{ref})}(x,y)}{\partial x^p}\right|}{\dfrac{\sum_{i=1}^{nm}\sum_{j=1}^{np}\left(\dfrac{\partial^p w_{i,j}^{(\mathrm{ref})}(x,y)}{\partial x^p}\right)}{nm \times np}}, \tag{24}$$

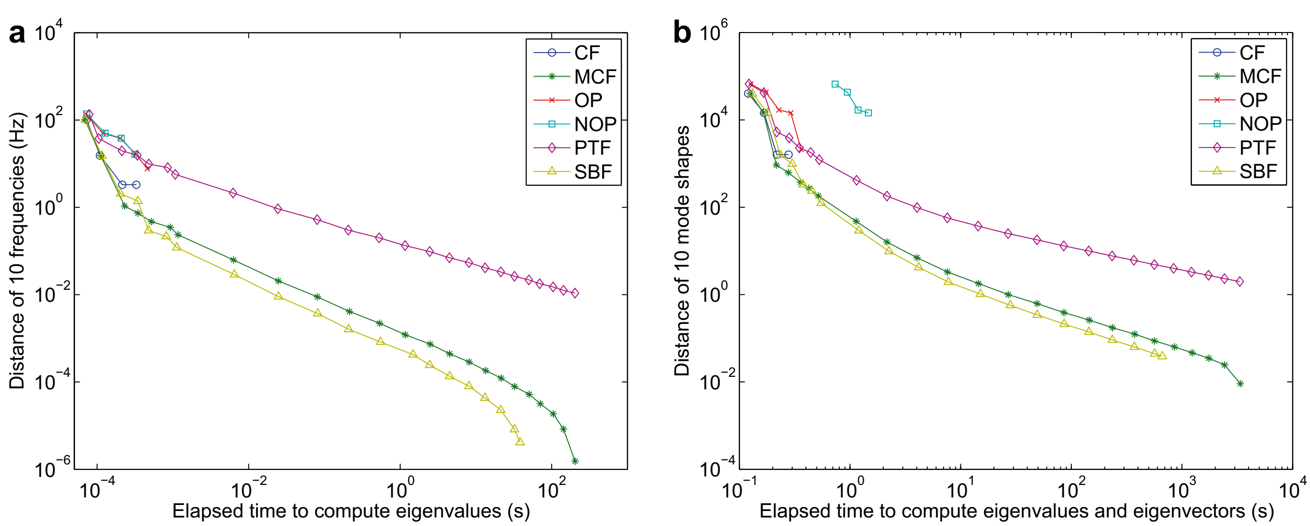

Fig. 10 Distance of (**a**) the frequencies and (**b**) the mode shapes to the references versus the elapsed time to compute (**a**) the eigenvalues and (**b**) the eigenvectors. (Color figure online)

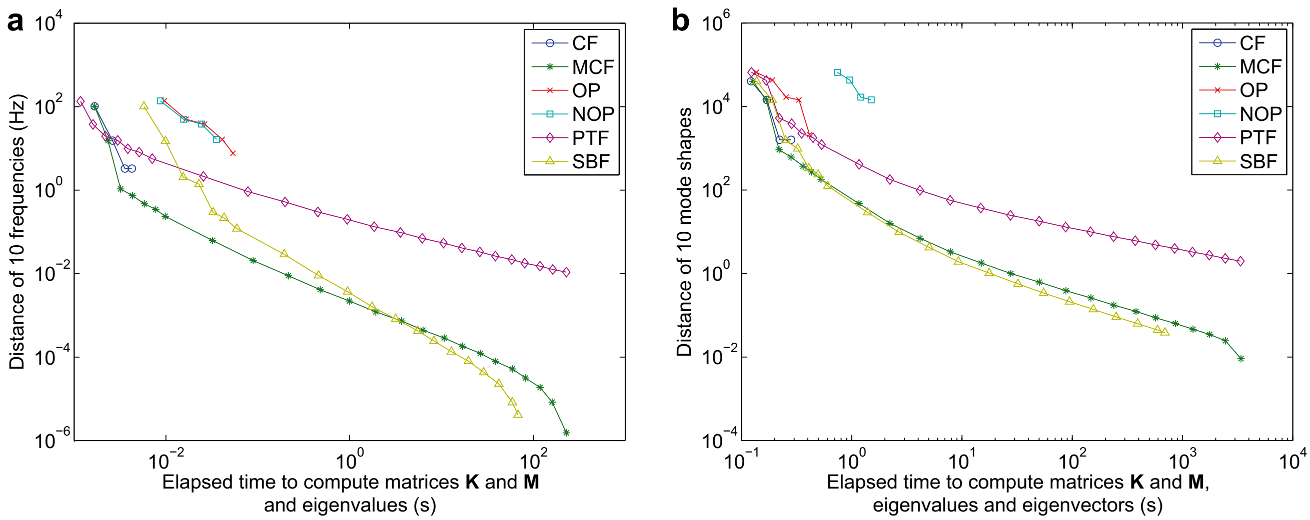

Fig. 11 Distance of (**a**) the frequencies and (**b**) the mode shapes to the references versus the elapsed time to compute (**a**) the matrices **K** and **M** and eigenvalues and (**b**) the matrices **K** and **M**, eigenvalues and eigenvectors. (Color figure online)

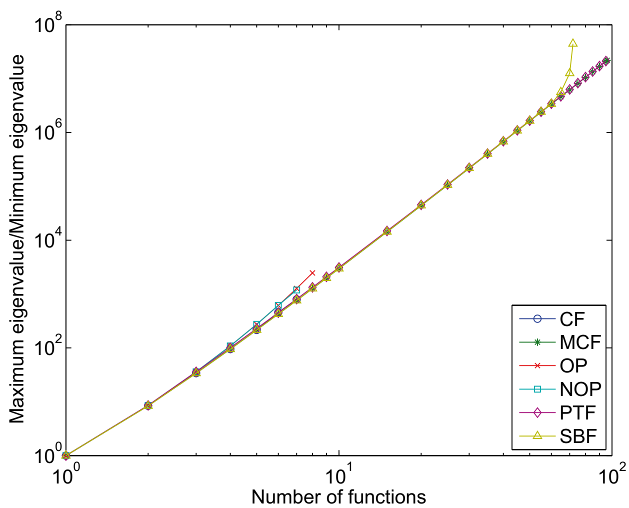

Fig. 12 Coefficient maximum eigenvalue/minimum eigenvalue versus the number of functions. (Color figure online)

being p the order of the derivative ($p = 1$, 2, 3 and 4). The derivatives are computed according to Eq. (9).

Figure 6 shows the distances d_f and d_m in Eqs. (22) and (23) versus the number of functions used in the computations. It can be seen that the SBF functions have a better convergence than the MCF functions, especially for the frequencies; the CF and MCF functions have similar behaviors, up to the point where the computations with the CF functions must be stopped. The PTF and OP functions are the ones presenting the worst convergence. The observed bad convergence of the OP functions seems to be in disagreement with the results presented in reference [54], where it was concluded that this kind of admissible functions have a better convergence than the CF. This is only true when only the convergence of the first frequency ($nm = 1$) is considered and not the convergence of all the first 10 ($nm = 10$).

To study the convergence of the derivatives, only the two sets of admissible functions with better convergence (MCF and SBF) were studied. Figure 7 shows the distances $d_m^{(p)}$ in Eq. (24) versus the number of functions, being $p = 1$, 2, 3 and 4. Like in Fig. 6b, the SBF functions present a better and faster convergence than the MCF functions for the first four derivative orders. We can also observe in this Figure that the higher the derivative order p, the slower the convergence, in accordance with results in reference [101].

4.3 Computational Time

Figure 8 shows the elapsed time to compute the matrices **K** and **M** [which involves the computation of the integrals in Eqs. (11) and (12)], the elapsed time to solve the eigenvalue problem with only the eigenvalues as outputs, and the elapsed time to solve the eigenvalue problem with the eigenvalues and eigenvectors as outputs versus the number of functions, using different sets of admissible functions. For the computation of

Table 2 Norm of the computed OP functions

i	1	2	3	4	5	6	7	8	9
$\int_0^a [p_i(x)]^2 dx$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0002	0.9950	1.9432

the matrices, the SBF, OP and NOP functions are considerably slower than the CF, MCF and PTF functions, which present similar behaviors. Indeed, the differences in the elapsed time is approximately one order of magnitude. It can be seen that for the computation of eigenvalues there is no significant differences among the sets of admissible functions, while for the eigenvalues and eigenvectors, the NOP functions present a higher elapsed time than the other functions.

4.4 Time Versus Accuracy

Linking the subjects of the two previous sections, Fig. 9 shows the distances d_f and d_m versus the elapsed time to compute the matrices **K** and **M**. It can be seen that the MCF functions are the best option, followed by the SBF (except the first points), CF, PTF, OP and NOP functions. However, for practical reasons, the elapsed time to solve the eigenvalue problem must also be taken in consideration. Although in Fig. 8 it is shown that this time is similar for equal M, the different convergencies of some admissible functions means that some of these functions need fewer terms in the series in order to obtain the same accuracy. Therefore, in practice, some functions will be faster than others. Figure 10 shows the distance d_f versus the elapsed time to compute the eigenvalues and the distance d_m versus the elapsed time to compute the eigenvectors. It can be seen in both cases that the SBF functions have a better performance than the MCF, CF, PTF, OP and NOP functions (in this order). Adding the elapsed time reported in Figs. 9 and 10, we obtain the total elapsed time to compute the matrices, eigenvalues and eigenvectors. The results can be seen in Fig. 11. For matrices and eigenvalues (Fig. 11a), it can be seen that the best choice are the MCF functions for lower values of M and the SBF functions for higher values. The CF, PTF, NOP and OP functions follow them. If, besides computing the eigenvalues, we also compute the eigenvectors (Fig. 11b), the SBF functions are the better option, except when using a few number of functions M, but with small differences in respect to the MCF functions. The NOP functions are clearly the worse option.

