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Onset of buoyancy-driven convection in isotropic porous media heated from below

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Abstract−A theoretical analysis of buoyancy-driven instability under transient basic fields is conducted in an initially quiescent, fluid-saturated, horizontal, isotropic porous layer. Darcy's law is employed to explain characteristics of fluid motion, and Boussinesq approximation is used to consider the density variation. Under the principle of exchange of stabilities, a stability analysis is conducted based on the linear stability analysis and energy method and their modifications. The critical condition of onset of buoyancy-driven convection is obtained as a function of the Darcy-Rayleigh number. The propagation theory and the modified energy method under the self-similar coordinate suggest reasonable stability criteria and support each other. The former one based on the linear stability theory predicts more stable results than the latter based on the energy method. The growth period for disturbances to grow seems to be required until the instabilities are detected experimentally.

Key words: Buoyancy-driven Convection, Porous Media, Propagation Theory, Energy Method

INTRODUCTION

Buoyancy-driven convection may occur in a fluid-saturated porous medium due to temperature or solute concentration gradient. Heating from below or increasing the solute concentration from above creates an unstable density profile. It is well known that the buoyancy-driven phenomena in porous media have a wide variety of engineering applications, such as geothermal reservoirs, agricultural product storage system, packed-bed catalytic reactors, the pollutant transport in underground and the heat removal of nuclear power plants. The current interest in the phenomena of natural convection in porous media is the enhanced carbon dioxide dissolution into the saline water confined within the geologically stable formations [1-3].

The analysis of convective instabilities in porous media begins with Horton-Rogers-Lapwood convection [4,5]. They examined thermally-driven convection and used the methods developed for convection in a homogeneous fluid. It was assumed that there was a linear increase in temperature with depth, appropriate for gradual heating or for a steady state, e.g., the naturally occurring geothermal gradients in the subsurface. However, in many experimental situations and field studies there is a relatively rapid change in temperature or solute concentration at one boundary. The basic profile of temperature or concentration before the onset of convection is then time-dependent. Dealing with startup, transient phenomena may eventually lead to a better understanding of the history-dependence of flows in this and other systems, something that a static analysis will not do. The related instability analysis has been conducted by using the frozen-time model [6], propagation theory [7], maximum-Rayleigh-number criterion [8] and amplification theory [9]. All of these methods have a parallel history of application in the Rayleigh-B?nard convection. The first two models are based on linear theory and yield the critical time as the parameter based on the quasi-static approximation. In the maximum-Rayleigh-number criterion the temperature profile is assumed to be linear within $Z=Z_{max}(t)$, and the onset time defined when newly defined transient-Rayleigh number reaches the conventional steady-state Darcy-Rayleigh number. $Z_{max}(t)$ is the vertical distance at which the transient-Rayleigh number has its maximum value. The last method is the initial value model; it requires the initial conditions at the time t=0 and the criterion to define manifest convection. Also, the stability of time-dependent base states has been investigated by energy method [6]. Recently, Ennis-King et al. [1], Xu et al. [2] and Riaz et al. [3] reexamined this problem with the connection to the enhanced carbon dioxide dissolution into the saline water confined within the geologically stable formations. Ennis-King et al. [1] corrected Catagirone's [6] energy method results and extended Catagirone's analysis into the anisotropic porous media. Energy methods give a lower bound for the onset of the instability, but give no information about the growth rate and wavenumber of the most dangerous disturbance. Riaz et al. [3] analyzed the onset of convection in porous media under the time-dependent concentration field in self-similar coordinate. They employed the dominant mode analysis without the quasi-steady state approximation and showed that the quasi-steady state approximation in self-similar coordinate, which is quite similar to the propagation theory, provides quite accurate results.

In the present study the onset of buoyancy-driven convection in isotropic porous media is investigated by conventional frozen-time model, the propagation theory we developed [5,10-12] and the modified energy method which is firstly introduced in the present study. Our predictions will be also compared with the existing theoretical results [1-3,6,8].

THEORETICAL ANALYSIS

1. Governing Equations

The system considered here is an initially quiescent, fluid-satu-

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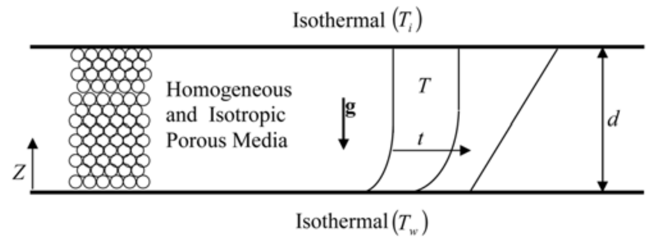

Fig. 1. Schematic diagram of system considered here.

rated, horizontal porous layer of depth d, as shown in Fig. 1. The solid substrate has a constant porosity ε and permeability K. And the interstitial fluid is characterized by thermal expansion coefficient β, density ρ, heat capacity $(\rho c)_f$ and kinematic viscosity ν. The porous medium is regarded as a homogeneous and isotropic fluid with heat capacity $(\rho c)^*=\varepsilon(\rho c)_f+(1-\varepsilon)(\rho c)_s$ and thermal conductivity $k^*=\varepsilon k_f+(1-\varepsilon)k_s$; here the subscripts f and s represent the fluid and solid phase, respectively. Before heating, the fluid layer is maintained at uniform temperature T_i for time t<0. For time t≥0 the lower boundary is heated with constant temperature T_w. For this system the governing equations of flow and temperature fields are expressed employing the Boussinesq approximation and Darcy's model [1]:

$$\nabla\cdot\mathbf{U}=0 \tag{1}$$

$$\frac{\mu}{K}\mathbf{U}=-\nabla P+\rho g \tag{2}$$

$$\left(\frac{\partial}{\partial t}+\frac{(\rho c)_f}{(\rho c)^*}\mathbf{U}\cdot\nabla\right)T=\alpha\nabla^2 T \tag{3}$$

$$\rho=\rho_i[1-\beta(T-T_i)] \tag{4}$$

where $\mathbf{U}$ is the velocity vector, T the temperature, P the pressure, μ the viscosity, $\alpha(=k^*/(\rho c)^*)$ the effective thermal diffusivity and $\mathbf{g}$ the gravitational acceleration. The important parameter to describe the present system is the Darcy-Rayleigh number Ra_D defined by

$$Ra_D=\frac{g\beta K\Delta T d}{\alpha\nu}\frac{(\rho c)_f}{(\rho c)^*}, \tag{5}$$

where $\Delta T=(T_w-T_i)$.

For a system of large Ra_D, the stability problem becomes transient and very difficult, and the critical time t_c to mark the onset of buoyancy-driven motion remains unsolved. For this transient stability analysis we define a set of nondimensionalized variables τ, z, θ_0 by using the scale of time d^2/α, length d and temperature ΔT. Then the basic conduction state is represented in dimensionless form by

$$\frac{\partial\theta_0}{\partial\tau}=\frac{\partial^2\theta_0}{\partial z^2} \tag{6}$$

with the following initial and boundary conditions,

$$\theta_0=0 \text{ at } t=0 \tag{7a}$$

$$\theta_0=1 \text{ at } z=0 \text{ and } \theta_0=0 \text{ at } z=1 \tag{7b}$$

The above equations can be solved by using conventional separation of variables technique or Laplace transform method as follows:

$$\theta_0=1-z-2\sum_{n=0}^{\infty}\frac{\sin(n\pi z)}{n\pi}\exp(-n^2\pi^2\tau), \tag{8a}$$

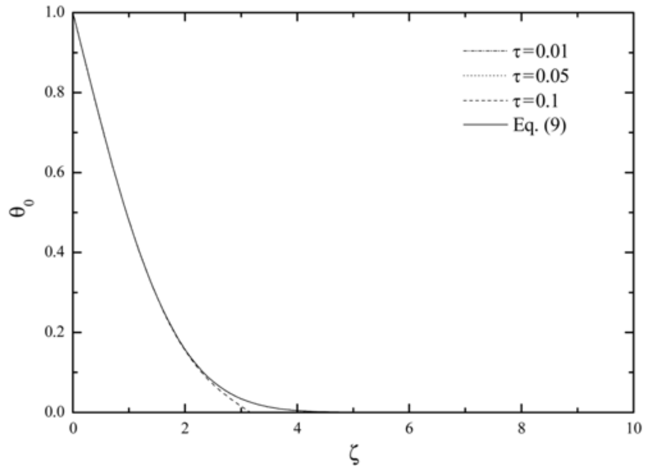

Fig. 2. Base temperature fields.

$$\theta_0=\sum_{n=0}^{\infty}\left\{\operatorname{erfc}\left(\frac{n}{\sqrt{\tau}}+\frac{\zeta}{2}\right)-\operatorname{erfc}\left(\frac{n+1}{\sqrt{\tau}}-\frac{\zeta}{2}\right)\right\}, \tag{8b}$$

where $\zeta=z/\sqrt{\tau}$. Eq. (8b) converges more rapidly than Eq. (8a) for a small time region. The evolution of the basic profiles of temperature with time is described in Fig. 2. For the deep-pool region of $\tau\leq 0.01$, the base temperature profiles reduced:

$$\theta_0=\operatorname{erfc}\left(\frac{\zeta}{2}\right), \tag{9}$$

The above Leveque-type soutions of Eq. (9) is in good agreement with the exact solutions of Eq. (8) in the region of $\tau<0.1$. For $\tau\leq 0.01$ Eq. (8b) with n=0 yields almost the same temperature profile as Eq. (9).

2. Propagation Theory

Under the linear stability theory the disturbances caused by the onset of thermal convection can be formulated, in dimensionless form, in terms of the temperature component θ_1 and the vertical velocity component by decomposing Eqs. (1)-(4):

$$\nabla^2 w_1=-\nabla_1^2\theta_1 \tag{10}$$

$$\frac{\partial\theta_1}{\partial\tau}+Ra_D w_1\frac{\partial\theta_0}{\partial z}=\nabla^2\theta_1, \tag{11}$$

where $\nabla^2=(\partial^2/\partial x^2)+(\partial^2/\partial y^2)+(\partial^2/\partial z^2)$ and $\nabla_1^2=(\partial^2/\partial x^2)+(\partial^2/\partial y^2)$. The velocity component has the scale of α/d and the concentration component has the scale of $\alpha\nu/(g\beta d^3)$. The proper boundary conditions are given by

$$w_1=\theta_1=0 \text{ at } z=0 \text{ and } z=1 \tag{12}$$

The boundary conditions represent no flow through the boundaries and the fixed temperature on the upper boundary and the mass flux condition of the lower boundary.

