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ORIGINAL PAPER

Recent Researches on Nonlocal Elasticity Theory in the Vibration of Carbon Nanotubes Using Beam Models: A Review

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Abstract Understanding dynamic behavior of carbon nanotubes has been of interest to researchers because of its practical applications. Recent studies show that nonlocal elasticity theory gives better results in the vibration of carbon nanotubes. The necessity of nonlocal elasticity theory, calibration of nonlocal parameter and application of nonlocal elasticity theory in various studies related to vibration of carbon nanotubes are discussed. This review emphasizes the application of nonlocal elasticity theory in the vibration of carbon nanotubes considering various types of complicating effects, nonlinearity, functionally graded material and different beam theories.

Keywords Nonlocal elasticity theory · Beam theories · Carbon nanotubes · Vibration

1 Introduction

Production of nanostructures has become one of the challenging area for the scientists and researchers. Some of the nanostructures are nanobeams, nanorods, nanoribons, nanoplates, nanocones, nanosheets and nanoshells etc. These materials have outstanding mechanical, electrical and thermal properties [1] resulting from their nanoscale dimensions. Carbon Nanotubes (CNTs) discovered by Ijima in 1991 is another type of nanostructures which have opened a new area in the field of nanotechnology. One may refer modelling and composite of carbon nanotubes in the review articles [2–4]. These structures have applications in the field of nanodevices, nanosensors, nanooscillators, nanocomposites and Nano-Optomechanical Systems (NOMS) etc. One of the interesting characteristics of carbon nanotubes is that they may be modeled as nanobeams. Hence static and dynamic behaviors of CNTs have become one of the interesting topic in the past five years. Static studies include bending and buckling analyses while dynamic studies include vibration and wave propagation analyses. This article focuses mainly on vibration of CNTs and a few on wave propagation in CNTs. There are mainly two types of studies related to vibration of carbon nanotubes. One is experimental studies [5–7] and other is theoretical modeling. Theoretical modeling is again categorized into atomistic modeling and continuum mechanics modeling. Atomistic modeling [8–11] is confined to problems at molecular or atomic motions. Some of these approaches are classical Molecular Dynamics (MD), Tight Binding Molecular Dynamics (TBMD) and Density Functional Theory (DFT). Conducting experiments at nanoscale size is quite expensive and time consuming. Hence, one needs huge computational tasks. So continuum modeling has played a significant role in nanostructures analysis. In this modeling, the structures are considered to be homogeneous and continuum and their intrinsic atomic structures are not taken into consideration. Hence accurate prediction of crystal lattice structure by this model is doubtful. Therefore atomistic-continuum modeling [12–14] came into existance which make up the difficulties found with atomistic modeling and continuum modeling. This linkage between continuum and atomic chirality was done by equating the molecular potential energy with the mechanical strain energy of a representative volume element of a continuum model. In addition to above, continuum modeling

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analysis is dependent on material constants. Continuum mechanics may be categorized into classical (local) continuum mechanics and nonlocal continuum mechanics. Various theories like rod, beam and shell are incorporated in the continuum mechanics to study vibration behavior of CNTs. Both analytical and numerical methods have been employed to study vibration of Single Walled Carbon NanoTubes (SWCNTs) and Multiwalled Carbon NanoTubes (MWCNTs) based on classical continuum mechanics [15–28] using various theories like Euler–Bernoulli, Timoshenko and some higher order beam theories. In these studies, various complicating effects have also been investigated. The significant influence caused by small scale effects such as electric force, chemical bond and van der waals forces is neglected when vibration of carbon nanotubes are investigated based on classical continuum models. Both experimental and atomistic simulation results show that at nano scale, these small length scale effect such as lattice spacing between individual atoms may not be neglected and the material will no longer be homogenized into a continuum. Study of nanostructures is a recent challenge with respect to the physical properties of the materials at micro/nano scales which is called the 'Quantum' or 'size' effect. The change in physical properties is due to the fact that micro/nanostructures at very small scales (molecular or atomic scale) have discrete nature. The system can no longer be modeled as a continuum when the dimensions of a system reduce to nanoscale. In this case, inter-atomic or intermolecular spacing of that system plays an important role. As such, the influence of long range inter-atomic and inter-molecular cohesive forces on the static and dynamic responses of the nanostructures becomes significant when the structures are at nanoscale and hence cannot be neglected. Ignoring the small scale effects in nanodesigning may cause completely incorrect solutions and hence gives an improper designs. So, small scale effects must be incorporated in the realistic design of the nanostructures [viz., nanoresonantors, nanoactuators, nanomachines and nanooptomechanical Systems] having nanoscale length. Hence, application of classical continuum model which ignores lattice spacing between individual atoms is doubtful. So various nonclassical continuum theories like strain gradient theory, couple stress theory [29, 30], micropolar theory and nonlocal elasticity theory are developed to incorporate size effect by introducing an intrinsic length scale. Among these theories, nonlocal elasticity theory which is a stress gradient elasticity theory proposed by Eringen [31, 32] has been widely applied in the vibration of nanotubes. According to this theory, the stress at a specific point depends on the strain tensors of the entire body. As a result, the constitutive relation is the spatial integral of weighted averages of the contribution of strain tensors of all points in the body to the stress tensor at a given point.

Nonlocal stress tensor σ at a point x is expressed as

$$\sigma = \int_V K(|x' - x|, \tau) t(x') dx' \tag{1}$$

where t(x) is the classical macroscopic stress tensor at a point x and is related to strain by Hooke's law as follows

$$t(x) = C(x) : \varepsilon(x) \tag{2}$$

where C is the fourth-order elasticity tensor. In Eq. (1), $K(|x' - x|, \tau)$ denotes nonlocal modulus, $|x' - x|$ the Euclidean distance, τ the material constant which depends on both internal length (lattice spacing) and external characteristic length (wavelength). Equation (1) denotes the weighted average of the contributions of the strain field of all points in the body to the stress field at a point.

