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# Scattering of shear waves by an elliptical cavity in a radially inhomogeneous isotropic medium

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**Abstract:** Complex function and general conformal mapping methods are used to investigate the scattering of elastic shear waves by an elliptical cylindrical cavity in a radially inhomogeneous medium. The conformal mappings are introduced to solve scattering by an arbitrary cavity for the Helmholtz equation with variable coefficient through the transformed standard Helmholtz equation with a circular cavity. The medium density depends on the distance from the origin with a power-law variation and the shear elastic modulus is constant. The complex-value displacements and stresses of the inhomogeneous medium are explicitly obtained and the distributions of the dynamic stress for the case of an elliptical cavity are discussed. The accuracy of the present approach is verified by comparing the present solution results with the available published data. Numerical results demonstrate that the wave number, inhomogeneous parameters and different values of aspect ratio have significant influence on the dynamic stress concentration factors around the elliptical cavity.

**Keywords:** radially inhomogeneous medium; Helmholtz equation with variable coefficient; power-law variation; elliptical cavity; dynamic stress concentration factor

## 1 Introduction

Dynamic stress analysis around the cavity in inhomogeneous medium, where the material properties vary continuously as a known function along the single spatial direction, is an important subject for the purpose of the design and security of engineering structures. Different from the classical theory of dynamic stress in homogeneous medium (Pao and Mow, 1973; Liu *et al.*, 1982), the material inhomogeneities bring about a new challenge and increase the complexity of the problem leading to restrictions in the available solutions. Therefore, there is a need to investigate in detail the problem of the dynamic response by the cavity in inhomogeneous medium.

More effects involving the dynamic stress concentration in functionally graded material have been executed in many investigations, including the fundamental solution and the numerical calculation. Most of the existing studies on inhomogeneous medium have focused on the material properties varying along the single coordinate axis, which are named as the functionally graded materials (Ashrafi *et al.*, 2013; Martin, 2009, 2011; Liu *et al.*, 2014).

Different from the case of the material property depending on the horizontal and vertical direction mentioned above, many studies related to radial functionally graded materials have emerged in recent years. Two-dimensional stress distributions around a circular hole in a functionally graded material plate were investigated by Yang *et al.* (2010), Yang and Gao (2013) under arbitrary constant loads and plane compressional waves, respectively, where the material properties are graded in the radial direction. In the same period, Mohammadi *et al.* (2011) considered the stress concentration factor around a circular hole in an infinite plate subjected to uniform biaxial tension and pure shear and an exponential function was used to define the radial variation of elastic property. For another different exponential variation of material properties in the radial direction, a finite element method is adopted to investigate the thermoelastic field in a circular rotation disk with a concentric circular hole under a thermal

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load and an inertia force (Afsar and Go, 2010).

However, few studies have focused on the problem of an elliptical cavity in an inhomogeneous medium. For exponential graded materials in which the shear modulus and density vary in one-dimensional coordinate axis with the same function form, Zhou *et al*. (2013) investigated the dynamic stress concentration around two elliptical holes by complex functions and complex mapping methods. A plane-strain problem of a cylinder with a circular outer surface and a non-circular (elliptical) inner surface was investigated, where the material properties vary exponentially in the radial direction (Nie and Batra, 2010). The present study focuses on dynamic stress concentration around an elliptical cavity in an infinite inhomogeneous medium, where the medium density varies radially with a power-law. First, the standard wave equation is derived by use of the conformal mapping method. Then, by truncating the infinite algebraic equation and requiring the good accuracy, the numerical results are given. Finally, the distributions of dynamic stress concentration around the elliptical cavity are graphed.

## 2 Description of the inhomogeneity

An infinite inhomogeneous medium with an elliptical cavity where $a$ and $b$ are the semi-major and semi-minor axes is shown in Fig. 1. The origin of the polar coordinate system is assumed to be located at the center of the elliptical cavity. The elastic medium is inhomogeneous and isotropic. It is assumed that the mass density varies continuously in the radial direction and approaches uniform values at distance far away from the origin, and the elastic modulus is constant. The time harmonic wave propagates with the incident angle $\alpha$ in a radially inhomogeneous medium. According to a power-law function, the variation of the medium density is assumed as

