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# Theory and application of equivalent transformation relationships between plane wave and spherical wave

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**Abstract:** Based on the governing equations and the equivalent models, we propose an equivalent transformation relationships between a plane wave in a one-dimensional medium and a spherical wave in globular geometry with radially inhomogeneous properties. These equivalent relationships can help us to obtain the analytical solutions of the elastodynamic issues in an inhomogeneous medium. The physical essence of the presented equivalent transformations is the equivalent relationships between the geometry and the material properties. It indicates that the spherical wave problem in globular geometry can be transformed into the plane wave problem in the bar with variable property fields , and its inverse transformation is valid as well. Four different examples of wave motion problems in the inhomogeneous media are solved based on the presented equivalent relationships. We obtain two basic analytical solution forms in Examples I and II, investigate the reflection behavior of inhomogeneous half-space in Example III, and exhibit a special inhomogeneity in Example IV, which can keep the traveling spherical wave in constant amplitude. This study implies that our idea makes solving the associated problem easier.

**Keywords:** elastodynamic issue, equivalent transformation relationship, governing equation, inhomogeneous medium

## 1 Introduction

The spherical wave is a common type of wave motion in a homogeneous medium. In general, the spherical wave can be caused by the spherical source, such as a spherical cavity subjected to a time-varying uniform internal pressure. In the homogeneous medium, the plane traveling wave's amplitude remains constant, but the spherical wave's amplitude decreases with the traveling distance. Then, the group velocity of the elastic wave in the homogeneous medium is constant, thus the wave travels the same distance in the equal time interval.

The situations are usually different in the inhomogeneous medium. The functionally graded material is a classical inhomogeneous medium. In the process of material production, the ratios of the components are controlled to be distributed continuously. Thus, the macroscopic material properties are graded. Therefore, the amplitude and group velocity of the traveling wave may both vary in the process of traveling.

The wave motion in inhomogeneous media is a classical topic in the theoretical research of elastic, seismic, acoustic, and electromagnetic waves. Many naturally occurring soils, such as flocculated clays, varved silts, or sands, are typically deposited via sedimentation over long periods. The effects of deposition, overburdening, and desiccation can cause soil media to exhibit both anisotropic and inhomogeneous deformability (Wang *et al.*, 2010). Because of the material's natural inhomogeneity, when the old concepts, theories, algorithms, and experimental measures are introduced and developed for homogeneous materials, they meet great difficulty. A lot of them are no longer suitable for inhomogeneous media, thus further work should be done for developing several better research methods. The wave motion problem in the inhomogeneous medium is commonly solved by the numerical techniques. Usually, the models for numerical simulating have to be simplified by some operations, such as the discretely layered model. For example, the model of a vertically inhomogeneous medium is simplified into several discretely homogeneous layers. The reverberation matrix method can be applied to solve the wave motion problem in the layered model where

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the layers in the model are all homogeneous. The result can be close to the exact solution when the layers are thin enough. This idea is often used for the asymptotic approach as well. The graded element method (or inhomogeneous element method) is a valuable numerical scheme for the problems in inhomogeneous media since the material property fields don't have to be simplified. Kim and Paulino (2002) presented the graded finite elements for the continuously nonhomogeneous isotropic and orthotropic materials within the framework of a generalized isoparametric formulation. In 2013, Wang *et al*. proposed the dynamic inhomogeneous finite element method for simulating the dynamic response in the inhomogeneous media, where the material properties (such as elastic modulus, Poisson ratio, and density) in the element are all graded, thus the error of discretization and the scale of calculation can be reduced greatly (Yang *et al.*, 2013; Wang *et al.*, 2013).

The mathematical background behind wave motion in inhomogeneous media involves the solution of partial differential equations with variable coefficients (Rangelov *et al.*, 2005). There is great difficulty to analytically study the wave motion in the inhomogeneous medium with arbitrary material property distributions, especially to obtain the exact solutions of the original partial differential equation (Hook, 1962; Watanabe and Takeuchi, 2003). Thus far, only the cases with a few special material property distributions have been solved analytically. The material property fields in most existing research can be categorized into three types: exponential function and associated forms (Wilson, 1942; Vrettos, 1990a, 1990b, 1990c, 1991a, 1991b); power function and associated forms (Hook, 1962; Hudson, 1962; Deresiewicz, 1962; Gazetas, 1980; Watanabe, 1982; Manolis and Bagtzoglou, 1992; Manolis and Shaw, 1992; Manolis *et al*., 2002, 2004; Dey *et al*., 1996a, 1996b), including linear elastic modulus distribution (Stoneley, 1934; Awojobi, 1972, 1973; Chattopadhya *et al*., 2012; Vardoulakis and Vrettos, 1988; Muravskii, 1997, 2000); and trigonometric functions and associated forms (Pekeris, 1935; Dey *et al*., 2000). Many methodologies can be applied to analytically solve the wave propagation in the inhomogeneous media. The geometrical optics method is able to solve the problems with high frequencies and large wave numbers, since the effect of inhomogeneity on the propagating wave is considerably diminished (Bahar, 1967; Zhu *et al*., 1995). The analytical solution of the wave propagation in continuously inhomogeneous media can be obtained by the complex function method. For example, the conformal mapping method and algebraic transformation technology can be employed to study the general solutions of the elastic wave propagation in the inhomogeneous medium (Manolis *et al*., 1999).

