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Complex complete quadratic combination method for damped system with repeated eigenvalues

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Abstract: A new response-spectrum mode superposition method, entirely in real value form, is developed to analyze the maximum structural response under earthquake ground motion for generally damped linear systems with repeated eigenvalues and defective eigenvectors. This algorithm has clear physical concepts and is similar to the complex complete quadratic combination (CCQC) method previously established. Since it can consider the effect of repeated eigenvalues, it is called the CCQC-R method, in which the correlation coefficients of high-order modal responses are enclosed in addition to the correlation coefficients in the normal CCQC method. As a result, the formulas for calculating the correlation coefficients of high-order modal responses are deduced in this study, including displacement, velocity and velocity-displacement correlation coefficients. Furthermore, the relationship between high-order displacement and velocity covariance is derived to make the CCQC-R algorithm only relevant to the high-order displacement response spectrum. Finally, a practical step-by-step integration procedure for calculating high-order displacement response spectrum is obtained by changing the earthquake ground motion input, which is evaluated by comparing it to the theory solution under the sine-wave input. The method derived here is suitable for generally linear systems with classical or non-classical damping.

Keywords: damped system; repeated eigenvalue; response spectrum; complex complete quadratic combination; correlation coefficient; high-order modal responses

1 Introduction

The dynamic responses of structures subjected to earthquake ground motion are usually calculated by the response-spectrum mode superposition method in the seismic design codes of many earthquake prone countries. Using the response-spectrum method for either a classically or non-classically damped MDOF linear system, the maximum structural response can be obtained by each mode of a set of modes which are used to represent the response. Based on the classically damped assumption, the square root of the sum of the squares (SRSS) method and complete quadratic combination (CQC) rule are proposed to calculate the dynamic response of structures (Caughey, 1960). However, for structures with strongly non-classical damping, the accuracy of the SRSS method or CQC rule becomes questionable (Clough and Mojtahedi, 1976; Veletsos and Ventura, 1986). For this reason, several modal combination rules accounting for the effect of non-classical damping are developed. For instance, Igusa *et al.* (1984) described the responses in terms of spectral moments and provided the formations of correlation coefficients among modes using filtered white noise process as inputs. Later on, Gupta and Jaw (1986) developed the response spectrum combination rules for non-classically damped systems by using the displacement and velocity response spectrum. Singh and Ghafory-Ashtiany (1986) formed a modified conventional SRSS approach where non-proportional damping effects can be properly included. Villaverde (1988) improved Rosenblueth's rule (1951) by including the effect of modal velocity responses. Maldonado and Singh (1991) proposed an improved response spectrum method for non-classically damped systems, which reduces the error associated with the truncation of high frequency modes without explicitly using them in the analysis. Zhou *et al.* (2004) derived the complex square root of the sum of the squares (CSRSS) method and the complex complete quadratic combination (CCQC) algorithm for a generally non-classically damped linear system, which were entirely in real value form and

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included three correlation coefficients of displacement, velocity and displacement-velocity. Moreover, in later research, the CCQC method was developed to consider the effects of over-damping and non-stationary input (Yu and Zhou, 2006; Zhou and Yu, 2008), and to estimate the structural response under multi-component excitation (Yu *et al*., 2005; Song *et al*., 2007).

However, these methods still need improvement. For example, for either classical or non-classical damping systems, all the combination rules mentioned above did not incorporate the effect of repeated eigenvalues in the formulation. With the growth of the structural size and complexity of the system, very close and repeated frequencies are no longer unusual and are sometimes inevitable (Michael, 2011). Moreover, it is worth considering whether the earthquake responses corresponding to the two equal frequencies can be offset from each other. In fact, it is based on this consideration that the tuned mass damper (TMD) was developed (Fujino and Abe, 1993). As a consequence, the methods dealing with the dynamic responses of the system with repeated eigenvalues are inevitably evolved (Katsuhiko, 2006; Yao and Gao, 2011; Li *et al*., 2013; Long *et al*., 2014). In our previous studies, the generally damped linear systems with repeated eigenvalues and defective eigenvectors, a hybrid decomposition approach was developed to calculate the dynamic response of the structure, which incorporates the merits of the modal superposition method and the residue matrix decomposition method, and does not need to consider the defective characteristics of the eigenvectors corresponding to repeated eigenvalues (Yu *et al*., 2012). However, the previous research only deduced dynamic responses of damped systems with repeated eigenvalues in the time domain but did not obtain the calculation algorithm based on earthquake response-spectrum, which limits its application in earthquake engineering.

The purpose of this study is to develop a response-spectrum mode superposition method which can consider the repeated-frequency characteristics based on the derived hybrid decomposition approach (Yu *et al*., 2012). Because this algorithm can consider the effect of repeated eigenvalues, it is called the CCQC-R rule, in which the correlation coefficients of high-order modal responses will be involved in addition to correlation coefficients in the normal CCQC method (Zhou *et al*., 2004). As a result, the formulas for calculating the correlation coefficients of high-order modal responses are deduced, including displacement, velocity and velocity-displacement correlation coefficients. Furthermore, the relationship between high-order displacement and velocity variance is derived to make the CCQC-R algorithm only relevant to the high-order displacement response-spectrum. Finally, the practical step-by-step method for calculating the high-order displacement response spectrum is discussed and tested for application in earthquake engineering.

2 Decomposition technology of damped linear systems with repeated eigenvalues

For a discrete system, with N degrees of freedom, the equations of motion in terms of nodal displacements are expressed as

$$\boldsymbol{M}\ddot{\boldsymbol{y}} + \boldsymbol{C}\dot{\boldsymbol{y}} + \boldsymbol{K}\boldsymbol{y} = -\boldsymbol{M}\boldsymbol{e}\ddot{y}_{\mathrm{g}}(t) \tag{1}$$

Here $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the $N \times N$ mass, damping and stiffness matrices, which are real symmetric matrices. Also, the damping matrix $\boldsymbol{C}$ should be a constant matrix due to consideration of the viscous damping. $\boldsymbol{y}$ is a $N \times 1$ nodal displacement vector which describes the dynamic response of the structure, and N is an arbitrarily large integer. $\boldsymbol{e}$ is a unit vector with dimension $N \times 1$, and $\ddot{y}_{\mathrm{g}}(t)$ is the arbitrary time history of ground acceleration. Equation (1) can also be rewritten as a group of first-order linear differential equations, that is

$$\dot{\boldsymbol{x}} = \boldsymbol{D}\boldsymbol{x} + \boldsymbol{b}\ddot{y}_{\mathrm{g}}(t) \tag{2}$$

in which

$$\boldsymbol{D} = \begin{bmatrix} -\boldsymbol{M}^{-1}\boldsymbol{C} & -\boldsymbol{M}^{-1}\boldsymbol{K} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} -\boldsymbol{e} \\ \boldsymbol{0} \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} \dot{\boldsymbol{y}} \\ \boldsymbol{y} \end{bmatrix} \tag{3}$$

and $\boldsymbol{I}$ is identity matrix with dimension $N \times N$.

The eigenvalues corresponding to the system expressed by Eq. (2) can be divided into two types: distinct eigenvalues and repeated eigenvalues. Correspondingly, the structural responses can also be divided into two groups according to distinct and repeated eigenvalues. And the response contributions from two groups can be calculated through different methods. Therefore, a hybrid decomposition method was deduced by making full use of the merits of the mode superposition method and residue matrix decomposition method (Yu *et al.*, 2012), which can be treated as an alternative expression form of the generalized complex mode analysis method.

Suppose the different conjugate eigenvalues of the system are $(\lambda_m, \bar{\lambda}_m)$ ($m = 1, 2, \cdots, z$) with multiplicity k_m ($k_m \geq 1$). Now separate the z_1-pairs distinct eigenvalues λ_m with $k_m = 1$ from the eigenvalues, and renumber the eigenvalues $(\lambda_m, \bar{\lambda}_m)$ ($m = 1, \cdots z_1, z_1 + 1, \cdots, z$) according to the corresponding multiplicity k_m, that is

$$\begin{cases} \lambda_m & (k_m = 1,\ m \leq z_1) \\ \lambda_m & (k_m \geq 2,\ z_1 + 1 < m \leq z) \end{cases}$$

The displacement responses of the structure in the time domain can then be written as

$$y(t) = y_{\mathrm{D}}(t) + y_{\mathrm{M}}(t) = \sum_{m=1}^{z_1} \{y_{\mathrm{D}}(t)\}_m + \sum_{m=z_1+1}^{z} \{y_{\mathrm{M}}(t)\}_m \tag{4}$$

in which the first part $y_{\mathrm{D}}(t)$ represents the linear combination of the displacement responses of the z_1

SDOF oscillators. The mode superposition method mentioned by Zhou *et al.* (2004) is recommended to calculate the first part in Eq. (4) to reduce the amount of computation, that is

$$y_{\mathrm{D}}(t)=\sum_{m=1}^{z_1}\widehat{\boldsymbol{A}}_{m,1}q_{m,1}(t)+\widehat{\boldsymbol{B}}_{m,1}\dot{q}_{m,1}(t) \tag{5}$$

in which the subscript $(m,1)$ represents the order and corresponding multiplicity of eigenvalue, respectively. In the case of distinct eigenvalues, the multiplicity is equal to 1. $\widehat{\boldsymbol{A}}_{m,1}$ and $\widehat{\boldsymbol{B}}_{m,1}$ are the generalized participation factors corresponding to the m-th distinct eigenvalue λ_m, which can be calculated based on the orthogonality of the eigenvectors provided by the characteristic equation, or be calculated based on the residue matrix $\boldsymbol{R}_{m,1}$ (Yu *et al.*, 2012), that is

$$\widehat{\boldsymbol{A}}_{m,1}=-2\alpha_m\,\mathrm{Re}\left(\boldsymbol{R}_{m,1}\right)\boldsymbol{Me}+2\beta_m\,\mathrm{Im}\left(\boldsymbol{R}_{m,1}\right)\boldsymbol{Me}$$

$$\widehat{\boldsymbol{B}}_{m,1}=-2\,\mathrm{Re}\left(\boldsymbol{R}_{m,1}\right)\boldsymbol{Me}$$

where $\alpha_m=\zeta_m\omega_m$ and $\beta_m=\omega_m\sqrt{1-\zeta_m^2}$ are the damping coefficient and the damped frequency of the m-th mode, and ω_m and ζ_m are the frequency and the corresponding damping ratio. Re (.) and Im (.) represent the operation of extracting the real and imaginary part of complex number.

