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Global Inverse Optimal Tracking Control of Underactuated Omni-directional Intelligent Navigators (ODINs)

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Abstract: This paper presents a design of optimal controllers with respect to a meaningful cost function to force an underactuated omni-directional intelligent navigator (ODIN) under unknown constant environmental loads to track a reference trajectory in two-dimensional space. Motivated by the vehicle's steering practice, the yaw angle regarded as a virtual control plus the surge thrust force are used to force the position of the vehicle to globally track its reference trajectory. The control design is based on several recent results developed for inverse optimal control and stability analysis of nonlinear systems, a new design of bounded disturbance observers, and backstepping and Lyapunov's direct methods. Both state- and output-feedback control designs are addressed. Simulations are included to illustrate the effectiveness of the proposed results.
Keywords: inverse optimality; optimal controller; global tracking; underactuated omni-directional intelligent navigator (ODIN); Lyapunov's direct method; backstepping method

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1 Introduction

Control of marine vessels is an active field, see for example (Fossen, 1994; 2002; Antonelli, 2006; Do and Pan, 2009), due to its theoretical challenges and important applications in practice. Most marine vessels are underactuated meaning that they have more degrees of freedom to be controlled than the number of independent control inputs. Different approaches to control of underactuated marine vessels are reviewed by Muske *et al*. (2010) and Paull *et al.* (2014).

The marine vehicle under consideration in this paper, see Fig. 1, is an ODIN, which has a spherical shape with only two horizontal thrusters (those thrusters marked with the red cross signs are not in use) along the surge direction while there are three degrees of freedom to be controlled. When an ODIN has all thrusters in use, various control algorithms were available (Fossen, 2002; Antonelli, 2006; Antonelli *et al.*, 2001). Several control schemes based on nontrivial coordinate transformations were available by Do *et al.* (2004a) for controlling an underactuated ODIN in two-dimensional space and in three-dimensional space (Do, 2013).

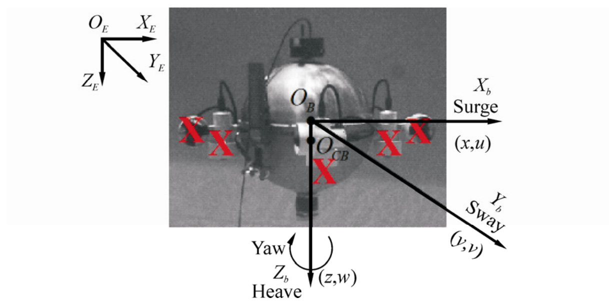

Fig. 1 Motion variables of an ODIN

A continuous time-invariant controller was developed by Godhavn *et al.* (1998) to achieve global exponential position tracking for underactuated ships. However, the ship orientation was not controlled. Output redefinition, input-output linearization and sliding mode techniques were used by Zhang *et al.* (2000) to obtain a local asymptotic result on path tracking for underactuated ships. The path following errors were first described in the Serret-Frenet frame, then a local path following controller was designed under constant ocean current disturbances (Encarnacao *et al.*, 2000). An application of the recursive technique for standard chain form systems (Jiang and Nijmeijer, 1999) was used by Pettersen and Nijmeijer (2001) to provide a high-gain, local exponential tracking result. By applying a cascade approach, a global tracking result was obtained (Lefeber *et al.*, 2003). Based on Lyapunov's direct method and the passivity approach, two tracking solutions were proposed by Jiang (2002). In these works (Jiang, 2002; Lefeber *et al.*, 2003; Pettersen and Nijmeijer, 2001), tracking a straight-line is excluded. The first controller was proposed by Pettersen and Lefeber (2001) to force an underactuated ship to track a straight-line. A solution was proposed to solve the trajectory tracking problem including a straight-line (Do *et al.*, 2002a). And a single controller was proposed by Do *et al.* (2002b) to solve both stabilization and tracking simultaneously, see also (Do and Pan, 2005) for how to deal with non-zero off-diagonal terms in the above articles. A nontrivial coordinate transformation was used by Behal *et al.* (2002) to transform the underactuated ship dynamics to a convenient form. In addition, Leonard (1995a; 1995b), Pettersen (1996),

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Silvestre *et al.* (2002) and Do *et al.* (2004b) have studied the problem of controlling underactuated underwater vehicles.

The controllers designed in all of the above works and others on control of marine vessels not listed here are not optimal in the sense that no meaningful cost function is resulted from their control designs. On the other hand, a direct optimal control design for a nonlinear system faces a formidable task of solving a Hamilton-Jacobi equation (Sepulchre *et al.*, 1997). Thus, this paper proposes a design of inverse optimal controllers that force the position and orientation of an ODIN under unknown constant environmental loads to track a reference trajectory. The proposed control design minimizes some meaningful cost function. To overcome difficulties caused by the underactuation, the yaw angle regarded as a virtual control plus the surge thrust force are used to control the position of the vehicle, bounded disturbance and state observers are developed to guarantee asymptotic estimate of the disturbances and states of the ODIN dynamics. To ensure that a meaningful cost function is minimized, the controls are designed in such a way that they do not cancel state (error) dynamics but dominate them instead.

2 Problem statement

2.1 ODIN dynamics

In hydrodynamics, it is common to assume a linear superposition so that wind and waves can be treated as generalized forces that can be directly added to nonlinear equations of motion but the generalized forces induced by ocean currents do not obey the linear superposition law (Fossen, 2011). Although ODIN is an underwater vehicle, we also consider wave and wind disturbances because they appear when the vehicle surfaces. Thus, equations of motion of an ODIN moving in a horizontal plane (heave, pitch and roll modes are neglected, so the gravitational and buoyancy terms do not appear in the equations of motion) need to be described by Fossen (2012):

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \boldsymbol{J}(\psi)\boldsymbol{v}\\ \dot{\boldsymbol{v}} &= \boldsymbol{M}_{RB}^{-1}[-\boldsymbol{C}_{RB}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{M}_A\dot{\boldsymbol{v}}_r - \boldsymbol{C}_A(\boldsymbol{v}_r)\boldsymbol{v}_r - \\ &\quad \boldsymbol{D}\boldsymbol{v}_r + \tau_{\text{wind}} + \tau_{\text{wave}} + \tau]\end{aligned} \tag{1}$$

where $\boldsymbol{\eta} = \mathrm{col}(x, y, \psi)$ with (x, y) denoting the (surge, sway) displacements of the center of mass, and ψ denoting the yaw angle of the vehicle coordinated in the earth-fixed frame $O_E X_E Y_E Z_E$, see Fig. 1. The vector $\boldsymbol{v} = \mathrm{col}(u, v, r)$ denotes the (surge, sway, yaw) velocities of the vehicle coordinated in the body-fixed frame $O_b X_b Y_b Z_b$. The relative velocity vector $\boldsymbol{v}_r$ is defined by $\boldsymbol{v}_r = \boldsymbol{v} - \boldsymbol{v}_c$ with $\boldsymbol{v}_c = \mathrm{col}(u_c, v_c, 0)$ being the ocean current velocity vector. The rotational matrix $\boldsymbol{J}(\psi)$, the vehicle and added mass inertia matrices $\boldsymbol{M}_{RB}$ and $\boldsymbol{M}_A$, the coriolis and centripetal matrices due to the vehicle inertia $\boldsymbol{C}_{RB}(\boldsymbol{v})$ and added mass $\boldsymbol{C}_A(\boldsymbol{v}_r)$, the damping matrix $\boldsymbol{D}$ are given by

$$\begin{aligned}\boldsymbol{J}(\psi) &= \begin{bmatrix}\cos(\psi) & -sin(\psi) & 0\\ \sin(\psi) & \cos(\psi) & 0\\ 0 & 0 & 1\end{bmatrix}\\ \boldsymbol{C}_{RB}(\boldsymbol{v}) &= m_{RB}\begin{bmatrix}0 & 0 & -v\\ 0 & 0 & u\\ v & -u & 0\end{bmatrix}\\ \boldsymbol{C}_A(\boldsymbol{v}_r) &= m_A\begin{bmatrix}0 & 0 & -v_r\\ 0 & 0 & u_r\\ v_r & -u_r & 0\end{bmatrix}\\ \boldsymbol{M}_{RB} &= \mathrm{diag}(m_{RB}, m_{RB}, J_{RB})\\ \boldsymbol{M}_A &= \mathrm{diag}(m_A, m_A, J_A)\\ \boldsymbol{D} &= \mathrm{diag}(d_l, d_l, d_r)\end{aligned} \tag{2}$$

where m_{RB} and m_A are the mass and added mass, J_{RB} and J_A are the inertia and added mass inertia, d_l and d_r are the damping coefficients in the surge and sway, and yaw directions, and $u_r = u - u_c$ and $v_r = v - v_c$. The control input vector is $\boldsymbol{\tau} = \mathrm{col}(\tau_u, 0, \tau_r)$. The force and moment vector $\tau_r := -\boldsymbol{M}_A\dot{\boldsymbol{v}}_r - \boldsymbol{C}_A(\boldsymbol{v}_r)\boldsymbol{v}_r - \boldsymbol{D}\boldsymbol{v}_r$ is the load due to the ocean current velocity $\boldsymbol{v}_c$. The wind force and moment vector is τ_{wind} and the wave-induced force and moment vector is τ_{wave}. In this paper, we make the following assumptions:

Assumption 2.1 The ocean currents are irrotational and but bounded in the earth-fixed coordinate. The wave and wind torque and moment vectors τ_{wind} and τ_{wave} are also constant and bounded in the earth-fixed coordinate. In particular, we have

$$\boldsymbol{v}_c = \boldsymbol{J}^{-1}(\psi)\boldsymbol{v}_{Ec}, \tau_{\text{wave}} = \boldsymbol{J}^{-1}(\psi)\tau_{E\text{wave}}, \tau_{\text{wind}} = \boldsymbol{J}^{-1}(\psi)\tau_{E\text{wind}} \tag{3}$$

where $\boldsymbol{v}_{Ec}$, $\tau_{E\text{wave}}$, and $\tau_{E\text{wind}}$ are constant in the earth-fixed coordinate.

Differentiating both sides of the first equation of (3) gives $\dot{\boldsymbol{v}}_c + \boldsymbol{J}^{-1}(\psi)\dot{\boldsymbol{J}}(\psi)\boldsymbol{v}_c = 0$. It is also verified that

$$\dot{\boldsymbol{v}}_c + \boldsymbol{M}_A^{-1}(\boldsymbol{C}_A(\boldsymbol{v}_c)\boldsymbol{v} + \boldsymbol{C}_A(\boldsymbol{v})\boldsymbol{v}_c) = \dot{\boldsymbol{v}}_c + \boldsymbol{J}^{-1}(\psi)\dot{\boldsymbol{J}}(\psi)\boldsymbol{v}_c$$

Thus,

$$\boldsymbol{M}_A\dot{\boldsymbol{v}}_c + (\boldsymbol{C}_A(\boldsymbol{v}_c)\boldsymbol{v} + \boldsymbol{C}_A(\boldsymbol{v})\boldsymbol{v}_c) = 0 \tag{4}$$

Let us define

$$-\boldsymbol{D}\boldsymbol{J}^{-1}(\psi)\boldsymbol{v}_c = \boldsymbol{J}^{-1}(\psi)\tau_{E\text{current}} \tag{5}$$

where $\tau_{E\text{current}}$ is referred to as the current induced forces and moments that are constant in the earth-fixed coordinate. Substituting $\boldsymbol{v}_r = \boldsymbol{v} - \boldsymbol{v}_c$ into $\boldsymbol{C}_A(\boldsymbol{v}_r)\boldsymbol{v}_r$ give

$$\boldsymbol{C}_A(\boldsymbol{v}_r)\boldsymbol{v}_r = \boldsymbol{C}_A(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{C}_A(\boldsymbol{v}_c)\boldsymbol{v} - \boldsymbol{C}_A(\boldsymbol{v})\boldsymbol{v}_c \tag{6}$$

where we have used $\boldsymbol{C}_A(\boldsymbol{v}_c)\boldsymbol{v}_c = 0$. Now substituting $\boldsymbol{v}_r = \boldsymbol{v} - \boldsymbol{v}_c$, Eqs. (4) and (6) into the second equation of (1), and using the last two equations of (3) and (5) together with the first equation of (1) result in

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \boldsymbol{J}(\psi)\boldsymbol{v} \\ \dot{\boldsymbol{v}} &= \boldsymbol{M}^{-1}[-\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v} + \boldsymbol{\tau} + \boldsymbol{J}^{-1}(\psi)\boldsymbol{\tau}_E] \end{aligned} \tag{7}$$

where, $\boldsymbol{M} = \boldsymbol{M}_{RB} + \boldsymbol{M}_A$, $\boldsymbol{C}(\boldsymbol{v}) = \boldsymbol{C}_{RB}(\boldsymbol{v}) + \boldsymbol{C}_A(\boldsymbol{v})$, and $\tau_E = \tau_{E\text{current}} + \tau_{E\text{wind}} + \tau_{E\text{wave}}$.