Above Eqs. (1) and (2) represent the nonlocal constitutive behavior of a Hookean solid. Since it is difficult to solve the integral constitutive relation, so equivalent differential form was proposed which is as follows [33]

$$\left(1 - \tau^2 l^2 \nabla^2\right)\sigma = t, \quad \tau = \frac{e_0 a}{l} \tag{3}$$

where e_0 is material constant, a the internal characteristic length and l the external characteristic length.

Most of the studies employed differential constitutive relation. Nonlocal constitutive equations contain internal length scale as a material parameter and give information about the forces between atoms. In case of nanostructures, effect of material constant (τ) could not be neglected since the magnitude of external characteristic scale is relatively of the same order as that of internal characteristic scale. Nonlocal effects considered in the nonlocal elasticity theory play an important role in the vibration analysis and is determined by the magnitude of nonlocal parameter $e_0 a$. Hence an accurate prediction of nonlocal parameter is needed which is done by either molecular dynamics or theoretical approach [34–41].

This paper provides a brief review on the application of nonlocal elasticity theory in the vibration and wave propagation analyses of CNTs using beam models. In all the studies mentioned here, one may see that size effect plays a crucial role in the vibration analysis of CNTs. Xu [42] emphasized that nonlocal impact on natural frequencies and vibrating modes is negligible for micro beams and plays a significant role in case of nanobeams. Importance of nonlocal effects [43] has been shown based on some of the choosen problems. In this study, it has been shown that microelectromechanical systems (MEMS) devices would not exhibit nonlocal effects while nanoscale devices exhibit. Hence nonlocal elasticity theory plays a vital role in the modeling and design of nanoelectromechanical systems (NEMS) devices [44, 45]. Since nonlocal elasticity theory proposed by Eringen, most of the studies have been investigated using this theory. As such, various types of beam theories have been employed. Some of the studies give exact solutions for simple boundary conditions viz., simply supported case. But, since it is somehow difficult to find exact solution for complex geometries with different boundary conditions, so numerical methods like Finite Element Method (FEM), Differential Quadrature (DQ) method, Rayleigh–Ritz method, Homotopy Perturbation method etc have been employed in the analysis. Seeing the practical application of CNTs in day today life, researchers have studied vibration analysis of CNTs considering geometry of CNTs (radius, length and number of layers), surrounding medium, temperature and boundary conditions etc. This article reviews all these studies done so far. Though Arash and Wang [46] have reviewed application of nonlocal elastic theories in the modeling of CNTs and graphene sheets. But to the best of our knowledge, there exists no paper which gives recent trends and works on the vibration of CNTs using beam models based on nonlocal elasticity theory. This article is arranged in the following manner: (1) different beam theories. In this section, we have considered both SWCNTs and MWCNTs and under this subsection, we have considered various complicating effects, nonlinear vibration, functionally graded and wave propagation. (2) Some of the studies based on rod models. (3) Future direction and lastly (4) conclusion.

One may note that various theories such as Euler–Bernoulli theory, Timoshenko beam theory, Reddy beam theory, Levinson beam theory and third order shear deformation beam theory etc are developed recently to study vibration of carbon nanotubes based on nonlocal elasticity theory. Accordingly, the subsequent paragraphs give the detail investigations in the above regard.

2 Euler–Bernoulli Beam Theory

Based on this theory, the displacement fields are given by Reddy [150]

$$\begin{aligned} u_1 &= u(x,t) - z\frac{\partial w}{\partial x} \\ u_2 &= 0 \\ u_3 &= w(x,t) \end{aligned} \tag{4}$$

where (u_1, u_2, u_3) are the displacements along x, y, z coordinates respectively, (u, w) are the axial and transverse displacements of the point $(x, 0)$ on the mid plane ($z = 0$) of the beam and t denotes time. Transverse shear and transverse normal strains are neglected in this theory. Strain displacement relation may be written as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \tag{5}$$

where ε_{xx} is the strain.

Governing equations of Euler–Bernoulli beam theory may be written as

$$\frac{\partial N}{\partial x} + f = m_0\frac{\partial^2 u}{\partial t^2} \tag{6}$$

$$\frac{\partial^2 M}{\partial x^2} + q - \bar{N}\frac{\partial^2 w}{\partial x^2} = m_0\frac{\partial^2 w}{\partial t^2} - m_2\frac{\partial^4 w}{\partial x^2 \partial t^2} \tag{7}$$

where $\bar{N}$ is applied axial compressive force, $M = \int_A z\sigma_{xx} dA$, $N = \int_A \sigma_{xx} dA$ and $(f(x, t), q(x, t))$ are the axial and transverse distributed forces. One may note that governing equations for free vibration of nanobeams may be obtained by setting $\bar{N}$ and q to zero.

In Eqs. (6) and (7), m_0 and m_2 are mass inertias and are defined as follows:

$$m_0 = \int_A \rho dA, \quad m_2 = \int_A \rho z^2 dA$$

where A and ρ denote cross sectional area and mass density of nanobeams respectively.

Some of the commonly used boundary conditions based on Euler–Bernoulli beam theory are given below:

Simply Supported Transverse displacement (w) is 0 and transverse shear force (Q_x) is unknown. In addition, the bending moment should be specified while slope is not specified.

Clamped Transverse displacement and slope are specified to be 0 whereas shear force and bending moment are unknown.

Free Transverse deflection as well as the slope are not specified whereas shear force and bending moment should be specified.

Based on this theory, various investigations have been carried out for SWCNTs and MWCNTs considering various complicating effects, nonlinearity and combination of functionally graded materials with nanotubes. Some wave propagation analysis of nanotubes has also been discussed.