And, $q_{m,1}(t)$ and $\dot{q}_{m,1}(t)$ in Eq. (5) can be calculated by solving the following equation

$$\ddot{q}_{m,1}(t)+2\zeta_m\omega_m\dot{q}_{m,1}(t)+\omega_m^2q_{m,1}(t)=-\ddot{y}_{\mathrm{g}}(t) \tag{6}$$

i.e.

$$q_{m,1}(t)=-\frac{1}{\beta_m}\int_0^t \mathrm{e}^{-\alpha_m(t-\tau)}\sin\left(\beta_m(t-\tau)\right)\ddot{y}_{\mathrm{g}}(\tau)\,\mathrm{d}\tau \tag{7}$$

and define the expression

$$h_{m,1}(t-\tau)=\frac{1}{\beta_m}\mathrm{e}^{-\alpha_m(t-\tau)}\sin\left(\beta_m(t-\tau)\right) \tag{8}$$

is the impulse response function, which is the displacement response of the SDOF oscillator due to unitary velocity at initial time $t=0$.

The second part $y_{\mathrm{M}}(t)$ represents the linear combination of the displacement responses of the $z-z_1$ coupled systems, in which $\{y_{\mathrm{M}}(t)\}_m$ is the response of the mth coupled system. According to the derived hybrid decomposition approach (Yu *et al*,. 2012), the response $y_{\mathrm{M}}(t)$ can be calculated by the residue matrix decomposition method, in which the coupled system corresponding to a repeated eigenvalue will be handled as a coupled-system, and no longer decomposed into smaller systems. Thus, it is not necessary to calculate the geometric multiplicity of the repeated eigenvalue λ_m and to determine the corresponding independent vectors and derived-vectors of the repeated eigenvalue, which is a time consuming task for a large system. The second part $y_{\mathrm{M}}(t)$ in Eq. (4) can be expressed as

$$y_{\mathrm{M}}(t)=\sum_{m=z_1+1}^{z}\sum_{j=1}^{k_m}\{y_{\mathrm{M}}(t)\}_{mj} \tag{9}$$

in which $\{y_{\mathrm{M}}(t)\}_{mj}$ is the jth order response corresponding to repeated eigenvalue λ_m with multiplicity k_m, the values of j varies from 1 to k_m. Based on the residue matrix decomposition method, Eq. (9) can be expressed as

$$\boldsymbol{y}_{\mathrm{M}}(t)=\sum_{m=z_1+1}^{z}\sum_{j=1}^{k_m}\left(\tilde{\boldsymbol{G}}_{m,j}q_{m,j-1}+\tilde{\boldsymbol{A}}_{m,j}q_{m,j}+\tilde{\boldsymbol{B}}_{m,j}\dot{q}_{m,j}\right) \tag{10}$$

where vectors $\tilde{\boldsymbol{G}}_{m,j}$, $\tilde{\boldsymbol{A}}_{m,j}$ and $\tilde{\boldsymbol{B}}_{m,j}$ are the generalized participation factors of the j-th order response corresponding to the repeated eigenvalue λ_m, which can be determined based on residue matrix $\boldsymbol{R}_{m,j}$ corresponding to the term $(\lambda-\lambda_m)^j$ of repeated eigenvalue λ_m (Yu *et al.*, 2012), that is

$$\tilde{\boldsymbol{G}}_{m,j}=\frac{2(j-1)}{\beta_m^{j-2}}\mathrm{Re}\left(\boldsymbol{R}_{m,j}\right)\boldsymbol{Me}$$

$$\tilde{\boldsymbol{A}}_{m,j}=-\frac{2}{\beta_m^{j-1}}\left[\alpha_m\,\mathrm{Re}\left(\boldsymbol{R}_{m,j}\right)\boldsymbol{Me}-\beta_m\,\mathrm{Im}\left(\boldsymbol{R}_{m,j}\right)\boldsymbol{Me}\right]$$

$$\tilde{\boldsymbol{B}}_{m,j}=-\frac{2}{\beta_m^{j-1}}\mathrm{Re}\left(\boldsymbol{R}_{m,j}\right)\boldsymbol{Me}$$

And $q_{m,j-1}(t)$ and $q_{m,j}(t)$ in Eq. (10) can be calculated by

$$q_{m,j-1}(t)=-\frac{1}{\beta_m}\int_0^t\left[\beta_m(t-\tau)\right]^{j-2}\mathrm{e}^{-\alpha_m(t-\tau)}\sin\left(\beta_m(t-\tau)\right)\ddot{y}_{\mathrm{g}}(\tau)\,\mathrm{d}\tau \tag{11}$$

$$q_{m,j}(t)=-\frac{1}{\beta_m}\int_0^t\left[\beta_m(t-\tau)\right]^{j-1}\mathrm{e}^{-\alpha_m(t-\tau)}\sin\left(\beta_m(t-\tau)\right)\ddot{y}_{\mathrm{g}}(\tau)\,\mathrm{d}\tau \tag{12}$$

and defined

$$h_{m,j}(t-\tau)=\frac{1}{\beta_m}\left[\beta_m(t-\tau)\right]^{j-1}\mathrm{e}^{-\alpha_m(t-\tau)}\sin\beta_m(t-\tau) \tag{13}$$

is the high-order impulse response function corresponding to the j-th order of the repeated root λ_m. Note that the impulse transfer function expressed by Eq. (13) has an additional dimensionless term $\left[\beta_m(t-\tau)\right]^{j-1}$ compared to Eq. (8) corresponding to distinct roots.

Because the structural responses of the distinct eigenvalues can be treated as the case when the multiplicity k_m of the eigenvalue is equal to 1, the

Eq. (4) can be generalized as the equation of the double-summation expression, i.e.

$$\boldsymbol{y}(t)=\sum_{m=1}^{z}\sum_{j=1}^{k_m}\left(\boldsymbol{G}_{m,j}q_{m,j-1}+\boldsymbol{A}_{m,j}q_{m,j}+\boldsymbol{B}_{m,j}\dot{q}_{m,j}\right) \quad (14)$$

in which

$$\boldsymbol{G}=\left[\boldsymbol{0},\cdots,\boldsymbol{0},\tilde{\boldsymbol{G}}_{z_1+1,1},\cdots,\tilde{\boldsymbol{G}}_{z_1+1,k_{z_1+1}},\cdots,\tilde{\boldsymbol{G}}_{z,1},\cdots,\tilde{\boldsymbol{G}}_{z,k_z}\right]$$

$$\boldsymbol{A}=\left[\widehat{\boldsymbol{A}}_{1,1},\cdots,\widehat{\boldsymbol{A}}_{z_1,1},\tilde{\boldsymbol{A}}_{z_1+1,1},\cdots,\tilde{\boldsymbol{A}}_{z_1+1,k_{z_1+1}},\cdots,\tilde{\boldsymbol{A}}_{z,1},\cdots,\tilde{\boldsymbol{A}}_{z,k_z}\right]$$

$$\boldsymbol{B}=\left[\widehat{\boldsymbol{B}}_{1,1},\cdots,\widehat{\boldsymbol{B}}_{z_1,1},\tilde{\boldsymbol{B}}_{z_1+1,1},\cdots,\tilde{\boldsymbol{B}}_{z_1+1,k_{z_1+1}},\cdots,\tilde{\boldsymbol{B}}_{z,1},\cdots,\tilde{\boldsymbol{B}}_{z,k_z}\right]$$

3 Response-spectrum mode superposition method for damped system with repeated eigenvalues

For the generally non-classically damped system with complex modes, it can be seen from Eq. (14) that the structural modal response not only depends upon the displacement response of the separated oscillator but also relates to the corresponding velocity response. Moreover, because the effect of repeated eigenvalues is considered, the high-order modal displacement and velocity are involved in Eq. (14) besides the one-order modal responses, which make the mode superposition method more difficult than that of classical one without repeated roots. However the response-spectrum mode superposition method of earthquake response for non-classically damped system with repeated eigenvalues is able to be deduced according to following steps.

According to the stationary random vibration theory, the deviation or mean square response of $\boldsymbol{y}(t)$ can be calculated by

$$E\left[\boldsymbol{y}^2(t)\right]=\sum_{n=1}^{z}\sum_{m=1}^{z}\sum_{i=1}^{k_n}\sum_{j=1}^{k_m}\begin{bmatrix}\boldsymbol{G}_{n,i}\boldsymbol{G}_{m,j}<q_{n,i-1}(t)q_{m,j-1}(t)>+\boldsymbol{A}_{n,i}\boldsymbol{A}_{m,j}<q_{n,i}(t)q_{m,j}(t)>+\\ \boldsymbol{B}_{n,i}\boldsymbol{B}_{m,j}<\dot{q}_{n,i}(t)\dot{q}_{m,j}(t)>+2\boldsymbol{A}_{n,i}\boldsymbol{G}_{m,j}<q_{n,i}(t)q_{m,j-1}(t)>+\\ 2\boldsymbol{B}_{n,i}\boldsymbol{G}_{m,j}<\dot{q}_{n,i}(t)q_{m,j-1}(t)>+2\boldsymbol{B}_{n,i}\boldsymbol{A}_{m,j}<\dot{q}_{n,i}(t)q_{m,j}(t)>\end{bmatrix} \quad (15)$$

in which the symbol $< >$ represents the operation for the calculation of average.