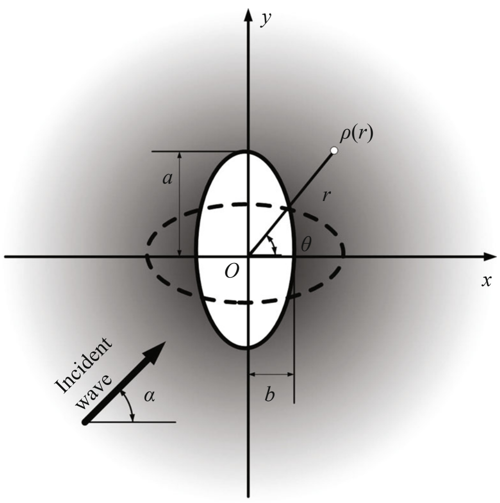



**Fig. 1 Model of a radially inhomogeneous elastic medium with an elliptical cavity: solid line ($a < b$); dashed line ($a < b$)**

$$\rho(r) = \rho_0 \beta^2 r^{2(\beta-1)} \tag{1}$$

where $\rho_0$ is the reference density, and $\beta$ is the inhomogeneous parameter of the radially inhomogeneous medium.

## 3 Transformation of the governing equation

Supposing harmonic response and neglecting body force, the governing equation for a shear wave problem is the Helmholtz equation with variable coefficient as

$$\nabla^2 \varphi(\boldsymbol{x}) + k^2(\boldsymbol{x})\varphi(\boldsymbol{x}) = 0 \tag{2}$$

where $\boldsymbol{x} = (x, y)$ is the position vector and $\nabla^2$ is the Laplacian, $k = \omega/c$ is the wave number, $\omega$ is the circular frequency of the displacement $\varphi$, $c = \sqrt{\mu/\rho}$ is the shear wave velocity, and $\rho$ and $\mu$ are the density and the shear modulus of the inhomogeneous medium, respectively.

Equation (2) written in the polar coordinates system $(r, \theta)$ takes the form

$$r^2 \frac{\partial^2 \varphi}{\partial r^2} + r\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial \theta^2} + r^2 k^2(r,\theta)\varphi = 0 \tag{3}$$

On the basis of the relation between $k$ and $\rho$, Eq. (1) can be rewritten as follows

$$k(r) = k_0 \cdot \beta r^{\beta-1} \tag{4}$$

where $k_0 = \omega/c_0$ is the reference wave number, and $c_0$ is the reference shear wave velocity. Thus, Eq. (3) can be expressed as

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \beta^2 r^{2(\beta-1)} k_0^2 \varphi = 0 \tag{5}$$

In polar coordinate systems $(r, \theta)$, the constitutive relations between stress components and the displacement are shown to be

$$\tau_{rz} = \mu\frac{\partial \varphi}{\partial r}, \quad \tau_{\theta z} = \mu\frac{1}{r}\frac{\partial \varphi}{\partial \theta} \tag{6}$$

Introducing complex variable systems $z = r\mathrm{e}^{\mathrm{i}\theta}$, Eq. (5) becomes

$$\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} + \frac{1}{4}\beta^2 (z\bar{z})^{\beta-1} k_0^2 \varphi = 0 \tag{7}$$

In a complex coordinate systems $(z, \bar{z})$, the corresponding stress components can be expressed as

$$\tau_{rz} = \mu\left[\frac{\partial\varphi}{\partial z}\mathrm{e}^{\mathrm{i}\theta} + \frac{\partial\varphi}{\partial\overline{z}}\mathrm{e}^{-\mathrm{i}\theta}\right] \tag{8}$$

$$\tau_{\theta z} = \mathrm{i}\mu\left[\frac{\partial\varphi}{\partial z}\mathrm{e}^{\mathrm{i}\theta} - \frac{\partial\varphi}{\partial\overline{z}}\mathrm{e}^{-\mathrm{i}\theta}\right] \tag{9}$$

To convert an elliptical cavity into the unit circle, the following conformal mapping is defined

$$z = w_1(\eta) = R(\eta + \frac{m}{\eta}) \tag{10}$$

where the parameters $R$ and $m$ are related to the major and minor axes by

$$R = (a+b)/2\,,\ m = (a-b)/(a+b) \tag{11}$$

In order to standardize the Helmholtz equation with a variable coefficient, a new transformation variable is introduced

$$\zeta = w_2(\eta) = z^{\beta} = \left[w_1(\eta)\right]^{\beta} \tag{12}$$

Substituting Eqs. (10) and (12) into Eq. (7) yields the governing equation

$$\frac{\partial^2\varphi}{\partial\zeta\partial\overline{\zeta}} + \frac{1}{4}k_0^2\varphi = 0 \tag{13}$$

Note that the above equation corresponds to the standard Helmholtz equation in the mapping plane.

