

# Rotation of the Earth as a Triaxial Rigid Body

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**Abstract** The Earth is taken as a triaxial rigid body, which rotates freely in the Euclidian space. The starting equations are the Euler dynamic equations, with  $A$  smaller than  $B$  and  $B$  smaller than  $C$ . The Euler equations are solved, and the numerical results are provided. In the calculations, the following parameters are used:  $(C-B)/A=0.003\ 273\ 53$ ;  $(B-A)/C=0.000\ 021\ 96$ ;  $(C-A)/B=0.003\ 295\ 49$ , and the mean angular velocity of the Earth's rotation,  $\omega=0.000\ 072\ 921\ 15$  rad/s. Calculations show that, besides the self-rotation of the Earth and the free Euler procession of its rotation, there exists the free nutation: the nutation angle, or the angle between the Earth's momentary rotation axis and the mean axis that periodically change with time. The free nutation is investigated.

**Keywords** Earth's rotation; triaxial rigid body problem; variation of Eulerian angles

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## Introduction

Conventionally, the Earth's rotation is described by the Euler (kinematic and dynamic) equations, taking the Earth as a biaxial rigid body. In this case, the Earth is rotationally symmetric, and two principal moments of inertia of the Earth,  $A$  and  $B$ , coincide with each other. Considering the free rotation of the Earth, the solutions of the Euler equations show that:

① there exists the self-rotation of the Earth, which is described by the variation of the self-rotation angle  $\psi$ ; ② there exists a free Euler procession of the Earth's rotation, which is described by the variation of the procession angle; ③ there is no nutation, which is described by the nutation angle  $\theta$ . In the real situation, the all principal moments of inertia of the Earth,  $A$ ,  $B$  and  $C$ , are different from each other<sup>[1]</sup>:  $A$  is smaller

than  $B$ , and  $B$  is smaller than  $C$ ; the difference between  $C$  and  $A$  is around  $0.003\ A$ , and the difference between  $B$  and  $A$  is around  $0.000\ 02\ A$ . Hence, it is interesting to explore the Earth's rotation when it is taken as a triaxial body.

## 1 Rotation of the Earth as triaxial rigid body

Let us choose an inertial system  $o-\xi\eta\zeta$  and the Earth-fixed system  $o-xyz$ , where the two systems have the common origin  $o$ . Then, the rotation of the Earth is uniquely described by three Eulerian angles  $\psi$ ,  $\phi$  and  $\theta$ , which denote the rotation angle, procession angle, and nutation angle, respectively (Fig.1).

Without considering the causes, the rotation of the Earth should satisfy the following Euler kinematic

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equations<sup>[2,3]</sup>:

$$\begin{aligned} \omega_1 &= \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \\ \omega_2 &= \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \\ \omega_3 &= \dot{\phi} \cos \theta + \dot{\psi} \end{aligned} \quad (1)$$

where  $\omega_i (i=1,2,3)$  denotes the component of the angular velocity of the Earth rotation along  $x^i (x^1 = x, x^2 = y, x^3 = z)$  axis. That means, given  $\omega_i$ , the Eulerian angles could be determined (at least numerically); inversely, given the Eulerian angles as the functions of time, the angular velocity components  $\omega_i$  could be determined.

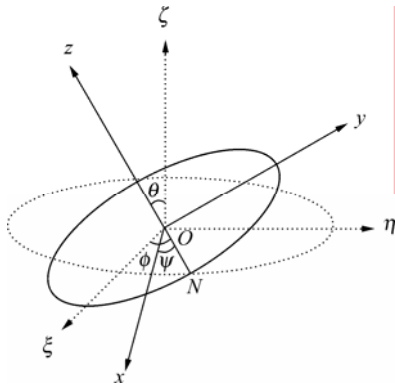


Fig.1 Three Eulerian angles

To explore the causes that control the rotation of the Earth, we have the Euler dynamic equations, which could be expressed as follows<sup>[4]</sup>:

$$I^{ij} \dot{\omega}_j + \varepsilon^i_{jk} \omega^j I^{kl} \omega_l = L^i \quad (2)$$

where  $I^{ij}$  is the second-order tensor of the moments of inertia;  $L^i \equiv L_i$  is the components of the external force moment exerted on the Earth;  $\dot{\omega}_j$  denotes the derivative of  $\omega_j$  with respect to time  $t$ ; and  $\varepsilon^i_{jk} \equiv \varepsilon_{ijk}$  (third-order tensor) is Levi-Civita symbol, satisfying the following conditions:  $\varepsilon_{ijk} = 1$ , if  $ijk$  is the even permutation of the order 1, 2, 3;  $\varepsilon_{ijk} = -1$ , if  $ijk$  is the odd permutation of the order 1, 2, 3;  $\varepsilon_{ijk} = 0$ , in other cases. It is noted that Einstein

summation convention<sup>[5]</sup> is applied throughout this paper: if and only if the super index and sub index are identical and appear in one term, one takes the summation from 1 to 3. The rotation of any rigid body could be described by Eqs.(1) and (2).

If the Earth-fixed coordinate  $o-xyz$  is chosen in such a way that  $ox, oy, oz$  axes coincide with the directions of the principal moments of inertia  $A, B, C$ , correspondingly, then one has  $I^{ij} = 0 (i \neq j)$ . Hence, set

$$I^{11} = A, I^{22} = B, I^{33} = C; I^{ij} = 0 (i \neq j) \quad (3)$$

Eq.(2) could be expressed as:

$$\begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= L_1 \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= L_2 \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= L_3 \end{aligned} \quad (4)$$

Considering the free rotation of rigid Earth, Eq.(4) is reduced as follows:

$$\begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= 0 \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= 0 \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= 0 \end{aligned} \quad (5)$$

or equivalently

$$\dot{\omega}_1 + \alpha\omega_2\omega_3 = 0 \quad (6)$$

$$\dot{\omega}_2 - \beta\omega_3\omega_1 = 0 \quad (7)$$

$$\dot{\omega}_3 + \gamma\omega_1\omega_2 = 0 \quad (8)$$

where  $\alpha, \beta, \gamma$  are small quantities, defined by following equations:

$$\begin{aligned} \alpha &= \frac{C - B}{A} \\ \beta &= \frac{C - A}{B} \\ \gamma &= \frac{B - A}{C} \end{aligned} \quad (9)$$

Especially,  $\alpha$  and  $\beta$  are in the order of  $3 \times 10^{-3}$ , and  $\gamma$  is in the order of  $2 \times 10^{-5}$ . Various studies have provided the values of  $\alpha, \beta, \gamma$ , which are listed in Table 1.

Table 1 Values of  $\alpha, \beta$  and  $\gamma$

Sources	$\alpha$	$\beta$	$\gamma$
Burša (1984)	$327\ 341 \times 10^{-8}$	$329\ 588 \times 10^{-8}$	$2.188\ 899\ 5 \times 10^{-5}$
Liu, et al.(1991)	(not given)	(not given)	$2.194\ 6 \times 10^{-5}$
Yoder(1995)	$328\ 448 \times 10^{-8}$	$330\ 655 \times 10^{-8}$	$2.207\ 434\ 09 \times 10^{-5}$
Groten(2000)	$(327\ 353 \pm 6) \times 10^{-8}$	$(329\ 549 \pm 6) \times 10^{-8}$	$(2\ 196 \pm 6) \times 10^{-8}$
WGS84(2000)	$327\ 519 \times 10^{-8}$	$329\ 423 \times 10^{-8}$	$1.904\ 053\ 80 \times 10^{-5}$
Wei(2005)	$(3\ 273.536\ 7 \pm 0.490\ 5) \times 10^{-6}$	$(3\ 295.494\ 9 \pm 0.493\ 8) \times 10^{-6}$	$(21.958\ 4 \pm 0.003\ 3) \times 10^{-6}$

In the present paper, the values provided by Groten are applied, which have minimal differences with the values provided by other authors<sup>[1,6-10]</sup>(Table 1).