There are many interesting theories and techniques in the research about the inhomogeneous medium's behaviors, such as the homogenization theory (Boutin and Auriault, 1993; Nicolas, 2010; Chen and Fish, 2001; Fish *et al*., 2012) and wave cloak techniques (Pendry *et al*., 2006; McManus *et al*., 2014; Zhou *et al*., 2008). Homogenization aims at deriving a homogenized description (governing partial differential equations and constitutive law) for the medium, based on the assumption that a statistically homogeneous medium represented by a representative volume element (RVE), or a material with periodic structure represented by a repeated unit cell (RUC), can be defined. Mathematical homogenization provides a rigorous definition of the homogenization process and the homogenized equations. It consists of setting the problem as a sequence of equations describing the heterogeneous material (Charalambakis, 2010). Using the freedom of design that metamaterials provide, wave fields can be redirected and can suggest a design strategy. Metamaterials which owe their properties to subwavelength details of structure rather than to their chemical composition can be impossible to find in nature (Pendry *et al*., 2006). Based on this technique, the properties of the materials and structures can be designed based on the engineer's purposes. The theory and technique mentioned above show that there are some equivalent transformation relationships between the microstructure and properties of material. These equivalent transformation relationships are of great significance and value in the theoretical study and engineering applications.

In this study, we investigate the equivalent transformation relationships between the plane wave in the one-dimensional medium and the spherical wave in the globular geometry with radial inhomogeneity. These equivalent relationships can make the analytical solving of the elastodynamic problem in inhomogeneous media easier. Based on these equivalent relationships, four basic examples are solved.

## 2 Governing equations

The equivalent relationships presented in this paper are obtained based on the governing equations.

### 2.1 Governing equation of spherical wave in radially inhomogeneous medium

The coordinate system shown in Fig. 1 is established for the spherical wave propagation. The undamped dynamic equilibrium equation can be written as follows for the spherical wave in the medium with spherically symmetrical material properties (the material properties are only dependent on radial coordinate $r$):

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D\frac{\partial u}{\partial r}) = \rho\ddot{u} \tag{1}$$

where $D = \lambda + 2\mu$ denotes the elastic coefficient of the medium; $\rho$ denotes the density of the medium; and $\lambda$ and $\mu$ are the Lamé coefficients.

If the elastic coefficient $D$ is constant, the dynamic

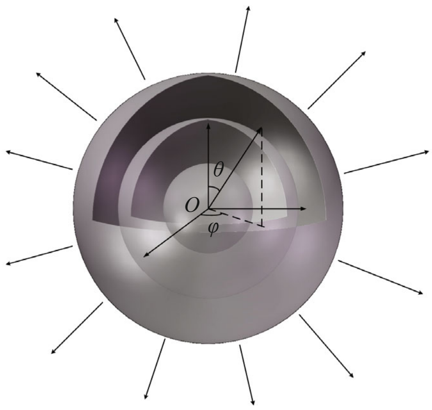



**Fig. 1 Coordinate system of spherical wave propagation**

equilibrium equation (Eq. (1)) is simplified into the following form:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) = \frac{1}{c_L^2}\ddot{u} \tag{2}$$

where the longitudinal wave velocity $c_L$ can be computed by the following expression.

$$c_L = \sqrt{\rho^{-1}D} \tag{3}$$

## 2.2 Governing equation of plane wave in bar with variable cross-section area

Consider a model (as shown in Fig. 2(a)) of the 1D isotropic bar with variable cross-section area and constant material properties. Figure 2(b) shows the infinitesimal body of this model.

The elastic wave in this bar model is governed by the following wave equation (Wang, 2006). As we know, the wavefront area in this model equals to the cross-section area of the bar.

$$\frac{1}{A}\frac{\partial(A\sigma)}{\partial x} = \rho_b\ddot{u} \tag{4}$$

where $\sigma$ is the stress; $\rho_b$ is the density of this model; and $A$ is the wave front area (in bar model, it also represents the section area).