4.5 Numerical Stability

To study the numerical stability, the coefficient maximum eigenvalue/minimum eigenvalue was computed, according to reference [129]. The results can be seen in Fig. 12. The coefficient is monotonically increasing with very similar results for all the admissible functions. In the case of SBF functions, we can see that for some values of M before they give wrong results, this coefficient increases more than the coefficients of other functions, revealing the instability which leads to complex frequencies with $M = 73$. The instability of CF and NOP functions can be explained looking at Fig. 5. The CF functions have a very large increase in their values from $M = 6$ to $M = 7$, while the NOP functions have very low values of the Frobenius norm. Both behaviors explain the wrong results provided by this type of functions with $M = 8$. A special mention must be made to the OP functions, because a wrong result was obtained with $M = 9$, despite no previous signs of instability. This fact can be explained by looking at Table 2 and Eqs. (25) and (26). Table 2 shows the norm of the computed polynomials $p_i(x)$. Although this norm must be one by definition [see Eqs. (16) and (17)], numerical errors can produce inaccurate values, as can be seen for $i = 7$ and $i = 8$, and especially $i = 9$, where the value is almost the double. The reason for this behavior can be seen in Eqs. (25) and (26), where the order of magnitude of each coefficient for the polynomial with $i = 8$ and $i = 9$ are shown:

$$p_8(x) \simeq -10^3x^2 + 10^4x^3 - 10^5x^4 + 10^6x^5 - 10^7x^6 + 10^7x^7 - 10^7x^8 + 10^7x^9 - 10^6x^{10} + 10^6x^{11} \quad (25)$$

$$p_9(x) \simeq 10^4x^2 - 10^5x^3 + 10^6x^4 - 10^7x^5 + 10^7x^6 - 10^8x^7 + 10^8x^8 - 10^8x^9 + 10^8x^{10} - 10^7x^{11} + 10^7x^{12} \quad (26)$$

For $i = 9$, these orders of magnitude, multiplied by two when the norm of the polynomial is computed (because of the square), implies a loss of information in the units range using double precision. Because the result of the integral, by definition, must be one, the error is in the same order of magnitude as the result, leading this large relative error to wrong results.

5 Conclusions

Six sets of admissible functions used in the Ritz method were tested, in order to study the performance of these functions for the computation of a large number of natural frequencies and mode shapes in terms of the numerical behavior: convergence, computational time, and stability. In terms of convergence, the SBF and MCF functions present the better results. Regarding the elapsed time to compute the matrices **K** and **M**, the SBF, OP and NOP functions are slower than the other three sets of admissible functions (CF, MCF and PTF), which present a similar trend with the number of functions M. For the computation of eigenvalues and eigenvectors the results are similar for the six sets of admissible functions. Considering a balance between time and accuracy, the MCF and

SBF functions are again the ones with better results. The CF, OP, NOP and SBF functions present evidences of numerical instability, while the MCF and PTF functions have a very good numerical behavior. Considering all the three numerical features, we conclude that the CF, OP, NOP and PTF functions should be discarded as appropriate for the computation of higher natural frequencies and mode shapes, the SBF functions can be used with some necessary cautions about the numerical stability, while the MCF functions present the better numerical behavior taking into account all the features studied.

Acknowledgements The authors greatly appreciate the financial support of FCOMP-01-0124-FEDER-010236 through Project Ref. FCT—Portugal PTDC/EME-PME/102095/2008 and Junta de Andalucía—Spain (Project of Excellence No. P11-TEP-7093). This work was also supported by FCT—Portugal, through IDMEC, under LAETA, project UID/EMS/50022/2013.

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

This Appendix presents the plots of admissible functions with $m = 1, 2, 10$ and 15 (Figs. 13, 14, 15, 16, 17 and 18). We see that the shapes of the functions with $m = 1$ and 2 are very similar (with the exception of the Non-Orthogonal Polynomials), but for $m = 10$ and 15 the shape varies substantially. It is also clear a breakdown of the Characteristic Function with $m = 15$ for large values of x (Fig. 13d). The symmetry of the Orthogonal Polynomials is lost for values of $m > 7$.

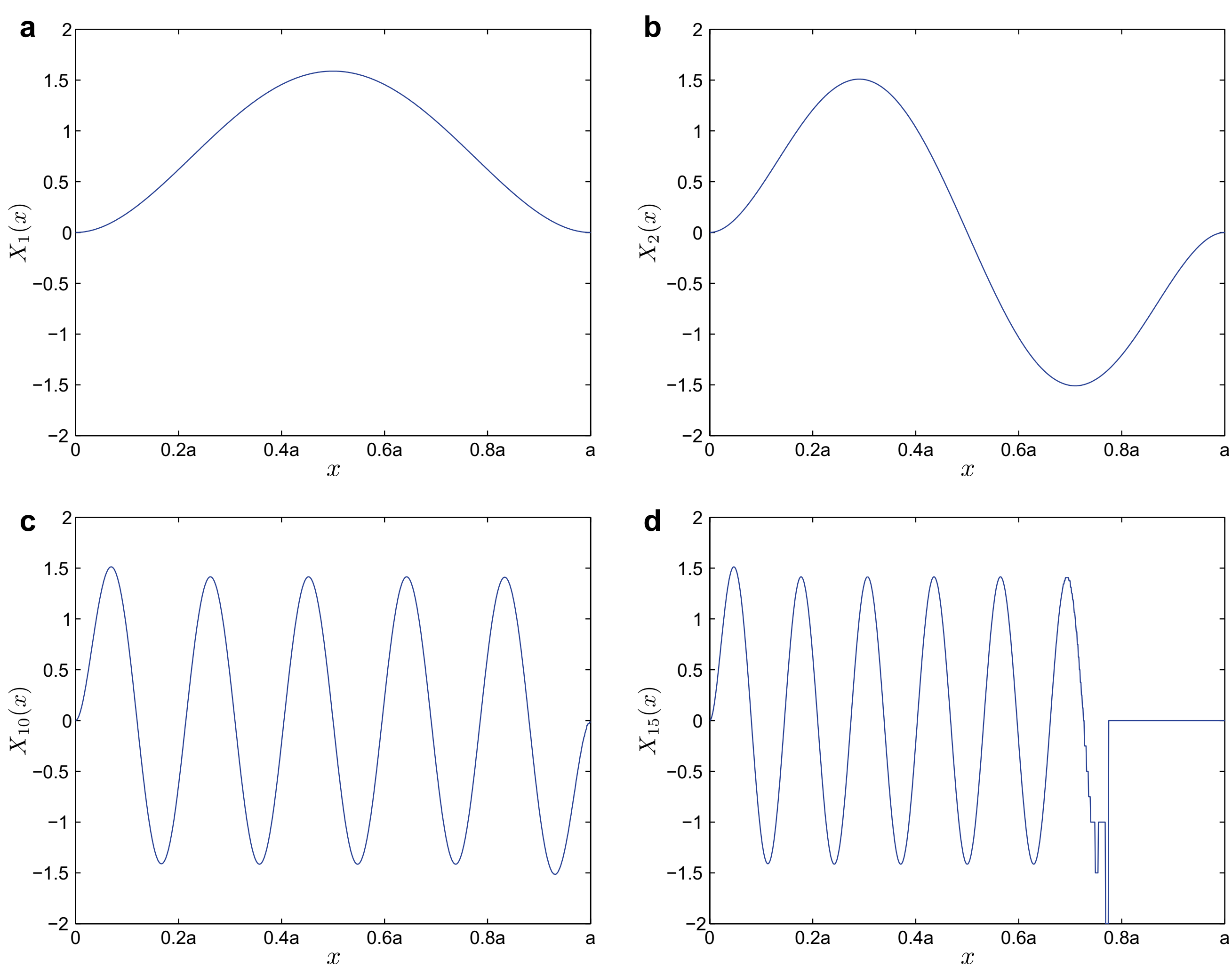

Fig. 13 Characteristic functions: (**a**) $m = 1$, (**b**) $m = 2$, (**c**) $m = 10$, and (**d**) $m = 15$