2.2 Control objective

In this paper, we consider the following control objective:

Control Objective 2.1

Suppose that $\boldsymbol{\tau}_E = \text{col}(\tau_{Eu}, \tau_{Ev}, \tau_{Er})$ is bounded in the sense that

$$\tau_{Eu} \in (\tau_{Eu}^{\min}, \tau_{Eu}^{\max}), \quad \tau_{Ev} \in (\tau_{Ev}^{\min}, \tau_{Ev}^{\max}), \quad \tau_{Er} \in (\tau_{Er}^{\min}, \tau_{Er}^{\max}) \tag{8}$$

where the constants $\bullet^{\min}$ and $\bullet^{\max}$ denote the maximum and minimum values of $\bullet$, respectively. Moreover, suppose that the reference position and yaw angle vector $\eta_d(t)$ is generated by the reference model

$$\begin{aligned} \dot{\boldsymbol{\eta}}_d &= \boldsymbol{J}(\psi_d)\boldsymbol{v}_d \\ \dot{\boldsymbol{v}}_d &= \boldsymbol{M}^{-1}[-\boldsymbol{C}(\boldsymbol{v}_d)\boldsymbol{v}_d - \boldsymbol{D}\boldsymbol{v}_d + \boldsymbol{\tau}_d] \end{aligned} \tag{9}$$

where all the symbols $\boldsymbol{\eta}_d = \text{col}(x_d, y_d, \psi_d)$, $\boldsymbol{v}_d = \text{col}(u_d, v_d, r_d)$, and $\boldsymbol{\tau}_d = \text{col}(\tau_{ud}, 0, \tau_{rd})$ have the same meaning as those defined in Eq. (7). The reference surge force τ_{ud} is supposed to satisfy the following condition

$$|\tau_{ud}| \geq \| \tau_{El}^{\max} \| + \varpi_0 \tag{10}$$

where $\tau_{El}^{\max} = \text{col}(\max(|\tau_{Eu}^{\min}|, |\tau_{Eu}^{\max}|), \max(|\tau_{Ev}^{\min}|, |\tau_{Ev}^{\max}|))$, and ϖ_0 is a strictly positive constant. In addition, τ_d is bounded and τ_{ud} is twice differentiable, i.e., there exist $\tau_{ud}^{\max}$, $\dot{\tau}_{ud}^{\max}$, $\ddot{\tau}_{ud}^{\max}$, and $\tau_{rd}^{\max}$ such that $|\tau_{ud}(t)| \leq \tau_{ud}^{\max}$, $|\dot{\tau}_{ud}(t)| \leq \dot{\tau}_{ud}^{\max}$, $|\ddot{\tau}_{ud}(t)| \leq \ddot{\tau}_{ud}^{\max}$ and $|\tau_{rd}(t)| \leq \tau_{rd}^{\max}$ for all $t \geq t_0 \geq 0$. For later use, we find the upper-bound of $|r_d(t)|$. As such, from the last equation of (9), we have $\dot{r}_d = -d_r r_d + \tau_{rd}$. We now consider the function $V_{rd} = \frac{1}{2} r_d^2$, whose derivative satisfies

$$\dot{V}_{rd} = -d_r r_d^2 + r_d \tau_{rd} \leq -\frac{d_r}{2} r_d^2 + \frac{1}{2d_r}\tau_{rd}^2 \leq -d_r V_{rd} + \frac{1}{2d_r}(\tau_{rd}^{\max})^2$$

Solving the above inequality gives

$$V_{rd}(t) \leq (V_{rd}(t_0) - \frac{1}{2d_r^2}(\tau_{rd}^{\max})^2)\text{e}^{-d_r(t-t_0)} + \frac{1}{2d_r^2}(\tau_{rd}^{\max})^2 \leq V_{rd}(t_0) + \frac{1}{2d_r^2}(\tau_{rd}^{\max})^2$$

which implies that $|r_d(t)| \leq \sqrt{(\tau_{rd}^{\max})^2 / d_r^2 + r_d^2(t_0)} := r_d^{\max}$ for all $t \geq t_0 \geq 0$.

Design τ and estimate laws for τ_E to force the position and yaw angle vector η of the vehicle whose dynamics are given by Eq. (7) to globally asymptotically track its reference trajectory vector η_d generated by Eq. (9) and to minimize a meaningful cost function of tracking errors and control inputs τ_u and τ_r.

3 Preliminaries

This section presents preliminary results on smooth saturation function, disturbance observer, disturbance-state observer, and inverse optimal stabilizer that will be used in the control design in the next section.

3.1 Smooth saturation function

Definition 3.1 The function $\sigma(x)$ is said to be a smooth saturation function if it is smooth and satisfies:

$$\begin{aligned} &1)\quad \sigma(x) = 0 \text{ if } x = 0, \quad \sigma(x)x > 0 \text{ if } x \neq 0 \\ &2)\quad \sigma(-x) = -\sigma(x), \quad (x - y)[\sigma(x) - \sigma(y)] \geq 0 \\ &3)\quad \lim_{x \to \pm\infty} \sigma(x) = \pm 1, \quad |\sigma(x)| \leq 1, \sigma'(x) > 0 \end{aligned} \tag{11}$$

for all $(x, y) \in \mathbb{R}^2$, where $\sigma'(x) = \text{d}\sigma(x)/\text{d}x$ and ε_0 is a positive constant. For the vector $\boldsymbol{x} = \text{col}(x_1, x_2, \ldots, x_n)$, the notation $\boldsymbol{\sigma}(\boldsymbol{x}) = \text{col}(\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n))$ denotes the smooth saturation function vector of the vector $\boldsymbol{x}$.

3.2 Disturbance observers

Lemma 3.1 Consider the following second-order nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2, u) + \theta \end{aligned} \tag{12}$$

where x_1 and x_2 are states, u is the control input, the unknown constant parameter θ is bounded, i.e., there exist constants $\theta^{\min}$ and $\theta^{\max}$ such that $\theta \in (\theta^{\min}, \theta^{\max})$. Suppose that x_1 and x_2 and the function $f(x_1, x_2, u)$ are available and that the system (12) is well defined for all $(x_1(t_0), x_2(t_0)) \in \mathbb{R}^2$, where $t_0 \geq 0$ is the initial time. The following disturbance observer

$$\begin{aligned} \hat{\theta} &= \frac{\theta^{\max} - \theta^{\min}}{2}\sigma(\xi + kx_2) + \frac{\theta^{\max} + \theta^{\min}}{2} \\ \dot{\xi} &= -k\left[\frac{\theta^{\max} - \theta^{\min}}{2}\sigma(\xi + kx_2) + \frac{\theta^{\max} + \theta^{\min}}{2}\right] - kf(x_1, x_2, u) \end{aligned} \tag{13}$$

where k is a positive constant, guarantees that $\hat{\theta}\in[\theta^{\min},\theta^{\max}]$, $\tilde{\theta}(t)=\theta-\hat{\theta}(t)$ globally asymptotically and locally exponentially converges to zero, and that $|\xi(t)+kx_2(t)|$ is bounded for all $t\geq t_0$ and for each initial condition $\xi(t_0)\in\mathbb{R}$.

Proof. See Appendix A.

Lemma 3.2 Consider the following second-order nonlinear system

$$\begin{aligned}\dot{x}_1&=x_2\\ \dot{x}_2&=-ax_2+f(x_1,u)+\theta\end{aligned}\tag{14}$$

where a is a positive constant, x_1 and x_2 are states, u is the control input, the unknown constant parameter θ is bounded, i.e., there exist constants $\theta^{\min}$ and $\theta^{\max}$ such that $\theta\in(\theta^{\min},\theta^{\max})$. Suppose that x_1 and the function $f(x_1,u)$ are available and that the system (14) is well defined for all $(x_1(t_0),x_2(t_0))\in\mathbb{R}^2$, $t_0\geq 0$. The following disturbance observer

$$\begin{aligned}\hat{\theta}&=\frac{\theta^{\max}-\theta^{\min}}{2}\sigma(\xi+k_1x_1+k_2\hat{x}_2)+\frac{\theta^{\max}+\theta^{\min}}{2}\\ \dot{\hat{x}}_2&=-a\hat{x}_2+f(x_1,u)+\hat{\theta}\\ \dot{\xi}&=-k_2\left[\frac{\theta^{\max}-\theta^{\min}}{2}\sigma(\xi+k_1x_1+k_2\hat{x}_2)+\frac{\theta^{\max}+\theta^{\min}}{2}\right]-\\ &\quad k_2(-a\hat{x}_2+f(x_1,u))-k_1\hat{x}_2\end{aligned}\tag{15}$$

where k_1 and k_2 are positive constants, guarantees that $\hat{\theta}\in[\theta^{\min},\theta^{\max}]$, $\tilde{\theta}(t)=\theta-\hat{\theta}(t)$ and $\tilde{x}_2(t)=x_2(t)-\hat{x}_2(t)$ globally asymptotically and locally exponentially converge to zero, and that $|\xi(t)+k_1x_1(t)+k_2\hat{x}_2(t)|$ is bounded for all $(\xi(t_0),\hat{x}_2(t_0))\in\mathbb{R}^2$ and $t\geq 0$.

Proof. See Appendix B

Remark 3.1 The main desired property of the disturbance observers proposed in Lemmas 3.1 and 3.2 in comparison with existing disturbance observers (e.g., Chen *et al*., 2000; Do and Pan, 2008; Mohammadi *et al*., 2013) is that the disturbance observers (13) and (15) guarantee pre-specified boundedness of $\hat{\theta}$, i.e., $\hat{\theta}\in[\theta^{\min},\theta^{\max}]$. This property is essential for the success of the control design, see the paragraph just below (46) in Section 4.

3.3 Inverse optimal stabilizer

Consider the following nonlinear system:

$$\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}\tag{16}$$

where $\boldsymbol{x}\in\mathbb{R}^n$ and $\boldsymbol{u}\in\mathbb{R}^m$ denote the state and control vectors, respectively. Moreover, $\boldsymbol{f}:\mathbb{R}^n\to\mathbb{R}^n$ and $\boldsymbol{G}:\mathbb{R}^n\to\mathbb{R}^{n\times m}$ are smooth, vector- and matrix-valued functions, respectively, with $\boldsymbol{f}(0)=0$.

Lemma 3.3 Moylan and Anderson (1973), Krstic and Tsiotras (1999) Suppose that the feedback control law

$$\boldsymbol{u}(\boldsymbol{x})=-\boldsymbol{R}^{-1}(\boldsymbol{x})(\frac{\partial V}{\partial\boldsymbol{x}}\boldsymbol{G}(\boldsymbol{x}))^{\mathrm{T}}\tag{17}$$

where $\boldsymbol{R}:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ is a positive definite matrix-valued function, stabilizes the system in (16) with respect to a positive definite radially unbounded Lyapunov function $V(\boldsymbol{x})$. Then the control law

$$\boldsymbol{u}^*(\boldsymbol{x})=-\beta\boldsymbol{R}^{-1}(\boldsymbol{x})(\frac{\partial V}{\partial\boldsymbol{x}}\boldsymbol{G}(\boldsymbol{x}))^{\mathrm{T}},\quad\beta\geq 2\tag{18}$$

is optimal with respect to the cost

$$\mathcal{J}=\int_{t_0}^{\infty}(\ell(\boldsymbol{x})+\boldsymbol{u}^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{x})\boldsymbol{u})\mathrm{d}t\tag{19}$$

where

$$\begin{aligned}\ell(\boldsymbol{x})&=-2\beta\frac{\partial V}{\partial\boldsymbol{x}}(\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}))+\\ &\quad\beta(\beta-2)\frac{\partial V}{\partial\boldsymbol{x}}\boldsymbol{G}(\boldsymbol{x})\boldsymbol{R}^{-1}(\boldsymbol{x})(\frac{\partial V}{\partial\boldsymbol{x}}\boldsymbol{G}(\boldsymbol{x}))^{\mathrm{T}}\end{aligned}\tag{20}$$

4 Control design

Let us define

$$\begin{aligned}&\boldsymbol{J}_l(\psi)=\begin{bmatrix}\cos(\psi)&-\sin(\psi)\\ \sin(\psi)&\cos(\psi)\end{bmatrix}\\ &\begin{cases}\boldsymbol{M}_l=(m_{RB}+m_A)\boldsymbol{I}_{2\times2},\quad\boldsymbol{D}_l=d_l\boldsymbol{I}_{2\times2}\\ \tau_l=\mathrm{col}(\tau_u,0),\quad\bar{\tau}_{El}=\mathrm{col}(\bar{\tau}_{Eu},\bar{\tau}_{Ev})\end{cases}\\ &\begin{cases}\boldsymbol{X}_1=\mathrm{col}(x,y),\quad\boldsymbol{X}_2=\boldsymbol{J}_l(\psi)\boldsymbol{v}_l\\ \boldsymbol{X}_{1d}=\mathrm{col}(x_d,y_d),\quad\boldsymbol{X}_{2d}=\boldsymbol{J}_l(\psi_d)\boldsymbol{v}_{dl}\end{cases}\end{aligned}\tag{21}$$

where $\boldsymbol{I}_{2\times2}$ is a 2×2 identity matrix. With the above definitions, we can write the vehicle dynamics (7) as the position subsystem P) and the heading subsystem H) in a cascade structure:

$$\begin{aligned}&\mathrm{P})\begin{cases}\dot{\boldsymbol{X}}_1=\boldsymbol{X}_2\\ \dot{\boldsymbol{X}}_2=\boldsymbol{M}_l^{-1}(-\boldsymbol{D}_l\boldsymbol{X}_2+\tau_{El}+\boldsymbol{J}_l(\psi)\tau_l)\end{cases}\\ &\mathrm{H})\begin{cases}\dot{\psi}=r\\ \dot{r}=\dfrac{1}{J}(-d_rr+\tau_{Er}+\tau_r)\end{cases}\end{aligned}\tag{22}$$

where $J=J_{RB}+J_A$. Similarly, the reference model (9) is rewritten as the position subsystem Pd) and the heading subsystem Hd) in a cascade structure:

$$\begin{aligned}&\mathrm{Pd})\begin{cases}\dot{\boldsymbol{X}}_{1d}=\boldsymbol{X}_{2d}\\ \dot{\boldsymbol{X}}_{2d}=\boldsymbol{M}_l^{-1}[-\boldsymbol{D}_l\boldsymbol{X}_{2d}+\boldsymbol{J}_l(\psi_d)\tau_{dl}]\end{cases}\\ &\mathrm{Hd})\begin{cases}\dot{\psi}_d=r_d\\ \dot{r}_d=\dfrac{1}{J}(-d_rr_d+\tau_{rd})\end{cases}\end{aligned}\tag{23}$$

A close look at Eqs. (22) and (23) suggests the design of the control input vector τ, i.e., τ_l and τ_r, in two stages.