The spherical wavefront area in the bar is associated with the radial coordinate $r$ (see Eq. (5)).

$$A = 4\pi r^2 \tag{5}$$

Consider the bar with the section area function as the same form of Eq. (5), where the variable $r$ in Eq. (5) is replaced with coordinate variable $x$ (variable $r$ is independent with variable $x$ in this operation). Substitute this section area function into wave equation (Eq. (4)), then

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2 D_b\frac{\partial u}{\partial x}\right) = \rho_b\ddot{u} \tag{6}$$

where $D_b$ (elastic coefficient) represents $E_b$ (Young's modulus) when the wave is in *P* mode; and $D_b$ represents $G_b$ (shear modulus) when the wave is in *S* mode.

Equations (6) and (1) have the same form. Based on the above derivation, this shows that the spherical elastic wave equation can be equivalent to the wave equation for the one-dimensional bar with section area variation. When the value of $D_b$ equals to $D$ (elastic coefficient mentioned in Section 2.1), the dynamic equilibrium equations (Eqs. (6) and (1)) are identical.

## 2.3 Governing equation of plane wave in bar with inhomogeneous material

Consider another model (as shown in Fig. 3(a)) of the 1D isotropic bar with the variable material properties and the constant cross-section area. Figure 3(b) exhibits the infinitesimal body of this model. The following wave equation is the governing equation for the wave motion in this model:

$$\frac{\partial}{\partial x}\left(D_b^*\frac{\partial u}{\partial x}\right) = \rho_b^*\ddot{u} \tag{7}$$

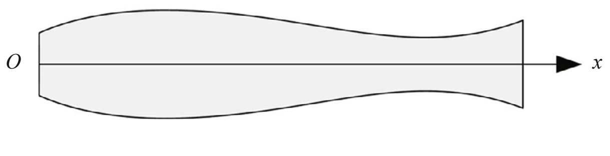
where $D_b^*$ (elastic coefficient) represents $E_b^*$ (Young's modulus) when the wave is in *P* mode; $D_b^*$ represents $G_b^*$ (shear modulus) when the wave is in *S* mode; and $\rho_b^*$ represents the density of this model.

Let

$$D_b^* = x_0^{-2}x^2 D_b$$
$$\rho_b^* = x_0^{-2}x^2\rho_b \tag{8}$$

=

where unit length $x_0$ is used for dimensionless of the coordinate $x$



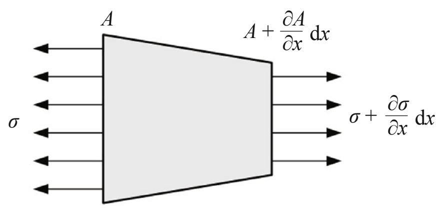
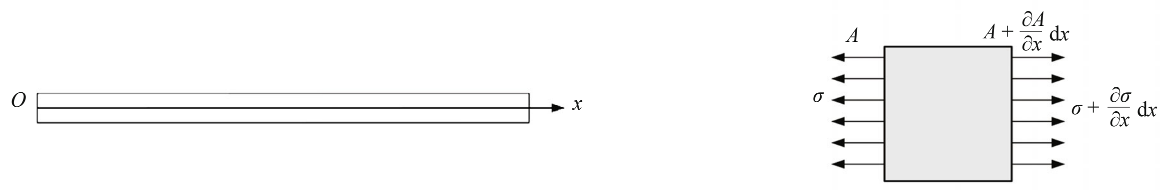




**Fig. 2 Bar model with variable cross-section area**




(a) Coordinate system (b) Infinitesimal body

**Fig. 3 Bar model with material property fields**

Substituting Eq. (8) into Eq. (7), the wave equation Eq. (7) is transformed into Eq. (6). This shows that the wave equation of this model (Eq. (7)) can be equivalent to the wave equation Eq. (6) when their material property fields satisfy Eq. (8). In consequence, the wave equation of this model (Eq. (7)) can also be equivalent to the spherical wave equation Eq. (1).

## 3 Equivalent transformation theory

Based on the above derivation, the equivalent spherical wave equation and plane wave equation have different material property profiles, but the forms of their wave velocity functions are the same. Thus, the necessary condition of the equivalent transformation is that the media have the same form of wave velocity profile.

### 3.1 Description of equivalent dynamic models

Consider three different models that satisfy the following material property fields:

(1) Model I: Globular model

$$\begin{aligned} D(r) &= \text{constant} \\ \rho(r) &= \text{constant} \end{aligned} \tag{9}$$

(2) Model II: Bar model with variable section area

$$\begin{aligned} D_\mathrm{b}(x) &= D(x) = \text{constant} \\ \rho_\mathrm{b}(x) &= \rho(x) = \text{constant} \\ A_\mathrm{b}(x) &= \alpha x^2 \end{aligned} \tag{10}$$

(3) Model III: Bar model with variable material properties

$$\begin{aligned} D_\mathrm{b}^*(x) &= \beta x_0^{-2} x^2 D_\mathrm{b} = \beta x_0^{-2} x^2 D(x) \\ \rho_\mathrm{b}^*(x) &= \beta x_0^{-2} x^2 \rho_\mathrm{b} = \beta x_0^{-2} x^2 \rho(x) \\ A_\mathrm{b}^*(x) &= A_0 \end{aligned} \tag{11}$$

where $A_\mathrm{b}$ and $A_\mathrm{b}^*$ are the section area functions of model II and III; $A_0$ is a non-zero constant; and $\alpha$ and $\beta$ are the undetermined coefficients.