According to the normal mode analysis, convective motion is assumed to exhibit the horizontal periodicity [13]. Then the perturbed quantities can be expressed as follows:

$$[w_1(\tau, x, y, z), \theta_1(\tau, x, y, z)]=[w_1(\tau, z), \theta_1(\tau, z)]\exp[i(a_x x+a_y y) \tag{13}$$

where "i" is the imaginary number. Substituting the above Eq. (13) into Eqs. (10)-(11) produces the usual amplitude functions in terms of the horizontal wave number $a=(a_x^2+a_y^2)^{1/2}$.

$$\left(\frac{\partial^2}{\partial z^2}-a^2\right)w_1=a^2\theta_1 \tag{14}$$

$$\frac{\partial\theta_1}{\partial\tau}=\left(\frac{\partial^2}{\partial z^2}-a^2\right)\theta_1-Ra_D w_1\frac{\partial\theta_0}{\partial z} \tag{15}$$

The propagation theory employed for finding the critical time τ_c is based on the assumption that disturbances at the onset of convection are propagated mainly within the penetration depth Δ and this is a proper length scaling factor. So, it is assumed that at the onset of convective motion the following scale analysis in terms of the thermal penetration depth $\Delta(\propto t^{1/2})$ is valid for perturbed quantities of Eqs. (2) and (3), respectively.

$$\rho_r g\beta T_1\sim\frac{\mu}{K}W_1,\ \ W_1\sim\frac{g\beta K}{\nu}T_1 \tag{16a}$$

$$\frac{\partial T_1}{\partial t}\sim W_1\frac{\partial T_0}{\partial Z}\sim\alpha_e\frac{\partial^2 T}{\partial Z^2}\sim\alpha_e\frac{T_1}{\Delta^2} \tag{16b}$$

Now, based on the above relations the following amplitude relation can be obtained:

$$w_1\sim\theta_1 \tag{17a}$$

$$Ra_D^* w_1\sim\theta_1 \tag{17b}$$

where Ra_D^* is the Rayleigh number based on the penetration depth Δ and temperature difference ΔT. With increasing Ra_D both the dimensionless onset time and the corresponding Δ become smaller and the characteristic value of $Ra_D^*(=Ra_D(\Delta/d))$ will become a constant. Therefore, taking the dimensional reasoning into account, we can obtain the following relation:

$$\left|\frac{w_1}{\theta_1}\right|\sim 1 \tag{18a}$$

$$Ra_D^*=Ra_D\delta\sim\text{constant} \tag{18b}$$

where $\delta(\propto\sqrt{\tau})$ is the dimensionless penetration depth. Eq. (18b) means that the Darcy-Rayleigh number based on penetration depth becomes a constant. Similar scaling-analysis can be found in various systems, such as solidification system [14,15], Marangoni-Bénard convection [16,17] and Taylor-vortex system [18,19]. There are many possible forms of dimensionless amplitude functions of disturbances like

$$[w_1^*(\tau,z),\theta_1^*(\tau,z)]=[\tau^n\overline{w}^*(\tau,z),\tau^n\overline{\theta}^*(\tau,z)], \tag{19}$$

which satisfy the relation of Eq. (18). Propagation theory is based on the scaling relations of Eq. (18). To determine n in Eq. (19), the momentary instability condition which was suggested by Shen [20] is employed. According to his conception the temporal growth rate of the kinetic energy of the perturbation velocity should exceed that of the basic velocity at the onset condition of secondary motion.

Buoyancy-driven convection sets in due to the buoyancy force, and therefore, the temporal growth rate of the perturbation energy (r_1) and the base energy (r_0) are defined as the root-mean-squared quantities of concentration components:

$$r_0=\frac{1}{E_0}\frac{dE_0}{d\tau}\ \text{and}\ r_1=\frac{1}{E_1}\frac{dE_1}{d\tau}, \tag{20a\&b}$$

Where E is the thermal energy integrated over the system volume Ω, $E=\int_\Omega\theta^2 d\Omega$. r_0 and r_1 have the meaning of the growth rates in the global (τ, z) coordinate. According to the momentary instability condition, the onset condition is determined at the time $r_0=r_1$. For the deep-pool region, the temporal growth rate of the base temperature (r_0) can be obtained by using the base temperareure of Eq. (8) as

$$r_0=\frac{1}{2\tau}\ \text{for}\ t\rightarrow 0. \tag{21}$$

For the case of n=0, which is already used in the dominant mode analysis of Riaz et al. [3], the relation of Eq. (19) with n=0 bounds the momentary instability conception for $\tau\rightarrow 0$, which will be discussed later. It is interesting that the dominant mode solution of Riaz et al.'s [3], i.e. $\theta^*=A_1(\tau)\zeta\exp(-\zeta^2/4)$ satisfied the condition $r_0=r_1$ of at the marginal codition of $(1/A_1)(\partial A_1/\partial\tau)=0$, here $A_1(\tau)$ is the time-dependent amplitude function.

It is well-known that a fundamental difficulty of this kind of problem is that the eigenfunctions of w_1 and θ_1 are localized in the boundary layer, while the eigenfunctions of the operator $\partial^2/\partial z^2$ are global modes [21]. Hence, they do not provide an appropriate basis for streamwise disturbances. For the boundary-layer systems of small critical time, we set $\overline{w}^*(\tau, z)=w^*(\tau, \zeta)$ and $\overline{\theta}^*(\tau, z)=\theta^*(\tau, \zeta)$. Here $\overline{w}^*$ and $\overline{\theta}^*$ are disturbances in global domain of (τ, z), while w^* and θ^* are those in boundary layer domain of (τ, ζ). Furthermore, it is assumed that $\partial\overline{\theta}^*/\partial\tau=-(\zeta/(2\tau))(\partial\overline{\theta}^*/\partial\zeta)$. This assumption means that $\partial\overline{\theta}^*/\partial\tau=0$ at the marginal stability conditions. Riaz et al. called this a quasi-steady steate approximation (QSSA) [3]. This means that the propagation theory is a frozen-time model in self-similar (τ, ζ) domain rather than global (τ, z) domain and satisfys the marginal stability condition of $\partial\overline{\theta}^*/\partial\tau=0$. And, in the propagation theory, the amplitude function of disturbances is assumed to be a function of ζ only. With the above reasoning the dimensionless amplitude functions of disturbances are given the following forms:

$$[w_1(\tau, z), \theta_1(\tau, z)]=[w^*(\zeta), \theta^*(\zeta)] \tag{22}$$

By using these relations the stability equation is obtained from Eqs. (10) and (11) as

$$(D^2-a^{*2})w^*=a^{*2}\theta^* \tag{23}$$

$$\left(D^2+\frac{1}{2}\zeta D-a^{*2}\right)\theta^*=Ra_D^* w^* D\theta_0 \tag{24}$$

with the following boundary conditions:

$$w^*=\theta^*=0\ \text{at}\ \zeta=0\ \text{and}\ \zeta=\infty \tag{25}$$

where, $a^*=a\sqrt{\tau}$, $Ra_D^*=Ra_D\sqrt{\tau}$ and $D=d/d\zeta$. It is assumed that a^* and Ra_D^* are the eigenvalues, and also the onset time of buoyancy-driven convection for a given Ra_D is unique under the principle of exchange of stabilities. This trend was predicted by Riaz et al. [3], Caltagirone [6], Tan et al. [8] and Elder [22,25] theoretically. The above procedure is the essence of our propagation theory.

In their dominant mode analysis, Riaz et al. [3] obtained the temporal growth rate by solving the following equation analytically:

$$\frac{\partial\theta^*}{\partial\tau}=\left(D^2+\frac{1}{2}\zeta D\right)\theta^*-Ra_D^* w^* D\theta_0 \tag{26}$$

with Eq. (23) and the boundary condition of Eq. (25). And they determined the onset condition where $\partial\theta^*/\partial\tau=0$. At this stability condition of $\partial\theta^*/\partial\tau=0$, Eq. (26) is a long-wave approximation of Eq. (24), which means the stability result of the dominant mode analysis is an approximation of that of propagation theory for the limiting case of $a^*\rightarrow 0$.

The conventional frozen-time model neglects the terms involving $\partial(\cdot)/\partial\tau$ in Eq. (15) in (τ, z) coordinate rather than (τ, ζ) coordinate. This results in $(D^2-a^{*2})w^*=a^{*2}\theta^*$ and $(D^2-a^{*2})\theta^*=Ra_D^* w^* D\theta_0$ instead of Eqs. (23) and (24).

3. Modified Energy Method

Consider the following temperature, velocity and pressure perturbations: $R T_1=T-T_0$, $\mathbf{U}_1=\mathbf{U}-\mathbf{U}_0$ and $P_1=P-P_0$ with $R=\sqrt{Ra_D}$. Let's introduce these perturbations into Eqs. (1)-(4). Then we can obtain the following dimensionless equations:

$$\nabla\cdot\mathbf{u}_1=0 \tag{27}$$

$$\mathbf{u}_1+\nabla p_1-R\theta_1\mathbf{k}=0 \tag{28}$$

$$\frac{\partial\theta_1}{\partial\tau}=\nabla^2\theta_1-Rw_1\frac{\partial\theta_0}{\partial z}-\mathbf{u}_1\cdot\nabla\theta_1 \tag{29}$$

under the following boundary conditions:

$$\mathbf{u}_1=\theta_1=0 \text{ at } z=0 \text{ and } z=1 \tag{30}$$

Here, the perturbations need not necessarily to be infinitesimal, and therefore Eq. (29) is different from Eq. (15) where the disturbances are assumed to be infinitesimal. In (τ, ζ) domain Eq. (29) can be expressed as

$$\frac{\partial\theta_1'}{\partial\tau}=\nabla^2\theta_1'-Rw_1\frac{\partial\theta_0}{\partial z}-\mathbf{u}_1\cdot\nabla\theta_1'+\frac{1}{2}\frac{\zeta}{\tau}\frac{\partial\theta_1'}{\partial\zeta} \tag{31}$$

by using the relation of $\partial\theta_1/\partial\tau=\partial\theta_1'/\partial\tau-1/2(\zeta/\tau)(\partial\theta_1'/\partial\zeta)$ which was used in the propagation theory. Now, multiply Eq. (28) by θ_1' and Eq. (31) by $\mathbf{u}_1$ and integrate over the volume Ω, then Eqs. (28) and (31) become

$$\int_\Omega\mathbf{u}_1\cdot\mathbf{u}_1 d\Omega+\int_\Omega\mathbf{u}_1\cdot\nabla p_1 d\Omega-R\int_\Omega\theta_1' w_1 d\Omega=0 \tag{32}$$

$$\int_\Omega\frac{1}{2}\frac{\partial\theta_1'^2}{\partial\tau}d\Omega=\int_\Omega\theta_1'\nabla^2\theta_1' d\Omega-R\int_\Omega w_1\theta_1'\frac{\partial\theta_0}{\partial z}d\Omega+\int_\Omega\frac{1}{2}\frac{\zeta}{\tau}\frac{\partial\theta_1'}{\partial\zeta}d\Omega \tag{33}$$

Using the divergence theorem, the following energy identities can be obtained

$$0=R\langle\theta_1 w_1\rangle-\langle|\mathbf{u}_1|^2\rangle \tag{34}$$

$$\frac{1}{2}\frac{\partial\langle|\theta_1|^2\rangle}{\partial\tau}=-\langle|\nabla\theta_1|^2\rangle-R\frac{1}{\sqrt{\tau}}\left\langle w\frac{\partial\theta_0}{\partial\zeta}\theta_1\right\rangle+\frac{1}{4\tau}\langle|\theta_1|^2\rangle \tag{35}$$

where the primes are dropped.