It can be concluded from Eq. (15) that, regarding the covariance between different modal responses for the damped system with repeated eigenvalues, the following situations are possible: (1) The covariance among the modal responses corresponding to a repeated eigenvalue. For instance, for the repeated root λ_n with multiplicity k_n, there are k_n modal responses, and each of them may have a different transfer function. Because these SDOF systems are subject to the same earthquake input, a relationship exists between modal responses corresponding to a repeated-root. (2) The covariance between the modal responses corresponding to a repeated-root and responses responding to distinct-roots. (3) The covariance between the modal responses corresponding to a repeated-root and responses of the other repeated-root. The key question in solving Eq. (15) is to calculate the covariance of the modal responses, such as $<q_{n,i}(t)q_{m,j}(t)>$, $<\dot{q}_{n,i}(t)\dot{q}_{m,j}(t)>$ and $<\dot{q}_{n,i}(t)q_{m,j}(t)>$, which are related to the high-order displacement and velocity responses. For damped systems without repeated eigenvalues, the covariances of displacement, velocity and displacement-velocity were provided by Zhou *et al*. (2004). Therefore, the focus herein is how to calculate the covariance among high-order modal responses corresponding to repeated roots.

3.1 Displacement correlation coefficient

Considering the expression given by Eq. (13) and noticing that the impulse response function is deterministic, the covariance $<q_{n,i}(t)q_{m,j}(t)>$, which is between the ith order modal displacement corresponding to the repeated eigenvalue λ_n and the jth order modal displacement corresponding to the repeated eigenvalue λ_m, can be expressed as

$$<q_{n,i}(t)q_{m,j}(t)>=\int_0^t h_{n,i}(t-\tau)h_{m,j}(t-\tau)<\ddot{y}_{gn}(\tau)\ddot{y}_{gm}(\tau)>\mathrm{d}\tau \quad (16)$$

Assuming ground motion excitation $\ddot{y}_{gm}(t)$ and $\ddot{y}_{gn}(t)$ involved in Eq. (16) are stationary white noise starting from 0, with zero mean value, results in $<\ddot{y}_{gn}(\tau)\ddot{y}_{gm}(\tau)>=2\pi S_0$, where S_0 is the severity of the ground motion $\ddot{y}_{gm}(t)$. Then,

$$<q_{n,i}(t)q_{m,j}(t)>=O_{ij}^{\mathrm{D}nm}(t)=2\pi S_0\int_0^t h_{n,i}(t-\tau)h_{m,j}(t-\tau)\mathrm{d}\tau \quad (17)$$

Actually, the earthquake ground motion excitations are non-stationary. Zhou and Yu (2008) deduced the covariances of displacement, velocity and displacement-velocity considering the non-stationarity of the earthquake by introducing the envelope function. However, in this study, the ground motion excitation is assumed as a white noise process because the complexity of repeated eigenvalues is considered.

Substituting Eq. (13) into Eq. (17),

$$O_{ij}^{\mathrm{D}nm}(t)=2\pi S_0\beta_n^{i-2}\beta_m^{j-2}\int_0^t(t-\tau)^l\mathrm{e}^{H(t-\tau)}\sin\beta_n(t-\tau)\sin\beta_m(t-\tau)\mathrm{d}\tau \quad (18)$$

here $H=-\alpha_{\mathrm{n}}-\alpha_{\mathrm{m}}$, $l=i+j-2$ ($i=1,\ldots,k_n$, $j=1,\ldots,k_m$), in which k_n and k_m are the multiplicity of repeated-eigenvalues λ_n and λ_m, respectively.

Let $t-\tau=s$, we have $\mathrm{d}(t-\tau)=\mathrm{d}s$ and $\mathrm{d}\tau=-\mathrm{d}s$, then Eq. (18) can be rewritten as

$$O_{ij}^{\mathrm{D}nm}(t)=-2\pi S_0\beta_n^{i-2}\beta_m^{j-2}\int_t^0 s^l\mathrm{e}^{Hs}\sin\beta_n s\sin\beta_m s\mathrm{d}s \quad (19)$$

For the convenience of calculation, Eq. (19) can be represented as the difference between the two cosine functions, that is

$$O_{ij}^{\mathrm{D}nm}(t)=\pi S_0\beta_n^{i-2}\beta_m^{j-2}\int_t^0 s^l \mathrm{e}^{Hs}\left(\cos L_2 s-\cos L_1 s\right)\mathrm{d}s$$
$$=\pi S_0\beta_n^{i-2}\beta_m^{j-2}\left[V_{l,2}(t)-V_{l,1}(t)\right] \tag{20}$$

here $L_1=\beta_n-\beta_m$, $L_2=\beta_n+\beta_m$. The deduction of $V_{l,k}(t)$ ($k=1,2$) is given in Appendix 1, in which the recursion formula of $V_{l,k}(t)=\int_t^0 s^l \mathrm{e}^{Hs}\cos L_k s\mathrm{d}s$ when $l=0,1,2,3,4$ are given.

When $t\to\infty$, the steady state solution of displacement covariance $O_{ij}^{\mathrm{D}nm}(t)$, called $I_{nm,ij}^{\mathrm{dd}}$, can be expressed as

$$I_{nm,ij}^{\mathrm{dd}}=O_{ij}^{\mathrm{D}nm}(t\to\infty)=\pi S_0\beta_n^{i-2}\beta_m^{j-2}\left[\bar{V}_{l,2}-\bar{V}_{l,1}\right] \tag{21}$$

where $\bar{V}_{l,k}$ is the steady state solution of $V_{l,k}(t\to\infty)$. Table 1 lists the calculation formula of $\bar{V}_{l,k}$ when $l=0,1,2,3,4$, in which the values L_k represents L_1 and L_2, respectively.

Substituting formula of $\bar{V}_{l,k}$ listed in Table 1 into Eq. (21), the steady state solution $I_{nm,ij}^{\mathrm{dd}}$ of the displacement covariance corresponding to a high-order response can be calculated. When $n\neq m$, the calculation formula of $I_{nm,ij}^{\mathrm{dd}}$ corresponding to three-repeated eigenvalue are listed in column 3 of Table 2, in which the possible combinations of (i,j) when $l=0,1,2,3,4$ are listed in column 2 of Table 2. It can be seen from these expressions that the displacement covariance is only related to the cosine function.

When $l=0$, $i=1$ and $j=1$, and substituting formula of $\bar{V}_{0,k}$ into expression of $I_{nm,ij}^{\mathrm{dd}}$, results in

$$I_{nm,11}^{\mathrm{dd}}=\frac{4\pi S_0\left(\zeta_n\omega_n+\zeta_m\omega_m\right)}{\left(\omega_n^2-\omega_m^2\right)^2+4\zeta_n\zeta_m\omega_n\omega_m\left(\omega_n^2+\omega_m^2\right)+4\omega_n^2\omega_m^2\left(\zeta_n^2+\zeta_m^2\right)} \tag{22}$$

Table 1 Calculation formula of $\bar{V}_{l,k}$ ($l=0,1,2,3,4$)

$l=0$	$\bar{V}_{0,\mathrm{k}}=\dfrac{H}{H^2+L_\mathrm{k}^2}$	$l=3$	$\bar{V}_{3,k}=-\dfrac{6\left(H^4-6H^2L_k^2+L_k^4\right)}{\left(H^2+L_k^2\right)^4}$
$l=1$	$\bar{V}_{1,k}=-\dfrac{H^2-L_k{}^2}{\left(H^2+L_k^2\right)^2}$	$l=4$	$\bar{V}_{4,k}=\dfrac{24H\left(H^4-10H^2L_k^2+5L_k^4\right)}{\left(H^2+L_k^2\right)^5}$
$l=2$	$\bar{V}_{2,k}=\dfrac{2H\left(H^2-3L_k^2\right)}{\left(H^2+L_k^2\right)^3}$		

Table 2 Covariances of high-order modal displacement response

l	(i,j)	$n\neq m$	$n=m$
0	$(i=1,j=1)$	$I_{nm,11}^{\mathrm{dd}}=\dfrac{\pi S_0}{\beta_n\beta_m}\left(\bar{V}_{0,2}-\bar{V}_{0,1}\right)$	$\dfrac{\pi S_0}{2}\dfrac{1}{\zeta_n\omega_n^3}$
1	$(i=1,j=2)$	$I_{nm,12}^{\mathrm{dd}}=\dfrac{\pi S_0}{\beta_n}\left(\bar{V}_{1,2}-\bar{V}_{1,1}\right)$	$\dfrac{\pi S_0\beta_n}{4}\dfrac{1+2\zeta_n^2}{\zeta_n^2\omega_n^4}$
	$(i=2,j=1)$	$I_{nm,21}^{\mathrm{dd}}=\dfrac{\pi S_0}{\beta_m}\left(\bar{V}_{1,2}-\bar{V}_{1,1}\right)$	
2	$(i=1,j=3)$	$I_{nm,13}^{\mathrm{dd}}=\dfrac{\pi S_0\beta_m}{\beta_n}\left(\bar{V}_{2,2}-\bar{V}_{2,1}\right)$	$\dfrac{\pi S_0}{4}\dfrac{1+\zeta_n^4\left(3-4\zeta_n^2\right)}{\zeta_n^3\omega_n^3}$
	$(i=2,j=2)$	$I_{nm,22}^{\mathrm{dd}}=\pi S_0\left(\bar{V}_{2,2}-\bar{V}_{2,1}\right)$	
	$(i=3,j=1)$	$I_{nm,31}^{\mathrm{dd}}=\dfrac{\pi S_0\beta_n}{\beta_m}\left(\bar{V}_{2,2}-\bar{V}_{2,1}\right)$	
3	$(i=2,j=3)$	$I_{nm,23}^{\mathrm{dd}}=\pi S_0\beta_m\left(\bar{V}_{3,2}-\bar{V}_{3,1}\right)$	$\dfrac{3\pi S_0\beta_n}{8}\dfrac{1-\zeta_n^4\left(8\zeta_n^4-8\zeta_n^2+1\right)}{\zeta_n^4\omega_n^4}$
	$(i=3,j=2)$	$I_{nm,32}^{\mathrm{dd}}=\pi S_0\beta_n\left(\bar{V}_{3,2}-\bar{V}_{3,1}\right)$	
4	$(i=3,j=3)$	$I_{nm,33}^{\mathrm{dd}}=\pi S_0\beta_n\beta_m\left(\bar{V}_{4,2}-\bar{V}_{4,1}\right)$	$\dfrac{3\pi S_0\beta_n^2}{4}\dfrac{1-\zeta_n^6\left(16\zeta_n^4-20\zeta_n^2+5\right)}{\zeta_n^5\omega_n^5}$

This expression is in accordance with the result corresponding to the distinct roots (Zhou *et al*., 2004). For the damped systems with repeated eigenvalues, $I^{\mathrm{dd}}_{nm,11}$ represents three cases: (1) The covariance of the modal displacement corresponding to different distinct-roots λ_n and λ_m; (2) The covariance between the modal displacement corresponding to a distinct-root λ_n and the 1-th order modal displacement corresponding to a repeated-root λ_m; (3) The covariance of the 1-th order modal displacement corresponding to different repeated-roots λ_n and λ_m.