In the first stage, the control τ_l and ψ, which is viewed as a virtual control, are designed to forces $\boldsymbol{X}_1$ to track $\boldsymbol{X}_{1d}$. In the second stage, the control τ_r is designed to force the virtual control of ψ to track its actual value.

4.1 Disturbance observers

We now design observers to estimate the disturbances τ_{El} and τ_{Er}. As such, applying Lemma 3.1 to Eq. (22) yields the following disturbance observers

$$\begin{cases}\hat{\tau}_{El} = D_{El}\sigma(\xi_{El} + \boldsymbol{K}_{El}\boldsymbol{M}_l\boldsymbol{X}_2) + \delta_{El} \\ \dot{\xi}_{El} = -\boldsymbol{K}_{El}(D_{El}\sigma(\xi_{El} + \boldsymbol{K}_{El}\boldsymbol{M}_l\boldsymbol{X}_2) + \delta_{El}) - \\ \quad \boldsymbol{K}_{El}[-\boldsymbol{D}_l\boldsymbol{X}_2 + \boldsymbol{J}_l(\psi)\tau_l]\end{cases} \quad (24)$$
$$\begin{cases}\hat{\tau}_r = \Delta_{Er}\sigma(\xi_r + k_{Er}Jr) + \delta_{Er} \\ \dot{\xi}_r = -k_{Er}(\Delta_{Er}\sigma(\xi_r + k_{Er}Jr) + \delta_{Er}) - k_{Er}(-d_r r + \tau_r)\end{cases}$$

where,

$$D_{El} = \mathrm{diag}\left(\frac{\tau_{Eu}^{\max} - \tau_{Eu}^{\min}}{2}, \frac{\tau_{Ev}^{\max} - \tau_{Ev}^{\min}}{2}\right), \quad \Delta_{Er} = \frac{\tau_{Er}^{\max} - \tau_{Er}^{\min}}{2}$$
$$\delta_{El} = \mathrm{col}\left(\frac{\tau_{Eu}^{\max} + \tau_{Eu}^{\min}}{2}, \frac{\tau_{Ev}^{\max} + \tau_{Ev}^{\min}}{2}\right), \quad \delta_{Er} = \frac{\tau_{Er}^{\max} + \tau_{Er}^{\min}}{2}$$

$\boldsymbol{K}_{El}$ is a diagonal positive definite matrix, and k_{Er} is a positive constant. Let $\tilde{\tau}_{El} = \tau_{El} - \hat{\tau}_{El}$ and $\tilde{\tau}_{Er} = \tau_{Er} - \hat{\tau}_{Er}$. It is obvious that

$$\begin{aligned}\dot{\tilde{\tau}}_{El} &= -\boldsymbol{K}_{El}D_{El}\sigma'(\xi_{El} + \boldsymbol{K}_{El}\boldsymbol{M}_l\boldsymbol{X}_2)\tilde{\tau}_{El} \\ \dot{\tilde{\tau}}_{Er} &= -k_{Er}\Delta_{Er}\sigma'(\xi_r + k_{Er}Jr)\tilde{\tau}_{Er}\end{aligned} \quad (25)$$

4.2 Stage I

4.2.1 Step 1

Define the following tracking errors

$$\begin{aligned}\boldsymbol{X}_{1e} &= \boldsymbol{X}_1 - \boldsymbol{X}_{1d} \\ \boldsymbol{X}_{2e} &= \boldsymbol{X}_2 - \alpha_1 \\ \psi_e &= \psi - \alpha_\psi\end{aligned} \quad (26)$$

where α_1 and α_ψ are referred to as the virtual controls of $\boldsymbol{X}_2$ and ψ, respectively. To design α_1 to stabilize the tracking error $\boldsymbol{X}_{1e}$ at the origin, we consider the following Lyapunov function candidate

$$V_1 = \frac{1}{2}\|\boldsymbol{X}_{1e}\|^2 \quad (27)$$

whose derivative the solutions of the first equation of (22) and the first equation of (23) with the use of Eq. (26) results in

$$\dot{V}_1 = \boldsymbol{X}_{1e}^{\mathrm{T}}(\alpha_1 + \boldsymbol{X}_{2e} - \dot{\boldsymbol{X}}_{1d}) \quad (28)$$

which suggests that we choose

$$\alpha_1 = -\boldsymbol{K}_1\boldsymbol{X}_{1e} + \dot{\boldsymbol{X}}_{1d} \quad (29)$$

where $\boldsymbol{K}_1$ is a diagonal positive definite matrix to be determined later. Substituting Eq. (29) into Eq. (28) yields

$$\dot{V}_1 = -\boldsymbol{X}_{1e}^{\mathrm{T}}\boldsymbol{K}_1\boldsymbol{X}_{1e} + \boldsymbol{X}_{1e}^{\mathrm{T}}\boldsymbol{X}_{2e} \quad (30)$$

Substituting (29) into the first equation of (22) with the use of the first equation of (23) yields

$$\dot{\boldsymbol{X}}_{1e} = -\boldsymbol{K}_1\boldsymbol{X}_{1e} + \boldsymbol{X}_{2e} \quad (31)$$

4.2.2 Step 2

Differentiating $\boldsymbol{X}_{2e}$ along the solutions of the second equations of (22), (23), and (29) yields

$$\begin{aligned}\dot{\boldsymbol{X}}_{2e} = \boldsymbol{M}_l^{-1}[&-\boldsymbol{D}_l(\boldsymbol{X}_{2e} - \boldsymbol{K}_1\boldsymbol{X}_{1e} + \boldsymbol{X}_{2d}) + \hat{\tau}_{El} + \\ &\tilde{\tau}_{El} + \boldsymbol{J}_l(\psi)\tau_l] + \boldsymbol{K}_1(-\boldsymbol{K}_1\boldsymbol{X}_{1e} + \boldsymbol{X}_{2e}) - \dot{\boldsymbol{X}}_{2d}\end{aligned} \quad (32)$$

To design τ_u and α_ψ that stabilize $\boldsymbol{X}_{2e}$ at the origin, we consider the following Lyapunov function candidate

$$V_2 = \rho_1 V_1 + \frac{m}{2}\|\boldsymbol{X}_{2e}\|^2 \quad (33)$$

where ρ_1 is a positive constant and $m = m_{RB} + m_A$. Differentiating both sides of Eq. (33) along the solutions of Eqs. (32) and (30) gives

$$\begin{aligned}\dot{V}_2 = &-\rho_1\boldsymbol{X}_{1e}^{\mathrm{T}}\boldsymbol{K}_1\boldsymbol{X}_{1e} + E_1 + \\ &\boldsymbol{X}_{2e}^{\mathrm{T}}[-\boldsymbol{J}_l(\psi_d)\tau_{dl} + \hat{\tau}_{El} + \boldsymbol{J}_l(\psi)\tau_l]\end{aligned} \quad (34)$$

where

$$\begin{aligned}E_1 = \rho_1\boldsymbol{X}_{1e}^{\mathrm{T}}\boldsymbol{X}_{2e} + \boldsymbol{X}_{2e}^{\mathrm{T}}[&-(\boldsymbol{D}_l - \boldsymbol{M}_l\boldsymbol{K}_1)\boldsymbol{X}_{2e} + \\ &(\boldsymbol{D}_l\boldsymbol{K}_1 - \boldsymbol{M}_l\boldsymbol{K}_1^2)\boldsymbol{X}_{1e}] + \boldsymbol{X}_{2e}^{\mathrm{T}}\tilde{\tau}_{El}\end{aligned} \quad (35)$$

and we have used $\boldsymbol{D}_l\boldsymbol{X}_{2d} + \boldsymbol{M}_l\dot{\boldsymbol{X}}_{2d} = \boldsymbol{J}_l(\psi_d)\tau_{dl}$. Based on Lemma 3.3, the stabilizing control τ_l is chosen such that

$$\boldsymbol{J}_l(\psi)\tau_l = -\bar{\boldsymbol{K}}_2\sigma(\boldsymbol{X}_{2e}) - \hat{\tau}_{El} + \boldsymbol{J}_l(\psi_d)\tau_{dl} := \mathrm{col}(\bar{\Omega}_1, \bar{\Omega}_2) \quad (36)$$

where $\bar{\boldsymbol{K}}_2 = \mathrm{diag}(\bar{k}_{21}, \bar{k}_{22})$ with $\bar{k}_{21}$ and $\bar{k}_{22}$ being positive constants to be specified later. Since $\tau_l = \mathrm{col}(\tau_u, 0)$ and $\tau_{dl} = \mathrm{col}(\tau_{ud}, 0)$, solving Eq. (36) for τ_u obtains

$$\begin{aligned}\tau_u &= \bar{\Omega}_1\cos(\psi) + \bar{\Omega}_2\sin(\psi) := \tau_{u1} + \tau_{u2} \\ \tau_{u1} &= -\bar{k}_{21}\sigma(X_{21e})\cos(\psi) - \bar{k}_{22}\sigma(X_{22e})\sin(\psi) \\ \tau_{u2} &= (-\hat{\tau}_{Eu} + \cos(\psi_d)\tau_{ud})\cos(\psi) + (-\hat{\tau}_{Ev} + \sin(\psi_d)\tau_{ud})\sin(\psi)\end{aligned} \quad (37)$$

where X_{21e} and X_{22e} are the first and second elements of $\boldsymbol{X}_{2e}$, i.e., $\boldsymbol{X}_{2e} = \mathrm{col}(X_{21e}, X_{22e})$.

Remark 4.1 The stabilizing control τ_u given in Eq. (37) consists of two parts. The part τ_{u1} is designed based on Lemma 3.3 while the part τ_{u2} is to handle $\hat{\tau}_{El}$, the estimate of disturbances, and the reference signal

$\boldsymbol{J}_l(\psi_d)\tau_{dl}$. In a normal application of the backstepping method, one would substitute the last equation of (26), i.e., $\psi=\psi_e+\alpha_\psi$, into Eq. (34) to obtain

$$\dot{V}=-\rho_1\boldsymbol{X}_{1e}^{\mathrm{T}}\boldsymbol{K}_1\boldsymbol{X}_{1e}+E_1+\boldsymbol{X}_{2e}^{\mathrm{T}}\left[-\boldsymbol{J}_l(\psi_d)\tau_{dl}+\hat{\tau}_{El}+\boldsymbol{J}_l(\alpha_\psi)\tau_l\right]+ \boldsymbol{X}_{2e}^{\mathrm{T}}\begin{bmatrix}(\cos(\psi_e)-1)\cos(\alpha_\psi)-\sin(\psi_e)\sin(\alpha_\psi) & 0\\ \sin(\psi_e)\cos(\alpha_\psi)+(\cos(\psi_e)-1)\sin(\alpha_\psi) & 0\end{bmatrix}\tau_l \tag{38}$$

Then the control τ_l would be chosen such that all the term in the square bracket in Eq. (38) are canceled, i.e., $\boldsymbol{J}_l(\alpha_\psi)\tau_l=-\boldsymbol{K}_2\sigma(\boldsymbol{X}_{2e})-\hat{\bar{\tau}}_{El}+\boldsymbol{J}_l(\psi_d)\tau_{dl}$ instead of Eq. (36). The above choice will result in the same virtual control α_ψ as in Eq. (46) but the actual control $\tau_u=\Omega_1\cos(\alpha_\psi)+\Omega_2\sin(\alpha_\psi)$ with Ω_1 and Ω_2 defined in Eq. (39), which is different from the inverse optimal one as in Eq. (40). This choice of τ_l (note that $\tau_l=\mathrm{col}(\tau_u,0)$) will not be amendable to obtain an (inverse) optimal control. This is because according to Lemma 3.3, the control τ_l should be chosen in the form of (17). Indeed, the choice of τ_l as in Eq. (36) is in the form of Eq. (17).