2.1 Single Walled Carbon NanoTubes (SWCNTs)

There are two types carbon nanotubes: Single Walled Carbon NanoTubes (SWCNTs) and Multi Walled Carbon NanoTubes (MWCNTs). SWCNTs have one single layer of graphene cylinders while MWCNTs have many layers. In this section, we will discuss recent investigations done for SWCNTs based on Euler–Bernoulli beam theory. One should keep in mind some of the important points [47] while applying nonlocal beam models in the vibration of CNTs. Authors [47] have derived governing equations for nonlocal Euler Bernoulli and Timoshenko beams. Then they have pointed out some of the facts which have been overlooked in some of the studies due to which inaccurate results may have been obtained.

Analytical method [97] and numerical methods like Ritz method [48–50] and finite element method [50, 51, 52] have been applied to solve governing equation of vibration of Euler–Bernoulli beams using nonlocal elasticity theory. Authors [99] found expression for frequency parameter (λ) of nonlocal simply supported Euler–Bernoulli beams as

$$\lambda = \frac{n^2\Pi^2}{\sqrt{1 + n^2\Pi^2\alpha^2}}$$

where n takes positive integral values, α is the scaling effect parameter and $\lambda = \frac{\rho A\omega^2 L^4}{EI}$. The parameters ρ, A, ω, L, E and I are density of nanotubes, cross sectional area, natural frequency, length of nanotube, Young's modulus and mass moment of inertia respectively.

As mentioned above, Ritz method and finite element method are two types of methods used by few researchers in their studies. The basis functions for FEM are taken element wise whereas the basis functions for Rayleigh–Ritz method are taken for the whole domain. Analytical and FEM solutions have been found only for some of the common boundary conditions. Till now, all the boundary conditions have not been taken into consideration since these solutions are quite difficult to solve. In this regard, Rayleigh–Ritz method is an efficient numerical method in handling all set of classical boundary conditions [50]. One may observe from the numerical and analytical results that frequency parameters decrease when scaling effect parameters increase [50, 53, 99]. Figure 1 shows the variation of frequency parameter with scaling effect parameter.

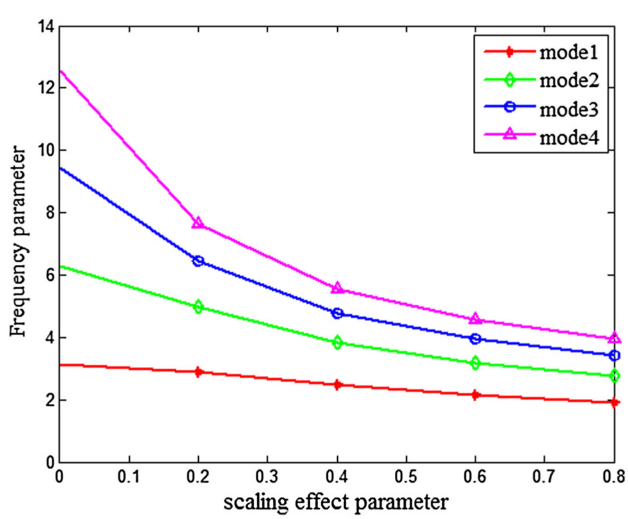

Fig. 1 Variation of frequency parameter of simply supported Euler Bernoulli nanobeams with scaling effect parameter

Next, in the following paragraph, we have discussed how the solutions are affected by surrounding medium, thermal effect, fluid conveying nanotubes and different shapes of nanotubes. The significance of nonlocal effects is emphasized in case of cracked nanobeams in which two segments are connected by rotational spring [54]. In case of cracked nanobeams, natural frequencies decrease when nonlocal parameter increases and the influence of nonlocal effect can be clearly seen in case of higher modes.

Sixth order compact finite difference method [55] has been used to study free vibration analysis of nanobeams embedded in an elastic medium using nonlocal Euler Bernoulli theory. Nanocantilever shows different results compared to other boundary conditions. Upto critical height ratio (CHR), nonlocal frequencies are larger than local frequencies and beyond CHR, the trend is exactly opposite [56].

Knowledge of dynamic behavior of rotating nanostructures is important in the production of nanomachines. So few studies have been analyzed for bending vibration characteristics of rotating nanocantilever beams [57, 58]. In Ref. [57], one may see that as rotational angular velocity increases, the small scale effect on the frequency response is increased in first mode while it is decreased in higher modes. Frequency parameters increase with rotating angular velocity [58] in both local and nonlocal elasticity models. In case of rotating nanotubes, fundamental frequency parameter increases with tensile axial preload and decreases with increase in compressive preload [59]. Nonlocal parameter is dependent on the aspect ratio except in slender nanotubes and plays an important role on the dynamic displacements [60, 61].

Vibration behavior of carbon nanotubes embedded in a pasternak medium has been analyzed by Soltani et al. [62] under a thermal environment. Bubnov–Galerkin method has been applied by Mustapha and Zhong [63] to study thermo-mechanical vibration of SWCNTs. The difference between local and nonlocal frequency is comparatively high at low temperature change [64]. Fundamental frequency and critical flow velocity increase with temperature change at low temperature while they decrease with temperature change [65] at high temperature. Waviness in the curved SWCNTs causes increase in natural frequency as compared to straight SWCNTs [66].

Vibration of viscous fluid conveying SWCNTs has been investigated by Lee and Chang [67] to examine the effects of nonlocal parameter, viscosity, aspect ratio and elastic medium on the fundamental frequency. Nonlocal Euler–Bernoulli beam theory has been applied to study vibration characteristics of nonuniform SWCNTs conveying fluid embedded in viscoelastic medium [68]. Small scale effect has significant effect on critical flow velocities for fluid conveying beams with nano length scale but this effect is neglected for microbeams [69]. Knowledge of vibration characteristics of fluid conveying nanotubes may help to design nanofluidic devices [70].