When $l=1$, there are two combinations of (i,j); they are $(i=1, j=2)$ and $(i=2, j=1)$. $I^{\mathrm{dd}}_{nm,12}$ represents two cases: (1) The covariance between the modal displacement corresponding to a distinct-root λ_n and the 2-th order modal displacement corresponding to a repeated-root λ_m; and (2) The covariance between the 1-th order modal displacement corresponding to a repeated-root λ_n and the 2-th order modal displacement corresponding to a repeated-root λ_m. Substituting formula of $\bar{V}_{1,k}$ into expression of $I^{\mathrm{dd}}_{nm,ij}$, results in

$$I^{\mathrm{dd}}_{nm,ij}=\frac{4\pi S_0\beta_n^{i-1}\beta_m^{j-1}\left[4(\omega_n\zeta_n+\omega_m\zeta_m)(\omega_n^3\zeta_n+\omega_m^3\zeta_m)+8\zeta_n\zeta_m\omega_n\omega_m(\omega_n\zeta_n+\omega_m\zeta_m)^2-(\omega_n^2-\omega_m^2)^2\right]}{\left[(\omega_n^2-\omega_m^2)^2+4\zeta_n\zeta_m\omega_n\omega_m(\omega_n^2+\omega_m^2)+4\omega_n^2\omega_m^2(\zeta_n^2+\zeta_m^2)\right]^2} \tag{23}$$

It can be seen that when the order of n and m is exchanged, only the term $\beta_n^{i-1}\beta_m^{j-1}$ in expression (23) is changed; that is, for two different eigenvalues, regardless of which is the repeated root, the displacement covariance is essentially the same. For the other combinations of (i, j), the expression of $I^{\mathrm{dd}}_{nm,ij}$ related to ω_n, ω_m, ζ_n and ζ_m can also be obtained by substituting $\bar{V}_{l,k}$ into the calculation formula of $I^{\mathrm{dd}}_{nm,ij}$ listed in Table 2.

When $n=m$, for a repeated-eigenvalue λ_m with multiplicity k_m, $I^{\mathrm{dd}}_{mm,ij}$ represents the covariance among k_m modal displacements, whose calculation formula are listed in column 4 of Table 2.

The displacement correlation coefficient $\rho^{\mathrm{dd}}_{nm,i,j}$ is defined next as

$$\rho^{\mathrm{dd}}_{nm,i,j}=\frac{I^{\mathrm{dd}}_{nm,ij}}{\sqrt{I^{\mathrm{dd}}_{nn,ij}}\sqrt{I^{\mathrm{dd}}_{mm,ij}}} \tag{24}$$

which is a standardized function of modal displacement covariance. In order to intuitively observe the variation of the displacement correlation coefficient, let $\zeta_m=0.05$, $\omega_m=2\pi$, and make the ratios of ζ_n/ζ_m and ω_n/ω_m change from 0.1 to 10. The variation of $\rho^{\mathrm{dd}}_{nm,i,j}$ ($i=1$, $j=1,2,3$) along with ζ_n/ζ_m and ω_n/ω_m are shown in Fig. 1. It can be seen that the variation of the displacement correlation coefficient $\rho^{\mathrm{dd}}_{nm,1,1}$ among one-order modal displacements is consistent with that of distinct eigenvalues (Yu and Zhou, 2006), but the variation of correlation coefficient $\rho^{\mathrm{dd}}_{nm,1,j}$ between one-order and high-order modal displacements is different from the changes of $\rho^{\mathrm{dd}}_{nm,1,1}$.

3.2 Velocity correlation coefficient

Based on Eq. (13), the one-order differential coefficient of a high-order impulse response function is calculated, that is

$$\dot{h}_{m,j}(t)=(j-1)(\beta_m t)^{j-2}\,\mathrm{e}^{\alpha_m t}\sin\beta_m t+\frac{(\beta_m t)^{j-1}\,\mathrm{e}^{\alpha_m t}}{\beta_m}\cdot(\alpha_m\sin\beta_m t+\beta_m\cos\beta_m t) \tag{25}$$

Then the velocity covariance $<\dot{q}_{n,i}(t)\dot{q}_{m,j}(t)>$ between the ith order modal velocity of a repeated eigenvalue λ_n and the jth order modal velocity of a repeated eigenvalue λ_m, can be expressed as

$$<\dot{q}_{n,i}(t)\dot{q}_{m,j}(t)>=O^{\mathrm{V}nm}_{ij}(t)=2\pi S_0\int_0^t\dot{h}_{n,i}(t-\tau)\dot{h}_{m,j}(t-\tau)\mathrm{d}\tau \tag{26}$$

Substituting Eq. (25) into Eq. (26), and let $t-\tau=s$, results in $\mathrm{d}\tau=-\mathrm{d}s$, then Eq. (26) can be rewritten as the integration about the sine and cosine function. The recursion formula of the integration

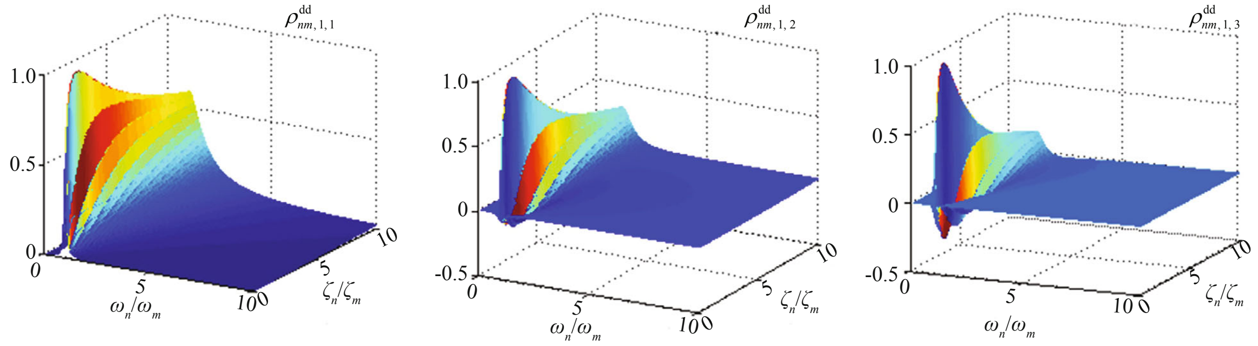

Fig. 1 Changes of displacement correlation coefficients $\rho^{\mathrm{dd}}_{nm,i,j}$ ($i=1$, $j=1,2,3$)

$U_{l,k}(t)=\int_t^0 s^l e^{Hs}\sin L_k s\mathrm{d}s$ when $l=0,1,2,3,4$ are also listed in Appendix 1, in which $L_1=\beta_n-\beta_m$, $L_2=\beta_n+\beta_m$. $l=i+j-2$ ($i=1,...,k_n$, $j=1,...,k_m$), in which k_n and k_m are the multiplicity of repeated-eigenvalues λ_n and λ_m, respectively.

When $t\to\infty$, the steady state solution of $V_{l,k}(t\to\infty)$ have been obtained, as is listed in Table 1. When $l=0,1,2,3,4$, the steady state solutions of $U_{l,k}(t\to\infty)$, expressed as $\bar{U}_{l,k}$ are shown in Table 3, in which the values L_k represents L_1 and L_2, respectively.

Based on expressions of $\bar{V}_{l,k}$ and $\bar{U}_{l,k}$, the steady state solution of the velocity covarianceis $O_{ij}^{\mathrm{V}nm}(t\to\infty)$ obtained and expressed as $I_{nm,ij}^{\mathrm{vv}}$. Table 4 gives the calculation formula of $I_{nm,ij}^{\mathrm{vv}}$ corresponding to three-repeated root that have nine combinations of (i,j), in which $V_{l-}^{*}=\bar{V}_{l,2}-\bar{V}_{l,1}$, $V_{l+}^{*}=\bar{V}_{l,2}+\bar{V}_{l,1}$, $U_{l-}^{*}=\bar{U}_{l,2}-\bar{U}_{l,1}$, $U_{l+}^{*}=\bar{U}_{l,2}+\bar{U}_{l,1}$ ($l=0,1,2,3,4$). It can be seen from these expressions that the velocity covariances are related to the sine and cosine function.

Similarly, when $l=0$, $i=1$ and $j=1$, substituting $\bar{V}_{0,k}$ and $\bar{U}_{0,k}$ into expression of $I_{nm,11}^{\mathrm{vv}}$, results in

$$I_{nm,11}^{\mathrm{vv}}=\frac{4\pi S_0\omega_n\omega_m\left(\zeta_n\omega_m+\zeta_m\omega_n\right)}{\left(\omega_n^2-\omega_m^2\right)^2+4\zeta_n\zeta_m\omega_n\omega_m\left(\omega_n^2+\omega_m^2\right)+4\omega_n^2\omega_m^2\left(\zeta_n^2+\zeta_m^2\right)} \tag{27}$$

This expression is in accordance with the result corresponding to the distinct roots (Zhou *et al*., 2004). For the damped systems with repeated eigenvalues, $I_{nm,11}^{\mathrm{vv}}$ represents three cases: (1) The covariance of modal velocity corresponding to different distinct-roots λ_n and λ_m; (2) The covariance between modal velocity corresponding to a distinct-root λ_n and the 1-th order modal velocity corresponding to a repeated-root λ_m; (3) The covariance among the 1-th order modal velocities corresponding to different repeated-roots λ_n and λ_m. When $l\neq 0$, the expression of $I_{nm,ij}^{\mathrm{vv}}$ related to ω_n, ω_m, ζ_n and ζ_m can also be derived by substituting formulas of $\bar{V}_{l,k}$ and $\bar{U}_{l,k}$ into the calculation formula of $I_{nm,ij}^{\mathrm{vv}}$ listed in Table 4.