### 3.2 Equivalent stiffness coefficient and lumped mass

In order to obtain the equivalent stiffness coefficient and the lumped mass, the equivalent dynamic models in Fig. 4(a) (one end of each model is fixed) are all simplified to the dynamic system shown in Fig. 4(b) which contains one spring component with an equivalent stiffness coefficient and one mass component with a lumped mass. The variable $f$ denotes the equivalent load. These three models have the same form of wave velocity function.

Equations (12)–(14) are the formulations for calculating the lumped mass of each model.

$$m = \int_V \rho \mathrm{d}V = \frac{4\pi r^3 \rho}{3} \tag{12}$$

$$m_\mathrm{b} = \int_0^x \rho_\mathrm{b} A_\mathrm{b} \mathrm{d}x = \frac{\alpha x^3 \rho_\mathrm{b}}{3} \tag{13}$$

$$m_\mathrm{b}^* = \int_0^x \rho_\mathrm{b}^* A_\mathrm{b}^* \mathrm{d}x = \frac{\beta x^3 \rho_\mathrm{b} A_0}{3x_0^2} \tag{14}$$

where $m$, $m_\mathrm{b}$, and $m_\mathrm{b}^*$ are the lumped mass of Models I, II, and III.

Equations (15)–(17) are the formulations for computing the equivalent compliance coefficient of each model.

$$s = \lim_{r_1 \to \varepsilon} \int_{r_1}^{r} \frac{1}{DA} \mathrm{d}r = \frac{r - \varepsilon}{4\pi D \varepsilon r} \tag{15}$$

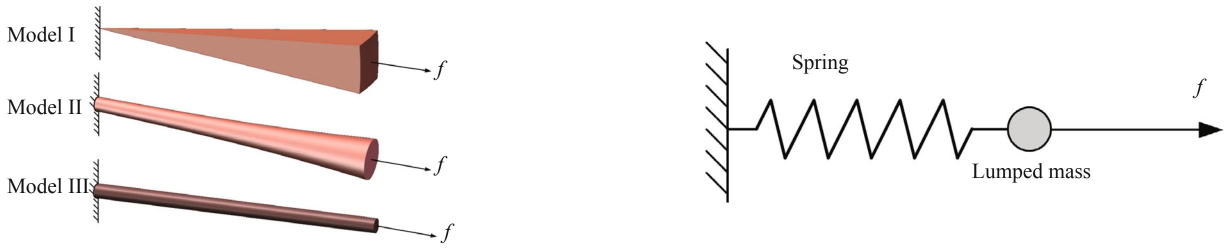
$$s_\mathrm{b} = \lim_{x_1 \to \varepsilon} \int_{x_1}^{x} \frac{1}{D_\mathrm{b} A_\mathrm{b}} \mathrm{d}x = \frac{x - \varepsilon}{\alpha D \varepsilon x} \tag{16}$$

$$s_\mathrm{b}^* = \lim_{x_1 \to \varepsilon} \int_{x_1}^{x} \frac{1}{D_\mathrm{b}^* A_\mathrm{b}^*} \mathrm{d}x = \frac{x_0^2 (x - \varepsilon)}{\beta D A_0 \varepsilon x} \tag{17}$$

where $s$, $s_\mathrm{b}$, and $s_\mathrm{b}^*$ denote the equivalent compliance coefficients of Models I, II, and III; and $\varepsilon$ denotes an arbitrary positive number which is close to zero.

Then, the equivalent stiffness coefficient of each model can be obtained as follows.

$$k^\mathrm{eq} = \frac{1}{s} = \frac{4\pi D \varepsilon r}{r - \varepsilon} \tag{18}$$




(a) Diagrams of models (b) Simplified dynamic system

**Fig. 4 Equivalent dynamic models**

$$k_{\mathrm{b}}^{\mathrm{eq}} = \frac{1}{s_{\mathrm{b}}} = \frac{\alpha D\varepsilon x}{x-\varepsilon} \tag{19}$$

$$k_{\mathrm{b}}^{\mathrm{eq}*} = \frac{1}{s_{\mathrm{b}}^{*}} = \frac{\beta D A_0 \varepsilon x}{x_0^2 \left(x-\varepsilon\right)} \tag{20}$$

where $k^{\mathrm{eq}}$, $k_{\mathrm{b}}^{\mathrm{eq}}$, and $k_{\mathrm{b}}^{\mathrm{eq}*}$ represent the equivalent stiffness coefficients of Models I, II, and III.