Since CNTs can undergo large deformations within the elastic limit, so nonlinear analysis is quite important. Some studies on nonlinear vibration based on nonlocal Euler–Bernoulli beam theory include effective resonant frequency of a cantilever SWCNTs with rippling deformation [71], nonlinear vibration of SWCNTs resting on an elastomeric substrate under thermal effects [72], nonlinear flexural vibrations of elastically supported nonuniform nanobeams [73], nonlinear vibration of simply supported SWCNTs embedded in Pasternak-type foundation [74] and nonlocal effect on nonlinear vibrations of Euler–Bernoulli beam [75].

Functionally graded materials (FGMs) have been graded to achieve specific thermal and mechanical properties to increase the functionality of the structure. Now a days, FGMs are used in nanoelecromechanical system to achieve high sensitivity and desired performance. Hence few studies [76, 77] are illustrated here related to functionally graded nanobeams. Wave propagation analysis in carbon nanotubes based on nonlocal elasticity theory has become one of the interesting area in the field of nanotechnology. Some studies on wave propagation in carbon nanotubes [78–80] have also been reported.

2.2 Multi Walled Carbon NanoTubes (MWCNTs)

One should note the difference between governing equations of DWCNTs and SWCNTs [47] based on Euler–Bernoulli beam theory. Multiwalled carbon nanotubes having length L [82] consist of n nanotubes of cylindrical shape. It lies on a Winkler foundation having foundation modulus as κ and is subjected to an axial stress σ_x.

Equations of motion for multiwalled carbon nanotubes are given by

$$D_1(w_1, w_2) = L_1(w_1) - c_{12}(w_2 - w_1) + \eta^2 c_{12}\left(\frac{\partial^2 w_2}{\partial x^2} - \frac{\partial^2 w_1}{\partial x^2}\right) = 0 \tag{8}$$

$$\begin{aligned} D_i(w_{i-1}, w_i, w_{i+1}) = {} & L_i(w_i) + c_{(i-1)(i)}(w_i - w_{i-1}) \\ & - c_{(i)(i+1)}(w_{i+1} - w_i) \\ & - \eta^2 c_{(i-1)(i)}\left(\frac{\partial^2 w_i}{\partial x^2} - \frac{\partial^2 w_{i-1}}{\partial x^2}\right) \\ & + \eta^2 c_{(i)(i+1)}\left(\frac{\partial^2 w_{i+1}}{\partial x^2} - \frac{\partial^2 w_i}{\partial x^2}\right) = 0 \\ & \text{for } i = 2, 3, \ldots, n-1 \end{aligned} \tag{9}$$

$$\begin{aligned} D_n(w_{n-1}, w_n) = {} & L_n(w_n) + c_{(n-1)(n)}(w_n - w_{n-1}) \\ & - \eta^2 c_{(n-1)(n)}\left(\frac{\partial^2 w_n}{\partial x^2} - \frac{\partial^2 w_{n-1}}{\partial x^2}\right) = f(x, t) \end{aligned} \tag{10}$$

where $f(x, t)$ is the external force acting on the outermost nanotube and $L_i(w_i)$ is the differential operator given by

$$\begin{aligned} L_i(w_i) = {} & EI_i \frac{\partial^4 w_i}{\partial x^4} + \rho A_i \frac{\partial^2 w_i}{\partial t^2} + A_i \sigma_x \frac{\partial^2 w_i}{\partial x^2} \\ & - \eta^2 A_i\left(\rho \frac{\partial^4 w_i}{\partial x^2 \partial t^2} + \sigma_x \frac{\partial^4 w_i}{\partial x^4}\right) + \delta_{in}\left(\kappa w_n - \kappa \eta^2 \frac{\partial^2 w_n}{\partial x^2}\right) \end{aligned} \tag{11}$$

Here, the index $i = 1, 2, \ldots, n$ refers the order of the nanotubes with the innermost nanotube indicated by $i = 1$ and the outermost nanotube indicated by $i = n$. Also, $w_i(x, t)$ is the transverse deflection of the ith nanotube, $0 \leq x \leq L$ and $t_1 \leq t \leq t_2$. In the above equations, δ_{in} is the Kronecker's delta, $\eta = e_0 a$ is the nonlocal parameter, E the Young's modulus, I_i and A_i are the moment of inertia and the cross sectional area of the ith nanotube respectively. The coefficient $c_{(i-1)(i)}$ is the interaction coefficient of van der Waals forces between the $(i-1)$ and ith nanotubes with $i = 2, \ldots, n$.

Variational formulations help to find solutions using various numerical methods like finite element method, Rayleigh–Ritz method etc. This also helps to derive natural and geometric boundary conditions correctly. Hence it is very necessary to obtain variational formulations. In this regard, semi inverse method developed by He [81] is one way to find. In this method, trial functions were used for deriving variational principles. Based on this method, some of the studies are discussed below. Semi inverse method [82, 83] has been applied to derive variational principles for the vibration of multiwalled carbon nanotubes based on nonlocal Euler–Bernoulli beam theory.

MWCNTs may behave similar to single beam in high values of nonlocal parameter. In higher modes, natural frequencies of multiple Euler–Bernoulli beams tend to frequencies of its constituent beams [84]. Amplitude ratios for nonlocal Euler–Bernoulli beam have been given for DWCNTs [85]. Higher modes of DWCNTs are dominated by van der walls interaction between the inner and outer nanotubes [86]. Natural frequencies and associated amplitude ratios (inner to outer tubes) are dependent on small length scale [87]. In double nanobeam system [88], the nonlocal effects are reduced with the increase in the stiffness of the coupling springs during the out-of-phase modes of vibration.