When $n=m$, for a repeated root λ_m with multiplicity k_m, the calculation formula of $I_{mm,ij}^{\mathrm{vv}}$ are listed in Table 5, which represent the covariances among k_m modal velocities of 9 kinds of combinations of (i, j).

The velocity correlation coefficient $\rho_{nm,i,j}^{\mathrm{vv}}$ is defined as

$$\rho_{nm,i,j}^{\mathrm{vv}}=\frac{I_{nm,i,j}^{\mathrm{vv}}}{\sqrt{I_{nn,i,j}^{\mathrm{vv}}}\sqrt{I_{mm,i,j}^{\mathrm{vv}}}} \tag{28}$$

which is a standardized function of modal velocity covariance. Similarly, let $\zeta_m=0.05$, $\omega_m=2\pi$, and make the ratios of ζ_n/ζ_m and ω_n/ω_m change from 0.1 to 10. The variation of $\rho_{nm,i,j}^{\mathrm{vv}}$ ($i=1$, $j=1,2,3$) are shown in Fig. 2. It can be seen that the variation of the correlation coefficient $\rho_{nm,1,1}^{\mathrm{vv}}$ among one-order modal velocity is consistent with that of distinct eigenvalues (Yu and Zhou, 2006), but the variation of correlation coefficient $\rho_{nm,i,j}^{\mathrm{vv}}$ when $i\neq 1$ or $j\neq 1$ is different from the changes of $\rho_{nm,1,1}^{\mathrm{vv}}$.

3.3 Velocity-displacement correlation coefficient

Based on Eq. (13) and Eq. (25), the velocity-displacement covariance $<\dot{q}_{n,i}(t)q_{m,j}(t)>$, which is between the *i*th order modal velocity response corresponding to a repeated-eigenvalue λ_n and the *j*th order modal displacement response corresponding to a repeated-eigenvalue λ_m, can be expressed as

$$<\dot{q}_{n,i}(t)q_{m,j}(t)>O_{ij}^{\mathrm{VD}nm}(t)=2\pi S_0\int_0^t \dot{h}_{n,i}(t-\tau)h_{m,j}(t-\tau)\mathrm{d}\tau \tag{29}$$

Similarly, let $t-\tau=s$, so that $\mathrm{d}\tau=-\mathrm{d}s$. Equation (29) can be rewritten as the expression about the integrations $V_{l,k}(t)=\int_t^0 s^l e^{Hs}\cos L_k s\mathrm{d}s$ and $U_{l,k}(t)=\int_t^0 s^l e^{Hs}\sin L_k s\mathrm{d}s$. Then the recursion formula of these integrations listed in Appendix 1 can be used to obtain the formula of $O_{ij}^{\mathrm{VD}nm}(t)$. When $t\to\infty$, the steady state solution of the velocity-displacement covariance, called $I_{nm,ij}^{\mathrm{vd}}$, can be calculated by introducing the formula of $\bar{V}_{l,k}$ and $\bar{U}_{l,k}$. The calculation formula of $I_{nm,ij}^{\mathrm{vd}}$ corresponding to different combinations of (i,j) are listed in Table 6, in which $V_{l-}^{*}=\bar{V}_{l,2}-\bar{V}_{l,1}$, $V_{l+}^{*}=\bar{V}_{l,2}+\bar{V}_{l,1}$

Table 3 Calculation formula of $\bar{U}_{l,k}$ ($l=0,1,2,3,4$)

$l=0$	$\bar{U}_{0,k}=-\frac{L_k}{H^2+L_k^2}$	$l=3$	$\bar{U}_{3,k}=\frac{24L_kH\left(H^2-L_k^2\right)}{\left(H^2+L_k^2\right)^4}$
$l=1$	$\bar{U}_{1,k}=\frac{2HL_k}{\left(H^2+L_k^2\right)^2}$	$l=4$	$\bar{U}_{4,k}=-\frac{24L_k\left(5H^4-10H^2L_k^2+L_k^4\right)}{\left(H^2+L_k^2\right)^5}$
$l=2$	$\bar{U}_{2,k}=-\frac{2L_k\left(3H^2-L_k^2\right)}{\left(H^2+L_k^2\right)^3}$		

Table 4 Covariances of high-order modal velocity responses when $n \neq m$

$l=0$	(i,j)	$n \neq m$
0	$(i=1, j=1)$	$I_{nm,11}^{\mathrm{vv}} = \pi S_0 \left(\frac{\alpha_n \alpha_m}{\beta_n \beta_m} V_{0-}^* - \frac{\alpha_n}{\beta_n} U_{0+}^* - \frac{\alpha_m}{\beta_m} U_{0-}^* - V_{0+}^* \right)$
1	$(i=1, j=2)$	$I_{nm,12}^{\mathrm{vv}} = \pi S_0 \left(\frac{\alpha_n}{\beta_n} V_{0-}^* - U_{0-}^* + \frac{\alpha_n \alpha_m}{\beta_n} V_{1-}^* - \alpha_m U_{1-}^* - \frac{\alpha_n \beta_m}{\beta_n} U_{1+}^* - \beta_m V_{1+}^* \right)$
	$(i=2, j=1)$	$I_{nm,21}^{\mathrm{vv}} = \pi S_0 \left(\frac{\alpha_m}{\beta_m} V_{0-}^* - U_{0+}^* + \frac{\alpha_n \alpha_m}{\beta_m} V_{1-}^* - \alpha_n U_{1+}^* - \frac{\alpha_m \beta_n}{\beta_m} U_{1-}^* - \beta_n V_{1+}^* \right)$
2	$(i=1, j=3)$	$I_{nm,13}^{\mathrm{vv}} = \pi S_0 \left[2\beta_m \left(\frac{\alpha_n}{\beta_n} V_{1-}^* - U_{1-}^* \right) + \alpha_m \beta_m \left(\frac{\alpha_n}{\beta_n} V_{2-}^* - U_{2-}^* \right) - \beta_m^2 \left(\frac{\alpha_n}{\beta_n} U_{2+}^* + V_{2+}^* \right) \right]$
	$(i=2, j=2)$	$I_{nm,22}^{\mathrm{vv}} = \pi S_0 \left[\begin{array}{c} V_{0-}^* + (\alpha_m + \alpha_n) V_{1-}^* - \beta_m U_{1+}^* - \beta_n U_{1-}^* + \alpha_n \alpha_m V_{2-}^* \\ - \beta_n \beta_m V_{2+}^* - \alpha_m \beta_n U_{2-}^* - \alpha_n \beta_m U_{2+}^* \end{array} \right]$
	$(i=3, j=1)$	$I_{nm,31}^{\mathrm{vv}} = \pi S_0 \left[2\beta_n \left(\frac{\alpha_m}{\beta_m} V_{1-}^* - U_{1+}^* \right) + \alpha_n \beta_n \left(\frac{\alpha_m}{\beta_m} V_{2-}^* - U_{2+}^* \right) - \beta_n^2 \left(\frac{\alpha_m}{\beta_m} U_{2-}^* + V_{2+}^* \right) \right]$
3	$(i=2, j=3)$	$I_{nm,23}^{\mathrm{vv}} = \pi S_0 \left[\begin{array}{c} 2\beta_m V_{1-}^* + \beta_m [\alpha_m + 2\alpha_n] V_{2-}^* - \beta_m (\beta_m U_{2+}^* + 2\beta_n U_{2-}^*) \\ + \alpha_m \beta_m (\alpha_n V_{3-}^* - \beta_n U_{3-}^*) - \beta_m^2 (\alpha_n U_{3+}^* + \beta_n V_{3+}^*) \end{array} \right]$
	$(i=3, j=2)$	$I_{nm,32}^{\mathrm{vv}} = \pi S_0 \left[\begin{array}{c} 2\beta_n V_{1-}^* + \beta_n [2\alpha_m + \alpha_n] V_{2-}^* - \beta_n (2\beta_m U_{2+}^* + \beta_n U_{2-}^*) \\ + \alpha_n \beta_n (\alpha_m V_{3-}^* - \beta_m U_{3+}^*) - \beta_n^2 (\alpha_m U_{3-}^* + \beta_m V_{3+}^*) \end{array} \right]$
4	$(i=3, j=3)$	$I_{nm,33}^{\mathrm{vv}} = \pi S_0 \beta_n \beta_m \left[\begin{array}{c} 4V_{2-}^* + 2(\alpha_m + \alpha_n) V_{3-}^* - 2(\beta_m U_{3+}^* + \beta_n U_{3-}^*) \\ - \beta_m (\alpha_n U_{4+}^* + \beta_n V_{4+}^*) - \alpha_m (\beta_n U_{4-}^* - \alpha_n V_{4-}^*) \end{array} \right]$

Table 5 Covariances of high-order modal velocity responses when $n = m$

$(i=1,\ j=1)$	$\frac{\pi S_0}{2} \frac{1}{\zeta_n \omega_n}$	$(i=2,\ j=2)$	$\frac{\pi S_0}{4} \frac{1-\zeta_n^4}{\omega_n \zeta_n^3}$
$(i=1,\ j=2)$ $(i=2,\ j=1)$	$\frac{\pi S_0}{4} \frac{\beta_n}{\zeta_n^2 \omega_n^2}$	$(i=2, j=3)$ $(i=3, j=2)$	$\frac{\pi S_0 \beta_n}{8} \frac{3+\zeta_n^2 \left(-2\zeta_n^4 + \zeta_n^2 - 2\right)}{\omega_n^2 \zeta_n^4}$
$(i=1,\ j=3)$ $(i=3,\ j=1)$	$\frac{\pi S_0}{4} \frac{\left(1-\zeta_n^2\right)^2}{\omega_n \zeta_n^3}$	$(i=3, j=3)$	$\frac{\pi S_0 \beta_n^2}{4} \frac{3+\zeta_n^2 \left(-4\zeta_n^6 + 3\zeta_n^4 - 2\right)}{\omega_n^3 \zeta_n^5}$

$U_{l-}^* = \bar{U}_{l,2} - \bar{U}_{l,1}$, $U_{l+}^* = \bar{U}_{l,2} + \bar{U}_{l,1}$ ($l = 0,1,2,3,4$). It is seen from these expressions that the velocity-displacement covariance is also related to the expression of sine and cosine function.