## 2 Solution

From Eqs.(6)-(8), one gets the following relations:

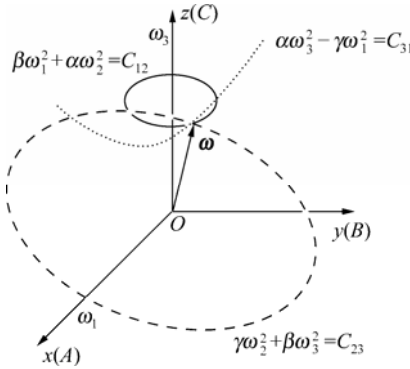
$$\beta\omega_1^2 + \alpha\omega_2^2 = C_{12} \quad (10)$$

$$\gamma\omega_2^2 + \beta\omega_3^2 = C_{23} \quad (11)$$

$$\alpha\omega_3^2 - \gamma\omega_1^2 = C_{31} \quad (12)$$

where  $C_{ij}$  are constants to be determined. If  $A = B$ , then  $\gamma = 0, \alpha = \beta$ , and in this case, from Eqs.(12) and (10) it holds that  $\omega_3 = \text{constant}$ , and  $\omega_1^2 + \omega_2^2 = C_{12}/\alpha$  holds invariant.

In the practical situation, we have  $\beta > \alpha \gg \gamma$  (Table 1). Eqs.(10), (11) and (12) denote ellipses and hyperbola, respectively, as shown by Fig.2.



**Fig.2 Orbit of angular velocity vector: combination of two ellipses and one hyperbola**

Based on the present situation of the Earth's rotation, it could be determined that  $\omega_1(0) = 1.144\ 579 \times 10^{-10}$  rad/s,  $\omega_2(0) = 0$ ,  $\omega_3(0) = 7.292\ 115 \times 10^{-5}$  rad/s. Then, from Eqs.(10)-(12), taking into account of Table 1, one gets:

$$\begin{aligned} C_{12} &= \beta\omega_1^2(0) \\ C_{23} &= \beta\omega_3^2(0) \\ C_{31} &= \alpha\omega_3^2(0) - \gamma\omega_1^2(0) \end{aligned} \quad (13)$$

From Eqs.(10) and (11), expressing  $\omega_1$  and  $\omega_3$  as the functions of  $\omega_2$ , and substituting  $\omega_1(\omega_2)$  and  $\omega_3(\omega_2)$  into Eq.(7), one gets:

$$\frac{d\omega_2}{dt} - \beta \sqrt{\frac{C_{23} - \gamma\omega_2^2}{\beta}} \sqrt{\frac{C_{12} - \alpha\omega_2^2}{\beta}} = 0 \quad (14)$$

From the above differential equation, one gets:

$$\int_{\omega_2(0)}^{\omega_2(t)} \frac{d\omega_2}{\sqrt{(C_{23} - \gamma\omega_2^2)(C_{12} - \alpha\omega_2^2)}} = t \quad (15)$$

After the above equation is solved, based on Eqs.(10) and (11), the solutions of Eqs.(6)-(8) could be found.