In the present system the dimensionless natural energy can be defined as a linear combination of Eqs. (34) and (35) with coupling constant $\lambda>0$:

$$E(\tau)=\frac{1}{2}\langle\mathbf{u}_1\rangle^2+\frac{1}{2}\lambda\langle\theta_1\rangle^2 \tag{36}$$

and the following energy identity can be derived:

$$\frac{1}{2}\frac{\partial(\lambda\langle|\theta_1|^2\rangle)}{\partial\tau}=-\lambda\langle|\nabla\theta_1|^2\rangle-R\frac{\lambda}{\sqrt{\tau}}\left\langle w\frac{\partial\theta_0}{\partial\zeta}\theta_1\right\rangle+\frac{\lambda}{4\tau}\langle\theta_1\rangle+R\langle w\theta_1\rangle-\langle|\mathbf{u}_1|^2\rangle \tag{37}$$

By setting $\theta_1=\hat{\theta}_1/\lambda^{1/2}$, the above energy identity can be expressed as

$$\frac{1}{2}\frac{\partial(\langle|\hat{\theta}|^2\rangle)}{\partial\tau}=-\left\langle|\nabla\hat{\theta}|^2+|\mathbf{u}_1|^2-\frac{1}{4\tau}|\hat{\theta}|^2\right\rangle+R\left\langle w\frac{\hat{\theta}}{\sqrt{\lambda}}-\frac{1}{\sqrt{\tau}}w\frac{\partial\theta_0}{\partial\zeta}\sqrt{\lambda}\hat{\theta}\right\rangle \tag{38}$$

The above relation can be represented as

$$\frac{dE}{d\tau}=RI-B=-B\left(1-\frac{I}{B}R\right) \tag{39}$$

where

$$I=\left\langle w\frac{\theta}{\sqrt{\lambda}}-\frac{1}{\sqrt{\tau}}w\frac{\partial\theta_0}{\partial\zeta}\sqrt{\lambda}\theta\right\rangle \tag{40}$$

$$B=\left\langle|\nabla\theta|^2+|\mathbf{u}_1|^2-\frac{1}{4\tau}|\theta|^2\right\rangle \tag{41}$$

where the hats are dropped.

The stability with respect to most dangerous disturbances is guaranteed under the condition of $dE/d\tau\le 0$ for all τ. The disturbance energy $E(\tau)$ decreases and the fluid layer is stable when $R<R_\lambda$ where

$$\frac{1}{R_\lambda}=\max_H\left(\frac{\langle w_1\theta_1\rangle}{\lambda^{1/2}}-\lambda^{1/2}\left\langle w_1\theta_1\frac{\partial\theta_0}{\partial z}\right\rangle\right) \tag{42}$$

under the condition of

$$B=\left\langle|\nabla\theta|^2+|\mathbf{u}_1|^2-\frac{1}{4\tau}|\theta|^2\right\rangle=1. \tag{43}$$

This maximum problem can be solved by the variational technique [23]. Eqs. (42) and (43) satisfy the condition of $r_0=r_1$ based on Eqs. (20) and (21), since $(^1(\langle|\theta_1|^2\rangle))/\partial\tau=1/2\tau(\langle|\theta_1|^2\rangle)$. In the usual manner, the following Euler-Lagrange equations can be obtained:

$$\nabla\cdot\mathbf{u}_1=0 \tag{27}$$

$$\frac{1}{2}R_\lambda\left(\frac{1}{\lambda^{1/2}}-\frac{\lambda^{1/2}}{\sqrt{\tau}}\frac{\partial\theta_0}{\partial\zeta}\right)\theta_1\mathbf{k}-\mathbf{u}_1-\nabla p_1=\mathbf{0} \tag{44}$$

$$0=\nabla^2\theta_1-\frac{1}{4\tau}\theta_1+\frac{1}{2}R\left(\frac{1}{\lambda^{1/2}}-\frac{\lambda^{1/2}}{\sqrt{\tau}}\frac{\partial\theta_0}{\partial\zeta}\right)w_1 \tag{45}$$

The fluid layer is strongly stable if $R=\sqrt{Ra_D}<\tilde{R}$, where

$$\tilde{R}=\max_\lambda R_\lambda \tag{46}$$

A detailed discussion on strong stability has been given in Homsy [24] and Caltagirone [9].

Taking the double curl on Eq. (44) and taking into account Eq. (27), the following equations can be obtained.

$$\nabla^2 w_1=\frac{1}{2}R_\lambda\left(\frac{1}{\lambda^{1/2}}-\frac{\lambda^{1/2}}{\sqrt{\tau}}\frac{\partial\theta_0}{\partial\zeta}\right)\nabla_1^2\theta_1 \tag{47}$$

$$\nabla^2\theta_1=\frac{1}{4\tau}\theta_1-\frac{1}{2}R\left(\frac{1}{\lambda^{1/2}}-\frac{\lambda^{1/2}}{\sqrt{\tau}}\frac{\partial\theta_0}{\partial\zeta}\right)w_1 \tag{48}$$

with the following boundary conditions

$$w_1=\theta_1=0 \text{ at } z=0 \text{ and } 1 \tag{49}$$

Under the normal mode analysis, the critical value $Ra_D(\tau_s)$ under which the disturbance energy decrease exponentially is given by

$$Ra_D^{1/2}=\max_\lambda\min_a R_\lambda \tag{50}$$

For any given τ, a and γ, the function R_γ may be determined by the solution method of a eigenvalue problem. By setting $\tilde{\theta}_1=((1/\lambda^{1/2})-(\lambda^{1/2}/\sqrt{\tau})(\partial\theta_0/\partial\zeta))\theta_1$, Eqs. (47) and (48) will be transformed as