### 3.3 Determination of unknown coefficients

Through the equivalent property technique, we are able to determine $\alpha$ and $\beta$.

Let

$$\begin{aligned} A_0 &= x_0^2 \\ x &= r \\ m &= m_{\mathrm{b}} = m_{\mathrm{b}}^{*} \\ k^{\mathrm{eq}} &= k_{\mathrm{b}}^{\mathrm{eq}} = k_{\mathrm{b}}^{\mathrm{eq}*} \end{aligned} \tag{21}$$

Then, the undetermined coefficients $\alpha$ and $\beta$ can be obtained as follows.

$$\begin{aligned} \alpha &= 4\pi \\ \beta &= 4\pi \end{aligned} \tag{22}$$

Therefore, two equivalent relationships can be given, where Eq. (23) denotes the equivalent transformation from Model I to II, and Eq. (24) denotes the equivalent transformation from Model I to III.

$$\begin{Bmatrix} D(r) \\ \rho(r) \end{Bmatrix} \rightarrow \begin{Bmatrix} D(x) \\ \rho(x) \\ A_{\mathrm{b}}(x) = 4\pi x^2 \end{Bmatrix} \tag{23}$$

$$\begin{Bmatrix} D(r) \\ \rho(r) \end{Bmatrix} \rightarrow \begin{Bmatrix} D_{\mathrm{b}}^{*}(x) = 4\pi x^2 \left(A_{\mathrm{b}}^{*}(x)\right)^{-1} D(x) \\ \rho_{\mathrm{b}}^{*}(x) = 4\pi x^2 \left(A_{\mathrm{b}}^{*}(x)\right)^{-1} \rho(x) \end{Bmatrix} \tag{24}$$

Their inverse transformations are valid as well. For example, the equivalent transformation from Model III to I is shown as Eq. (25).

$$\begin{Bmatrix} D_{\mathrm{b}}^{*}(x) \\ \rho_{\mathrm{b}}^{*}(x) \\ A_{\mathrm{b}}^{*}(x) \end{Bmatrix} \rightarrow \begin{Bmatrix} D(r) = \left(4\pi r^2\right)^{-1} A_{\mathrm{b}}^{*}(r) D_{\mathrm{b}}^{*}(r) \\ \rho(r) = \left(4\pi r^2\right)^{-1} A_{\mathrm{b}}^{*}(r) \rho_{\mathrm{b}}^{*}(r) \end{Bmatrix} \tag{25}$$

These formulations (Eqs. (23)–(25)) are useful for finding the solution of the wave propagation in inhomogeneous media.

### 3.4 Unified model description

In order to present the physical and mathematical explanation on the equivalent transformations, a unified model is constructed in this section.

The following two functions are introduced to replace the material property fields and section area functions (or wavefront area functions) of the equivalent models.

$$g(q) = \begin{cases} D(q)A(q) = 4\pi q^2 D(q) & \text{Model I} \\ D_b(q) A_{\mathrm{b}}(q) & \text{Model II} \\ D_{\mathrm{b}}^{*}(q) A_{\mathrm{b}}^{*}(q) & \text{Model III} \end{cases} \tag{26}$$

$$h(q) = \begin{cases} \rho(q)A(q) = 4\pi q^2 \rho(q) & \text{Model I} \\ \rho_{\mathrm{b}}(q) A_{\mathrm{b}}(q) & \text{Model II} \\ \rho_{\mathrm{b}}^{*}(q) A_{\mathrm{b}}^{*}(q) & \text{Model III} \end{cases} \tag{27}$$

where the variable $q$ is the coordinate of the unified model.

Then, the wave equations of the equivalent dynamic models can be written as a general form (Eq. (28)) for the unified descriptions.

$$\frac{\partial}{\partial q}\left(g(q)\frac{\partial \phi}{\partial q}\right) - h(q)\ddot{\phi} = 0 \tag{28}$$

In order to study the harmonic wave motion, let

$$\phi(q,t) = W(q)\exp(-\mathrm{i}\omega t) \tag{29}$$

where i is square root of -1; $\omega$ denotes frequency; and $t$ denotes time.