Set

$$x = \sqrt{\frac{\alpha}{C_{12}}} \omega_2 \quad (16)$$

then we have:

$$\frac{1}{\sqrt{\alpha C_{23}}} \int_{x(0)}^{x(t)} \frac{d \arcsin x}{\sqrt{1 - \frac{\gamma C_{12}}{\alpha C_{23}} x^2}} = t \quad (17)$$

Set

$$m = \frac{\gamma C_{12}}{\alpha C_{23}}, \quad \varphi = \arcsin x \quad (18)$$

one has:

$$\int_{\varphi(0)}^{\varphi(t)} \frac{d\varphi}{\sqrt{1 - m \sin^2 \varphi}} = t \quad (19)$$

where  $\varphi(0) = 0$ , due to the fact that  $\omega_2(0) = 0$ . It is noted that it holds that  $0 < m < 1$ . Eq.(19) is the first-kind of a non-complete ellipsoidal integration which has no analytical solution. However, one could give sequences  $\varphi_i(t)$  to determine the corresponding time series  $t_i$ , and consequently  $\varphi - t$  curve is determined based on Eq.(19), and then the  $\omega_2 - t$  curve is also determined. Furthermore, based on Eqs.(10) and (11), the  $\omega_1 - t$  and  $\omega_3 - t$  curves are determined.

## 3 Numerical results and analytical conclusions

Based on Eq.(19), as well as Eqs.(10) and (11), the numerical calculations provide the  $\omega_i - t (i=1,2,3)$  curves (as shown by Fig.2) and main information about the angular velocity components (the values of maximum, minimum, and mean, as shown by Table 2).

In the sequel, the Eulerian angles will be investigated. From Eq.(1) one derives the following relation:

$$\begin{aligned} \dot{\phi} &= \frac{\omega_1 \sin \psi + \omega_2 \cos \psi}{\sin \theta} \\ \dot{\theta} &= \omega_1 \cos \psi - \omega_2 \sin \psi \\ \dot{\psi} &= \omega_3 - \cot \theta (\omega_1 \sin \psi + \omega_2 \cos \psi) \end{aligned} \quad (20)$$

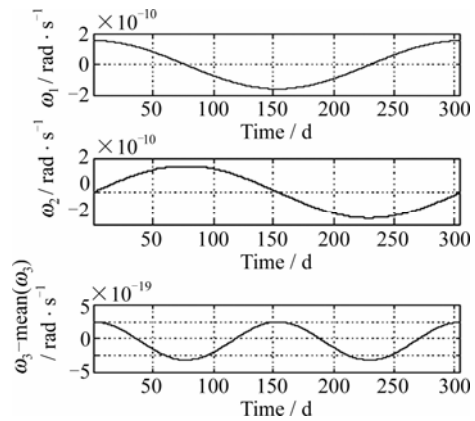
**Table 2 Information of three angular velocity components /rad · s<sup>-1</sup>**

	$\omega_1$	$\omega_2$	$\omega_3$
Max	$1.144\ 579\ 343\ 902\ 996 \times 10^{-10}$	$1.148\ 412\ 048\ 908\ 373 \times 10^{-10}$	$7.292\ 115\ 000\ 000\ 060 \times 10^{-5}$
Min	$-1.144\ 579\ 343\ 902\ 996 \times 10^{-10}$	$-1.148\ 412\ 048\ 908\ 373 \times 10^{-10}$	$7.292\ 115\ 000\ 000\ 000 \times 10^{-5}$
Mean	$1.132\ 967\ 280\ 689\ 162 \times 10^{-12}$	$1.785\ 767\ 039\ 139\ 498 \times 10^{-14}$	$7.292\ 115\ 000\ 000\ 035 \times 10^{-5}$

under the assumption that  $\sin \theta \neq 0$ . Noting that  $\dot{\phi} \equiv d\phi/dt \approx \Delta\phi/\Delta t$ , etc., the differential equation Eq.(20) could be directly solved using difference approach, i.e., by the following model:

$$\begin{aligned} \phi_{n+1} &= \tau \frac{\omega_1^{(n)} \sin \psi_n + \omega_2^{(n)} \cos \psi_n}{\sin \theta_n} + \phi_n \\ \theta_{n+1} &= \tau(\omega_1^{(n)} \cos \psi_n - \omega_2^{(n)} \sin \psi_n) + \theta_n \\ \psi_{n+1} &= \tau[\omega_3^{(n)} - \cot \theta_n (\omega_1^{(n)} \sin \psi_n + \omega_2^{(n)} \cos \psi_n)] + \psi_n \end{aligned} \quad (21)$$

where  $\tau \equiv t_{n+1} - t_n$  is the time interval (e.g.,  $\tau = 10$  min);  $\phi_n, \psi_n, \theta_n$  and  $\omega_i^{(n)} (i=1,2,3)$  correspond to the time  $t = t_n$ . The results calculated are shown by Fig.3 and Table 3.