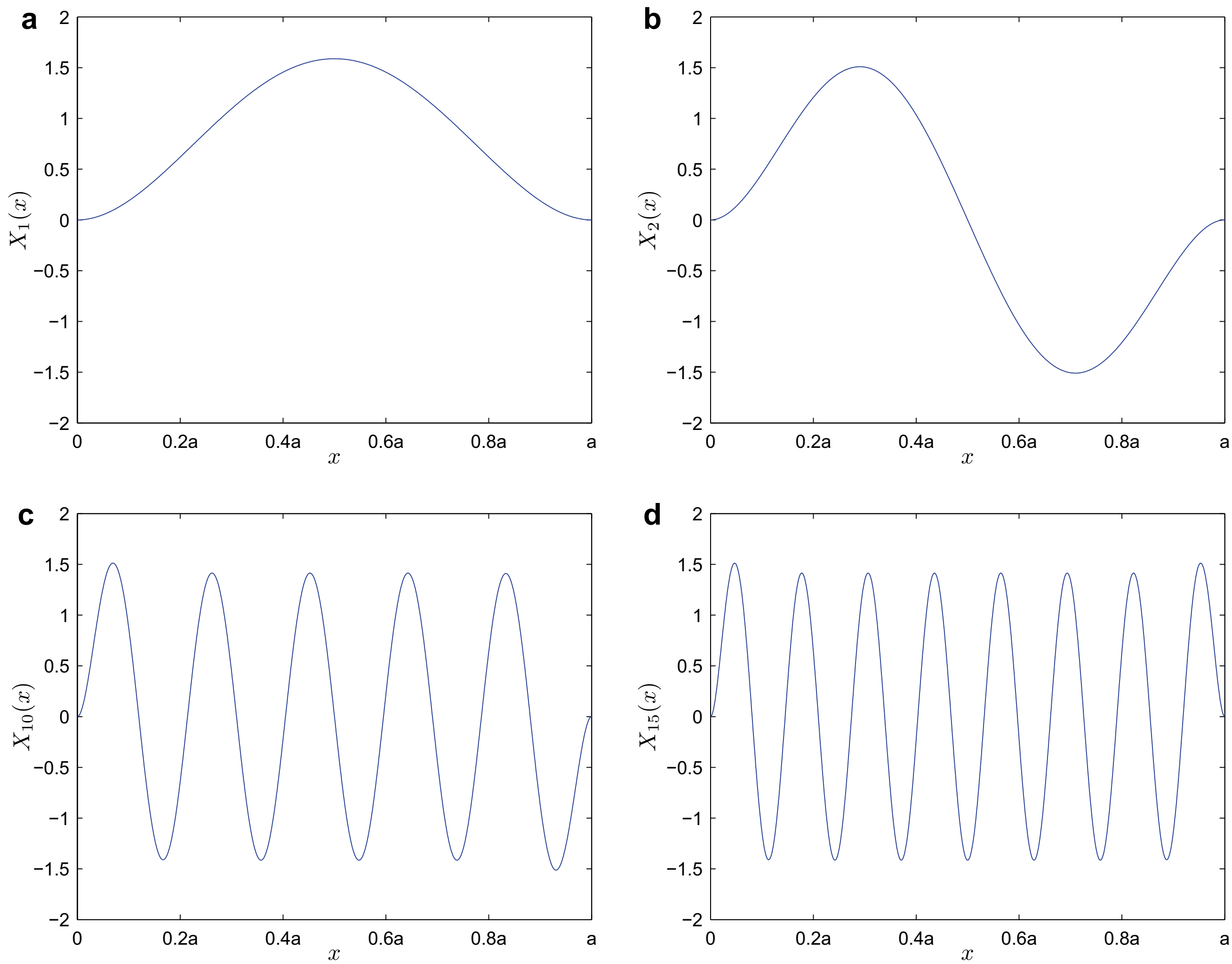

Fig. 14 Modified characteristic functions: (**a**) $m = 1$, (**b**) $m = 2$, (**c**) $m = 10$, and (**d**) $m = 15$

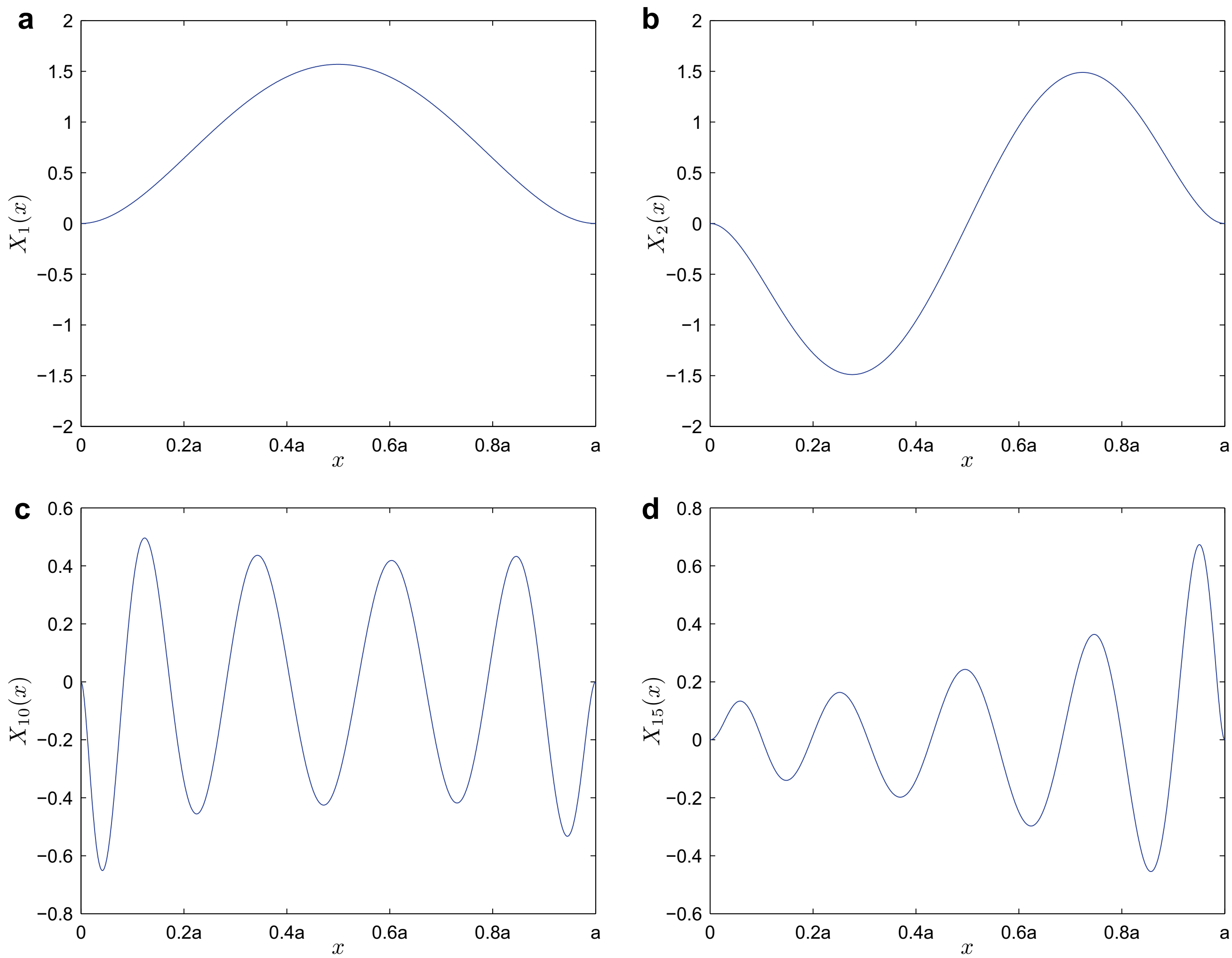

Fig. 15 Orthogonal polynomials: (**a**) $m = 1$, (**b**) $m = 2$, (**c**) $m = 10$, and (**d**) $m = 15$