Then, we have

$$\frac{\partial}{\partial q}\left(g(q)\frac{\partial W}{\partial q}\right)+\omega^2 h(q)W=0 \tag{30}$$

The above wave equations (Eqs. (28) and (30)) both govern the one-dimensional wave motion. The functions (Eqs. (26) and (27)) have the specific physical meaning. The function $g(q)$ denotes the stiffness distribution normal to the wavefront; the function $h(q)$ denotes the linear density distribution normal to the wavefront. The wave speed profile can be obtained by Eq. (31).

$$c=\sqrt{h^{-1}\mathrm{g}} \tag{31}$$

In Models II and III, the wavefront is flat, thus the distributions are along the direction of wave propagation; in Model I, the wavefront is spherical, thus the distributions must be radial (or spherically symmetrical). When the models are equivalent, the stiffness distribution (Eq. (26)) and linear density distribution (Eq. (27)) of each model are the same. Then, the equivalent formulations (Eqs. (23)–(25)) can be proved tenable.

The stiffness distribution (Eq. (26)) and density distribution (Eq. (27)) include both the wavefront area and the material properties. The same distributions of stiffness coefficient and linear density lead to the same dynamic behavior and solution form. Therefore, the physical essence of the equivalent transformation relationships presented in this paper is the equivalent relationships between the geometry (such as the area of wavefront) and the material properties.

## 4 Application examples

Based on the equivalent relationships, the analytical approaches can be given easily.

### 4.1 Example I (From Model III to I)

Consider a semi-infinite bar with the unit cross-section area as shown in Fig. 3(a), where the material property fields are denoted as follows.

$$\begin{aligned} D_\mathrm{b}^* &= \xi x_0^{-2}x^2 \\ \rho_\mathrm{b}^* &= \zeta x_0^{-2}x^2 \end{aligned},\quad x>0 \tag{32}$$

where $\xi$ and $\zeta$ are both constant.

The general solution of the spherical wave motion in an infinite homogeneous medium is written as Eq. (33) (Achenbach, 1973), when the original point of this medium is subjected to a harmonic load.

$$W=\frac{1}{4\pi\xi R}\exp\left(\mathrm{i}kR\right) \tag{33}$$

where $R$ denotes the distance to the original point. The wave number $k$ can be expressed as follows.

$$k=\sqrt{\xi^{-1}\zeta\omega^2} \tag{34}$$

Based on the inverse transformation from III to I (Eq. (25)), the corresponding solution of the bar model can be given directly as Eq. (35), where the bar is subjected to an equivalent harmonic load.

$$W=\frac{1}{\xi x}\exp(\mathrm{i}kx) \tag{35}$$

Equation (35) serves as a basic solution form in this paper. Its detail application is discussed in Example III.

### 4.2 Example II (From model I to III)

We focus on an infinite radially inhomogeneous medium with the globular geometry as shown in Fig. 5(a). The material property distributions are defined as the following forms.

$$\begin{aligned} D &= \xi r_0^2 r^{-2} \\ \rho &= \zeta r_0^2 r^{-2} \end{aligned} \tag{36}$$

where unit length $r_0$ is used for dimensionless of the coordinate $r$.

Based on the equivalent transformation relationship (Eq. (24)), the equivalent material property fields (Eq. (37)) are determined for the equivalent Model III.

$$\begin{aligned} D_\mathrm{b}^* &= 4\pi\xi \\ \rho_\mathrm{b}^* &= 4\pi\zeta \end{aligned} \tag{37}$$

Observing Eq. (37), the equivalent material properties are homogeneous. We assume a harmonic spherical wave $\exp(-\mathrm{i}\omega t)$ propagating from $r_1$ to $+\infty$. Then, the analytical approach of the displacement field can be given as Eq. (38):

$$W=\exp(\mathrm{i}k(r-r_1)) \tag{38}$$

where

$$k=\sqrt{\xi^{-1}\zeta\omega^2} \tag{39}$$

Equation (38) serves as a basic solution form in this paper, too. Its detail application is discussed in Example IV.

### 4.3 Example III (From Model III to I)

Next we pay attention to an inhomogeneous elastic half-space as shown in Fig. 5(b). The material property distributions as follow depend on the depth $z$.


(a) Example II (b) Example III

**Fig. 5 Models**

$$D=\left(a_1z+b_1z_0\right)^2,\quad \begin{cases}a_i\ge 0\\ b_i>0\end{cases},\quad i=1,2 \qquad (40)$$
$$\rho=(a_2z+b_2z_0)^2$$

where $z_0$ is unit length of $z$-axis; $a_1$, $a_2$, $b_1$, and $b_2$ are all constant; and $a_1b_2=a_2b_1$.