Bending vibration of coupled nanobeam system [89] under prestressed condition has also been studied using nonlocal elasticity theory. Forced vibration of elastically connected double walled carbon nanotubes carrying a moving nanoparticle has been investigated using nonlocal Euler–Bernoulli beam theory [90]. Longitudinal magnetic effect on DWCNTs has been analyzed analytically [91, 92] based on nonlocal Euler–Bernoulli theory. This study shows that synchronous vibration phases of DWCNTs are greatly influenced by nonlocal effects than asynchronous vibration phases. Effect of small length scale on natural frequencies of DWCNTs conveying fluid has been discussed by Wang [93]. First three vibration frequencies of DWCNTs conveying fluid are lower than those of SWCNTs [94].

Harmonic balance method and Davidon–Fletcher–Powell method are applied to study nonlinear free vibration of double walled carbon nanotubes based on nonlocal Euler–Bernoulli theory [95] and it may be noticed that noncoaxial vibration amplitudes considering only nonlinear van der Waals forces are larger than considering both geometric nonlinearity and nonlinear van der Waals forces. Incremental harmonic balance method has been adopted to study nonlinear vibrations of embedded MWCNTs under thermal environment [96]. Vibration characteristics of DWCNTs subjected to initial stress [97] have been analyzed based on the above theory. Small scale effect also plays an important role in case of wave propagation analysis in DWCNTs under temperature field [98].

3 Timoshenko Beam Theory

Timoshenko beam theory is based on constant shear stress assumption. Therefore, shear correction factor is included to compensate the error.

The displacement fields are based on [150]

$$\begin{aligned} u_1 &= u(x,t) + z\phi(x,t) \\ u_2 &= 0 \\ u_3 &= w(x,t) \end{aligned} \tag{12}$$

where ϕ denote the rotation of cross section.

Strain displacement relations are given by

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} + z\frac{\partial \phi}{\partial x} \\ \gamma_{xz} &= \phi + \frac{\partial w}{\partial x} \end{aligned} \tag{13}$$

where γ_{xz} is transverse shear strain.

Governing equations of Timoshenko beam theory are given by

$$\begin{aligned} \frac{\partial Q}{\partial x} + q - \bar{N}\frac{\partial^2 w}{\partial x^2} &= m_0\frac{\partial^2 w}{\partial t^2} \\ \frac{\partial M}{\partial x} - Q &= m_2\frac{\partial^2 \phi}{\partial t^2} \end{aligned} \tag{14}$$

where $Q = \int_A \sigma_{xz} dA$. For other notations one may refer Sect. 2.

Some of well known boundary conditions based on Timoshenko beam theory are given below:

Simply Supported Transverse displacement is 0 and transverse shear force is unknown. In addition, the bending moment is specified while rotation is not specified.

Clamped Transverse deflection as well as the rotation are assumed to be 0. Shear force and bending moment are unknown.

Free Transverse deflection as well as the rotation are not specified. Shear force and bending moment are specified.

3.1 Single Walled Carbon NanoTubes (SWCNTs)

As in Euler–Bernoulli beams, one should also carefully handle while applying nonlocal elasticity theory in Timoshenko beams [47]. In this study, it is seen that Timoshenko beam model predicts closer results with molecular dynamics (MD) results than Euler–Bernoulli beam model. Analytical [99, 100] and numerical solutions [101, 102] for vibration of nonlocal Timoshenko nanobeams show that frequency parameters are greatly affected by nonlocal parameter. Frequency parameters decrease with increase in nonlocal parameter (Fig. 2). Effect of nonlocal parameter on the natural frequencies and mode shapes has been shown in [103]. Meshless method with radial basis function has been used in the bending, buckling and free vibration of nonlocal Timoshenko nanobeams. They found that global collocation is used for bending and free vibration while local collocation is used for buckling

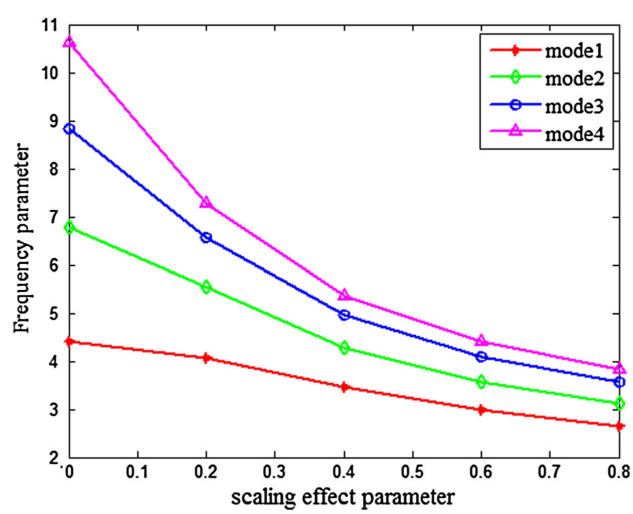

Fig. 2 Variation of frequency parameter of clamped-clamped Timoshenko nanobeams with scaling effect parameter

analysis [104]. Vibration of nanotubes embedded in an elastic medium has been investigated based on nonlocal Timoshenko beam theory [105]. Nonlocal parameter has prominent effect particularly in case of cantilever carbon nanotubes which will flutter at critical values of material constant [106]. Nonlocal Timoshenko beam models yield better fundamental frequencies of shorter SWCNTs than local beam models [107].