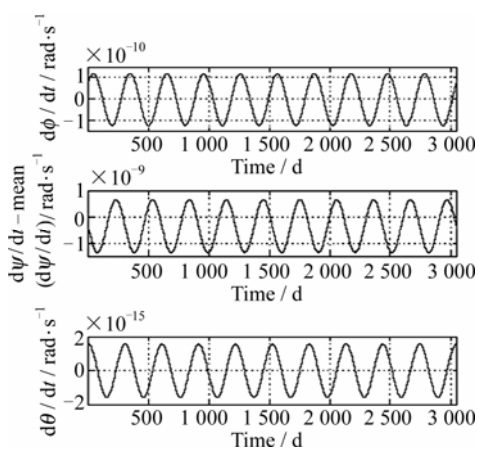
**Fig.3  $\omega_i - t$  curves: relations between angular velocity components and time**

**Table 3 Information of variation of Eulerian angles /rad · s<sup>-1</sup>**

	$d\phi/dt$	$d\psi/dt$	$d\theta/dt$
Max	$6.294\ 536\ 031\ 967\ 882 \times 10^{-11}$	$7.292\ 115\ 052\ 419\ 944 \times 10^{-5}$	$8.725\ 983\ 626\ 235\ 586 \times 10^{-16}$
Min	$-6.294\ 536\ 031\ 967\ 882 \times 10^{-11}$	$7.292\ 115\ 052\ 419\ 938 \times 10^{-5}$	$-8.725\ 983\ 626\ 235\ 586 \times 10^{-16}$
Mean	$-1.855\ 655\ 505\ 327\ 499 \times 10^{-12}$	$7.292\ 115\ 052\ 419\ 941 \times 10^{-5}$	$1.176\ 572\ 904\ 729\ 431 \times 10^{-18}$

With a longer period, the results might be summarized by Figs.4 and 5. In Fig.5,  $\text{mean}(\theta) = 1.569\ 747 \times 10^{-6}$  rad.

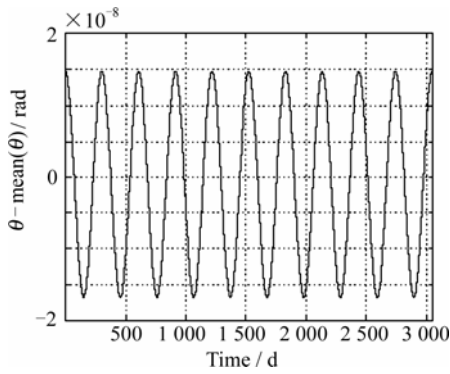
From Figs.4-6 and Tables 2 and 3, the following conclusions could be drawn out.



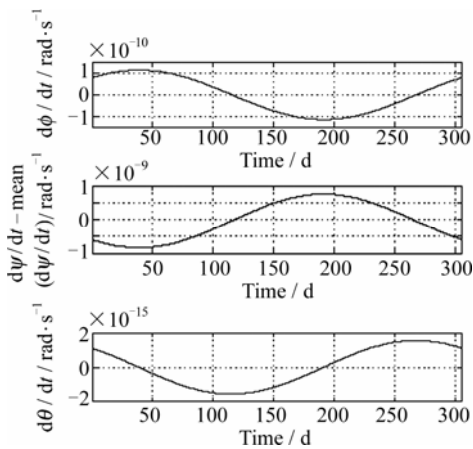
**Fig.4 Curves of variations of Eulerian angular velocity with respect to time**