An inverse optimal control $\tau_l^*=\mathrm{col}(\tau_u^*,0)$ is obtained from the stabilizing control $\tau_l=\mathrm{col}(\tau_u,0)$ given in Eq. (37) as follows

$$\boldsymbol{J}_l(\psi)\tau_l^*=-\boldsymbol{K}_2\sigma(\boldsymbol{X}_{2e})-\hat{\bar{\tau}}_{El}+\boldsymbol{J}_l(\psi_d)\tau_{dl}:=\mathrm{col}(\Omega_1,\Omega_2) \tag{39}$$

where $\boldsymbol{K}_2=\beta_{u1}\bar{\boldsymbol{K}}_2, \beta_{u_1}\geq 2$. Solving Eq. (39) results in

$$\begin{aligned}&\tau_u^*=\Omega_1\cos(\psi)+\Omega_2\sin(\psi):=\tau_{u1}^*+\tau_{u2}\\&\tau_{u1}^*=-k_{21}\sigma(X_{21e})\cos(\psi)-k_{22}\sigma(X_{22e})\sin(\psi)\\&\tau_{u2}=(-\hat{\tau}_{Eu}+\cos(\psi_d)\tau_{ud})\cos(\psi)+(-\hat{\tau}_{Ev}+\sin(\psi_d)\tau_{ud})\sin(\psi)\end{aligned} \tag{40}$$

It is easy to verify that

$$|\tau_u^*|\leq k_{21}+k_{22}+\max(|\tau_{Eu}^{\min}|,|\tau_{Eu}^{\max}|)+\max(|\tau_{Ev}^{\min}|,|\tau_{Ev}^{\max}|)+2\tau_{ud}^{\max}:=\tau_u^{*\max} \tag{41}$$

Substituting Eq. (40) into Eq. (34) results in

$$\begin{aligned}\dot{V}_2=&-\rho_1\boldsymbol{X}_{1e}^{\mathrm{T}}\boldsymbol{K}_1\boldsymbol{X}_{1e}+E_1+\boldsymbol{X}_{2e}^{\mathrm{T}}\{-\boldsymbol{J}_l(\psi_d)\tau_{dl}+\hat{\tau}_{El}+\\&\begin{bmatrix}\Omega_1\cos^2(\psi)+\Omega_2\sin(\psi)\cos(\psi)\\ \Omega_1\sin(\psi)\cos(\psi)+\Omega_2\sin^2(\psi)\end{bmatrix}\}=\\&-\rho_1\boldsymbol{X}_{1e}^{\mathrm{T}}\boldsymbol{K}_1\boldsymbol{X}_{1e}+E_1+\boldsymbol{X}_{2e}^{\mathrm{T}}\{-\boldsymbol{J}_l(\psi_d)\tau_{dl}+\hat{\tau}_{El}+\\&\begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix}+\begin{bmatrix}\Omega_1(\cos^2(\psi)-1)+\Omega_2\sin(\psi)\cos(\psi)\\ \Omega_1\sin(\psi)\cos(\psi)+\Omega_2(\sin^2(\psi)-1)\end{bmatrix}\}=\\&-\rho_1\boldsymbol{X}_{1e}^{\mathrm{T}}\boldsymbol{K}_1\boldsymbol{X}_{1e}-\boldsymbol{X}_{2e}^{\mathrm{T}}\boldsymbol{K}_2\sigma(\boldsymbol{X}_{2e})+E_1+\\&\boldsymbol{X}_{2e}^{\mathrm{T}}\begin{bmatrix}\Omega_1(\cos^2(\psi)-1)+\Omega_2\sin(\psi)\cos(\psi)\\ \Omega_1\sin(\psi)\cos(\psi)+\Omega_2(\sin^2(\psi)-1)\end{bmatrix}\end{aligned} \tag{42}$$

We now detail the last term in the right-hand side of Eq. (42). As such, substituting the last equation of Eq. (26) into the last term in the right-hand side of Eq. (42) with a note that $\tau_u^*=\Omega_1\cos(\psi)+\Omega_2\sin(\psi)$, see Eq. (40), gives

$$\begin{aligned}&\boldsymbol{X}_{2e}^{\mathrm{T}}\begin{bmatrix}\Omega_1(\cos^2(\psi)-1)+\Omega_2\sin(\psi)\cos(\psi)\\ \Omega_1\sin(\psi)\cos(\psi)+\Omega_2(\sin^2(\psi)-1)\end{bmatrix}=\boldsymbol{X}_{2e}^{\mathrm{T}}\begin{bmatrix}-\Omega_1+\cos(\psi)\tau_u^*\\ -\Omega_2+\sin(\psi)\tau_u^*\end{bmatrix}=\\&\boldsymbol{X}_{2e}^{\mathrm{T}}\begin{bmatrix}-\Omega_1+\cos(\alpha_\psi)\tau_u^*+((\cos(\psi_e)-1)\cos(\alpha_\psi)-\sin(\psi_e)\sin(\alpha_\psi))\tau_u^*\\ -\Omega_2+\sin(\alpha_\psi)\tau_u^*+((\cos(\psi_e)-1)\sin(\alpha_\psi)+\sin(\psi_e)\cos(\alpha_\psi))\tau_u^*\end{bmatrix}=\\&\boldsymbol{X}_{2e}^{\mathrm{T}}\begin{bmatrix}A_1\\ A_2\end{bmatrix}+\boldsymbol{X}_{2e}^{\mathrm{T}}\begin{bmatrix}\sin(\alpha_\psi)(-\sin(\alpha_\psi)\Omega_1+\cos(\alpha_\psi)\Omega_2)\\ \cos(\alpha_\psi)(\sin(\alpha_\psi)\Omega_1-\cos(\alpha_\psi)\Omega_2)\end{bmatrix}\end{aligned} \tag{43}$$

where

$$\begin{aligned}A_1=&(\tau_u^*+\Omega_1\cos(\alpha_\psi))\cdot\\&[(\cos(\psi_e)-1)\cos(\alpha_\psi)-\sin(\psi_e)\sin(\alpha_\psi)]\\A_2=&(\tau_u^*+\Omega_2\sin(\alpha_\psi))\cdot\\&[(\cos(\psi_e)-1)\sin(\alpha_\psi)+\sin(\psi_e)\cos(\alpha_\psi)]\end{aligned} \tag{44}$$

and we have used

$$\begin{aligned}\tau_u^*=&\Omega_1\cos(\psi)+\Omega_2\sin(\psi)=\\&\Omega_1\cos(\alpha_\psi)+\Omega_2\sin(\alpha_\psi)+\Omega_1(-\sin(\alpha_\psi)\Omega_1+\\&\cos(\alpha_\psi)\Omega_2)+\Omega_2(\sin(\alpha_\psi)\Omega_1-\cos(\alpha_\psi)\Omega_2)\end{aligned}$$

Substituting Eq. (43) into Eq. (42) yields

$$\begin{aligned}\dot{V}_2=&-\rho_1\boldsymbol{X}_{1e}^{\mathrm{T}}\boldsymbol{K}_1\boldsymbol{X}_{1e}-\boldsymbol{X}_{2e}^{\mathrm{T}}\boldsymbol{K}_2\sigma(\boldsymbol{X}_{2e})+E_1+\\&\boldsymbol{X}_{2e}^{\mathrm{T}}\begin{bmatrix}A_1\\ A_2\end{bmatrix}+\boldsymbol{X}_{2e}^{\mathrm{T}}\begin{bmatrix}\sin(\alpha_\psi)(-\sin(\alpha_\psi)\Omega_1+\cos(\alpha_\psi)\Omega_2)\\ \cos(\alpha_\psi)(\sin(\alpha_\psi)\Omega_1-\cos(\alpha_\psi)\Omega_2)\end{bmatrix}\end{aligned} \tag{45}$$

We now choose α_ψ such that the last term in Eq. (45) is equal to zero, i.e.,

$$\begin{aligned}&-\sin(\alpha_\psi)\Omega_1+\cos(\alpha_\psi)\Omega_2=0\Rightarrow\\&\alpha_\psi=\psi_d+\arctan\left[\frac{-\Omega_1\sin(\psi_d)+\Omega_2\cos(\psi_d)}{\Omega_1\cos(\psi_d)+\Omega_2\sin(\psi_d)}\right]\end{aligned} \tag{46}$$

From (39), we have

$$\Omega_1\cos(\psi_d)+\Omega_2\sin(\psi_d)=\tau_{ud}+(k_{21}\sigma(X_{21e})-\hat{\tau}_{Eu})\cos(\psi_d)+(k_{22}\sigma(X_{22e})-\hat{\tau}_{Ev})\sin(\psi_d)$$

Thus, the condition for $|\Omega_1\cos(\psi_d)+\Omega_2\sin(\psi_d)|>\varepsilon_0^{\lozenge}$ with $\varepsilon_0^{\lozenge}$ being a positive constant, i.e., for α_ψ to be a smooth function, is

$$\|(k_{21}\sigma(X_{21e})-\hat{\tau}_{Eu}),(k_{22}\sigma(X_{22e})-\hat{\tau}_{Ev})\|<|\tau_{ud}|$$

which can be written as

$$\| k_{21} + \max(|\tau_{Eu}^{\max}|,|\tau_{Eu}^{\min}|), k_{22} + \max(|\tau_{Ev}^{\max}|,|\tau_{Ev}^{\min}|) \| < |\tau_{ud}| - \varpi_0^{\lozenge} \tag{47}$$

where $\varpi_0^{\lozenge}$ is a positive constant and we have used the facts that $|\sigma(\bullet)| \le 1$ for all $\bullet \in \mathbb{R}$ and $\hat{\tau}_{Eu} \in [\tau_{Eu}^{\min}, \tau_{Eu}^{\max}]$ and $\hat{\tau}_{Ev} \in [\tau_{Ev}^{\min}, \tau_{Ev}^{\max}]$, which are guaranteed by the disturbance observers (24). Since τ_{ud} is assumed to satisfy the condition (10), there always exist positive constants k_{21} and k_{22} such that the condition (47) holds. The condition (47) also implies that there exists a positive constant ε_0 such that

$$\Omega_1^2 + \Omega_2^2 \ge \varepsilon_0^2 \tag{48}$$

Substituting Eq. (46) into Eq. (42) results in

$$\dot{V}_2 = -\rho_1 \boldsymbol{X}_{1e}^{\mathrm{T}} \boldsymbol{K}_1 \boldsymbol{X}_{1e} - \boldsymbol{X}_{2e}^{\mathrm{T}} \boldsymbol{K}_2 \sigma(\boldsymbol{X}_{2e}) + E_2 \tag{49}$$

where

$$E_2 = E_1 + \boldsymbol{X}_{2e}^{\mathrm{T}} \mathrm{col}(A_1, A_2) := E_1 + E_2^* \tag{50}$$

Substituting Eqs. (40) and (46) into Eq. (32) gives

$$\dot{\boldsymbol{X}}_{2e} = \boldsymbol{M}_l^{-1}[-\boldsymbol{K}_2\sigma(\boldsymbol{X}_{2e}) - \boldsymbol{D}_l(\boldsymbol{X}_{2e} - \boldsymbol{K}_1\boldsymbol{X}_{1e}) + \tilde{\boldsymbol{\tau}}_{El} + \mathrm{col}(A_1, A_2)] + \boldsymbol{K}_1(-\boldsymbol{K}_1\boldsymbol{X}_{1e} + \boldsymbol{X}_{2e}) := \boldsymbol{f}_2 \tag{51}$$

4.3 Stage 2

4.3.1 Step 1

Define

$$r_e = r - \alpha_r \tag{52}$$

where α_r is a virtual control of r. With a note that α_ψ is a smooth function of ψ_d, τ_{ud}, $\hat{\tau}_{El}$, and $\boldsymbol{X}_{2e}$, differentiating both sides of the third equation of (26) along the solutions of the third equations of (22) and (23), (51), and (52) results in

$$\dot{\psi}_e = \alpha_r + r_e - \frac{\partial \alpha_\psi}{\partial \psi_d} r_d - \frac{\partial \alpha_\psi}{\partial \tau_{ud}} \dot{\tau}_{ud} - \frac{\partial \alpha_\psi}{\partial \hat{\tau}_{El}} \dot{\hat{\tau}}_{El} - \frac{\partial \alpha_\psi}{\partial \boldsymbol{X}_{2e}} \boldsymbol{f}_2 \tag{53}$$

where $\boldsymbol{f}_2$ is defined in Eq. (51). To design α_r, we consider the following Lyapunov function candidate

$$V_3 = V_2 + \frac{\rho_2}{2} \psi_e^2 \tag{54}$$

whose derivative along the solutions of (49) and (53) is

$$\dot{V}_3 = -\rho_1 \boldsymbol{X}_{1e}^{\mathrm{T}} \boldsymbol{K}_1 \boldsymbol{X}_{1e} - \boldsymbol{X}_{2e}^{\mathrm{T}} \boldsymbol{K}_2 \sigma(\boldsymbol{X}_{2e}) + \rho_2 \psi_e (\alpha_r - \frac{\partial \alpha_\psi}{\partial \psi_d} r_d - \frac{\partial \alpha_\psi}{\partial \tau_{ud}} \dot{\tau}_{ud}) + E_3 \tag{55}$$

where

$$E_3 = E_2 + \rho_2 \psi_e (r_e - \frac{\partial \alpha_\psi}{\partial \hat{\tau}_{El}} \dot{\hat{\tau}}_{El} - \frac{\partial \alpha_\psi}{\partial \boldsymbol{X}_{2e}} \boldsymbol{f}_2) := E_2 + E_3^* \tag{56}$$