For nonlocal elastic structures, quadratic functionals could not be obtained directly using minimum of potential energy. They can be obtained from weak form of governing equations [108]. Timoshenko beam model may be used for short CNT biosensors [109]. Frequencies of embedded SWCNTs are dependent on the nonlocal parameter and also on the stiffness of the surrounding elastic medium [110]. Mechanically based approach [111] for nonolocal elasticity has been discussed to study dynamics of nonlocal Timoshenko beam model. Nonlocal viscoelastic model (extension of nonlocal elasticity model) was used to study vibration of damped Timoshenko beams [112]. The nonlocal viscoelastic constitutive relation for Timoshenko beam is as follows

$$\begin{aligned} \sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} &= E\left(1 + \tau_d \frac{\partial}{\partial t}\right) z \frac{\partial \theta}{\partial x} \\ \sigma_{xz} - (e_0 a)^2 \frac{\partial^2 \sigma_{xz}}{\partial x^2} &= G\left(1 + \tau_d \frac{\partial}{\partial t}\right)\left(\theta + \frac{\partial w}{\partial x}\right) \end{aligned} \tag{15}$$

where E is the Young's modulus, G the shear modulus, τ_d the viscous damping coefficient, $e_0 a$ the nonlocal parameter, w the transverse deflection and θ the rotation of cross section. Nondimensional fundamental frequency increases with angular velocity [113]. Dynamic analysis of embedded SWCNTs has been studied under a moving nanoparticle [114]. Vibration of nanobeams has been analyzed to examine effect of initial stress [115]. Effect of nonlocal parameter on the frequency is greatly affected by temperature change and frequencies decrease with thickness of CNTs [116]. Piezoelectric materials have electromechanical coupling effect due to mechanical deformations under the application of electrical loads or electrical deformation under the influence of mechanical loads. So these materials have application in smart structures or systems. It is therefore important to study piezoelectric materials [117]. Vibration characteristics also depend on the chirality of zigzag carbon nanotubes [118].

Nonlocal Timoshenko beam theory is used to study thermal vibration response of embedded SWCNTs [119]. This study shows that at low temperature change, the effect of Winkler's constant is negligible on nonlocal frequency. At low temperature, frequencies including thermal effect are larger than excluding thermal effect while at high temperature, the trend is opposite [120]. For designing CNT-based fluidic devices, it is necessary to know vibration property of fluid flow inside CNTs. Nonlocal Timoshenko beam theory has been used to study flexural vibration of viscoelastic carbon nanotubes which convey fluids and has been embedded in viscous fluid [121]. It is observed here that the critical flow velocity including the effect of external fluid is lesser than that without considering the effect of external fluids. Timoshenko beam theory has been taken into consideration for examining surface effects on frequency analysis of nanotubes [122]. Investigations on nonlinear vibration based on nonlocal Timoshenko beam theory include nonlinear vibration of an embedded curved single walled carbon nanotube under a harmonic load [123], nonlinear free vibration of single walled carbon nanotube [124] and nonlinear vibration of piezoelectric nanobeams [125].

3.2 Multi Walled Carbon NanoTubes (MWCNTs)

Governing equations for multiwalled carbon nanotubes based on nonlocal Timoshenko beam theory are given as follows [126]

$$D_{a1}(w_1, \varphi_1, w_2) = L_{a1}(w_1, \varphi_1) - c_{12}\Delta w_{21} + \eta^2 c_{12} \frac{\partial^2 \Delta w_{21}}{\partial x^2} = 0 \tag{16}$$

$$D_{b1}(w_1, \varphi_1) = L_{b1}(w_1, \varphi_1) = 0 \tag{17}$$

$$\begin{aligned} D_{a2}(w_1, w_2, \varphi_2, w_3) = {} & L_{a2}(w_2, \varphi_2) + c_{12}\Delta w_{21} \\ & - c_{23}\Delta w_{32} \\ & + \eta^2\left(-c_{12}\frac{\partial^2 \Delta w_{21}}{\partial x^2} + c_{23}\frac{\partial^2 \Delta w_{32}}{\partial x^2}\right) = 0 \end{aligned} \tag{18}$$

$$D_{b2}(w_2, \varphi_2) = L_{b2}(w_2, \varphi_2) = 0 \tag{19}$$

$$\begin{aligned} & D_{ai}(w_{i-1}, w_i, \varphi_i, w_{i+1}) \\ & = L_{ai}(w_i, \varphi_i) + c_{(i-1)(i)}\Delta w_{(i)(i-1)} - c_{(i)(i+1)}\Delta w_{(i+1)(i)} \\ & \quad - \eta^2 c_{(i-1)(i)} \frac{\partial^2 \Delta w_{(i)(i-1)}}{\partial x^2} + \eta^2 c_{(i)(i+1)} \frac{\partial^2 \Delta w_{(i+1)(i)}}{\partial x^2} = 0 \end{aligned} \tag{20}$$

$$D_{bi}(w_i, \varphi_i) = L_{bi}(w_i, \varphi_i) = 0 \tag{21}$$

where $i = 3, 4, \ldots, n-1$

$$\begin{aligned} D_{an}(w_{n-1}, w_n, \varphi_n) = {} & L_{an}(w_n, \varphi_n) + c_{(n-1)(n)}\Delta w_{(n)(n-1)} \\ & - \eta^2 c_{(n-1)(n)} \frac{\partial^2 \Delta w_{(n)(n-1)}}{\partial x^2} \\ = {} & f(x, t) \end{aligned} \tag{22}$$

$$D_{bn}(w_n, \varphi_n) = L_{bn}(w_n, \varphi_n) = 0 \tag{23}$$

where $L_{ai}(w_i, \varphi_i)$ and $L_{bi}(w_i, \varphi_i)$ are given as follows:

$$L_{ai}(w_i,\varphi_i)=\rho A_i\frac{\partial^2 w_i}{\partial t^2}-\rho A_i\eta^2\frac{\partial^4 w_i}{\partial x^2\partial t^2}+k_sGA_i\frac{\partial}{\partial x}\left(\varphi_i-\frac{\partial w_i}{\partial x}\right) +A_i\sigma_x\frac{\partial^2 w_i}{\partial x^2}-A_i\sigma_x\eta^2\frac{\partial^4 w_i}{\partial x^4}+\delta_{in}\left(kw_n-k\eta^2\frac{\partial^2 w_n}{\partial x^2}\right) \tag{24}$$

$$L_{bi}(w_i,\varphi_i) = \rho I_i\frac{\partial^2 \varphi_i}{\partial t^2} - \rho I_i\eta^2\frac{\partial^4 \varphi_i}{\partial x^2\partial t^2} + k_sGA_i\left(\varphi_i - \frac{\partial w_i}{\partial x}\right) - EI_i\frac{\partial^2 \varphi_i}{\partial x^2} \tag{25}$$