A plane harmonic SH wave is propagating in this medium, where the incident angle is $\theta$. The incident wave field satisfies the following condition:

$$W^{(i)}\Big|_{z=0}=\exp\left(\mathrm{i}kx\sin\theta\right) \qquad (41)$$

where the wave numbers satisfy the following relationships.

$$\frac{k_x}{\sin\theta}=\frac{k_z}{\cos\theta}=k_x^2+k_z^2=k=\frac{\omega}{c}=\left|b_1^{-1}b_2\right|\omega \qquad (42)$$

The wave equation of this example can be written as follows.

$$\frac{\partial}{\partial z}\left(D\frac{\partial W}{\partial z}\right)+D\frac{\partial^2W}{\partial x^2}+\omega^2\rho W=0 \qquad (43)$$

We set the solution of incident wave as the following formulation (Eq. (44)):

$$W^{(i)}=\eta\left(z\right)\exp\left(\mathrm{i}k_xx\right) \qquad (44)$$

where $\eta$(z) is a undetermined function only associated with the coordinate $z$. The function $\eta(z)$ satisfies the following condition.

$$\eta\left(0\right)=1 \qquad (45)$$

Substitute Eq. (44) into Eq. (43), then we have

$$\frac{\partial}{\partial z}\left(D\frac{\partial\eta}{\partial z}\right)+(\omega\cos\theta)^2\rho\eta=0 \qquad (46)$$

If $a_i>0$, apply the coordinate transformation $z'=z+b_1z_0/a_1$. Then, the material property fields become the following forms.

$$D\left(z'\right)=a_1^2z'^2$$
$$\rho\left(z'\right)=a_2^2z'^2 \qquad (47)$$

Then, the solution of Eq. (46) can be given as Eq. (48) based on the basic solution form Eq. (35) and the condition Eq. (45).

$$\eta\left(z\right)=\frac{b_1z_0}{a_1z+b_1z_0}\exp\left(-\mathrm{i}k_zz\right) \qquad (48)$$

If $a_i=0$, Eq. (48) can also serve as the exact solution satisfying the wave equation Eq. (46) and the condition Eq. (45). Thus, the solution of incident wave field is obtained as Eq. (49).

$$W^{(i)}=\frac{b_1z_0}{a_1z+b_1z_0}\exp(\mathrm{i}(k_xx-k_zz)) \qquad (49)$$

The surface of half space is traction-free. This boundary condition can be written as follows.

$$\sigma_{yz}\Big|_{z=o}=\frac{\partial(W^{(i)}+W^{(r)})}{\partial z}\Bigg|_{z=0}\exp\left(\mathrm{i}\omega t\right)=0 \qquad (50)$$

According to Snell's law, we set the reflected wave field as the following expression.

$$W^{(r)}=\frac{\gamma b_1z_0}{a_1z+b_1z_0}\exp\left(\mathrm{i}\left(k_xx+k_zz\right)\right) \qquad (51)$$

Then, the reflection coefficient $\gamma$ can be determined as the following form based on the traction-free boundary condition (Eq. (50)). Seeing Eq. (52), the reflection coefficient $\gamma$ is dependent on the vertical wave number $k_z$.

$$\gamma=\frac{ib_1k_z+a_1}{ib_1k_z-a_1} \qquad (52)$$

Therefore, the surface's displacement field can be expressed as Eq. (53). A similar case can be seen in the work of Wuttke *et al*., where the $z$ axis in that work is in the opposite direction and the range of $a_i$ is different (Wuttke *et al*., 2015).

$$W\Big|_{z=0}=\frac{2\mathrm{i}b_1k_z}{\mathrm{i}b_1k_z-a_1}\exp\left(\mathrm{i}k_xx\right) \qquad (53)$$

Observing Eq. (53), the surface displacement field can be viewed as a wave propagating in the $x$ direction with apparent wave number $k\sin\theta$ and apparent phase velocity $c/\sin\theta$. Except for the case of grazing incidence, the apparent phase velocity exceeds $c$. The amplitude of surface displacement field can be given as follows.

$$\left|W\right|_{z=0} = \left|\frac{2ib_1k_z}{ib_1k_z - a_1}\right| = \frac{2b_1k\cos\theta}{\sqrt{b_1^2k^2\cos^2\theta + a_1^2}} \tag{54}$$

Viewing Eq. (54), the surface displacement amplitude is dependent on the wave number $k$, the incident angle $\theta$ and the ratio $a_i/b_i$. When $k$ decreases or $a_i/b_i$ increases, the amplitude decreases; when $\theta$ decreases, the amplitude increases. The possible range of the amplitude is from 0 to 2. Figure 6(a) exhibits the surface displacement amplitude at different $k$ and $a_1/b_1$; Figure 6(b) denotes the surface displacement amplitude at different $k$ and $\theta$. These two figures show the following: the effect of $a_1/b_1$ and $\theta$ can be almost ignored when $k$ is small enough; the effect of $\theta$ on displacement amplitude becomes obvious when $\theta$ is close to $\pi/2$; and the effect of $a_1/b_1$ on displacement amplitude is not very obvious.