The Eq. (55) suggests that we choose the virtual control α_r as follows

$$\alpha_r = -k_3 \psi_e + \frac{\partial \alpha_\psi}{\partial \psi_d} r_d + \frac{\partial \alpha_\psi}{\partial \tau_{ud}} \dot{\tau}_{ud} \tag{57}$$

where k_3 is a positive constant. Substituting Eq. (57) into Eq. (55) gives

$$\dot{V}_3 = -\rho_1 \boldsymbol{X}_{1e}^{\mathrm{T}} \boldsymbol{K}_1 \boldsymbol{X}_{1e} - \boldsymbol{X}_{2e}^{\mathrm{T}} \boldsymbol{K}_2 \sigma(\boldsymbol{X}_{2e}) - \rho_2 k_3 \psi_e^2 + E_3 \tag{58}$$

Substituting Eq. (57) into Eq. (53) results in

$$\dot{\psi}_e = -k_3 \psi_e + r_e - \frac{\partial \alpha_\psi}{\partial \hat{\tau}_{El}} \dot{\hat{\tau}}_{El} - \frac{\partial \alpha_\psi}{\partial \boldsymbol{X}_{2e}} \boldsymbol{f}_2 := f_3 \tag{59}$$

4.3.2 Step 2

With a note that α_r is a smooth function of ψ_d, r_d, τ_{ud}, $\dot{\tau}_{ud}$, $\hat{\tau}_{El}$, $\boldsymbol{X}_{2e}$, and ψ_e, differentiating both sides of (52) along the solutions of the last equations of (22) and (23), (51), and (59) results in

$$\dot{r}_e = \frac{1}{J}(-d_r r + \tau_{Er} + \tau_r) - \frac{\partial \alpha_r}{\partial \psi_d} r_d - \frac{\partial \alpha_r}{\partial r_d} \frac{1}{J}(-d_r r_d + \tau_{rd}) - \frac{\partial \alpha_r}{\partial \tau_{ud}} \dot{\tau}_{ud} - \frac{\partial \alpha_r}{\partial \dot{\tau}_{ud}} \ddot{\tau}_{ud} - \frac{\partial \alpha_r}{\partial \hat{\tau}_{El}} \dot{\hat{\tau}}_{El} - \frac{\partial \alpha_r}{\partial \boldsymbol{X}_{2e}} \boldsymbol{f}_2 - \frac{\partial \alpha_r}{\partial \psi_e} f_3 \tag{60}$$

To design the actual control τ_r, we consider the following Lyapunov function candidate

$$V_4 = V_3 + \frac{\rho_3}{2} J r_e^2 \tag{61}$$

where ρ_3 is a positive constant. Differentiating both sides of (61) along the solutions of (58) and (60) results in

$$\dot{V}_4 = -\rho_1 \boldsymbol{X}_{1e}^{\mathrm{T}} \boldsymbol{K}_1 \boldsymbol{X}_{1e} - \boldsymbol{X}_{2e}^{\mathrm{T}} \boldsymbol{K}_2 \sigma(\boldsymbol{X}_{2e}) - \rho_2 k_3 \psi_e^2 + \rho_3 r_e [-d_r (\frac{\partial \alpha_\psi}{\partial \psi_d} r_d + \frac{\partial \alpha_\psi}{\partial \tau_{ud}} \dot{\tau}_{ud}) + \hat{\tau}_{Er} + \tau_r - J \frac{\partial \alpha_r}{\partial \psi_d} r_d - \frac{\partial \alpha_r}{\partial r_d}(-d_r r_d + \tau_{rd}) - J \frac{\partial \alpha_r}{\partial \tau_{ud}} \dot{\tau}_{ud} - J \frac{\partial \alpha_r}{\partial \dot{\tau}_{ud}} \ddot{\tau}_{ud}] + E_4 \tag{62}$$

where,

$$E_4 = E_3 + \rho_3 J r_e [\frac{1}{J}(-d_r (r_e - k_3 \psi_e) + \tilde{\tau}_{Er}) - \frac{\partial \alpha_r}{\partial \hat{\tau}_{El}} \dot{\hat{\tau}}_{El} - \frac{\partial \alpha_r}{\partial \boldsymbol{X}_{2e}} \boldsymbol{f}_2 - \frac{\partial \alpha_r}{\partial \psi_e} f_3] := E_3 + E_4^* \tag{63}$$

From Eq. (62), we choose the stabilizing control τ_r as follows:

$$\begin{aligned} &\tau_r = \tau_{r1} + \tau_{r2} \\ &\tau_{r1} = -k_4 r_e \\ &\tau_{r2} = d_r (\frac{\partial \alpha_\psi}{\partial \psi_d} r_d + \frac{\partial \alpha_\psi}{\partial \tau_{ud}} \dot{\tau}_{ud}) - \hat{\tau}_{Er} + J \frac{\partial \alpha_r}{\partial \psi_d} r_d + \\ &\quad \frac{\partial \alpha_r}{\partial r_d}(-d_r r_d + \tau_{rd}) + J \frac{\partial \alpha_r}{\partial \tau_{ud}} \dot{\tau}_{ud} + J \frac{\partial \alpha_r}{\partial \dot{\tau}_{ud}} \ddot{\tau}_{ud} \end{aligned} \tag{64}$$

where k_4 is a positive constant. The control stabilizing τ_r given in Eq. (64) consists of two parts. The part τ_{r1} is designed based on Lemma 3.3 while the part τ_{r2} is to handle $\hat{\tau}_{Er}$, the estimate of disturbance, and the reference signals. An inverse optimal control τ_r^* is obtained from the stabilizing τ_r as follows:

$$\begin{aligned}&\tau_r^* = \tau_{r1}^* + \tau_{r2}\\ &\tau_{r1}^* = \beta_{r1}\tau_{r1}, \beta_{r1} \geq 2\end{aligned} \tag{65}$$

where τ_{r1} and τ_{r2} are defined in Eq. (64). Substituting Eq. (64) into Eq. (62) yields

$$\begin{aligned}\dot{V}_4 = &-\rho_1 \boldsymbol{X}_{1e}^{\mathrm{T}} \boldsymbol{K}_1 \boldsymbol{X}_{1e} - \boldsymbol{X}_{2e}^{\mathrm{T}} \boldsymbol{K}_2 \sigma(\boldsymbol{X}_{2e}) -\\ &\rho_2 k_3 \psi_e^2 - \rho_3 k_4 r_e^2 + E_4\end{aligned} \tag{66}$$

Substituting Eq. (64) into Eq. (60) results in

$$\begin{aligned}\dot{r}_e = &\frac{1}{J}[-k_4 r_e - d_r(r_e - k_3\psi_e) + \tilde{\tau}_{Er}] -\\ &\frac{\partial \alpha_r}{\partial \hat{\tau}_{El}}\dot{\hat{\tau}}_{El} - \frac{\partial \alpha_r}{\partial \boldsymbol{X}_{2e}}\boldsymbol{f}_2 - \frac{\partial \alpha_r}{\partial \psi_e} f_3\end{aligned} \tag{67}$$

The control design has been completed. We present the main results in the following theorem.

Theorem 4.1 Under the assumptions listed in Control Objective 2.1, the controls τ_u^* and τ_r^* given in Eqs. (40) and (65), and update laws for $\hat{\tau}_{El}$ and $\hat{\tau}_{Er}$ given in Eq. (24) solve Control Objective 2.1 as long as the control gains are chosen such that the condition (47) holds, and $k_1^* = \rho_1\lambda_m(\boldsymbol{K}_1) - c_1$, $k_2^* = -c_2$, $k_3^* = \rho_2 k_3 - c_3$, and $k_4^* = \rho_3 k_4 - c_4$ are positive constants with c_i, $i = 1,2,...,5$ defined below:

$$\begin{aligned}
c_1 = &\frac{\lambda_M(\boldsymbol{D}_l\boldsymbol{K}_1 + \boldsymbol{M}_l\boldsymbol{K}_1^2 + \rho_1\boldsymbol{I})}{4\varepsilon_{11}} + \frac{\rho_2(k_{21}+k_{22})}{m\varepsilon_0}\varepsilon_{34} + \frac{3\rho_3 J(k_{21}+k_{22})(\tau_{ud}^{\max} r_d^{\max} + \dot{\tau}_{ud}^{\max})}{m\varepsilon_0^2}\varepsilon_{47} + \frac{\rho_3 J k_3(k_{21}+k_{22})}{m\varepsilon_0^2}\varepsilon_{47}\\
c_2 = &-[\lambda_m(\boldsymbol{D}_l - \boldsymbol{M}_l\boldsymbol{K}_1) - \lambda_M(\boldsymbol{D}_l\boldsymbol{K}_1 + \boldsymbol{M}_l\boldsymbol{K}_1^2 + \rho_1\boldsymbol{I})\varepsilon_{11} - \varepsilon_{12}] + \varepsilon_{21} + \frac{\rho_2(k_{21}+k_{22})}{m\varepsilon_0}\varepsilon_{33} +\\
&\frac{3\rho_3 J(k_{21}+k_{22})(\tau_{ud}^{\max} r_d^{\max} + \dot{\tau}_{ud}^{\max})}{m\varepsilon_0^2}\varepsilon_{46} + \frac{\rho_3 J k_3(k_{21}+k_{22})}{m\varepsilon_0^2}\varepsilon_{46}\\
c_3 = &\frac{(\tau_u^{*\max})^2}{2\varepsilon_{21}} + 2\rho_2\varepsilon_{31} + \frac{\rho_2(k_{21}+k_{22})}{m\varepsilon_0}2\tau_u^{*\max} + \varepsilon_{32} + \frac{\lambda_M^2(\boldsymbol{K}_2 + \boldsymbol{D}_l + m\boldsymbol{K}_1)}{4\varepsilon_{33}} + \frac{\lambda_M(\boldsymbol{D}_l\boldsymbol{K}_1 + m\boldsymbol{K}_1^2)}{4\varepsilon_{34}} +\\
&\rho_3 d_r k_3 \varepsilon_{41} + \frac{3\rho_3 J(k_{21}+k_{22})(\tau_{ud}^{\max} r_d^{\max} + \dot{\tau}_{ud}^{\max})}{m\varepsilon_0^2}\varepsilon_{44} + \frac{\rho_3 J k_3(k_{21}+k_{22})}{m\varepsilon_0^2}\varepsilon_{44}\\
c_4 = &\frac{\rho_2}{4\varepsilon_{31}} - \rho_3 d_r + \frac{\rho_3 d_r k_3}{4\varepsilon_{41}} + \rho_3 J\varepsilon_{42} + \frac{6(r_d^{\max} + \dot{\tau}_{ud}^{\max})}{\varepsilon_0^2} \times \lambda_M(\boldsymbol{K}_{El}D_{El})\rho_3 J\varepsilon_{43} + \frac{3\rho_3 J(k_{21}+k_{22})(\tau_{ud}^{\max} r_d^{\max} + \dot{\tau}_{ud}^{\max})}{m\varepsilon_0^2}\times\\
&[\frac{\tau_u^{*\max}}{\varepsilon_{44}} + \varepsilon_{45} + \frac{\lambda_M^2(\boldsymbol{K}_2 + \boldsymbol{D}_l + m\boldsymbol{K}_1)}{4\varepsilon_{46}} + \frac{\lambda_M^2(\boldsymbol{D}_l\boldsymbol{K}_1 + m\boldsymbol{K}_1^2)}{4\varepsilon_{47}}] + \rho_3 J k_3[\frac{k_3^2}{4\varepsilon_{48}} + (1+\varepsilon_{49})] +\\
&\frac{\rho_3 J k_3(k_{21}+k_{22})}{m\varepsilon_0^2}[\frac{\tau_u^{*\max}}{\varepsilon_{44}} + \varepsilon_{46} + \varepsilon_{45} + \frac{\lambda_M^2(\boldsymbol{K}_2 + \boldsymbol{D}_l + m\boldsymbol{K}_1)}{4\varepsilon_{46}} + \frac{\lambda_M^2(\boldsymbol{D}_l\boldsymbol{K}_1 + m\boldsymbol{K}_1^2)}{4\varepsilon_{47}}]
\end{aligned} \tag{68}$$

where $\lambda_m(\bullet)$ and $\lambda_M(\bullet)$ denote the minimum and maximum eigenvalues of $\bullet$, respectively, and ε_{ij} with $i = 1,2,...,4$ and $j = 1,2,...,9$ are positive constants. The conditions (47) hold, and $k_1^* = \rho_1\lambda_m(\boldsymbol{K}_1) - c_1$, $k_2^* = -c_2$, $k_3^* = \rho_2 k_3 - c_3$, and $k_4^* = \rho_3 k_4 - c_4$ are made to be positive constants by choosing sufficiently small $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$, and sufficiently large k_3 and k_4. Particularly, the below results hold:

1) The closed loop system consisting of (25), (31), (51), (59) and (67) is forward complete.