In the above equations, φ is the angle of rotation, Δw_{ij} is the difference operator defined as $\Delta w_{ij} = w_i - w_j$. Also, δ_{in} is the Kronecker's delta, $f(x, t)$ the forcing function, G the shear modulus and k_s the shear correction factor. The coefficient $c_{(i)(i-1)}$ is the interaction coefficient of van der Waals forces between the $(i-1)$ and ith nanotubes with $i = 2, \ldots, n$. It may be noted that, other parameters are defined in Sect. 2.2. Various studies of MWCNTs viz., Free vibration analysis of embedded double walled carbon nanotubes [127], semi inverse method to derive variational principles for MWCNTs [128], vibration of in-plane loaded double walled carbon nanotubes [129], vibration of carbon nanotubes using nonlocal continuum mechanics [130], nonlinear vibrations of embedded multiwalled carbon nanotubes under thermal effect [131], nonlinear free vibration of embedded double walled carbon nanotubes [132], transverse wave propagation in DWCNTs [130], flexural wave propagation in DWCNTs [133, 134] have been analysed based on nonlocal Timoshenko beam theory.

4 Modified Nonlocal Beam Model

Most of the studies on carbon nanotubes are based on partial elasticity theory and hence results obtained may be inaccurate. In the partial nonlocal elasticity model, higher boundary conditions for a higher order differential equation do not exist. Based on Eringen's nonlocal elasticity theory, an exact nonlocal beam model [135] has been developed where infinite order governing equation and the corresponding higher order boundary conditions are taken into consideration. In this model, effective nonlocal bending moment is derived which is an infinite series of higher-order nonlocal bending moments. In the equilibrium condition, classical bending moment should be replaced with effective nonlocal bending moment instead of nonlocal bending moment.

Nonlocal beam model was first used by Lee and Chang [136] for studying nonlocal effect on single walled carbon nanotubes conveying fluid. In their study, they saw that natural frequencies decrease with nonlocal parameter and critical flow velocities do not change with nonlocal parameter. Later it was pointed out that the equation used by them should be improved and hence they got inaccurate results [137]. At the same time, various studies have also been carried out based on partial nonlocal models. Using this exact nonlocal stress model, the vibration properties and stability of nanotubes conveying fluid are examined [138]. They found natural frequencies induce higher natural frequencies and the critical flow velocity with increasing nonlocal parameter. Hence exact nonolocal stress model gives different results to that of partially nonlocal stress model. Some of the works based on this theory are transverse vibration of simply supported nanobeams with an initial axial tension [139], transverse vibration of simply supported nanobeams subjected to initial axial force [140], dynamic behaviour of axially moving nanobeams [141], modeling of one dimensional nanobeam by exact variational approach [142], free vibration of thick nanostructures [143], transverse vibrations of a nanobeam subjected to a variable initial axial force [144], torsional vibration of axially moving nanotubes [145], thermal elasticity for nanobeam deformation [146], initial axial tension on free vibration of cantilever nanobeams [147] and wave propagation in carbon nanotubes [148, 149].

5 Other Beam Theories

5.1 Single Walled Carbon NanoTubes (SWCNTs)

Other than Euler–Bernoulli and Timshenko beam theories, investigations have also been carried out using other beam theories like parabolic shear deformation beam theory (RBT) of Reddy, general exponential shear deformation beam theory (ABT) of Aydogdu, Levinson beam theory, refined theory of Thai and sinusoidal shear deformation theory etc. Here some of them are cited below.

The displacement fields of Reddy beam theory are given by Reddy [150]

$$\begin{aligned} u_1 &= u(x,t) + z\phi(x,t) - c_1z^3\left(\phi + \frac{\partial w}{\partial x}\right) \\ u_2 &= 0 \\ u_3 &= w(x,t) \end{aligned} \tag{26}$$

where $c_1 = \frac{4}{(3h^2)}$ and h is the height of the beam. One may refer Sect. 2 for other notations.

Governing equations of this theory are as follows:

$$-m_0\frac{\partial^2 w}{\partial t^2}-c_1m_4\frac{\partial^3\phi}{\partial x\partial t^2}+c_1^2m_6\left(\frac{\partial^3\phi}{\partial x\partial t^2}+\frac{\partial^4 w}{\partial x^2\partial t^2}\right)+c_1\frac{\partial^2 P}{\partial x^2}=0 \tag{27}$$
$$-\hat{m_2}\frac{\partial^2\phi}{\partial t^2}+c_1\hat{m_4}\left(\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^3 w}{\partial x\partial t^2}\right)+\frac{\partial\hat{M}}{\partial x}=0$$

where $m_i=\int_A\rho z^i dA,\ \ i=0,4,6$ are mass inertias, $\hat{m_2}=m_2-c_1m_4,\ \hat{m_4}=m_4-c_1m_6,\ M=\int_A z\sigma_{xx}dA,\ P=\int_A z^3\sigma_{xx}dA,\ Q=\int_A\sigma_{xz}dA,\ R=\int_A z^2\sigma_{xz}dA,\ N=\int_A\sigma_{xx}dA,\ \hat{M}=M-c_1P,\ \hat{Q}=Q-c_2R$ and $c_2=\frac{4}{h^2}$.