### 4.4 Example IV (From Model I to III)

Now we consider an infinite globular geometry with the radial inhomogeneity as shown in Fig. 5(a). The material property fields are defined by the following expressions (Eq. (55)):

$$\begin{aligned} D &= r_0^2\left(a_1r + b_1r_0\right)^{-2} \\ \rho &= r_0^2\left(a_2r + b_2r_0\right)^{-2} \end{aligned}, \quad \begin{cases} a_i \ge 0 \\ b_i \ge 0 \end{cases}, \quad i = 1,2 \tag{55}$$

where $a_1b_2 = a_2b_1$.

A harmonic spherical wave $\exp(-i\omega t)$ propagates from $r_1$ to $+\infty$.

(1) When $a_i = 0$ and $b_i \ne 0$, the medium is homogeneous. Thus, the displacement field can be written as Eq. (56):

$$W = \frac{r_1}{r}\exp\left(ik\left(r - r_1\right)\right) \tag{56}$$

where

$$k = \left|b_1^{-1}b_2\right|\omega \tag{57}$$

(2) When $a_i \ne 0$, apply the coordinate transformation $r' = r + b_1r_0/a_1$. Then, we have

$$\begin{aligned} D\left(r'\right) &= a_1^{-2}r_0^2r'^{-2} \\ \rho\left(r'\right) &= a_2^{-2}r_0^2r'^{-2} \end{aligned} \tag{58}$$

Since Eq. (58) is similar to Eq. (36), the equivalent material parameters of the equivalent Model III can be determined as constant. Then, the analytic solution can be given as Eq. (59) based on the basic solution form Eq. (38).

$$W = \exp\left(ik\left(r - r_1\right)\right) \tag{59}$$

where

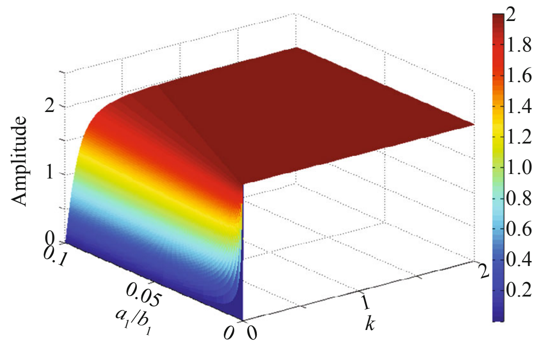
$$k = \left|a_2^{-1}a_1\right|\omega \tag{60}$$

In consequence, there are two possible types of displacement fields in this example. The one case is that the amplitude of the spherical wave decreases with the factor $r^{-1}$ (see Eq. (56)). Another one is an interesting case: the amplitude remains constant (see Eq. (59)), even if $a_i$ is extremely close to zero.

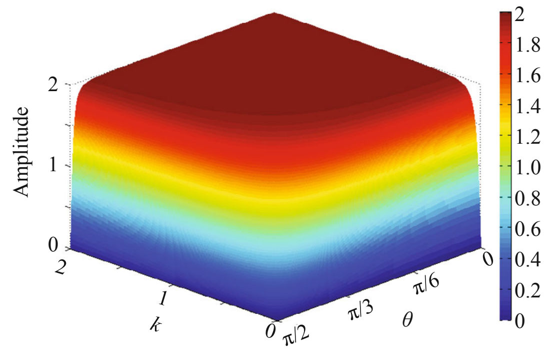
## 5 Conclusions

Through the description of the previous sections, several conclusions can be obtained.

In this paper, we present the equivalent transformation relationships between the plane wave in one-dimensional media and the spherical wave in globular geometry with the radial inhomogeneity to make the analytical solving




(a)




(b)

**Fig. 6 Surface displacement amplitude (a) at different wave number $k$ and ratio $a_1/b_1$ (incident angle $\theta = \pi/4$) (b) at different wave number $k$ and incident angle $\theta$ (ratio $a_1/b_1 = 0.02$)**

easier. By the theoretical derivations, it's proved that the spherical wave problem in this study is able to be equivalent to the plane wave problem in bar, and the inverse transformations are valid as well. The unknown analytical solutions can be given directly based on the known analytical solutions of the associated equivalent models.

We solve four different examples of wave motion problems in the inhomogeneous media based on the presented equivalent relationships. We obtain two basic analytical solution forms in Examples I and II; investigate the wave reflection on the traction-free surface of inhomogeneous half-space in Example III; and exhibit a special material property fields in Example IV which can keep the traveling spherical wave in constant amplitude.

The physical essence of presented equivalent transformations is the equivalent relationships between the geometry and the material properties. The findings in this study can potentially contribute to the development of the acoustic, electromagnetic, elastic wave theories in inhomogeneous medium.

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