$$(D^2-a^{*2})w_1=-a^{*2}\theta_1 \tag{51}$$

$$(D^2-a^{*2})\theta_1=\frac{1}{4}\theta_1-\frac{1}{2}R^2\sqrt{\tau}\left(\sqrt{\frac{\tau^{1/2}}{\lambda}}-\sqrt{\frac{\lambda}{\tau^{1/2}}}D\theta_0\right)^2 w_1 \tag{52}$$

where $D=d/d\zeta$ and $a^*=a\sqrt{\tau}$, and the tildes are dropped. By setting $R^{*2}=R^2\sqrt{\tau}$, $\lambda^*=\lambda/\sqrt{\tau}$, $w^*=w_1$ and $\theta^*=1/\sqrt{R^*}((1/\sqrt{\lambda^*})-\sqrt{\lambda^*}D\theta_0)^{-1}\theta_1$, the following stability equations can be derived:

$$(D^2-a^{*2})w^*=-\frac{1}{2}\sqrt{R^*}\left(\sqrt{\frac{1}{\lambda^*}}-\sqrt{\lambda^*}D\theta_0\right)a^{*2}\theta^* \tag{53}$$

$$(D^2-a^{*2})\theta^*=\frac{1}{4}\theta^*-\frac{1}{2}\sqrt{R^*}\left(\sqrt{\frac{1}{\lambda^*}}-\sqrt{\lambda^*}D\theta_0\right)w^* \tag{54}$$

The strong stability condition is to find

$$\sqrt{Ra_D^*}=\max_{\lambda^*}\min_{a^*}R^* \tag{55}$$

to satisfy Eqs. (53) and (54) under the following boundary conditions:

$$w^*=\theta^*=0 \text{ at } \zeta=0 \text{ and } 1/\sqrt{\tau} \tag{56}$$

The conventional energy method[1,2] neglects $-1/2(\zeta/\tau)(\partial\theta_1'/\partial\zeta)$ term in Eq. (31) in (τ, z) coordinate rather than (τ, ζ) coordinate. This results in $(D^2-a^{*2})w^*=-(1/2)\sqrt{R^*}((1/\sqrt{\lambda^*})-\sqrt{\lambda^*}D\theta_0)a^{*2}\theta^*$ and $(D^2-a^{*2})\theta^*=-(1/2)\sqrt{R^*}((1/\sqrt{\lambda^*})-\sqrt{\lambda^*}D\theta_0)w^*$.

RESULTS AND DISCUSSION

From propagation theory, the following marginal conditions may be obtained:

$$Ra_D\tau_{c,P}^{1/2}=12.94 \text{ and } a_c\sqrt{\tau_{c,P}}=0.90 \text{ for } \tau\to 0 \tag{57}$$

based on the marginal stability curve of Fig. 3. The solution procedure is summarized in the Appendix. This criterion is quite close to the results from the dominant mode analysis of Riaz et al. [3]. At the critical conditions illustrated above, the amplitude functions of w^* and θ^* are featured in Fig. 4, wherein the quantities have been normalized by the corresponding maximum magnitude θ^*_{max}. It is seen that incipient temperature disturbances are confined mainly within the dimensionless concentration penetration depth, but velocity disturbances are driven more upward over the thermal penetration depth. Based on the distribution of temperature disturbance, the relation of $r_0=r_1$ can be obtained with the aid of Eq. (21) at $\tau=\tau_c$. As shown in Figs. 3 and 4, the propagation theory under QSSA in (τ, ζ)-coordinate represents the dominant mode analysis without QSSA in (τ, ζ)-coordinate.

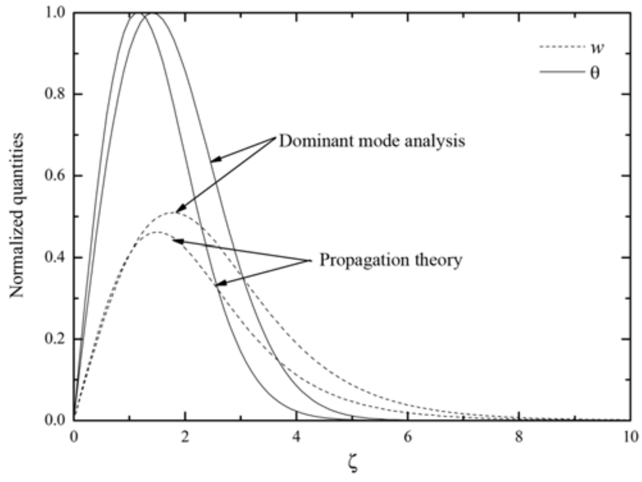

Fig. 4. Distribution of disturbances quantities based on the propagation theory and the dominant mode analysis.

Now, the domain of time is extended to $\tau_c>0.01$ by keeping Eqs. (23) and (24) and using Eq. (8b). In the condition of Eq. (25) the upper boundary $\zeta\to\infty$ is replaced with z=1, i.e., $\zeta=1/\sqrt{\tau_c}$ and in Eqs. (23) and (24) Ra_D^* and a^* are replaced with $Ra_D\sqrt{\tau_c}$ and $a\sqrt{\tau_c}$. Also, in Eq. (8b) τ is replaced with τ_c but ζ is maintained. Since τ_c is the fixed parameter, the resulting stability equations are a function of ζ only and the physics of Eq. (22) is still alive. For a given τ_c the minimum Ra_D-value and its corresponding wavenumber a_c are obtained. The solution procedure is almost the same as that in the previous section. The results are summarized in Fig. 5, wherein those obtained from the conventional frozen-time model are also