When $l=0$, $i=1$ and $j=1$, and substituting the expressions of $\bar{V}_{0,k}$ and $\bar{U}_{0,k}$ into $I_{nm,1,1}^{\mathrm{vd}}$, results in

$$I_{nm,11}^{\mathrm{vd}} = \frac{2\pi S_0 \left(\omega_m^2 - \omega_n^2\right)}{\left(\omega_n^2 - \omega_m^2\right)^2 + 4\zeta_n \zeta_m \omega_n \omega_m \left(\omega_n^2 + \omega_m^2\right) + 4\omega_n^2 \omega_m^2 \left(\zeta_n^2 + \zeta_m^2\right)} \tag{30}$$

This expression is also accordance with the result corresponding to the distinct roots (Zhou *et al*., 2004). For the damped systems with repeated eigenvalues, $I_{nm,11}^{\mathrm{vd}}$ represents three cases: (1) The covariance between modal velocity corresponding to a distinct-root λ_n and modal displacement corresponding to a distinct-root λ_m; (2) The covariance between modal velocity corresponding to a distinct-root λ_n and the 1-th order modal displacement corresponding to a repeated-root λ_m; and (3) The covariance between 1-th order modal velocity corresponding to a repeated-root λ_n and 1-th order modal displacement corresponding to another

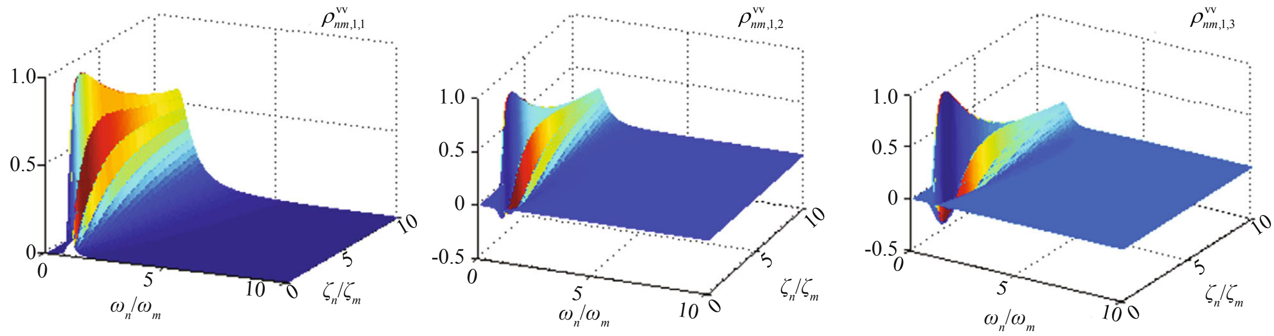

Fig. 2 Changes of velocity correlation coefficients $\rho^{vv}_{nm,i,j}$ ($i=1$, $j=1,2,3$)

Table 6 Covariances of high-order modal velocity-displacement responses

l	(i,j)	$n \neq m$	$n = m$
0	$(i=1, j=1)$	$I^{vd}_{nm,11} = \pi S_0 \dfrac{\alpha_n V^*_{0-} - \beta_n U^*_{0-}}{\beta_n \beta_m}$	0
1	$(i=1, j=2)$	$I^{vd}_{nm,12} = \dfrac{\pi S_0}{\beta_n}\left[\alpha_n V^*_{1-} - \beta_n U^*_{1-}\right]$	$\dfrac{\pi S_0}{\beta_n}\dfrac{\left(\zeta_n^2-1\right)}{4\omega_n\zeta_n}$
	$(i=2, j=1)$	$I^{vd}_{nm,12} = \dfrac{\pi S_0}{\beta_m}\left[V^*_{0-} + \alpha_n V^*_{1-} - \beta_n U^*_{1-}\right]$	$-\dfrac{\pi S_0}{\beta_n}\dfrac{\left(\zeta_n^2-1\right)}{4\zeta_n\omega_n}$
2	$(i=1, j=3)$	$I^{vd}_{nm,13} = \dfrac{\pi S_0 \beta_m}{\beta_n}\left(\alpha_n V^*_{2-} - \beta_n U^*_{2-}\right)$	$\pi S_0\left(\dfrac{2\zeta_n^4-\zeta_n^2-1}{4\zeta_n^2\omega_n^2}\right)$
	$(i=2, j=2)$	$I^{vd}_{nm,22} = \pi S_0\left(V^*_{1-} + \alpha_n V^*_{2-} - \beta_n U^*_{2-}\right)$	0
	$(i=3, j=1)$	$I^{vd}_{nm,31} = \dfrac{\pi S_0 \beta_n}{\beta_m}\left(2V^*_{1-} + \alpha_n V^*_{2-} - \beta_n U^*_{2-}\right)$	$-\pi S_0\left(\dfrac{2\zeta_n^4-\zeta_n^2-1}{4\zeta_n^2\omega_n^2}\right)$
3	$(i=2, j=3)$	$I^{vd}_{nm,23} = \pi S_0 \beta_m\left[V^*_{2-} + \alpha_n V^*_{3-} - \beta_n U^*_{3-}\right]$	$\pi S_0 \beta_n \dfrac{4\zeta_n^6-3\zeta_n^4-1}{8\zeta_n^3\omega_n^3}$
	$(i=3, j=2)$	$I^{vd}_{nm,32} = \pi S_0 \beta_n\left[2V^*_{2-} + \alpha_n V^*_{3-} - \beta_n U^*_{3-}\right]$	$-\pi S_0 \beta_n \dfrac{4\zeta_n^6-3\zeta_n^4-1}{8\zeta_n^3\omega_n^3}$
4	$(i=3, j=3)$	$I^{vd}_{nm,33} = \pi S_0 \beta_n \beta_m\left[2V^*_{3-} + \alpha_n V^*_{4-} - \beta_n U^*_{4-}\right]$	0

repeated-root λ_m. When $l \neq 0$, the expression of $I^{vd}_{nm,ij}$ related to ω_n, ω_m, ζ_n and ζ_m can also be calculated by substituting the expressions of $\bar{V}_{l,k}$ and $\bar{U}_{l,k}$ into the calculation formula of $I^{vd}_{nm,ij}$ listed in Table 6.

When $n = m$, $I^{vd}_{mm,ij}$ represent the variances between modal velocity and displacements corresponding to a repeated-eigenvalue λ_m, which are listed in column 4 of Table 6.

Next, the velocity-displacement correlation coefficient $\rho^{vd}_{nm,i,j}$ is defined as

$$\rho^{vd}_{nm,i,j} = \frac{I^{vd}_{nm,ij}}{\sqrt{I^{vv}_{nn,ij}}\sqrt{I^{dd}_{mm,ij}}} \tag{31}$$

which is a standardized function of the modal velocity-displacement covariance. Similarly, let $\zeta_m = 0.05$, $\omega_m = 2\pi$, and make the ratios of ζ_n/ζ_m and ω_n/ω_m change from 0.1 to 10. The variation of $\rho^{vd}_{nm,i,j}$ ($i=1$, $j=1,2,3$) along with ζ_n/ζ_m and ω_n/ω_m are shown in Fig. 3. It can be seen that the velocity-displacement correlation coefficient $\rho^{vd}_{nm,1,1}$ when $i=1$ and $j=1$ is consistent with that of distinct eigenvalues (Yu and Zhou, 2006), but the variation of correlation coefficient $\rho^{vd}_{nm,i,j}$ when $i \neq 1$ or $j \neq 1$ is different from the changes of $\rho^{vd}_{nm,1,1}$.

3.4 Discussion of the relationship between displacement and velocity covariance

The relationship between modal displacement and velocity covariances of a linear SDOF system when $n = m$ are discussed in this subsection. Based on the expressions of displacement covariance $I^{dd}_{nn,ij}$ and velocity covariance $I^{vv}_{nn,ij}$,

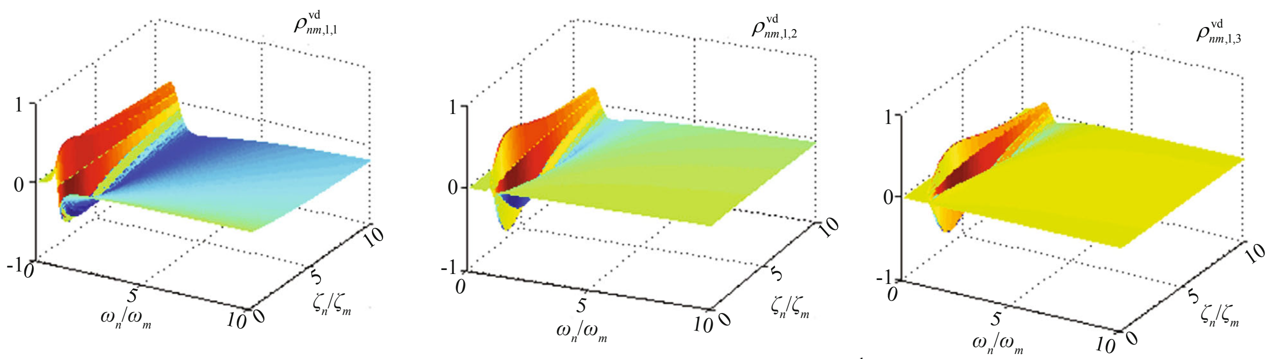

Fig. 3 Changes of velocity-displacement correlation coefficient $\rho^{vd}_{nm,i,j}$ ($i=1$, $j=1,2,3$)

$$I^{vv}_{nn,ij} = \gamma_{n,ij}\omega_n^2 I^{dd}_{nn,ij} \tag{32}$$

here n represents the order of concerned mode, and $\gamma_{n,ij}$ is the scale factor. Because $I^{vv}_{nn,ij}$ and $I^{dd}_{nn,ij}$ are only for steady state responses under stationary white noise input, this relationship is accurate. Based on the expressions of $I^{vv}_{nn,ij}$ and $I^{dd}_{nn,ij}$ listed in Table 2 and Table 5, respectively, the scale factor $\gamma_{n,ij}$ can be obtained, as shown in Table 7.