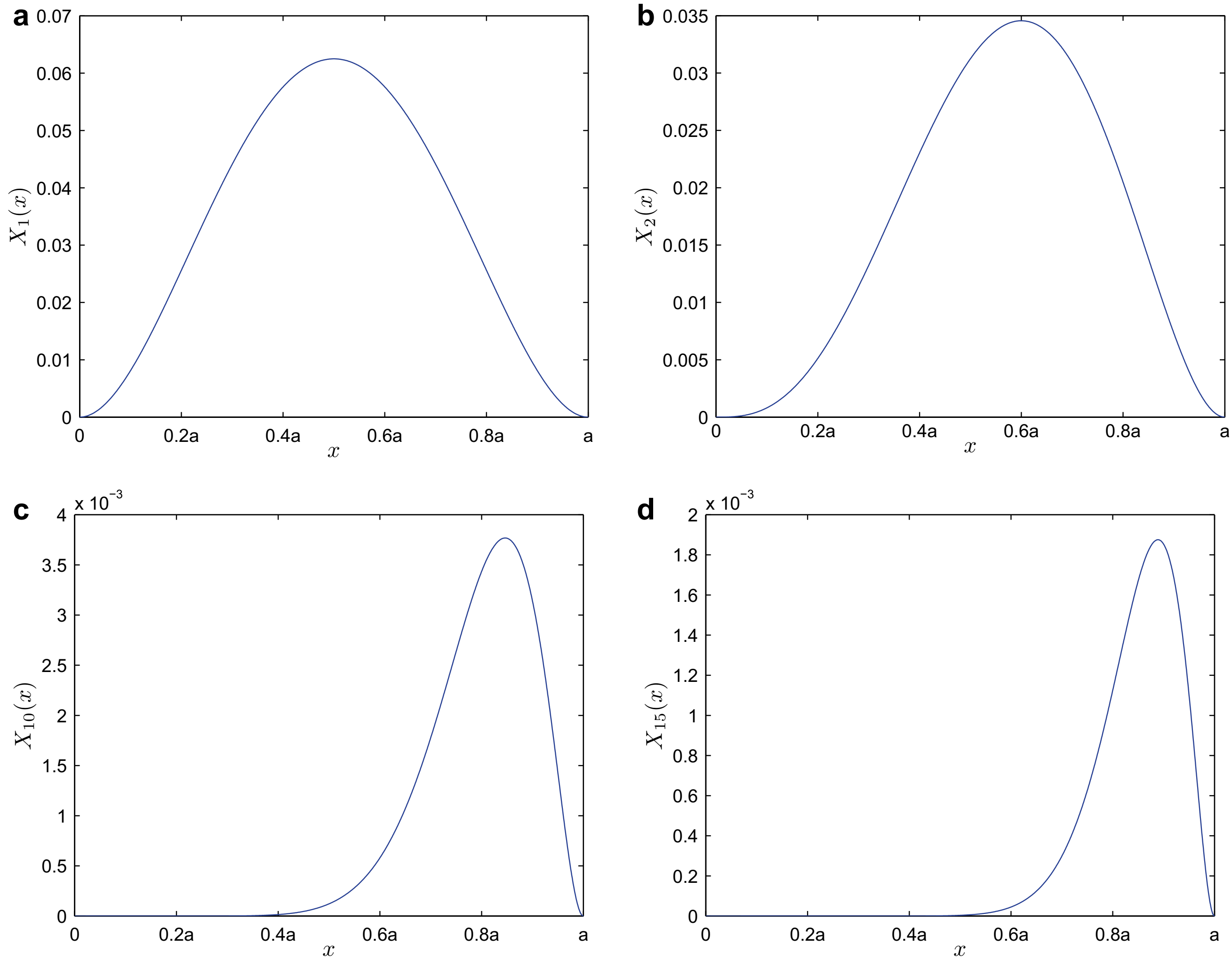

Fig. 16 Non-orthogonal polynomials: (**a**) $m = 1$, (**b**) $m = 2$, (**c**) $m = 10$, and (**d**) $m = 15$

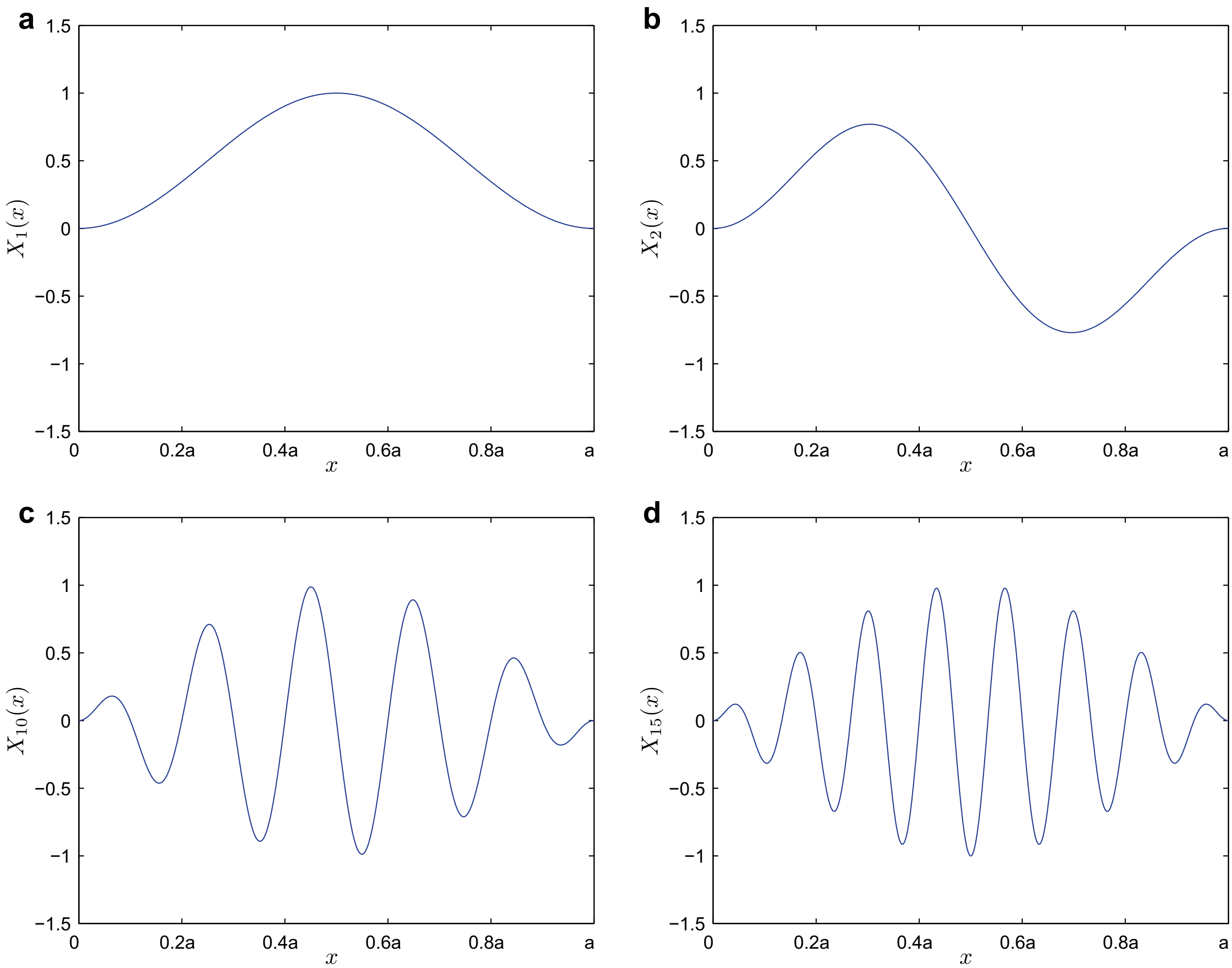

Fig. 17 Product of trigonometric functions: (**a**) $m = 1$, (**b**) $m = 2$, (**c**) $m = 10$, and (**d**) $m = 15$

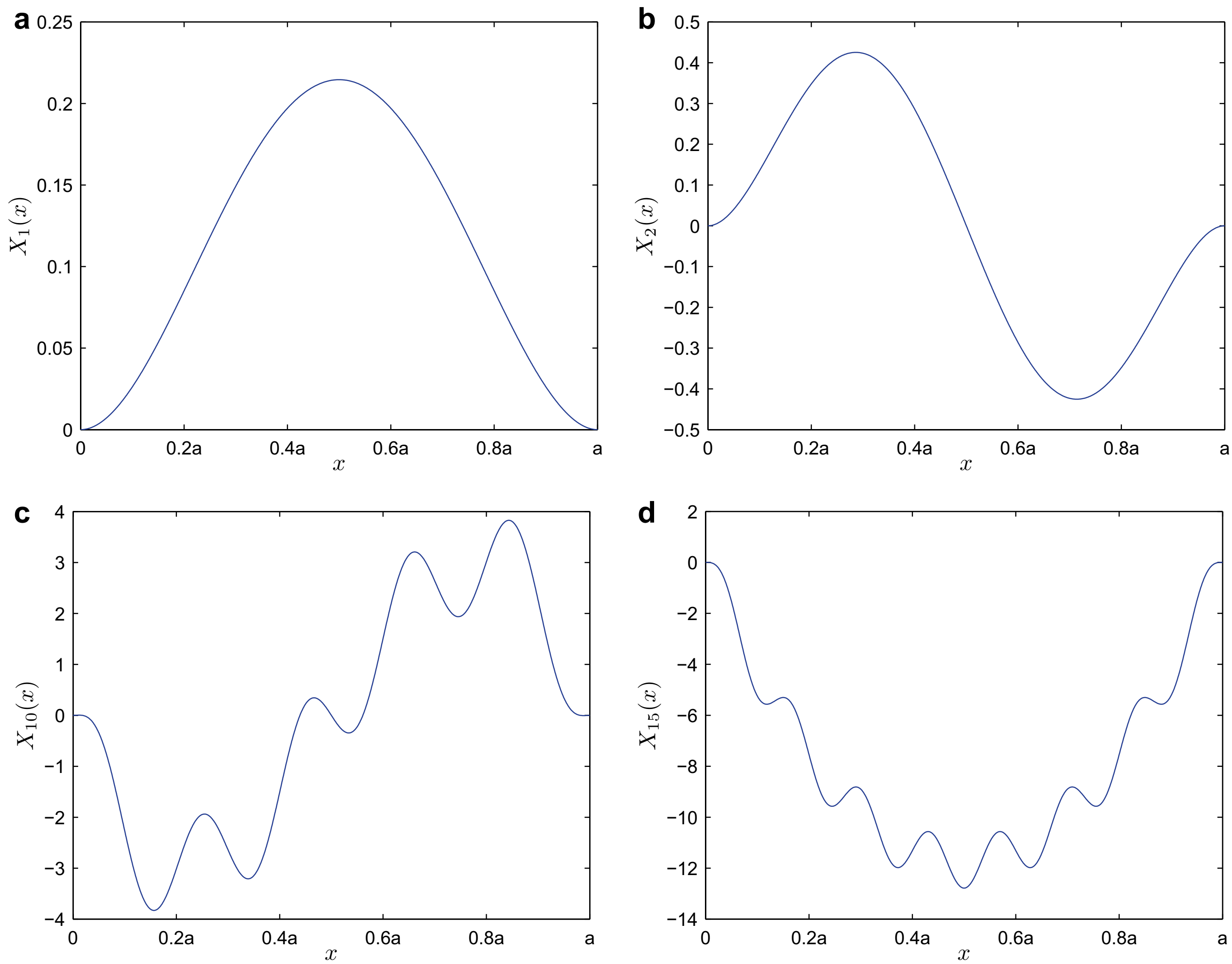

Fig. 18 Static beam functions: (**a**) $m = 1$, (**b**) $m = 2$, (**c**) $m = 10$, and (**d**) $m = 15$