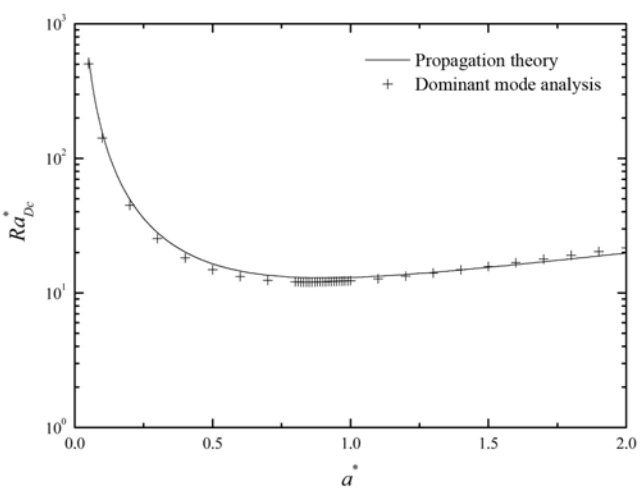

Fig. 3. Marginal stability curves based on the propagation theory.

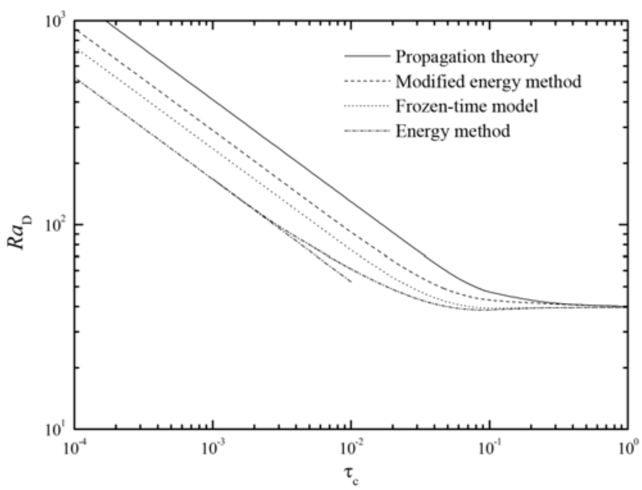

Fig. 5. The critical times for a given based on the various methods.

shown. For τ_c<0.01 the former ones are the same as those of the deep-pool system (Eq. (57)). For large τ_c they approach the frozen-time model since the basic concentration profile becomes linear. It is known that the terms involving $\partial(\cdot)/\partial\tau$ in Eq. (15) stabilize the system. It is interesting that propagation theory yields smoothly the stability criteria over the whole domain of time, and the results based on QSSA represent those of dominant mode analysis without QSSA.

Based on the modified energy method in (τ, ζ)-coordinate, a new strong stability condition is obtained as,

$$Ra_D\,\tau_{c,M}^{1/2}=9.16 \quad \text{for } \tau\rightarrow 0. \tag{58}$$

It is interesting that based on Eqs. (57) and (58), for the limiting case of $\tau\rightarrow 0$, $\tau_{c,P}$=2.00$\tau_{c,M}$ can be obtained, as shown in Fig. 5 and Table 1. And, the same relation might be applied between the frozen-time model and the conventional energy method. This means that it takes more time for the infinitesimal disturbances to grow in the propagation theory/frozen-time model than for the finite disturbance in the modified energy method/energy method. Usually, the linear stability theory predicts higher stability limits than the energy method. To check and confirm our solution method, the strong stability results of the energy method are compared with those of Ennis-King et al. [1]. As shown in Fig. 6, our results are quite similar to theirs. Furthermore, our results show the subcritical region which can be shown in Ennis-King et al. [1]. It is interesting that there exists a subcritical region, where the critical values of Ra_D are lower than the well-known value of $4\pi^2$, which does not exist in the present propagation theory, modified energy method and the frozen-time model. The minimum value of the critical Ra_D is 38.46 at τ=0.09.

For the present system, Elder [22] suggested the following simple and crude prediction:

$$Ra_D\delta_*=4\pi^2, \tag{59}$$

where δ_* is the thickness of the heated layer of order $2\sqrt{\tau}$. This re- lation predicts $\tau_c=400Ra_D^{-2}$. Recently, Tan et al. [8] suggested a sim- ple instability analysis assuming that at the onset of convection the following relation is maintained, based on the original Horton-Rogers-Lapwood convection:

$$\text{Maximum of}\left\{\frac{g\beta K Z^2}{\varepsilon\nu\alpha_s}\left(\frac{\partial T_0}{\partial Z}\right)\right\}=4\pi^2, \tag{60}$$

which is satisfied by $\partial T_0/\partial Z=0.83\Delta T/\sqrt{\alpha t}$ at $Z_{max}=2\sqrt{\alpha t}$ from Eq. (9). This concept is quite similar to Elder's suggestion of Eq. (59) and Z_{max} corresponds to δ_* in Eq. (59). Tan et al.'s suggestion of Eq. (60) results in $\tau_c=2262Ra_D^{-2}$. Eq. (60) seems to correspond to the upper bound of critical Rayleigh number, wherein the concentration profile assumed to be linear within $Z=Z_{max}$. But it is interesting that common physics is involved in the above results: Ra_D^*~ constant, which is adopted in the present propagation theory.

The stability boundaries are summarized in Fig. 7. And the stability limit based on the propagation theory is higher than those suggested by using the energy method and linear amplification theory (given by Fig. 2 of Ennis-King et al. [1]) and quite lower than that of Tan et al. [6]. However, for the region of large τ, the propagation theory approaches the results obtained by using the frozen-time model, the energy method and the linear amplification theory. The propagation theory represents the dominant mode for short time region and the frozen-time model for long time region quite well. It is interesting that the modified energy method shows quite similar results with those of the amplification theory.