It can be seen that the expression of $\gamma_{n,ij}$ is only related to damping ratio ζ_n. When $i=1$ and $j=1$, $\gamma_{n,11}=1$, which is consistent with the result corresponding to the distinct eigenvalues (Zhou *et al*., 2004). However, when $i \neq 1$ or $j \neq 1$, $\gamma_{n,ij}$ varies from 1 to 0 along with increasing of the damping ratio ζ_n, as is shown in Fig. 4, in which ζ_n changes from 0.02 to 0.9. Since the value of the scale factor $\gamma_{n,ij}$ is small when the damping ratio ζ_n is relatively large, the scale factor should be calculated according to the formula listed in Table 7 when $i \neq 1$ or $j \neq 1$.

3.5 Complex complete quadratic combination algorithm considering repeated eigenvalues

According to the definition of Eq. (24), Eq. (28) and Eq. (31), and considering the relationship of modal displacement and velocity covariances when $n=m$, as listed in Eq. (32), Eq. (15) can be rewritten as

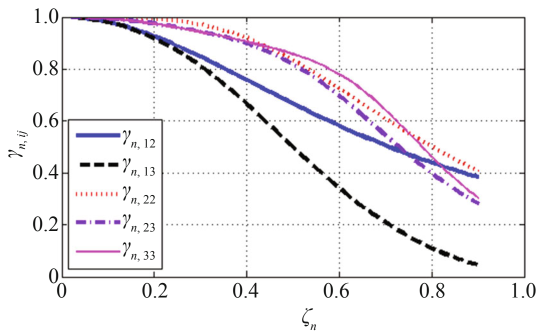

Fig. 4 Changes of scale factor $\gamma_{n,ij}$ with damping ratio ζ_n

Table 7 Scale factor between displacement and velocity covariance when $n=m$

l	(i,j)	$\gamma_{n,ij}$
0	$(i=1,\ j=1)$	1
1	$(i=1,\ j=2)$ $(i=2,\ j=1)$	$\dfrac{1}{1+2\zeta_n^2}$
2	$(i=1,\ j=3)$ $(i=3,\ j=1)$	$\dfrac{(1-\zeta_n^2)^2}{1+\zeta_n^4(3-4\zeta_n^2)}$
	$(i=2,\ j=2)$	$\dfrac{1-\zeta_n^4}{1+\zeta_n^4(3-4\zeta_n^2)}$
3	$(i=2, j=3)$ $(i=3, j=2)$	$\dfrac{1+\zeta_n^2(-2\zeta_n^4+\zeta_n^2-2)/3}{1-\zeta_n^4(8\zeta_n^4-8\zeta_n^2+1)}$
4	$(i=3, j=3)$	$\dfrac{1+\zeta_n^2(-4\zeta_n^6+3\zeta_n^4-2)/3}{1-\zeta_n^6(16\zeta_n^4-20\zeta_n^2+5)}$

$$
\begin{aligned}
&E\left[\boldsymbol{y}^2(t)\right] \\
&=\sum_{n=1}^{z}\sum_{m=1}^{z}\sum_{i=1}^{r_n}\sum_{j=1}^{r_m}\cdot \\
&\left\{\begin{aligned}
&\boldsymbol{G}_{n,i}\boldsymbol{G}_{m,j}\rho^{\mathrm{dd}}_{nm,i-1,j-1}\langle q^2_{n,i-1}(t)\rangle^{1/2}\langle q^2_{m,j-1}(t)\rangle^{1/2}+\\
&\left[\boldsymbol{A}_{ni}\boldsymbol{A}_{mj}\rho^{\mathrm{dd}}_{nm,i,j}+\sqrt{\gamma_{n,ij}\gamma_{m,ij}}\,\omega_n\omega_m\boldsymbol{B}_{ni}\boldsymbol{B}_{mj}\rho^{\mathrm{vv}}_{nm,i,j}+2\sqrt{\gamma_{n,ij}}\,\omega_n\boldsymbol{B}_{n,i}\boldsymbol{A}_{m,j}\rho^{\mathrm{vd}}_{nm,i,j}\right]\cdot\\
&\langle q^2_{n,i}(t)\rangle^{1/2}\langle q^2_{m,j}(t)\rangle^{1/2}+\\
&2\left[\boldsymbol{A}_{n,i}\boldsymbol{G}_{m,j}\rho^{\mathrm{dd}}_{nm,i-1,j}+\sqrt{\gamma_{n,ij}}\,\omega_n\boldsymbol{B}_{n,i}\boldsymbol{G}_{m,j}\rho^{\mathrm{vd}}_{nm,i-1,j}\right]\langle q^2_{n,i}(t)\rangle^{1/2}\langle q^2_{m,j-1}(t)\rangle^{1/2}
\end{aligned}\right\}
\end{aligned}
\tag{33}
$$

If it is assumed as usual that the maximum response $\left|\boldsymbol{y}(t)\right|_{\max}$ is proportional to the root of the mean square responses, the following closed-form formula of the complex mode response-spectrum superposition for the calculation of maximum response of the non-classically damped system with repeated eigenvalues, i.e., the complex complete quadratic combination considering repeated eigenvalues (CCQC-R) algorithm, is deduced as

$$
\begin{aligned}
&\left|\boldsymbol{y}(t)\right|_{\max} \\
&=\sum_{n=1}^{z}\sum_{m=1}^{z}\sum_{i=1}^{r_n}\sum_{j=1}^{r_m}\cdot \\
&\left\{\begin{aligned}
&\boldsymbol{G}_{n,i}\boldsymbol{G}_{m,j}\rho^{\mathrm{dd}}_{nm,i-1,j-1}\left|q_{n,i-1}(t)\right|_{\max}\left|q_{m,j-1}(t)\right|_{\max}+\\
&\left[\boldsymbol{A}_{ni}\boldsymbol{A}_{mj}\rho^{\mathrm{dd}}_{nm,i,j}+\sqrt{\gamma_{n,ij}\gamma_{m,ij}}\,\omega_n\omega_m\boldsymbol{B}_{ni}\boldsymbol{B}_{mj}\rho^{\mathrm{vv}}_{nm,i,j}+2\sqrt{\gamma_{n,ij}}\,\omega_n\boldsymbol{B}_{n,i}\boldsymbol{A}_{m,j}\rho^{\mathrm{vd}}_{nm,i,j}\right]\cdot\\
&\left|q_{n,i}(t)\right|_{\max}\left|q_{m,j}(t)\right|_{\max}+\\
&2\left[\boldsymbol{A}_{n,i}\boldsymbol{G}_{m,j}\rho^{\mathrm{dd}}_{nm,i-1,j}+\sqrt{\gamma_{n,ij}}\,\omega_n\boldsymbol{B}_{n,i}\boldsymbol{G}_{m,j}\rho^{\mathrm{vd}}_{nm,i-1,j}\right]\left|q_{n,i}(t)\right|_{\max}\left|q_{m,j-1}(t)\right|_{\max}
\end{aligned}\right\}^{1/2}
\end{aligned}
\tag{34}
$$

where $\left|q_{m,j}(t)\right|_{\max}$ are high-order displacement response spectrum corresponding to the *j*th order response of the repeated-eigenvalue λ_m, which can be defined as

$$
\begin{aligned}
&\left|q_{m,j}(t)\right|_{\max}= \\
&\left|-\frac{1}{\beta_m}\int_0^t\left[\beta_m(t-\tau)\right]^{j-1}\mathrm{e}^{-\alpha_m(t-\tau)}\sin\left(\beta_m(t-\tau)\right)\ddot{y}_{\mathrm{g}}(\tau)\,\mathrm{d}\tau\right|_{\max}
\end{aligned}
\tag{35}
$$

Compared to the calculation of the displacement response spectrum corresponding to the distinct eigenvalue, the impulse transfer function in Eq. (35) has an additional dimensionless term $\left[\beta_m(t-\tau)\right]^{j-1}$. The practice calculation method is discussed in section 4.

4 Practice calculation procedure of high-order response spectrum

For the Duhamel integration shown in Eq. (35), the high-order dynamic response can be solved by using the trapezoidal rule and Simpson's rule (Clough and Penzien, 1993). However, in earthquake engineering, the structural responses are usually calculated using the step-by-step integration procedure, such as the Newmark-β method, Wilson-θ method and the piecewise exact method. Compared to the Duhamel integration in Eq. (7), there is an additional dimensionless term $\left[\beta_m(t-\tau)\right]^{j-1}$ in Eq. (35). If the Duhamel integration in Eq. (35) can be transformed into the expression of Eq. (7), the familiar step-by-step integration can be used to solve the high-order dynamic response.