References

1. Afshari M, Inman DJ (2013) Continuous crack modeling in piezoelectrically driven vibrations of an Euler-Bernoulli beam. J Vib Control 19(3):341–355. doi:10.1177/1077546312437803
2. Ansari R, Rouhi H (2011) Analytical treatment of the free vibration of single-walled carbon nanotubes based on the nonlocal Flugge shell theory. J Eng Mater Technol 134(1):011008. doi:10.1115/1.4005347
3. Ansari R, Rouhi H (2012) Nonlocal Flugge shell model for thermal buckling of multi-walled carbon nanotubes with layerwise boundary conditions. J Therm Stress 35(4):326–341. doi:10.1080/01495739.2012.663683
4. Ansari R, Sahmani S, Rouhi H (2011) Axial buckling analysis of single-walled carbon nanotubes in thermal environments via the Rayleigh-Ritz technique. Comput Mater Sci 50(10):3050–3055. doi:10.1016/j.commatsci.2011.05.027
5. Ansari R, Sahmani S, Rouhi H (2011) Rayleigh-Ritz axial buckling analysis of single-walled carbon nanotubes with different boundary conditions. Phys Lett A 375(9):1255–1263. doi:10.1016/j.physleta.2011.01.046
6. Aref AJ, Alampalli S, He Y (2001) Ritz-based static analysis method for fiber reinforced plastic rib core skew bridge superstructure. J Eng Mech-ASCE 127(5):450–458. doi:10.1061/(ASCE)0733-9399(2001) 127:5(450)
7. Arshad SH, Naeem MN, Sultana N, Iqbal Z, Shah AG (2011) Effects of exponential volume fraction law on the natural frequencies of FGM cylindrical shells under various boundary conditions. Arch Appl Mech 81(8):999–1016. doi:10.1007/s00419-010-0460-5
8. Ashton JE (1969a) Analysis of anisotropic plates II. J Compos Mater 3(3):470–479. doi:10.1177/002199836900300311
9. Ashton JE (1969b) Natural modes of free-free anisotropic plates. Shock Vib Bull 39(4):93–99
10. Ashton JE (1970) Anisotropic plate analysis-boundary conditions. J Compos Mater 4(2):162–171. doi:10.1177/002199837000400201
11. Ashton JE, Anderson JD (1969) The natural modes of vibration of boron-epoxy plates. Shock Vib Bull 39(4):81–91

12. Ashton JE, Waddoups ME (1969) Analysis of anisotropic plates. J Compos Mater 3(1):148–165. doi:10.1177/002199836900300111
13. Askari E, Daneshmand F, Amabili M (2011) Coupled vibrations of a partially fluid-filled cylindrical container with an internal body including the effect of free surface waves. J Fluid Struct 27(7):1049–1067. doi:10.1016/j.jfluidstructs.2011.04.010
14. Bae CH, Kwak MK, Koo JR (2012) Free vibration analysis of a hanged clamped-free cylindrical shell partially submerged in fluid: the effect of external wall, internal shaft, and flat bottom. J Sound Vib 331(17):4072–4092. doi:10.1016/j.jsv.2012.04.020
15. Bambill DV, Felix DH, Rossit CA (2006) Natural frequencies of thin, rectangular plates with holes or orthotropic "patches" carrying an elastically mounted mass. Int J Solids Struct 43(14–15):4116–4135. doi:10.1016/j.ijsolstr.2005.03.051
16. Bassily SF, Dickinson SM (1972) Buckling and lateral vibration of rectangular plates subject to inplane loads-A Ritz approach. J Sound Vib 24(2):219–239. doi:10.1016/0022-460X(72)90951-0
17. Bassily SF, Dickinson SM, (1973) Corrigendum: Buckling and lateral vibration of rectangular plates subject to inplane loads-A Ritz approach: (S. F. Bassily and S. M. Dickinson, (1972) J Sound Vib 24, 219–239). J Sound Vib 29(4):505–508. doi:10.1016/S0022-460X(73)80066-5
18. Bassily SF, Dickinson SM (1975) On the use of beam functions for problems of plates involving free edges. J Appl Mech 42(4):858–864. doi:10.1115/1.3423720
19. Bert CW, Mayberry BL (1969) Free vibrations of unsymmetrically laminated anisotropic plates with clamped edges. J Compos Mater 3(2):282–293. doi:10.1177/002199836900300207
20. Berthelot JM (2006) Damping analysis of laminated beams and plates using the Ritz method. Compos Struct 74(2):186–201. doi:10.1016/j.compstruct.2005.04.031
21. Berthelot JM, Sefrani Y (2004) Damping analysis of unidirectional glass and Kevlar fibre composites. Compos Sci Technol 64(9):1261–1278. doi:10.1016/j.compscitech.2003.10.003
22. Bhat RB (1985a) Natural frequencies of rectangular plates using characteristic orthogonal polynomials in Rayleigh-Ritz method. J Sound Vib 102(4):493–499. doi:10.1016/S0022-460X(85)80109-7
23. Bhat RB (1985) Plate deflections using orthogonal polynomials. J Eng Mech-ASCE 111(11):1301–1309. doi:10.1061/(ASCE)0733-9399(1985) 111:11(1301)
24. Bhat RB (1986) Transverse vibrations of a rotating uniform cantilever beam with tip mass as predicted by using beam characteristic orthogonal polynomials in the Rayleigh-Ritz method. J Sound Vib 105(2):199–210. doi:10.1016/0022-460X(86)90149-5
25. Bhat RB (1987a) Beam characteristic orthogonal polynomials with fractional power increments. J Sound Vib 112(3):556–558. doi:10.1016/S0022-460X(87)80121-9
26. Bhat RB (1987b) Flexural vibration of polygonal plates using characteristic orthogonal polynomials in two variables. J Sound Vib 114(1):65–71. doi:10.1016/S0022-460X(87)80234-1
27. Bhat RB (1987c) Rayleigh-Ritz method with separate deflection expressions for structural segments. J Sound Vib 115(1):174–177. doi:10.1016/0022-460X(87)90500-1
28. Bhat RB, Laura PAA, Gutierrez RG, Cortinez VH, Sanzi HC (1990) Numerical experiments on the determination of natural frequencies of transverse vibrations of rectangular plates of non-uniform thickness. J Sound Vib 138(2):205–219. doi:10.1016/0022-460X(90)90538-B
29. Blevins RD (1979) Formulas for natural frequency and mode shape. Van Nostrand Reinhold, New York
30. Bose T, Mohanty AR (2014) Detection and monitoring of side crack in a rectangular plate using mobility. J Vib Control. doi:10.1177/1077546314534285
31. Bose T, Mohanty AR (2015) Sound radiation response of a rectangular plate having a side crack of arbitrary length, orientation, and position. J Vib Acoust 137(2):021,019–021,019, doi:10.1115/1.4029449
32. Chai GB (1993) Free vibration of rectangular isotropic plates with and without a concentrated mass. Comput Struct 48(3):529–533. doi:10.1016/0045-7949(93)90331-7
33. Chai GB (1994) Buckling of generally laminated composite plates with various edge support conditions. Compos Struct 29(3):299–310. doi:10.1016/0263-8223(94)90026-4
34. Chai GB (1995) Frequency analysis of a S-C-S-C plate carrying a concentrated mass. J Sound Vib 179(1):170–177. doi:10.1006/jsvi.1995.0011
35. Chai GB (1995b) Frequency analysis of rectangular isotropic plates carrying a concentrated mass. Comput Struct 56(1):39–48. doi:10.1016/0045-7949(94)00533-9
36. Chai GB, Khong PW (1993) The effect of varying the support conditions on the buckling of laminated composite plates. Compos Struct 24(2):99–106. doi:10.1016/0263-8223(93)90031-K
37. Chai GB, Low KH, Lim TM (1995) Tension effects on the natural frequencies of centre-loaded clamped beams. J Sound Vib 181(4):727–736. doi:10.1006/jsvi.1995.0168
38. Chakraverty S, Bhat RB, Stiharu I (1999) Recent research on vibration of structures using boundary characteristic orthogonal polynomials in the Rayleigh-Ritz method. Shock Vib Dig 31(3):187–194
39. Cheung KY, Zhou D (1999a) The free vibrations of tapered rectangular plates using a new set of beam functions with the Rayleigh-Ritz method. J Sound Vib 223(5):703–722. doi:10.1006/jsvi.1998.2160
40. Cheung YK, Zhou D (1999b) Eigenfrequencies of tapered rectangular plates with intermediate line supports. Int J Solids Struct 36(1):143–166. doi:10.1016/S0020-7683(97)00272-2
41. Cheung YK, Zhou D (1999c) The free vibrations of rectangular composite plates with point-supports using static beam functions. Compos Struct 44(2–3):145–154. doi:10.1016/S0263-8223(98)00122-6
42. Cheung YK, Zhou D (2000a) Vibrations of moderately thick rectangular plates in terms of a set of static Timoshenko beam functions. Comput Struct 78(6):757–768. doi:10.1016/S0045-7949(00)00058-4
43. Cheung YK, Zhou D (2000b) Vibrations of rectangular plates with elastic intermediate line-supports and edge constraints. Thin Wall Struct 37(4):305–331. doi:10.1016/S0263-8231(00)00015-X
44. Cheung YK, Zhou D (2001a) Free vibrations of rectangular unsymmetrically laminated composite plates with internal line supports. Comput Struct 79(20–21):1923–1932. doi:10.1016/S0045-7949(01)00096-7
45. Cheung YK, Zhou D (2001b) Vibration analysis of symmetrically laminated rectangular plates with intermediate line supports. Comput Struct 79(1):33–41. doi:10.1016/S0045-7949(00)00108-5
46. Cheung YK, Zhou D (2003) Vibration of tapered Mindlin plates in terms of static Timoshenko beam functions. J Sound Vib 260(4):693–709. doi:10.1016/S0022-460X(02)01008-8
47. Chiba M, Sugimoto T (2003) Vibration characteristics of a cantilever plate with attached spring-mass system. J Sound Vib 260(2):237–263. doi:10.1016/S0022-460X(02)00921-5
48. Ciancio PM, Rossit CA, Laura PAA (2007) Approximate study of the free vibrations of a cantilever anisotropic plate carrying a