Thus, Eqs. (8) and (9) can be written as

$$\tau_{rz} = \beta\mu\left[\frac{\partial\varphi}{\partial\zeta}z^{\beta-1}\mathrm{e}^{\mathrm{i}\theta} + \frac{\partial\varphi}{\partial\overline{\zeta}}\overline{z}^{\beta-1}\mathrm{e}^{-\mathrm{i}\theta}\right] \tag{14}$$

$$\tau_{\theta z} = \mathrm{i}\beta\mu\left[\frac{\partial\varphi}{\partial\zeta}z^{\beta-1}\mathrm{e}^{\mathrm{i}\theta} - \frac{\partial\varphi}{\partial\overline{\zeta}}\overline{z}^{\beta-1}\mathrm{e}^{-\mathrm{i}\theta}\right] \tag{15}$$

## 4 Fields of displacements and stresses

In the complex $\zeta$ -plane, the incident wave is given by

$$\varphi^{(\mathrm{i})}\left(\zeta,\overline{\zeta}\right) = \varphi_0\exp\left[\mathrm{i}k_0\left(\zeta\mathrm{e}^{-\mathrm{i}\alpha} + \overline{\zeta}\mathrm{e}^{\mathrm{i}\alpha}\right)/2\right] \tag{16}$$

where $\varphi_0$ is the amplitude of the incident waves.

The expression of the scattering waves satisfying Eq. (13) can be expressed as

$$\varphi^{(\mathrm{s})}\left(\zeta,\overline{\zeta}\right) = \sum_{n=-\infty}^{\infty} A_n\mathrm{H}_n^{(1)}\left(k_0|\zeta|\right)\left\{\frac{\zeta}{|\zeta|}\right\}^n \tag{17}$$

The whole wave field in an inhomogeneous infinite medium is the superposition of the incident wave and the scattering wave, which can be written as

$$\varphi = \varphi^{(\mathrm{i})} + \varphi^{(\mathrm{s})} \tag{18}$$

Substituting Eq. (16) into Eqs. (14) and (15), the stress components of the incident wave are

$$\tau_{rz}^{(\mathrm{i})} = \frac{\mathrm{i}\beta\mu k_0\varphi_0}{2}\left[z^{\beta-1}\mathrm{e}^{\mathrm{i}(\theta-\alpha)} + \overline{z}^{\beta-1}\mathrm{e}^{\mathrm{i}(\alpha-\theta)}\right]\exp\left[\frac{\mathrm{i}k_0}{2}\left(\zeta\mathrm{e}^{-\mathrm{i}\alpha} + \overline{\zeta}\mathrm{e}^{\mathrm{i}\alpha}\right)\right] \tag{19}$$

$$\tau_{\theta z}^{(\mathrm{i})} = -\frac{\beta\mu k_0\varphi_0}{2}\left[z^{\beta-1}\mathrm{e}^{\mathrm{i}(\theta-\alpha)} - \overline{z}^{\beta-1}\mathrm{e}^{\mathrm{i}(\alpha-\theta)}\right]\exp\left[\frac{\mathrm{i}k_0}{2}\left(\zeta\mathrm{e}^{-\mathrm{i}\alpha} + \overline{\zeta}\mathrm{e}^{\mathrm{i}\alpha}\right)\right] \tag{20}$$

Substitution of Eq. (17) into Eqs. (14) and (15), leads to the stress components of the scattering wave

$$\tau_{rz}^{(\mathrm{s})} = \frac{\beta\mu k_0}{2}\sum_{n=-\infty}^{\infty} A_n\left\{\mathrm{H}_{n-1}^{(1)}\left(k_0|\zeta|\right)\left[\frac{\zeta}{|\zeta|}\right]^{n-1}\cdot z^{\beta-1}\mathrm{e}^{\mathrm{i}\theta} - \mathrm{H}_{n+1}^{(1)}\left(k_0|\zeta|\right)\left[\frac{\zeta}{|\zeta|}\right]^{n+1}\cdot\overline{z}^{\beta-1}\mathrm{e}^{-\mathrm{i}\theta}\right\} \tag{21}$$

$$\tau_{\theta z}^{(\mathrm{s})} = \frac{\mathrm{i}\beta\mu k_0}{2}\sum_{n=-\infty}^{\infty} A_n\left\{\mathrm{H}_{n-1}^{(1)}\left(k_0|\zeta|\right)\left[\frac{\zeta}{|\zeta|}\right]^{n-1}\cdot z^{\beta-1}\mathrm{e}^{\mathrm{i}\theta} + \mathrm{H}_{n+1}^{(1)}\left(k_0|\zeta|\right)\left[\frac{\zeta}{|\zeta|}\right]^{n+1}\cdot\overline{z}^{\beta-1}\mathrm{e}^{-\mathrm{i}\theta}\right\} \tag{22}$$