2) All the parameter estimates are within their limits, i.e., $\hat{\tau}_{Eu}(t) \in [\tau_{Eu}^{\min}, \tau_{Eu}^{\max}]$, $\hat{\tau}_{Ev}(t) \in [\tau_{Ev}^{\min}, \tau_{Ev}^{\max}]$, and $\hat{\tau}_{Er}(t) \in [\tau_{Er}^{\min}, \tau_{Er}^{\max}]$ for all $t \geq t_0$.

3) All the tracking errors $\boldsymbol{X}_{1e}(t)$, $\boldsymbol{X}_{2e}(t)$, $\psi_e(t)$, $r_e(t)$, and the estimate errors $\tilde{\tau}_{El}(t)$ and $\tilde{\tau}_{Er}(t)$ globally asymptotically and locally exponentially converge to zero.

4) The controls τ_u^* and τ_r^* are optimal in the sense that a meaningful cost function, see Appendix C.3, penalizing on the tracking errors and the controls is minimized.

Proof. See Appendix C.

5 Extension to output-feedback control design

In Section 4, all the states of the ODIN were assumed to be available for the control design. In this section, we assume that only position and yaw angle vector $\boldsymbol{\eta}$ is available for feedback. As such, we apply Lemma 3.2 to design observers to estimate the states $\boldsymbol{\nu}$ (via $\boldsymbol{X}_2$ and r), and the disturbances τ_{El} and τ_{Er} as follows

$$\begin{cases}\hat{\tau}_{El} = D_{El}\sigma(\xi_{El} + \boldsymbol{K}_{1El}\boldsymbol{X}_1 + \boldsymbol{K}_{2El}\boldsymbol{M}_l\hat{\boldsymbol{X}}_2) + \delta_{El}\\ \dot{\xi}_{El} = -\boldsymbol{K}_{2El}[D_{El}\sigma(\xi_{El} + \boldsymbol{K}_1\boldsymbol{X}_{1El} + \boldsymbol{K}_{2El}\boldsymbol{M}_l\hat{\boldsymbol{X}}_2) + \delta_{El}] -\\ \qquad \boldsymbol{K}_{2El}[-\boldsymbol{D}_l\hat{\boldsymbol{X}}_2 + \boldsymbol{J}_l(\psi)\tau_l] - \boldsymbol{K}_{1El}\hat{\boldsymbol{X}}_2\\ \dot{\hat{\boldsymbol{X}}}_2 = \boldsymbol{M}_l^{-1}[-\boldsymbol{D}_l\hat{\boldsymbol{X}}_2 + \hat{\tau}_{El} + \boldsymbol{J}_l(\psi)\tau_l]\end{cases} \tag{69a}$$

$$\begin{cases} \hat{\tau}_r = \Delta_{Er}\sigma(\xi_r + k_{1Er}\psi + k_{2Er}J\hat{r}) + \delta_{Er} \\ \dot{\xi}_r = -k_{2Er}[\Delta_{Er}\sigma(\xi_r + k_{1Er}\psi + k_{2Er}J\hat{r}) + \delta_{Er}] - \\ \quad k_{2Er}(-d_r\hat{r} + \tau_r) - k_{1Er}\hat{r} \\ \dot{\hat{r}} = -\dfrac{1}{J}(-d_r\hat{r} + \hat{\tau}_{Er} + \tau_r) \end{cases} \tag{69b}$$

where D_{El} and δ_{El} are defined just below Eq. (24), $\boldsymbol{K}_{1El}$ and $\boldsymbol{K}_{2El}$ are diagonal positive definite matrices, and k_{1Er} and k_{2Er} are positive constants. Let $\tilde{\tau}_{El} = \tau_{El} - \hat{\tau}_{El}$, $\tilde{\boldsymbol{X}}_2 = \boldsymbol{X}_2 - \hat{\boldsymbol{X}}_2$, $\tilde{\tau}_{Er} = \tau_{Er} - \hat{\tau}_{Er}$, and $\tilde{r} = r - \hat{r}$. It is obvious that

$$\begin{aligned} &\dot{\tilde{\tau}}_{El} = -\boldsymbol{K}_{2El}D_{El}\sigma'(\xi_{El} + \boldsymbol{K}_{1El}\boldsymbol{X}_1 + \boldsymbol{K}_{2El}\boldsymbol{M}_l\hat{\boldsymbol{X}}_2)\tilde{\boldsymbol{X}}_2 \\ &\dot{\tilde{\boldsymbol{X}}}_2 = -\boldsymbol{M}_l^{-1}(-\boldsymbol{D}_l\tilde{\boldsymbol{X}}_2 + \tilde{\tau}_{El}), \\ &\dot{\tilde{\tau}}_{Er} = -k_{2Er}\Delta_{Er}\sigma'(\xi_r + k_{1Er}\psi + k_{2Er}J\hat{r})\tilde{r} \\ &\dot{\tilde{r}} = -\frac{1}{J}(-d_r\tilde{r} + \tilde{\tau}_{Er}) \end{aligned} \tag{70}$$

Lemma 3.2 shows that the estimate errors $\tilde{\tau}_{El}(t)$, $\tilde{\boldsymbol{X}}_2(t)$, $\tilde{\tau}_{Er}(t)$, and $\tilde{r}(t)$ globally asymptotically and locally exponentially converge to zero. Therefore with an observation that all the virtual and actual controls designed in Section 4 are either bounded or linearly dominated, the state-feedback control design in Section 4 is directly applied to the output-feedback case with the ODIN's equations of motion (22) are replaced by

$$\begin{aligned} &\text{P)}\begin{cases} \dot{\boldsymbol{X}}_1 = \hat{\boldsymbol{X}}_2 + \tilde{\boldsymbol{X}}_2 \\ \dot{\hat{\boldsymbol{X}}}_2 = \boldsymbol{M}_l^{-1}[-\boldsymbol{D}_l\hat{\boldsymbol{X}}_2 + \hat{\tau}_{El} + \boldsymbol{J}_l(\psi)\tau_l] \end{cases} \\ &\text{H)}\begin{cases} \dot{\psi} = \hat{r} + \tilde{r} \\ \dot{\hat{r}} = \dfrac{1}{J}(-d_r\hat{r} + \hat{\tau}_{Er} + \tau_r) \end{cases} \end{aligned} \tag{71}$$

6 Simulations

In this section, we present some simulation results to illustrate the effectiveness of the output-feedback control design outlined in the previous section. The ODIN's parameters are taken as m_{RB}=125 kg, m_A=62.5 kg, J_{RB}=8 kg/m^2, J_A=4 kg/m^2, d_l =468 m/s^2, and d_r =30 kg/(s·m^2). The reference trajectory η_d is generated by Eq. (9) with the initial values $\boldsymbol{\eta}_d(0) = \text{col}(0,0,0)$ and $\boldsymbol{v}_d(0) = \text{col}(0,0,0)$. The reference force τ_{ud} is chosen as $\tau_{ud} = 10(m_{RB} + m_A)$ and τ_{rd} is chosen such that $\tau_{rd} = 0$ for $t \le 12$ s and $\tau_{rd} = 1.33(J_{RB} + J_A)$ for $t > 12$ s. This means that the reference trajectory is a straight-line for $t \le 12$ s and is a circle for $t > 12$ s. The initial values of the ODIN are $\boldsymbol{\eta}(0) = \text{col}(-5,5,0.5)$ and $\boldsymbol{v} = \text{col}(0,0,0)$. The waves, wind and ocean currents are assumed such that

$$\tau_{Eu} = \frac{1}{3}(\tau_{Eu}^{\min} + \tau_{Eu}^{\max}),\ \tau_{Ev} = \frac{1}{3}(\tau_{Ev}^{\min} + \tau_{Ev}^{\max}), \tau_{Er} = \frac{1}{3}(\tau_{Er}^{\min} + \tau_{Er}^{\max})$$

with

$$\begin{aligned} &\tau_{Eu}^{\min} = 0.5(m_{RB} + m_A),\quad \tau_{Eu}^{\max} = 1.5(m_{RB} + m_A) \\ &\tau_{Ev}^{\min} = 0.2(m_{RB} + m_A),\quad \tau_{Ev}^{\max} = 0.8(m_{RB} + m_A) \\ &\tau_{Er}^{\min} = 0.5(J_{RB} + J_A),\quad \tau_{Er}^{\max} = 1.5(J_{RB} + J_A) \end{aligned}$$

The control and update gains are chosen as $\beta_{u1} = 2$, $\beta_{r1} = 2$, $\boldsymbol{K}_1 = 0.5\boldsymbol{I}_{2\times2}$, $\boldsymbol{K}_2 = \boldsymbol{I}_{2\times2}$, $k_3 = 4$, $k_4 = 8$, $\boldsymbol{K}_{1El} = \boldsymbol{K}_{2El} = 5\boldsymbol{I}_{2\times2}$, and $k_{1Er} = k_{2Er} = 5$. The saturation function $\sigma(\bullet)$ is chosen as $\tanh(\bullet)$. It is checked that the condition (47) holds, the constants defined in Eq. (68) are positive. Simulation results are plotted in Fig. 2. It is seen from Figs. 2(a)–2(h) that all the tracking and observer errors converge to zero and that all the parameter estimates are within their pre-specified ranges thanks to the state and disturbance observer (69). Fig. 2(i) plots the cost function W, which is minimized by the proposed control design, given by

$$\begin{aligned} W = &-2\beta\frac{\partial V}{\partial \boldsymbol{x}}[\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})] + \beta(\beta-2)\frac{\partial V}{\partial \boldsymbol{x}}\times \\ &\boldsymbol{G}(\boldsymbol{x})\boldsymbol{R}^{-1}(\boldsymbol{x})[\frac{\partial V}{\partial \boldsymbol{x}}\boldsymbol{G}(\boldsymbol{x})]^{\mathrm{T}} + \boldsymbol{u}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{R}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}) \end{aligned} \tag{72}$$

i.e., the function inside the integral (19), where $\beta = \beta_{u1} = \beta_{r1}$, and $(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{G}(\boldsymbol{x}), \boldsymbol{u}(\boldsymbol{x}))$ are defined in Eqs. (90) and (93), see Appendix C.3. It is seen from Fig. 2(i) that the cost function W converges to a non-zero value, which represents the value due to the controls τ_{u2} and τ_{r2}, see Eqs. (40) and (64). As mentioned in Remark 4.1 and the paragraph just below Eq. (64) in Section 4, the controls τ_{u2} and τ_{r2} are to handle the disturbance estimates and reference signals.

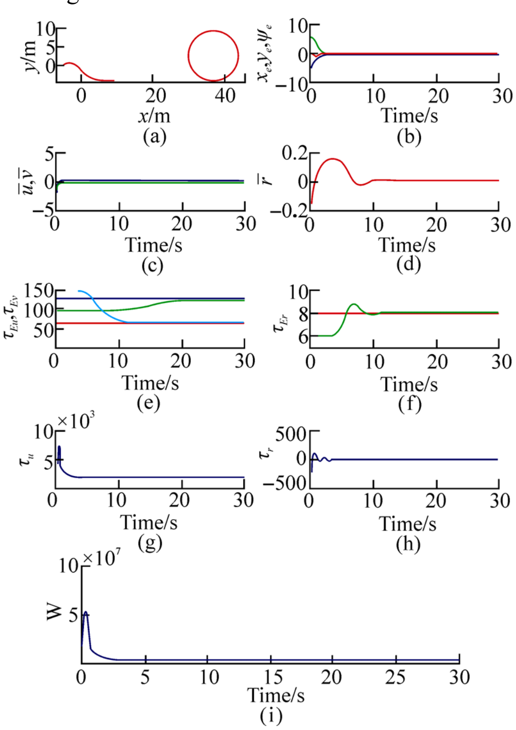

Fig. 2 Simulation results under the proposed output-feedback control design

7 Conclusions

This paper has designed both state- and output-feedback inverse optimal trajectory tracking controllers for an underactuated ODIN under unknown constant environmental loads. The keys are to the success of the proposed control designs include 1) bounded disturbance and state observers, 2) the use of the yaw angle regarded as a virtual control, and 3) the design of non-canceling virtual and actual controls. The results of this paper motivate redesign of existing controllers in these studies (Fossen, 2002; Antonelli, 2006; Antonelli *et al.*, 2001; Do *et al.*, 2002b; 2004a; Do, 2013; Zhang *et al.*, 2000; Jiang, 2002; Lefeber *et al.*, 2003; Pettersen and Nijmeijer, 2001), for (underactuated) ocean vehicles so that optimality can be achieved.