concentrated mass. J Sound Vib 302(3):621–628. doi:10.1016/j.jsv.2006.11.027
49. Courant R (1943) Variational methods for the solution of problems of equilibrium and vibrations. Bull Am Math Soc 49(1):1–23. doi:10.1090/S0002-9904-1943-07818-4
50. Cupial P (1997) Calculation of the natural frequencies of composite plates by the Rayleigh-Ritz method with orthogonal polynomials. J Sound Vib 201(3):385–387. doi:10.1006/jsvi.1996.0802
51. Dasgupta A, Huang KH (1997) A layer-wise analysis for free vibrations of thick composite spherical panels. J Compos Mater 31(7):658–671. doi:10.1177/002199839703100702
52. Deobald LR, Gibson RF (1988) Determination of elastic constants of orthotropic plates by a modal analysis/Rayleigh-Ritz technique. J Sound Vib 124(2):269–283. doi:10.1016/S0022-460X(88)80187-1
53. Dickinson SM (1978) The buckling and frequency of flexural vibration of rectangular isotropic and orthotropic plates using Rayleigh's method. J Sound Vib 61(1):1–8. doi:10.1016/0022-460X(78)90036-6
54. Dickinson SM, Di Blasio A (1986) On the use of orthogonal polynomials in the Rayleigh-Ritz method for the study of the flexural vibration and buckling of isotropic and orthotropic rectangular plates. J Sound Vib 108(1):51–62. doi:10.1016/S0022-460X(86)80310-8
55. Dickinson SM, Li EKH (1982) On the use of simply supported plate functions in the Rayleigh-Ritz method applied to the flexural vibration of rectangular plates. J Sound Vib 80(2):292–297. doi:10.1016/0022-460X(82)90199-7
56. Dowell EH (1984) On asymptotic approximations to beam model shapes. J Appl Mech 51(2):439. doi:10.1115/1.3167639
57. Eringen AC (1983) On differential equations of nonlocal elasticity and solutions of screw dislocations and surface waves. J Appl Phys 54(9):4703–4710. doi:10.1063/1.332803
58. Eringen AC, Edelen DGB (1972) On nonlocal elasticity. Int J Eng Sci 10:233–248
59. Fasana A, Marchesiello S (2001) Rayleigh-Ritz analysis of sandwich beams. J Sound Vib 241(4):643–652. doi:10.1006/jsvi.2000.3311
60. Felgar RP (1950) Formulas for integrals containing characteristic functions of a vibrating beam. Tech. rep., University of Texas. Bureau of Engineering Research. Circular
61. Frederiksen PS (1995) Single-layer plate theories applied to the flexural vibration of completely free thick laminates. J Sound Vib 186(5):743–759. doi:10.1006/jsvi.1995.0486
62. Gallego A, Moreno-García P, Casanova CF (2013) Modal analysis of delaminated composite plates using the finite element method and damage detection via combined Ritz/2D-wavelet analysis. J Sound Vib 332(12):2971–2983. doi:10.1016/j.jsv.2013.01.012
63. Gander M, Kwok F (2012) Chladni figures and the Tacoma bridge: motivating PDE eigenvalue problems via vibrating plates. SIAM Rev 54(3):573–596. doi:10.1137/10081931X
64. Gander M, Wanner G (2012) From Euler, Ritz, and Galerkin to modern computing. SIAM Rev 54(4):627–666. doi:10.1137/100804036
65. Ganesan N, Engels RC (1992) Hierarchical Bernoulli-Euler beam finite elements. Comput Struct 43(2):297–304. doi:10.1016/0045-7949(92)90146-Q
66. Gartner JR, Cobb EC (1988) Natural frequencies and biplanar response of generalized rotating spindle systems. Robot Cim-Int Manuf 4(1–2):165–174. doi:10.1016/0736-5845(88)90073-7
67. Gartner JR, Olgac N (1982) Improved numerical computation of uniform beam characteristic values and characteristic functions. J Sound Vib 84(2):481–489. doi:10.1016/S0022-460X(82)80029-1
68. Gaul L (2014) From Newton's principia via Lord Rayleigh's theory of sound to finite elements. In: Stein E (ed) The history of theoretical, material and computational mechanics - mathematics meets mechanics and engineering. Lecture notes in applied mathematics and mechanics, vol 1. Springer, Berlin, pp 385–398. doi:10.1007/978-3-642-39905-3_21
69. Grossi RO, Bhat RB (1995) Natural frequencies of edge restrained tapered rectangular plates. J Sound Vib 185(2):335–343. doi:10.1006/jsvi.1995.0382
70. Hearmon RFS (1959) The frequency of flexural vibration of rectangular orthotropic plates with clamped or supported edges. J Appl Mech 26(4):537–540
71. Hosseini-Hashemi S, Karimi M, Rokni DH (2010) Hydroelastic vibration and buckling of rectangular Mindlin plates on Pasternak foundations under linearly varying in-plane loads. Soil Dyn Earthq Eng 30(12):1487–1499. doi:10.1016/j.soildyn.2010.06.019
72. Hosseini-Hashemi S, Karimi M, Taher HRD (2010b) Vibration analysis of rectangular Mindlin plates on elastic foundations and vertically in contact with stationary fluid by the Ritz method. Ocean Eng 37(2–3):174–185. doi:10.1016/j.oceaneng.2009.12.001
73. Hu XX, Sakiyama T, Lim CW, Xiong Y, Matsuda H, Morita C (2004) Vibration of angle-ply laminated plates with twist by Rayleigh-Ritz procedure. Comput Method Appl M 193(9–11):805–823. doi:10.1016/j.cma.2003.08.003
74. Huang CS, Leissa AW (2009) Vibration analysis of rectangular plates with side cracks via the Ritz method. J Sound Vib 323(3–5):974–988. doi:10.1016/j.jsv.2009.01.018
75. Huang CS, Leissa AW, Chan CW (2011) Vibrations of rectangular plates with internal cracks or slits. Int J Mech Sci 53(6):436–445. doi:10.1016/j.ijmecsci.2011.03.006
76. Huang CS, Leissa AW, Li RS (2011b) Accurate vibration analysis of thick, cracked rectangular plates. J Sound Vib 330(9):2079–2093. doi:10.1016/j.jsv.2010.11.007
77. Huang CS, McGee OG III, Chang MJ (2011) Vibrations of cracked rectangular FGM thick plates. Compos Struct 93(7):1747–1764. doi:10.1016/j.compstruct.2011.01.005
78. Huang CS, Yang PJ, Chang MJ (2012) Three-dimensional vibration analyses of functionally graded material rectangular plates with through internal cracks. Compos Struct 94(9):2764–2776. doi:10.1016/j.compstruct.2012.04.003
79. Huang CS, McGee OG III, Wang KP (2013) Three-dimensional vibrations of cracked rectangular parallelepipeds of functionally graded material. Int J Mech Sci 70:1–25. doi:10.1016/j.ijmecsci.2012.05.009
80. van Hulzen JR, Schitter G, den Hof PMJV, van Eijk J (2012) Dynamics, load balancing, and modal control of piezoelectric tube actuators. Mechatron 22(3):282–294. doi:10.1016/j.mechatronics.2011.10.003
81. Ilanko S (2009) Comments on "The historical bases of the Rayleigh and Ritz methods". J Sound Vib 319(1–2):731–733. doi:10.1016/j.jsv.2008.06.001
82. Ip KH, Tse PC, Lai TC (1998) Material characterization for orthotropic shells using modal analysis and Rayleigh-Ritz models. Compos Part B 29(4):397–409. doi:10.1016/S1359-8368(97)00037-1
83. Isvandzibaei MR, Jamaluddin H, Raja Hamzah R (2014) Frequency analysis of multiple layered cylindrical shells under lateral pressure with asymmetric boundary conditions. Chinese J Mech Eng 27(1):23–31. doi:10.3901/CJME.2014.01.023
84. Jeong KH, Kang HS (2013) Free vibration of multiple rectangular plates coupled with a liquid. Int J Mech Sci 74:161–172. doi:10.1016/j.ijmecsci.2013.05.011