1) The axis of the maximum inertia momentum deviates from the spinning axis while the latter ro-

tates around the former. The variation of the spinning angle  $\psi$  varies periodically. On the other hand, the inertial frame including the axes of the maximum, the medium and the minimum inertia momentums travel revolutionary path around the spinning axis, so that the three inertial axes rotate about the instantaneous spinning axis. In this situation, although the maximum is close to the spinning while the medium and the minimum far from the spinning, the three inertial axes possess spinning as seen in Table 2. It can be seen clearly that the spinning angular velocity of the axis of the maximum is extreme at the observed rotation angular velocity of the Earth with a tiny deviation that is less than the spinning velocity of the Earth waving as expressed in elliptic function. Meanwhile, the minimum and the medium have spinning angular velocities that do not vanish but vary from being almost absolutely equal to crossing the zeros as average. The extreme absolute angular velocities of the two axes are almost equal to 46.46 parts of the spinning velocity of the Earth.



**Fig.5 Curves of variation of nutation angle with respect to time**



**Fig.6 Curves of variations of Eulerian angular velocity with respect to time**

2) The variation ratios of the three Euler angles act similarly to angular velocity components. The third Euler angle  $\psi$  describing the spin deviates from the spinning velocity with waving around the average. The other two Euler angles, the nutation angle  $\theta$  and the free Euler precession angle  $\phi$  oscillate from the minus to the positive angular velocities, crossing the zero at average. But this time, the two do not have the approximate absolute maximum velocities with Euler angle  $\theta$  almost equal to that of the minimum or the medium, while the value of Euler angle  $\phi$  varies between the minus and the plus of about 32.56 parts of the spinning velocity of the Earth.

3) Considering the inertia momentum axes, one can easily determine that the axis of the maximum inertia momentum, though close enough to the spinning axis, can be indeed seen as identical to the spinning so that the Chandler wobble obtained from the rotation solution may almost be exactly the same as the former solution. But the oscillation of the rotation of the axis of

the maximum inertia momentum implies that the true solution of the Euler rotation equations must be composite.

4) The axis of the minimum inertia momentum may spin clockwise and anticlockwise in the process of frequency 46.46 parts of the rotation velocity of the Earth. This may show that this axis has a wobble similar to that of the axis of the maximum. On the other hand, the Length-of-Day fluctuates because of the clockwise and anti-clockwise spinning of the axis of the minimum. The Eigen frequency of the spinning of the axis further illustrates that the fluctuation period of the effect for the Length-of-Day variations may be 46.46 d, which is almost exactly identical to the observed 40-50 d fluctuations in Length-of-Day.

5) The axis of the medium inertia momentum may spin clockwise and anti-clockwise in the process of frequency 46.46 parts of the rotation velocity of the Earth. This consequence may be seen as similar to that of the minimum. Further result of the spinning of the medium axis may introduce a special solution to Euler rotation equations with a hyperbolic function separated from elliptic function. The hyperbolic function solution for Euler rotation equations appears as a one-way motion of an inverted pendulum so that one can see on the surface of the Earth that the pole wanders westward in a secular trend, which is suggested in Reference [11] as the observed secular polar motion.

6) The rotation solution of a triaxial Earth may be separated as three pendulums, where two stable wobbles appear as two terms of periodic polar motions and one unstable one-way motion appears as a secular polar motion. It is important that the three motions are coupled as an exact solution simultaneously. One can observe Earth rotation with Chandler wobble and a decadal free polar motion of period 14.6 a as well as a secular wander for the pole. Meantime, the rotation of the triaxial Earth with a liquid core might explain the geomagnetic polarity as a reversal from the same mechanism of a tumbling spinning pole<sup>[12]</sup>.

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- GPS
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