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Appendix A: Proof of Lemma 3.1

We first show that $|\xi(t)+kx_2(t)|$ is bounded for all $t \ge t_0 \ge 0$. As such, let $X = \xi + kx_2$ whose derivative along the solutions of (12) and (13) satisfies

$$\dot{X} = -k\frac{\theta^{\max}-\theta^{\min}}{2}\sigma(X) + k(\theta - \frac{\theta^{\max}+\theta^{\min}}{2}) \quad \text{(A1)}$$

Since $\theta \in (\theta^{\min}, \theta^{\max})$, we have $|\theta - \frac{\theta^{\max}+\theta^{\min}}{2}| < \frac{\theta^{\max}-\theta^{\min}}{2}$. Using this inequality plus the fact that $\sigma(X)$ is a smooth saturation function as defined in Definition 3.1, it is seen from Eq. (A1) that $|X(t)|$ is bounded for all $t \ge t_0 \ge 0$ and that $|X(t)|$ converges to a constant less that 1. Now, differentiating $\tilde{\theta}(t)$ along the solutions of (12) and (13) gives

$$\begin{aligned}\dot{\tilde{\theta}} = \frac{\theta^{\max}-\theta^{\min}}{2}\sigma'(\xi+kx_2)\{-k[\frac{\theta^{\max}-\theta^{\min}}{2}\sigma(\xi+kx_2) + \\ \frac{\theta^{\max}+\theta^{\min}}{2}] - kf(x_1,x_2,u) + k(f(x_1,x_2,u)+\theta)\} = \quad \text{(A2)} \\ -k\frac{\theta^{\max}-\theta^{\min}}{2}\sigma'(\xi+kx_2)\tilde{\theta}\end{aligned}$$

Since $|\xi(t)+kx_2(t)|$ is bounded for all $t \ge t_0 \ge 0$, we have $\sigma'(\xi+kx_2) < 0$ for all $t \ge t_0 \ge 0$. Thus, the last equation of (A2) yields global asymptotic convergence of $\tilde{\theta}(t)$ to zero. Local exponential convergence of $\tilde{\theta}(t)$ to zero follows by using the fact that there exists a constant $\delta > 0$ such that $\sigma'(\xi+kx_2) < \delta$ for $t \ge T$, where T is a constant larger or equal to t_0 since we have already proved that $|X(t)|$ is bounded for all $t \ge t_0 \ge 0$ and that $|X(t)|$ converges to a constant less that 1. Finally, since $|\sigma(\xi(t)+kx_2(t))| \le 1$, the first equation of (13) ensures that $\hat{\theta} \in [\theta^{\min}, \theta^{\max}]$. □

Appendix B: Proof of Lemma 3.2

The first equation of (15) ensures that $\hat{\theta} \in [\theta^{\min}, \theta^{\max}]$. Let $X = \xi + k_1x_1 + k_2\hat{x}_2$. Differentiating $\tilde{\theta}$, $\tilde{x}_2$ and X along the solutions of (14) and (15) yields

$$\begin{aligned}&\dot{\tilde{\theta}} = -k_1\frac{\theta^{\max}-\theta^{\min}}{2}\sigma'(\xi+k_1x_1+k_2\hat{x}_2)\tilde{x}_2 \\ &\dot{\tilde{x}}_2 = -a\tilde{x}_2 + \tilde{\theta} \qquad \text{(B1)} \\ &\dot{X} = -k_2\frac{\theta^{\max}-\theta^{\min}}{2}\sigma(X) - k_2\frac{\theta^{\max}+\theta^{\min}}{2} + k_2\hat{\theta} + k_1\tilde{x}_2\end{aligned}$$

The following change of variables

$$\begin{aligned}&\varphi = \sigma^{-1}\left[\frac{2}{\theta^{\max}-\theta^{\min}}(\theta - \frac{\theta^{\max}+\theta^{\min}}{2})\right] \\ &\hat{\varphi} = \sigma^{-1}\left[\frac{2}{\theta^{\max}-\theta^{\min}}(\hat{\theta} - \frac{\theta^{\max}+\theta^{\min}}{2})\right]\end{aligned} \qquad \text{(B2)}$$

where $\sigma^{-1}(\bullet)$ denotes the inverse function of $\sigma(\bullet)$, transforms Eq. (B1) to

$$\begin{aligned}&\dot{\tilde{\varphi}} = -k_1\tilde{x}_2 \\ &\dot{\tilde{x}}_2 = -a\tilde{x}_2 + \frac{\theta^{\max}-\theta^{\min}}{2}[\sigma(\varphi)-\sigma(\hat{\varphi})] \qquad \text{(B3)} \\ &\dot{X} = -k_2\frac{\theta^{\max}-\theta^{\min}}{2}\sigma(X) - k_2\frac{\theta^{\max}+\theta^{\min}}{2} + k_2\hat{\theta} + k_1\tilde{x}_2\end{aligned}$$

where $\tilde{\varphi} = \varphi - \hat{\varphi}$, $\frac{\mathrm{d}\sigma^{-1}(\bullet)}{\mathrm{d}\bullet} = \frac{1}{\sigma'(\sigma^{-1}(\bullet))}$ and the first equation of (15) have been used. Consider the following Lyapunov function candidate

$$V_1 = \frac{\theta^{\max}-\theta^{\min}}{2k_1}\int_0^{\tilde{\varphi}}\frac{\sigma(\varphi)-\sigma(\hat{\varphi})}{\varphi-\hat{\varphi}}\chi\mathrm{d}\chi + \frac{1}{2}\tilde{x}_2^2 + \frac{1}{2}X^2 \quad \text{(B4)}$$

whose derivative along the solutions of (B3) satisfies

$$\begin{aligned}\dot{V}_1 &= -a\tilde{x}_2^2 - k_2\frac{\theta^{\max}-\theta^{\min}}{2}\sigma(X)X + \\ & X(-k_2\frac{\theta^{\max}+\theta^{\min}}{2} + k_2\hat{\theta}) + k_1\tilde{x}_2 X \le \\ & k_2\frac{\theta^{\max}-\theta^{\min}}{2}|X| + k_2\frac{\theta^{\max}-\theta^{\min}}{2}|X| + k_1|\tilde{x}_2||X| \le \\ & \frac{1}{2}X^2 + k_1^2\tilde{x}_2^2 + k_2^2(\theta^{\max}-\theta^{\min})^2 \le \varepsilon_1 V_1 + \varepsilon_2\end{aligned} \tag{B5}$$

where $\varepsilon_1 = \max(2k_1^2, 1)$, $\varepsilon_2 = k_2^2(\theta^{\max}-\theta^{\min})^2$, and we have used $|\hat{\theta} - \frac{\theta^{\max}+\theta^{\min}}{2}| \le \frac{\theta^{\max}-\theta^{\min}}{2}$. Due to Eqs. (B4) and (B5), the system (B3) is well defined. Now, consider the Lyapunov function candidate

$$V_2 = \frac{\theta^{\max}-\theta^{\min}}{2k_1}\int_0^{\tilde{\varphi}}\frac{\sigma(\varphi)-\sigma(\hat{\varphi})}{\varphi-\hat{\varphi}}\chi\,\mathrm{d}\chi + \frac{1}{2}\tilde{x}_2^2 \tag{B6}$$

whose derivative along the solutions of the first two equations of (B3) is $\dot{V}_2 = -a\tilde{x}_2^2$. Global asymptotic and local exponential convergence of $\tilde{\varphi}(t)$ and $\tilde{x}_2(t)$ to zero follows from the expression of V_2, $\dot{V}_2 = -a\tilde{x}_2^2$, Barbalat's lemma, and linearization of the first two equations of (B3) around the origin. This in turn implies global asymptotic and local convergence of $\tilde{\theta}(t)$ to zero from Eq. (B2) and the fact that the smooth saturation function $\sigma(\bullet) = 0$ if only $\bullet = 0$. Since we have already proved that $\tilde{x}_2(t)$ and $\tilde{\theta}(t)$ globally asymptotically and locally exponentially converge to zero, the proof of boundedness of $|\xi(t) + k_1 x_1(t) + k_2\hat{x}_2(t)|$ follows the same lines as in that of Lemma 3.1 using the last equation of (B3). □

Appendix C: Proof of Theorem 4.1

To prove Theorem 4.1, we need to calculate the upper bound of E_4 defined in Eq. (63). To do so, we calculate the following partial derivatives:

$$\begin{aligned}\frac{\partial\alpha_\psi}{\partial\bullet} &= \frac{-\frac{\partial\Omega_1}{\partial\bullet}\Omega_2 + \frac{\partial\Omega_2}{\partial\bullet}\Omega_1}{\Omega_1^2+\Omega_2^2} \\ \frac{\partial\alpha_r}{\partial\Diamond} &= \frac{\partial}{\partial\Diamond}(\frac{\partial\alpha_\psi}{\partial\psi_d})r_d + \frac{\partial}{\partial\Diamond}(\frac{\partial\alpha_\psi}{\partial\tau_{ud}})\dot{\tau}_{ud}\end{aligned} \tag{C1}$$

where $\bullet$ stands for $\hat{\tau}_{El}$, $\boldsymbol{X}_{2e}$, ψ_d, and τ_{ud}, and $\Diamond$ stands for $\boldsymbol{X}_{2e}$ and $\hat{\tau}_{El}$. Using Eq. (C1) and completion of squares, a tedious but simple calculation results in upper-bounds of E_1, and E_i^*, $i = 2,...,4$ as follows:

$$\begin{aligned}E_1 &\le -[\lambda_m(\boldsymbol{D}_l - \boldsymbol{M}_l\boldsymbol{K}_1) - \lambda_M(\boldsymbol{D}_l\boldsymbol{K}_1 + \boldsymbol{M}_l\boldsymbol{K}_1^2 + \rho_1\boldsymbol{I})\varepsilon_{11} - \varepsilon_{12}]\times\|\boldsymbol{X}_{2e}\|^2 + \frac{\lambda_M(\boldsymbol{D}_l\boldsymbol{K}_1 + \boldsymbol{M}_l\boldsymbol{K}_1^2 + \rho_1\boldsymbol{I})}{4\varepsilon_{11}}\|\boldsymbol{X}_{1e}\|^2 + \frac{1}{4\varepsilon_{42}}\|\tilde{\tau}_{El}\|^2 \\ E_2^* &\le \varepsilon_{21}\|\boldsymbol{X}_{2e}\|^2 + \frac{(\tau_u^{*\max})^2}{2\varepsilon_{21}}\psi_e^2 \\ E_3^* &\le 2\rho_2\varepsilon_{31}\psi_e^2 + \frac{\rho_2}{4\varepsilon_{31}}r_e^2 + \frac{\rho_2}{\varepsilon_0^2\varepsilon_{31}}\lambda_M(\boldsymbol{K}_{El}D_{El})\|\tilde{\tau}_{El}\|^2 + \frac{\rho_2(k_{21}+k_{22})}{m\varepsilon_0}\times[2\tau_u^{*\max}\psi_e^2 + \varepsilon_{32}\psi_e^2 + \\ &\quad \frac{1}{4\varepsilon_{32}}\|\tilde{\tau}_{El}\|^2 + \varepsilon_{33}\|\boldsymbol{X}_{2e}\|^2 + \varepsilon_{34}\|\boldsymbol{X}_{1e}\|^2 + \frac{\lambda_M^2(\boldsymbol{K}_2 + \boldsymbol{D}_l + m\boldsymbol{K}_1)}{4\varepsilon_{33}}\psi_e^2 + \frac{\lambda_M(\boldsymbol{D}_l\boldsymbol{K}_1 + m\boldsymbol{K}_1^2)}{4\varepsilon_{34}}\psi_e^2] \\ E_4^* &\le -\rho_3 d_r r_e^2 + \rho_3 d_r k_3(\varepsilon_{41}\psi_e^2 + \frac{r_e^2}{4\varepsilon_{41}}) + \rho_3 J(\varepsilon_{42}r_e^2 + \frac{\tilde{\tau}_{Er}^2}{4\varepsilon_{42}}) + \frac{6(r_d^{\max} + \dot{\tau}_{ud}^{\max})}{\varepsilon_0^2}\lambda_M(\boldsymbol{K}_{El}D_{El})\rho_3 J(\varepsilon_{43}r_e^2 + \frac{1}{\varepsilon_{43}}\|\tilde{\tau}_{El}\|^2) + \\ &\quad \frac{3\rho_3 J(k_{21}+k_{22})(\tau_{ud}^{\max}r_d^{\max} + \dot{\tau}_{ud}^{\max})}{m\varepsilon_0^2}[\varepsilon_{44}\psi_e^2 + \frac{\tau_u^{*\max}}{\varepsilon_{44}}r_e^2 + \varepsilon_{45}r_e^2 + \frac{1}{4\varepsilon_{45}}\|\tilde{\tau}_{El}\|^2 + \varepsilon_{46}\|\boldsymbol{X}_{2e}\|^2 + \frac{\lambda_M^2(\boldsymbol{K}_2 + \boldsymbol{D}_l + m\boldsymbol{K}_1)}{4\varepsilon_{46}}r_e^2 + \\ &\quad \varepsilon_{47}\|\boldsymbol{X}_{1e}\|^2 + \frac{\lambda_M^2(\boldsymbol{D}_l\boldsymbol{K}_1 + m\boldsymbol{K}_1^2)}{4\varepsilon_{47}}r_e^2] + \rho_3 Jk_3[\varepsilon_{48}\psi_e^2 + \frac{k_3^2 r_e^2}{4\varepsilon_{48}} + (1+\varepsilon_{49})r_e^2 + \frac{\lambda_M^2(\boldsymbol{K}_{El}D_{El})}{\varepsilon_0^2\varepsilon_{49}}\|\tilde{\tau}_{El}\|^2 + \varepsilon_{46}\|\boldsymbol{X}_{2e}\|^2] + \\ &\quad \frac{\rho_3 Jk_3(k_{21}+k_{22})}{m\varepsilon_0^2}\times[\varepsilon_{44}\psi_e^2 + \frac{\tau_u^{*\max}}{\varepsilon_{44}}r_e^2 + \varepsilon_{45}r_e^2 + \frac{1}{\varepsilon_{45}}\|\tilde{\tau}_{El}\|^2 + \varepsilon_{46}\|\boldsymbol{X}_{2e}\|^2 + \frac{\lambda_M^2(\boldsymbol{K}_2 + \boldsymbol{D}_l + m\boldsymbol{K}_1)}{4\varepsilon_{46}}r_e^2 + \\ &\quad \varepsilon_{47}\|\boldsymbol{X}_{1e}\|^2 + \frac{\lambda_M^2(\boldsymbol{D}_l\boldsymbol{K}_1 + m\boldsymbol{K}_1^2)}{4\varepsilon_{47}}r_e^2]\end{aligned} \tag{C2}$$