85. Kato Y, Honma T (1998) The Rayleigh-Ritz solution to estimate vibration characteristics of building floors. J Sound Vib 211(2):195–206. doi:10.1006/jsvi.1997.1362
86. Kim CS, Young PG, Dickinson SM (1990) On the flexural vibration of rectangular plates approached by using simple polynomials in the Rayleigh-Ritz method. J Sound Vib 143(3):379–394. doi:10.1016/0022-460X(90)90730-N
87. Kim YW (2005) Temperature dependent vibration analysis of functionally graded rectangular plates. J Sound Vib 284(3–5):531–549. doi:10.1016/j.jsv.2004.06.043
88. Kollar LP, Veres IA (2001) Buckling of rectangular orthotropic plates subjected to biaxial normal forces. J Compos Mater 35(7):625–635. doi:10.1177/002199801772662109
89. Kong S (2013) Size effect on pull-in behavior of electrostatically actuated microbeams based on a modified couple stress theory. Appl Math Model 37(12–13):7481–7488. doi:10.1016/j.apm.2013.02.024
90. Kwak MK, Koo JR, Bae CH (2011) Free vibration analysis of a hung clamped-free cylindrical shell partially submerged in fluid. J Fluid Struct 27(2):283–296. doi:10.1016/j.jfluidstructs.2010.11.005
91. Lai TC, Ip KH (1996) Parameter estimation of orthotropic plates by bayesian sensitivity analysis. Compos Struct 34(1):29–42. doi:10.1016/0263-8223(95)00128-X
92. Lam KY, Amrutharaj G (1995) Natural frequencies of rectangular stepped plates using polynomial beam functions with subsectioning. Appl Acoust 44(4):325–340. doi:10.1016/0003-682X(94)00030-Y
93. Lam KY, Chun L (1994) Analysis of clamped laminated plates subjected to conventional blast. Compos Struct 29(3):311–321. doi:10.1016/0263-8223(94)90027-2
94. Lam KY, Hung KC (1990) Vibration study on plates with stiffened openings using orthogonal polynomials and partitioning method. Comput Struct 37(3):295–301. doi:10.1016/0045-7949(90)90321-R
95. Lam KY, Loy CT (1995) Effects of boundary conditions on frequencies of a multi-layered cylindrical shell. J Sound Vib 188(3):363–384. doi:10.1006/jsvi.1995.0599
96. Lam KY, Hung KC, Chow ST (1989) Vibration analysis of plates with cutouts by the modified Rayleigh-Ritz method. Appl Acoust 28(1):49–60. doi:10.1016/0003-682X(89)90030-3
97. Lee D-C, Wang C-S, Lee L-T (2011) The natural frequency of elastic plates with void by Ritz-method. Stud in Math Sci 2(1):36–50. doi:10.3968/j.sms.1923845220120201.009
98. Lee LT, Lee DC (1997) Free vibration of rectangular plates on elastic point supports with the application of a new type of admissible function. Comput Struct 65(2):149–156. doi:10.1016/S0045-7949(96)00426-9
99. Leissa AW (1973) The free vibration of rectangular plates. J Sound Vib 31(3):257–293. doi:10.1016/S0022-460X(73)80371-2
100. Leissa AW (2005) The historical bases of the Rayleigh and Ritz methods. J Sound Vib 287(45):961–978. doi:10.1016/j.jsv.2004.12.021
101. Leissa AW, Shihada SM (1995) Convergence of the Ritz method. Appl Mech Rev 48(11S):S90–S95. doi:10.1115/1.3005088
102. Leung AYT (1988) Integration of beam functions. Comput Struct 29(6):1087–1094. doi:10.1016/0045-7949(88)90332-X
103. Leung AYT (1990) Recurrent integration of beam functions. Comput Struct 37(3):277–282. doi:10.1016/0045-7949(90)90319-W
104. Liew KM, Hung KC, Lim MK (1993a) Method of domain decomposition in vibrations of mixed edge anisotropic plates. Int J Solids Struct 30(23):3281–3301. doi:10.1016/0020-7683(93)90114-M
105. Liew KM, Hung KC, Lim MK (1993b) Roles of domain decomposition method in plate vibrations: Treatment of mixed discontinuous periphery boundaries. Int J Mech Sci 35(7):615–632. doi:10.1016/0020-7403(93)90005-F
106. Liew KM, Hung KC, Lim MK (1994) On the use of the domain decomposition method for vibration of symmetric laminates having discontinuities at the same edge. J Sound Vib 178(2):243–264. doi:10.1006/jsvi.1994.1481
107. Liew KM, Hung KC, Lim MK (1995) Vibration of Mindlin plates using boundary characteristic orthogonal polynomials. J Sound Vib 182(1):77–90. doi:10.1006/jsvi.1995.0183
108. Lim SP, Senthilnathan NR, Lee KH (1989) Rayleigh-Ritz vibration analysis of thick plates by a simple higher order theory. J Sound Vib 130(1):163–166. doi:10.1016/0022-460X(89)90527-0
109. Rayleigh L (1911) XXIV. On the calculation of Chladni's figures for a square plate. Philos Mag 22(128):225–229. doi:10.1080/14786440808637121
110. Low KH, Chai GB, Ng CK (1993a) Experimental and analytical study of the frequencies of an S-C-S-C plate carrying a concentrated mass. J Vib Acoust 115(4):391–396. doi:10.1115/1.2930362
111. Low KH, Lim TM, Chai GB (1993b) Experimental and analytical investigations of vibration frequencies for centre-loaded beams. Comput Struct 48(6):1157–1162. doi:10.1016/0045-7949(93)90448-M
112. Low KH, Chai GB, Tan GS (1997) A comparative study of vibrating loaded plates between the Rayleigh-Ritz and experimental methods. J Sound Vib 199(2):285–297. doi:10.1006/jsvi.1996.0633
113. Low KH, Chai GB, Lim TM, Sue SC (1998) Comparisons of experimental and theoretical frequencies for rectangular plates with various boundary conditions and added masses. Int J Mech Sci 40(11):1119–1131. doi:10.1016/S0020-7403(98)00013-7
114. Loy CT, Lam KY (1997) Vibration of cylindrical shells with ring support. Int J Mech Sci 39(4):455–471. doi:10.1016/S0020-7403(96)00035-5
115. Maheri MR, Adams RD (2003) Modal vibration damping of anisotropic FRP laminates using the Rayleigh-Ritz energy minimization scheme. J Sound Vib 259:17–29. doi:10.1006/jsvi.2002.5151
116. Meleshko VV (1997) Bending of an elastic rectangular clamped plate: exact versus 'engineering' solutions. J Elast 48(1):1–50. doi:10.1023/A:1007472709175
117. Messina A, Soldatos KP (1999) Ritz-type dynamic analysis of cross-ply laminated circular cylinders subjected to different boundary conditions. J Sound Vib 227(4):749–768. doi:10.1006/jsvi.1999.2347
118. Morales CA, Ramírez JF (2002) Further simplest-expression integrals involving beam eigenfunctions and derivatives. J Sound Vib 253(2):518–522. doi:10.1006/jsvi.2001.4011
119. Moreno-García P (2012) Simulación y ensayos de vibraciones en placas de material compuesto de fibra de carbono y detección de daño mediante la respuesta en frecuencia y la transformada wavelet. PhD thesis, Universidad de Granada, Spain. http://hdl.handle.net/10481/23283 **(in Spanish)**
120. Moreno-García P, Castro E, Romo-Melo L, Gallego A, Roldán A (2012) Vibration tests in CFRP plates for damage detection via non-parametric signal analysis. Shock Vib 19(5):857–865. doi:10.1155/2012/385835
121. Moreno-García P, Lopes H, Araújo dos Santos JV, Maia NMM (2012b) Damage localisation in composite laminated plates using higher order spatial derivatives. In: Topping B (ed) Proceedings of the eleventh international conference on computational structures technology. Civil-Comp Press, Stirlingshire, UK. doi:10.4203/ccp.99.75