The term $\left[\beta_m(t-\tau)\right]^{j-1}$ of Eq. (35) is expanded into a binomial expression, i.e.

$$
\begin{aligned}
\left[\beta_m(t-\tau)\right]^{j-1}&=\beta_m^{j-1}(t-\tau)^{j-1}= \\
&\beta_m^{j-1}\left\{\sum_{x=0}^{j-1}\frac{(j-1)!}{(j-1-x)!\,x!}t^x(-\tau)^{j-1-x}\right\}
\end{aligned}
\tag{36}
$$

After substituting this formula into Eq. (35),

$$
\begin{aligned}
&\left|q_{m,j}(t)\right|_{\max}= \\
&\left|-\frac{1}{\beta_m}\int_0^t\mathrm{e}^{-\alpha_m(t-\tau)}\sin\left(\beta_m(t-\tau)\right)\ddot{y}^{j}_{\mathrm{gG}}(t-\tau)\,\mathrm{d}\tau\right|_{\max}
\end{aligned}
\tag{37}
$$

in which

$$
\begin{aligned}
\ddot{y}^{j}_{\mathrm{gG}}(t-\tau)&=\beta_m^{j-1}(t-\tau)^{j-1}\,\ddot{y}_{\mathrm{g}}(\tau)= \\
&\sum_{x=0}^{j-1}\frac{(j-1)!}{(j-1-x)!\,x!}(\beta_m t)^x(-\tau\beta_m)^{j-1-x}\,\ddot{y}_{\mathrm{g}}(\tau)
\end{aligned}
\tag{38}
$$

Then the form of Eq. (37) is identical to the Eq. (7) by changing the input of acceleration input. In order to verify the practice calculation method mentioned above, the high-order response of the SDOF system is solved through the superposition numerical integration method and step-by-step integration procedure. The calculation results are then compared with the theory solution.

Suppose $\alpha_m=6.2915$, $\beta_m=10.0735$, the input is $\ddot{y}_{\mathrm{g}}(t)=\sin\beta_m t$ (Gal), and the time interval is $\Delta t=0.01\,\mathrm{s}$. Now let the order number $j=2$, and then the corresponding response is the second-order response. It is easy to obtain a theory solution of displacement response shown in Eq. (35) under the sine-wave input. Figure 5 displays the theory solutions of the displacement time history, where the peak value of the corresponding displacement is 0.0136 cm.

In order to examine the various numerical calculation methods, the Duhamel integration shown in Eq. (35) is solved first by using the trapezoidal rule. The results illustrate that the displacement response obtained from

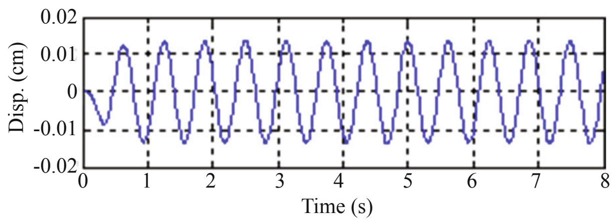

Fig. 5 Theory solution of displacement response under the sine-wave input

the trapezoidal rule matches the theory data well. In addition, the step-by-step integration procedure and the piecewise exact method (Clough and Penzien, 1993) are selected to calculate the same example, in which the input is changed according to Eq. (38). The results show that the piecewise exact method obtained an identical displacement response as that of the theory.

5 Conclusion

According to the theoretical analysis and numerical investigation performed in this study, some important results and conclusion are obtained as follows:

(1) For the generally damped linear MDOF system with repeated eigenvalues, a new response spectrum mode superposition method considering the effect of repeated roots is deduced based on a hybrid decomposition method in the time domain. This algorithm has clear physical concepts and is similar to the previously established complex complete quadratic combination (CCQC) method. Since it can consider the effect of repeated roots, it is called the CCQC-R rule, in which the correlation coefficients of high-order modal responses are involved in addition to correlation coefficients in the normal CCQC method. Moreover, the method derived in this study is suitable for generally damped systems with classical or non-classical damping.

(2) Based on the stationary random vibration theory, the formula for calculating the high-order modal responses correlation coefficients are deduced, including displacement, velocity and velocity-displacement correlation coefficients. Moreover, the relationship between displacement and velocity covariances is derived to make the CCQC-R algorithm relevant only to the high-order displacement response-spectrum.

(3) The practical method for calculating high-order displacement response spectrum is discussed, in which the earthquake input is changed and step-by-step integration procedures, such as Newmark- β method, Wilson- θ method and the piecewise exact method can be used to solve the high-order structural responses. The derived method is evaluated by comparing it to the theory solution obtained under sine-wave input.

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Appendix 1 The recursion formula of simple integration
Basic formula

$$U_{l,k}=\int\tau^l e^{\lambda\tau}\sin L_k\tau d\tau=\frac{\tau^l e^{\lambda\tau}}{\lambda^2+L_k^2}\left(\lambda\sin L_k\tau-L_k\cos L_k\tau\right)-\frac{l}{\lambda^2+L_k^2}\left(\lambda U_{l-1,k}-L_kV_{l-1,k}\right) \tag{A1}$$

$$V_{l,k}=\int\tau^l e^{\lambda\tau}\cos L_k\tau d\tau=\frac{\tau^l e^{\lambda\tau}}{\lambda^2+L_k^2}\left(\lambda\cos L_k\tau+L_k\sin L_k\tau\right)-\frac{l}{\lambda^2+L_k^2}\left(\lambda V_{l-1,k}+L_kU_{l-1,k}\right) \tag{A2}$$

When $l=0,1,2,3,4$, $U_{l,k}$ and $V_{l,k}$ can be written as

$$U_{l,k}=\int\tau^l e^{\lambda\tau}\sin L_k\tau d\tau=\frac{e^{\lambda\tau}}{\lambda^2+L_k^2}\left(A_l(\tau)\sin L_k\tau-B_l(\tau)\cos L_k\tau\right) \tag{A3}$$

$$V_{l,k}=\int\tau^l e^{\lambda\tau}\cos L_k\tau d\tau=\frac{e^{\lambda\tau}}{\lambda^2+L_k^2}\left(A_l(\tau)\cos L_k\tau+B_l(\tau)\sin L_k\tau\right) \tag{A4}$$

in which $A_l(\tau)$ and $B_l(\tau)$ can be expressed as

(1) when $l=0$:

$$A_0(\tau)=\lambda;B_0(\tau)=L_k \tag{A5}$$

(2) when $l=1$:

$$A_1(\tau)=\tau\lambda-\frac{\lambda^2-L_k^2}{\lambda^2+L_k^2};\ B_1(\tau)=\tau L_k-\frac{2\lambda L_k}{\lambda^2+L_k^2} \tag{A6}$$

(3) when $l=2$:

$$A_2(\tau)=\lambda\tau^2-\frac{2\tau\left(\lambda^2-L_k^2\right)}{\lambda^2+L_k^2}+\frac{2\lambda\left(\lambda^2-3L_k^2\right)}{\left(\lambda^2+L_k^2\right)^2};\ B_2(\tau)=L_k\tau^2-\frac{4\tau\lambda L_k}{\lambda^2+L_k^2}+\frac{2L_k\left(3\lambda^2-L_k^2\right)}{\left(\lambda^2+L_k^2\right)^2} \tag{A7}$$

(4) when $l=3$:

$$\begin{aligned}A_3(\tau)&=\lambda\tau^3-\frac{3}{\lambda^2+L_k^2}\left[\tau^2\left(\lambda^2-L_k^2\right)-\frac{2\lambda\tau\left(\lambda^2-3L_k^2\right)}{\lambda^2+L_k^2}+\frac{2\left(\lambda^4-6\lambda^2L_k^2+L_k^4\right)}{\left(\lambda^2+L_k^2\right)^2}\right]\\B_3(\tau)&=L_k\tau^3-\frac{6L_k}{\lambda^2+L_k^2}\left[\lambda\tau^2-\frac{\tau\left(3\lambda^2-L_k^2\right)}{\lambda^2+L_k^2}+\frac{4\lambda\left(\lambda^2-L_k^2\right)}{\left(\lambda^2+L_k^2\right)^2}\right]\end{aligned} \tag{A8}$$

(5) when $l=4$:

$$A_4(\tau)=\lambda\tau^4-\frac{4}{\lambda^2+L_k^2}\left\{\tau^3\left(\lambda^2-L_k^2\right)-\frac{3}{\lambda^2+L_k^2}\left[\lambda\tau^2\left(\lambda^2-3L_k^2\right)-\frac{2\tau\left(\lambda^4-6L_k{}^2\lambda^2+L_k^4\right)}{\lambda^2+L_k^2}+\frac{2\lambda\left(\lambda^4-10\lambda^2L_k^2+5L_k^4\right)}{\left(\lambda^2+L_k^2\right)^2}\right]\right\}$$

$$B_4(\tau)=\tau^4L_k-\frac{4L_k}{\lambda^2+L_k^2}\left\{2\lambda\tau^3-\frac{3}{\lambda^2+L_k^2}\left[\tau^2\left(3\lambda^2-L_k^2\right)-\frac{8\lambda\tau\left(\lambda^2-L_k^2\right)}{\lambda^2+L_k^2}+\frac{2\left(5\lambda^4-10\lambda^2L_k^2+L_k^4\right)}{\left(\lambda^2+L_k^2\right)^2}\right]\right\} \tag{A9}$$

Based on the above recursion formula, we can have

$$U_{l,k}(t)=\int_0^t\tau^l\mathrm{e}^{\lambda\tau}\sin L_k\tau\mathrm{d}\tau=\frac{\mathrm{e}^{\lambda t}}{\lambda^2+L_k^2}\left(A_l(t)\sin L_kt-B_l(t)\cos L_kt\right)+\frac{B_l(0)}{\lambda^2+L_k^2} \tag{A10}$$

$$V_{l,k}(t)=\int_0^t\tau^l\mathrm{e}^{\lambda\tau}\cos L_k\tau\mathrm{d}\tau=\frac{\mathrm{e}^{\lambda t}}{\lambda^2+L_k^2}\left(A_l(t)\cos L_kt+B_l(t)\cos L_kt\right)-\frac{A_l(0)}{\lambda^2+L_k^2} \tag{A11}$$

When $t\to\infty$, the Eqs. (A10) -(A11) can be written as:

$$U_{l,k}(\infty)=\int_0^\infty\tau^l\mathrm{e}^{\lambda\tau}\sin L_k\tau\mathrm{d}\tau=\frac{B_l(0)}{\lambda^2+L_k^2} \tag{A12}$$

$$V_{l,k}(\infty)=\int_0^\infty\tau^l\mathrm{e}^{\lambda\tau}\cos L_k\tau\mathrm{d}\tau=-\frac{A_l(0)}{\lambda^2+L_k^2} \tag{A13}$$

where $A_l(0)$ and $B_l(0)$ can be obtained by using Eq. (A5)−(A9).