By definition, $E_4 = E_1 + \sum_{i=2}^4 E_i^*$. Thus, we have from Eq. (C2) that

$$\begin{aligned}E_4 &\le c_1\|\boldsymbol{X}_{1e}\|^2 + c_2\|\boldsymbol{X}_{2e}\|^2 + c_3\psi_e^2 + \\ &\quad c_4 r_e^2 + c_5\|\tilde{\tau}_{El}\| + c_6\tilde{\tau}_{Er}^2\end{aligned} \tag{C3}$$

where c_i with $i = 1,2,...,4$ are given in (68), and

$$\begin{aligned}c_5 &= \frac{1}{4\varepsilon_{42}} + \frac{\rho_2}{\varepsilon_0^2\varepsilon_{31}}\lambda_M(\boldsymbol{K}_{El}D_{El}) + \frac{\rho_2(k_{21}+k_{22})}{m\varepsilon_0}\frac{1}{4\varepsilon_{32}} + \\ &\quad \frac{6(r_d^{\max} + \dot{\tau}_{ud}^{\max})}{\varepsilon_0^2}\lambda_M(\boldsymbol{K}_{El}D_{El})\rho_3 J\frac{1}{\varepsilon_{43}} + \frac{\rho_3 Jk_3(k_{21}+k_{22})}{m\varepsilon_0^2}\frac{1}{\varepsilon_{45}} + \\ &\quad \frac{3\rho_3 J(k_{21}+k_{22})(\tau_{ud}^{\max}r_d^{\max} + \dot{\tau}_{ud}^{\max})}{m\varepsilon_0^2}\frac{1}{4\varepsilon_{45}} + \rho_3 Jk_3\frac{\lambda_M^2(\boldsymbol{K}_{El}D_{El})}{\varepsilon_0^2\varepsilon_{49}} \\ c_6 &= \frac{\rho_3 J}{4\varepsilon_{42}}\end{aligned} \tag{C4}$$

Substituting Eq. (C3) into Eq. (66):

$$\begin{aligned}\dot{V}_4 \quad &\le -k_1^* \| \boldsymbol{X}_{1e} \|^2 - \boldsymbol{X}_{2e}^{\mathrm{T}} \boldsymbol{K}_2 \sigma(\boldsymbol{X}_{2e}) - k_2^* \| \boldsymbol{X}_{2e} \|^2 - \\ &k_3^* \psi_e^2 - k_4^* r_e^2 + c_5 \| \tilde{\boldsymbol{\tau}}_{El} \|^2 + c_6 \tilde{\tau}_{Er}^2\end{aligned} \tag{C5}$$

C.1 Forward completeness of the closed loop system and boundedness of parameter estimates

We consider the following Lyapunov function candidate

$$V_\Sigma = V_4 + \frac{1}{2} \| \tilde{\boldsymbol{\tau}}_{El} \|^2 + \frac{1}{2} \tilde{\tau}_{Er}^2 \tag{C6}$$

whose derivative along the solutions of (C5) and (25) satisfies

$$\begin{aligned}\dot{V}_\Sigma &\le -k_1^* \| \boldsymbol{X}_{1e} \|^2 - \boldsymbol{X}_{2e}^{\mathrm{T}} \boldsymbol{K}_2 \sigma(\boldsymbol{X}_{2e}) - k_2^* \| \boldsymbol{X}_{2e} \|^2 - k_3^* \psi_e^2 - \\ &k_4^* r_e^2 + c_5 \| \tilde{\boldsymbol{\tau}}_{El} \|^2 + c_6 \tilde{\tau}_{Er}^2 - \tilde{\boldsymbol{\tau}}_{El}^{\mathrm{T}} \boldsymbol{K}_{El} D_{El} \sigma'(\xi_{El} + \\ &\boldsymbol{K}_{El} \boldsymbol{M}_l \boldsymbol{X}_2) \tilde{\boldsymbol{\tau}}_{El} - k_{Er} \Delta_{Er} \sigma'(\xi_r + k_{Er} Jr) \tilde{\tau}_{Er}^2 \le \\ &c_5 \| \tilde{\boldsymbol{\tau}}_{El} \|^2 + c_6 \tilde{\tau}_{Er}^2 \le \alpha V_\Sigma\end{aligned} \tag{C7}$$

where $\alpha = 2\max(c_5, c_6)$, and we have used Property 3) of the saturation function, see (11), i.e., $\sigma'(\xi_{El} + \boldsymbol{K}_{El} \boldsymbol{M}_l \boldsymbol{X}_2)$ is nonnegative positive definite and $\sigma'(\xi_r + k_{Er} Jr) > 0$. Thus, the closed loop system consisting of (25), (31), (51), (59) and (67) is forward complete. Since all the parameter estimates are designed as in Eq. (24), Lemma 3.1 ensures that they are within their limits, i.e., $\hat{\tau}_{Eu}(t) \in [\tau_{Eu}^{\min}, \tau_{Eu}^{\max}]$, $\hat{\tau}_{Ev}(t) \in [\tau_{Ev}^{\min}, \tau_{Ev}^{\max}]$ and $\hat{\tau}_{Er}(t) \in [\tau_{Er}^{\min}, \tau_{Er}^{\max}]$ for all $t \ge t_0$.

C.2 Convergence of tracking and estimate errors

Since we have already proved that the closed loop system consisting of (25), (31), (51), (59) and (67) is forward complete, we can now consider the tracking error system consisting of (31), (51), (59) and (67), and the estimate error system (25) separately. Proof of Lemma 3.2 shows that the estimate errors $\tilde{\boldsymbol{\tau}}_{El}(t)$ and $\tilde{\tau}_{Er}(t)$ globally asymptotically and locally exponentially converge to zero. Thus, there exist class $\mathcal{K}$ functions $\gamma_{El}(\| \tilde{\boldsymbol{\tau}}_{El}(t_0) \|)$ and $\gamma_{Er}(| \tilde{\tau}_{Er}(t_0) |)$ such that

$$\begin{aligned}\| \tilde{\boldsymbol{\tau}}_{El}(t) \| &\le \gamma_{El}(\| \tilde{\boldsymbol{\tau}}_{El}(t_0) \|) \mathrm{e}^{-\delta_{El}(t-t_0)} \\ | \tilde{\tau}_{Er}(t) | &\le \gamma_{Er}(| \tilde{\tau}_{Er}(t_0) |) \mathrm{e}^{-\delta_{Er}(t-t_0)}\end{aligned} \tag{C8}$$

where δ_{El} and δ_{Er} are positive constants depending on the initial values $\tilde{\boldsymbol{\tau}}_{El}(t_0)$ and $\tilde{\tau}_{Er}(t_0)$. Substituting Eq. (C8) into Eq. (C5) results in

$$\begin{aligned}\dot{V}_4 &\le -k_1^* \| \boldsymbol{X}_{1e} \|^2 - \boldsymbol{X}_{2e}^{\mathrm{T}} \boldsymbol{K}_2 \sigma(\boldsymbol{X}_{2e}) - k_2^* \| \boldsymbol{X}_{2e} \|^2 - k_3^* \psi_e^2 - k_4^* r_e^2 + \\ &c_5 (\gamma_{El}(\| \tilde{\boldsymbol{\tau}}_{El}(t_0) \|) \mathrm{e}^{-\delta_{El}(t-t_0)})^2 + c_6 (\gamma_{Er}(| \tilde{\tau}_{Er}(t_0) |) \mathrm{e}^{-\delta_{Er}(t-t_0)})^2\end{aligned} \tag{C9}$$

which readily shows that the tracking errors $\boldsymbol{X}_{1e}(t)$, $\boldsymbol{X}_{2e}(t)$, $\psi_e(t)$, and $r_e(t)$ globally asymptotically and locally exponentially converge to zero.

C.3 Optimality

Let us define

$$\boldsymbol{x} = \mathrm{col}(\tilde{\boldsymbol{\tau}}_{El}, \tilde{\tau}_{Er}, \boldsymbol{X}_{1e}, \boldsymbol{X}_{2e}, \psi_e, r_e)$$

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} -\boldsymbol{K}_{El} D_{El} \sigma'(\xi_{El} + \boldsymbol{K}_{El} \boldsymbol{M}_l \boldsymbol{X}_2) \tilde{\boldsymbol{\tau}}_{El} \\ -k_{Er} \Delta_{Er} \sigma'(\xi_r + k_{Er} Jr) \tilde{\tau}_{Er} \\ -\boldsymbol{K}_1 \boldsymbol{X}_{1e} + \boldsymbol{X}_{2e} \\ \boldsymbol{M}_l^{-1} [-\boldsymbol{D}_l (\boldsymbol{X}_{2e} - \boldsymbol{K}_1 \boldsymbol{X}_{1e}) + \tilde{\boldsymbol{\tau}}_{El} + \mathrm{col}(A_1, A_2)] + \\ \quad \boldsymbol{K}_1 (-\boldsymbol{K}_1 \boldsymbol{X}_{1e} + \boldsymbol{X}_{2e}) \\ -k_3 \psi_e + r_e - \dfrac{\partial \alpha_\psi}{\partial \hat{\boldsymbol{\tau}}_{El}} \dot{\hat{\boldsymbol{\tau}}}_{El} - \dfrac{\partial \alpha_\psi}{\partial \boldsymbol{X}_{2e}} \boldsymbol{f}_2 \\ \dfrac{1}{J} (-d_r (r_e - k_3 \psi_e) + \tilde{\tau}_{Er}) - \dfrac{\partial \alpha_r}{\partial \hat{\boldsymbol{\tau}}_{El}} \dot{\hat{\boldsymbol{\tau}}}_{El} \\ \quad - \dfrac{\partial \alpha_r}{\partial \boldsymbol{X}_{2e}} \boldsymbol{f}_2 - \dfrac{\partial \alpha_r}{\partial \psi_e} f_3 \end{bmatrix} \tag{C10}$$

$$\boldsymbol{G}(\boldsymbol{x}) = \mathrm{diag}(\boldsymbol{0}, 0, \boldsymbol{0}, \boldsymbol{M}_l^{-1} \cos(\psi), \boldsymbol{M}_l^{-1} \sin(\psi), 0, 1/J)$$

$$\boldsymbol{u} = \mathrm{col}(\boldsymbol{0}, 0, \boldsymbol{0}, \tau_{u1}, \tau_{u1}, 0, \tau_{r1})$$

We rewrite the closed loop system consisting of (25), (31), (51), (59) and (67) as follows

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x}) \boldsymbol{u} \tag{C11}$$

where we haven't substituted τ_{u1} and τ_{r1} given in Eqs. (37) and (64) into Eq. (C11). It is seen from the second equations of (40) and (64) that the controls τ_{u_1} and τ_{r1} are of the form Eq. (18), i.e.,

$$\begin{aligned}\tau_{u1} &= -\boldsymbol{R}_{u1}^{-1} \mathrm{col}(X_{21e} \cos(\psi), X_{22e} \sin(\psi)) \\ \boldsymbol{R}_{u1}^{-1} &= \bar{\boldsymbol{K}}_2 \mathrm{diag}\left(\frac{\sigma(X_{21e})}{X_{21e}}, \frac{\sigma(X_{22e})}{X_{22e}}\right) \\ \tau_{r1} &= -R_{r1}^{-1} r_e, R_{r1}^{-1} = k_4 / \rho_3\end{aligned} \tag{C12}$$

Since $\boldsymbol{K} = \beta_{u1} \bar{\boldsymbol{K}}_2$, the control gains chosen such that k_i^*, $i = 1, 2, \ldots, 4$ are positive will also cover the case when τ_{u1} given in Eq. (37). Thus, by Lemma 3.3 the controls τ_u^* and τ_r^* given in Eqs. (40) and (65) are optimal in the sense that the cost function defined in Eqs. (19) and (20) with $\boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{G}(\boldsymbol{x})$, $\kappa(\boldsymbol{x})$ are defined in Eq. (C10),

$$\boldsymbol{R}(\boldsymbol{x}) = \mathrm{diag}(\boldsymbol{0}, 0, \boldsymbol{0}, \boldsymbol{R}_{u1}, 0, R_{r1}) \tag{C13}$$

with $\boldsymbol{R}_{u1}$ and R_{r1} are defined in Eq. (C12), and $V = V_\Sigma$. □