Displacement fields of Levinson beam theory are same as that of Reddy beam theory but governing equations are same as that of Timoshenko beam theory. In this theory, Levinson used a vector approach to derive equilibrium equations while Reddy used variational consistent approach for deriving equilibrium equations.

Equations of motion, variational statements and analytical solutions have been given for bending, buckling and vibration of beams using different nonlocal beam theories like Euler–Bernoulli, Timoshenko, Reddy and Levinson beam theories [150]. A generalized nonlocal beam theory [151] has also been introduced to study bending, buckling and free vibration of nanobeams. After the general formulation, other well-known beam theories like Euler–Bernoulli, Timoshenko, Reddy and Aydogdu are found as a special case without repeating derivation of governing equations. As such, generalized nonlocal beam theory is given as below

$$\begin{aligned}U(x,z,t)&=u(x,t)-z\frac{\partial w}{\partial x}+f(z)u_1(x,t)\\V(x,z,t)&=0\\W(x,z,t)&=w(x,t)\end{aligned} \tag{28}$$

where $f(z)=0$ for Euler–Bernoulli beam theory, $f(z)=z$ for Timoshenko beam theory, $f(z)=z(1-\frac{4z^2}{3h^2})$ for Reddy beam theory and $f(z)=(z)(3)^{\frac{-2\frac{z}{h^2}}{ln3}}$ for ABT.

In Eq. (14), (U, V, W) are displacements along x, y, z directions respectively, u_1 the rotation of cross section and (u, w) are the axial and transverse displacements respectively. They also gave analytical solutions for simply supported nanobeams.

A nonlocal shear deformation beam theory [152] and nonlocal sinusoidal shear deformation beam theory [153] have been applied for bending, buckling and vibration of nanobeams. Nonlocal sinusoidal shear deformation beam theory includes quadratic variation of shear strains and shear stresses through the thickness of the beam and has some similarities with Euler–Bernoulli beam theory. Vibration of nanotube structures under excitation of a moving nanoparticle [154] has been investigated by nonlocal Euler–Bernoulli, nonlocal Timoshenko and nonlocal higher order beam theories. Nonlocal Rayleigh beam model [155] has been applied to study flap wise vibration characteristics of a rotating single walled carbon nanotube embedded in an elastic medium and has been solved by Differential Quadrature method. Nonlocal Rayleigh (NLR) beam has been used to study vibration of SWCNTs that is axially loaded and embedded in an elastic medium [156]. Effects of boundary conditions and initial axial forces have been investigated for transverse vibration of single walled nanotube structures embedded in an elastic medium based on Euler Bernoulli, Timoshenko and other higher order beam models [157]. Forced vibration analysis of functionally graded nanobeams has been investigated using generalized beam theory [158].

5.2 Multi Walled Carbon NanoTubes (MWCNTs)

Vibration of double walled carbon nanotubes subjected to a moving nanoparticle has been investigated by Kiani [159] using nonlocal Euler–Bernoulli, nonlocal Timoshenko and nonlocal higher order beam theories. Nonlocal Rayleigh beam theory [160] has been used to study free transverse vibration of elastically supported double walled carbon nanotubes subjected to axially varying magnetic fields.

6 Rod Models

Differential Quadrature method has been applied [161] to investigate axial vibration of tapered nanorod using nonlocal elasticity theory. Free axial vibration of nanorod in terms of displacement is given by Danesh et al. [161]

$$EA(x)\frac{\partial^2 u(x,t)}{\partial x^2}=\left(1-(e_0a)^2\frac{\partial^2}{\partial x^2}\right)m(x)\frac{\partial^2 u(x,t)}{\partial t^2} \tag{29}$$

where $A(x)$ is cross sectional area, $m(x)$ the mass per unit length, ρ the density, $u(x,\ t)$ the axial displacement and E the elastic modulus.

Nonlocal elasticity has been employed to study axial vibration of SWCNTs embedded in an elastic medium [162]. Axial vibration of SWCNTs based mass sensor [163] has been proposed using nonlocal elasticity theory. Some of the other works based on rod models are axial vibration of nanorod [164], axial wave propagation in coupled nanorod [165], axial vibration of carbon nanotube heterojunctions [166], longitudinal vibration of SWCNTs with attached buckyballs [167], longitudinal vibration of double nanorod system [168], longitudinal vibration of nanorod with internal long range interactions [169] and torsional vibration of carbon nanotube with attached buckyball systems [170].

7 Future Directions and Conclusions

Application of nonlocal elasticity theory may be extended to other types of nanostructures incluiding various complicating effects such as carbon nanotubes carrying nanoparticles subjected to external force and carbon nanotubes carrying fluid under magnetic field. Few works have been done using rod models and hybrid nonlocal continuum model. Future studies may include application of hybrid nonlocal model using beam or rod theories. Most of the recent studies employ analytical solutions for simply supported boundary condition. Some of the efficient numerical methods like homotopy perturbation method, discrete convolution technique, Adomian decomposition may also be applied in the analysis. One may develop analytical solutions for other boundary conditions. It is seen that most of the studies include analysis by numerical or analytical methods. One may also develop solutions using theoretical proofs.

Present article includes importance of nonlocal elasticity theory in the vibration of carbon nanotubes. Various investigations of vibration of carbon nanotubes based on nonlocal elasticity theory have been cited and discussed. In this article, authors have discussed vibration of SWCNTs and MWCNTs using different beam theories including nonlinear vibration, complicating effects and functionally graded materials. Some of the references related with wave propagation analysis in the CNTs have also been cited and discussed. Lastly, few analysis based on rod models have also been included. Some of the future directions have also been pointed out.

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