To validate the theoretical analysis, the predictions of t_c should be compared with experimental observations. Unfortunately, Elder [25] only states the Darcy-Rayleigh number of his experiments and does not give all the data necessary to calculate t_c. He only reported

Table 1. Comparison of the critical conditions in the form of τ_c= ARa_D^{-2} for the limiting case of $\tau\rightarrow 0$

Method	A
Propagation theory	167.44
Frozen-time model	55.65
Modified energy method	83.72
Energy method	27.83 (~30*)
Amplification theory [1,2]	~75
Elder [22]	400
Tan et al. [8]	2262

*Value suggested by Ennis-King et al. [1]

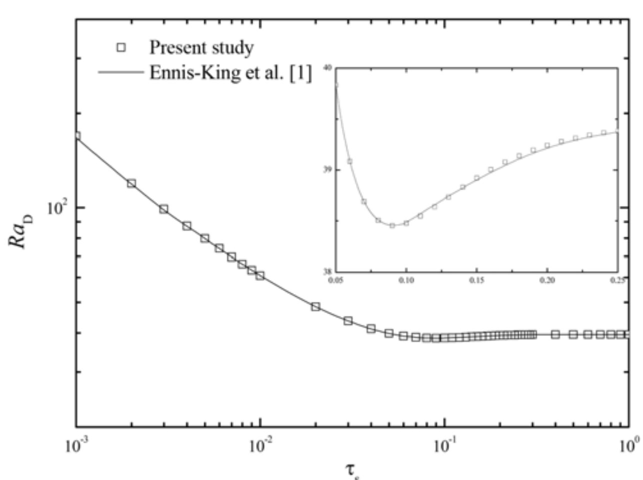

Fig. 6. Comparison with Ennis-King et al.'s [1] result based on the energy method. The inset clearly shows a subcritical region.

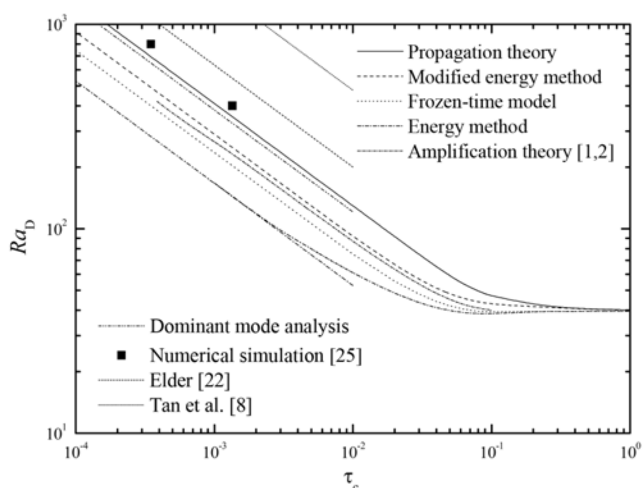

Fig. 7. Comparison of the critical times based on the various predictions.

the relation of a^*~constant. Foster [26] commented that with correct dimensional relations the relation of $t_o \cong 4t_c$ would be kept for the case of horizontal fluid layers heated isothermally from below. This means that a growth period for disturbances to grow is required until they are detected experimentally. Therefore, it seems evident that the predicted onset time t_c is smaller than the detection time t_o. This means that a fastest growing mode of instabilities, which set in at $t=t_c$, will grow with time until manifest motion is first detected experimentally. As commented by Riaz et al. [3], since, for the boundary-layer systems of small τ, (τ, ζ)-domain is more appropriate than (τ, z)-domain, the critical time τ_c might be set to $\tau_{c,P}$ based on the propagation theory. The validity $t_o \cong 4t_{c,P}$ of requires further study, but this relation is kept in the various transient diffusive systems [10,12,16-19]. It seems evident that convective motion is very weak during $t_c \le t \le t_o$ since the related transport is well represented by the diffusion state.

CONCLUSIONS

The critical condition to mark the onset of convective motion in an initially quiescent, horizontal isotropic porous layer has been analyzed by using the dominant mode analysis, the propagation theory, the frozen-time model and also the energy method and its modification. The resulting stability criteria compare reasonably well with previous theoretical predictions. The present result shows that the propagation theory can be applied to the stability analysis of diffusive systems without loss of generality.

APPENDIX

To find eigenvalues and eigenfunctions for differential equations, several methods such as compound matrix method and shooting method are proposed [27]. The stability Eqs. (35)-(37) based on the propagation theory are solved by employing the outward shooting scheme. To integrate these stability equations the proper values of Dw^* and $D\theta^*$ at $\zeta=0$ are assumed for a given a^*. Since the stability equations and their boundary conditions are all homogeneous, the value of $Dw^*(0)$ can be assigned arbitrarily and the value of the parameter Ra_D^* is assumed. This procedure can be understood easily by taking into account the characteristics of eigenvalue problems. After all the values at $\zeta=0$ are provided, this eigenvalue problem can proceed numerically.

Integration is performed from $\zeta=0$ to a fictitious upper boundary with the fourth-order Runge-Kutta-Gill method. If the guessed values of Ra_D^* and $D\theta^*(0)$ are correct, w^* and θ^* will vanish at the upper boundary. To improve the initial guesses the Newton-Raphson iteration is used. When convergence is achieved, the upper boundary for computation is increased by a predetermined value and the above procedure is repeated. Since the temperature disturbances decay exponentially outside the thermal penetration depth, the incremental change of Ra_D^* also decays fast with increasing the fictitious upper boundary thickness. This behavior enables us to extrapolate the eigenvalue to an infinite depth. For the isotropic case, the maginal stability curve is comapred with that of the dominant mode analysis in Fig. 3. Similar procedure can be applied to the energy method and modified energy method.

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