122. Moreno-García P, Araújo dos Santos JV, Lopes H (2014) A new technique to optimize the use of mode shape derivatives to localize damage in laminated composite plates. Compos Struct 108:548–554. doi:10.1016/j.compstruct.2013.09.050
123. Moreno-García P, Lopes H, Araújo dos Santos JV (2015) Application of higher order finite differences to damage localization in laminated composite plates. Compos Struct. doi:10.1016/j.compstruct.2015.08.011
124. Muthukumaran P, Bhat R, Stiharu I (1999) Boundary conditioning technique for structural tuning. J Sound Vib 220(5):847–859. doi:10.1006/jsvi.1998.1991
125. Muthukumaran P, Demirli K, Stiharu I, Bhat R (2000) Boundary conditioning for structural tuning using fuzzy logic approach. Comput Struct 74(5):547–557. doi:10.1016/S0045-7949(99)00063-2
126. Nallim LG, Grossi RO (2003) On the use of orthogonal polynomials in the study of anisotropic plates. J Sound Vib 264(5):1201–1207. doi:10.1016/S0022-460X(02)01523-7
127. Nallim LG, Oller S (2008) An analytical-numerical approach to simulate the dynamic behaviour of arbitrarily laminated composite plates. Compos Struct 85(4):311–325. doi:10.1016/j.compstruct.2007.10.031
128. Nallim LG, Martinez SO, Grossi RO (2005) Statical and dynamical behaviour of thin fibre reinforced composite laminates with different shapes. Comput Method Appl Mech Eng 194(17):1797–1822. doi:10.1016/j.cma.2004.06.009
129. Oosterhout G, van der Hoogt P, Spiering R (1995) Accurate calculation methods for natural frequencies of plates with special attention to the higher modes. J Sound Vib 183(1):33–47. doi:10.1006/jsvi.1995.0237
130. Pandey MD, Sherbourne AN (1991) Buckling of anisotropic composite plates under stress gradient. J Eng Mech-ASCE 117(2):260–275. doi:10.1061/(ASCE)0733-9399(1991)117:2(260)
131. Pandey MD, Sherbourne AN (1993) Stability analysis of inhomogeneous, fibrous composite plates. Int J Solids Struct 30(1):37–60. doi:10.1016/0020-7683(93)90131-P
132. Pao YC, Peterson KA (1988) Contour-plot simulation of vibrational and buckling mode shapes of composite plates. J Compos Mater 22(10):935–954. doi:10.1177/002199838802201003
133. Parashar SK, Kumar A (2015) Three-dimensional analytical modeling of vibration behavior of piezoceramic cylindrical shells. Arch Appl Mech 85(5):641–656. doi:10.1007/s00419-014-0977-0
134. Pirnat M, Cepon G, Boltezar M (2014) Structural-acoustic model of a rectangular plate-cavity system with an attached distributed mass and internal sound source: theory and experiment. J Sound Vib 333(7):2003–2018. doi:10.1016/j.jsv.2013.11.044
135. Plunkett R (1963) Natural frequencies of uniform and non-uniform rectangular cantilever plates. J Mech Eng Sci 5(2):146–156. doi:10.1243/JMES_JOUR_1963_005_020_02
136. Rango RF, Bellomo FJ, Nallim LG (2015) A variational Ritz formulation for vibration analysis of thick quadrilateral laminated plates. Int J Mech Sci 104:60–74. doi:10.1016/j.ijmecsci.2015.09.018
137. Reddy JN (1984) A simple higher-order theory for laminated composite plates. J Appl Mech 51(4):745–752. doi:10.1115/1.3167719
138. Rinaldi G, Packirisamy M, Stiharu I (2006) Boundary characterization of microstructures through thermo-mechanical testing. J Micromech Microeng 16(3):549. doi:10.1088/0960-1317/16/3/010
139. Rinaldi G, Packirisamy M, Stiharu I (2007) Quantitative boundary support characterization for cantilever MEMS. Sensors 7(10):2062–2079. doi:10.3390/s7102062
140. Rinaldi G, Packirisamy M, Stiharu I (2008) Boundary characterization of MEMS structures through electro-mechanical testing. Sensor Actuat A-Phys 143(2):415–422. doi:10.1016/j.sna.2007.08.032
141. Ritz W (1909a) Theorie der transversalschwingungen einer quadratische platte mit freien rändern. Ann Phys-Leipzig 333(4):737–786. doi:10.1002/andp.19093330403
142. Ritz W (1909) Über eine neue methode zur lösung gewisser variationsprobleme der mathematischen physik. J Reine Angew Math 135:1–61. doi:10.1515/crll.1909.135.1
143. Rouhi H, Bazdid-Vahdati M, Ansari R (2015) Rayleigh-Ritz vibrational analysis of multiwalled carbon nanotubes based on the nonlocal Flugge shell theory. J Compos 2015:11. doi:10.1155/2015/750392
144. Araújo dos Santos JV, Reddy JN (2012) Free vibration and buckling analysis of beams with a modified couple-stress theory. Int J Appl Mech 4(3):1250,026. doi:10.1142/S1758825112500263
145. Savoye P (2011) On the benefits of exposing mathematics majors to the Rayleigh-Ritz procedure. PRIMUS 21(6):554–566. doi:10.1080/10511970903474455
146. Sharma CB (1978) Calculation of integrals involving characteristic beam functions. J Sound Vib 56(4):475–480. doi:10.1016/0022-460X(78)90289-4
147. Smith ST, Bradford MA, Oehlers DJ (1999) Numerical convergence of simple and orthogonal polynomials for the unilateral plate buckling problem using the Rayleigh-Ritz method. Int J Numer Meth Eng 44(11):1685–1707
148. Soldatos KP, Messina A (1998) Vibration studies of cross-ply laminated shear deformable circular cylinders on the basis of orthogonal polynomials. J Sound Vib 218(2):219–243. doi:10.1006/jsvi.1998.1769
149. Soldatos KP, Messina A (2001) The influence of boundary conditions and transverse shear on the vibration of angle-ply laminated plates, circular cylinders and cylindrical panels. Comput Method Appl Mech Eng 190(18–19):2385–2409. doi:10.1016/S0045-7825(00)00242-5
150. Song X, Han Q, Zhai J (2015) Vibration analyses of symmetrically laminated composite cylindrical shells with arbitrary boundaries conditions via rayleigh-ritz method. Compos Struct 134:820–830. doi:10.1016/j.compstruct.2015.08.134
151. Song X, Zhai J, Chen Y, Han Q (2015) Traveling wave analysis of rotating cross-ply laminated cylindrical shells with arbitrary boundaries conditions via rayleigh-ritz method. Compos Struct 133:1101–1115. doi:10.1016/j.compstruct.2015.08.015
152. Sun S, Cao D, Han Q (2013) Vibration studies of rotating cylindrical shells with arbitrary edges using characteristic orthogonal polynomials in the rayleigh-ritz method. Int J Mech Sci 68:180–189. doi:10.1016/j.ijmecsci.2013.01.013
153. Timoshenko S (1937) Vibration problems in engineering, 2nd edn. D. Van Nostrand Company Inc, New York
154. Timoshenko S, Woinowsky-Krieger S (1959) Theory of plates and shells. McGraw-Hill, Auckland
155. Warburton GB (1954) The vibration of rectangular plates. P I Mech Eng 168(1):371–384. doi:10.1243/PIME_PROC_1954_168_040_02
156. Warburton GB, Edney SL (1984) Vibrations of rectangular plates with elastically restrained edges. J Sound Vib 95(4):537–552. doi:10.1016/0022-460X(84)90236-0
157. Williamson F Jr (1980) Richard Courant and the finite element method: a further look. Hist Math 7(4):369–378. doi:10.1016/0315-0860(80)90001-4
158. Yang F, Chong ACM, Lam DCC, Tong P (2002) Couple stress based strain gradient theory for elasticity. Int J Solids Struct 39(10):2731–2743. doi:10.1016/S0020-7683(02)00152-X

159. Yoon H, Youn BD, Kim HS (2016) Kirchhoff plate theory-based electromechanically-coupled analytical model considering inertia and stiffness effects of a surface-bonded piezoelectric patch. Smart Mater Struct 25(2):025,017. doi:10.1088/0964-1726/25/2/025017
160. Young D (1950) Vibration of rectangular plates by the Ritz method. J Appl Mech 17(4):448–453
161. Young D, Felgar RP (1949) Tables of characteristic functions representing normal modes of vibration of a beam. Tech. rep., Engineering Research Series, No.44, University of Texas, Austin, Texas. http://hdl.handle.net/2152/6001
162. Young PG (2000) Application of a three-dimensional shell theory to the free vibration of shells arbitrarily deep in one direction. J Sound Vib 238(2):257–269. doi:10.1006/jsvi.2000.3103
163. Young PG, Dickinson SM (1993) On the free flexural vibration of rectangular plates with straight or curved internal line supports. J Sound Vib 162(1):123–135. doi:10.1006/jsvi.1993.1106
164. Zhou D (1994) The application of a type of new admissible function to the vibration of rectangular plates. Comput Struct 52(2):199–203. doi:10.1016/0045-7949(94)90272-0
165. Zhou D (1995) Natural frequencies of elastically restrained rectangular plates using a set of static beam functions in the Rayleigh-Ritz method. Comput Struct 57(4):731–735. doi:10.1016/0045-7949(95)00066-P
166. Zhou D (1996) Natural frequencies of rectangular plates using a set of static beam functions in Rayleigh-Ritz method. J Sound Vib 189(1):81–87. doi:10.1006/jsvi.1996.0006
167. Zhou D (2001) Vibrations of Mindlin rectangular plates with elastically restrained edges using static Timoshenko beam functions with the Rayleigh-Ritz method. Int J Solids Struct 38(32–33):5565–5580. doi:10.1016/S0020-7683(00)00384-X
168. Zhou D (2002) Vibrations of point-supported rectangular plates with variable thickness using a set of static tapered beam functions. Int J Mech Sci 44(1):149–164. doi:10.1016/S0020-7403(01)00081-9
169. Zhou D, Cheung YK (2000a) The free vibration of a type of tapered beams. Comput Method Appl M 188(1–3):203–216. doi:10.1016/S0045-7825(99)00148-6
170. Zhou D, Cheung YK (2000b) Vibrations of tapered Timoshenko beams in terms of static Timoshenko beam functions. J Appl Mech 68(4):596–602. doi:10.1115/1.1357164
171. Zhou D, Lo SH, Au FTK, Cheung YK (2002) Vibration analysis of rectangular Mindlin plates with internal line supports using static Timoshenko beam functions. Int J Mech Sci 44(12):2503–2522. doi:10.1016/S0020-7403(02)00188-1