## 5 Boundary condition and dynamic stress concentration factor (DSCF)

Without loss of generality, the case that the elliptical cavity is free of traction is investigated. The boundary condition is that the radial shear stress is equal to zero, i.e.

$$\tau_{rz} = \tau_{rz}^{(\mathrm{i})} + \tau_{rz}^{(\mathrm{s})} = 0;\quad \eta = \mathrm{e}^{\mathrm{i}\theta}\ \text{on}\ \theta = 0, \pi \tag{23}$$

Substituting Eqs. (19) and (21) into Eq. (23), results in

$$\sum_{n=-\infty}^{\infty} A_n\Theta_n = \Theta \tag{24}$$

where

$$\Theta_n = \mathrm{H}_{n-1}^{(1)}\left(k_0|\zeta|\right)\left[\frac{\zeta}{|\zeta|}\right]^{n-1}\cdot z^{\beta-1}\mathrm{e}^{\mathrm{i}\theta} - \mathrm{H}_{n+1}^{(1)}\left(k_0|\zeta|\right)\left[\frac{\zeta}{|\zeta|}\right]^{n+1}\cdot \bar{z}^{\beta-1}\mathrm{e}^{-\mathrm{i}\theta} \tag{25}$$

$$\Theta = -\mathrm{i}\varphi_0\left[z^{\beta-1}\mathrm{e}^{\mathrm{i}(\theta-\alpha)} + \bar{z}^{\beta-1}\mathrm{e}^{\mathrm{i}(\alpha-\theta)}\right]\exp\left[\mathrm{i}k_0\left(\zeta\mathrm{e}^{-\mathrm{i}\alpha} + \bar{\zeta}\mathrm{e}^{\mathrm{i}\alpha}\right)/2\right] \tag{26}$$

Multiplying both sides of Eq. (24) with $\mathrm{e}^{-\mathrm{i}m\theta}$ and integrating on the interval $(-\pi,\pi)$, an infinity algebraic equations set with respect to $A_n$ is found as follows

$$\sum_{n=-\infty}^{\infty} A_n\Theta_{mn} = \Theta_m \qquad m = n = 0,\ \pm1,\ \pm2,\ldots \tag{27}$$

where

$$\Theta_{mn} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Theta_n\mathrm{e}^{-\mathrm{i}m\theta}\mathrm{d}\theta,\quad \Theta_m = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Theta\mathrm{e}^{-\mathrm{i}m\theta}\mathrm{d}\theta \tag{28}$$

This study focuses on the dynamic stress concentration in the vicinity of the elliptical cavity, since the maximum occurs in this region because of the cavity. The DSCF is defined as the ratio of the stress $\tau_{\theta z}$ to the amplitude $\tau_0$. For the time harmonic elastic shear waves, the DSCF is given by

$$\tau_{\theta z}^{*} = \left|\tau_{\theta z}/\tau_0\right| \tag{29}$$

where

$$\tau_0 = \mu\beta k_0\varphi_0/2 \tag{30}$$

Substituting Eqs. (20) and (22) into Eq. (29) yields the final expression as follows

$$\tau_{\theta z}^{*} = -\left[z^{\beta-1}\mathrm{e}^{\mathrm{i}(\theta-\alpha)} - \bar{z}^{\beta-1}\mathrm{e}^{\mathrm{i}(\alpha-\theta)}\right]\exp\left[\mathrm{i}k_0\left(z^{\beta}\mathrm{e}^{-\mathrm{i}\alpha} + \bar{z}^{\beta}\mathrm{e}^{\mathrm{i}\alpha}\right)/2\right] + \frac{\mathrm{i}}{\varphi_0}\sum_{n=-\infty}^{\infty}A_n\left\{\mathrm{H}_{n-1}^{(1)}\left(k_0\left|z^{\beta}\right|\right)\left[\frac{z^{\beta}}{\left|z^{\beta}\right|}\right]^{n-1}\cdot z^{\beta-1}\mathrm{e}^{\mathrm{i}\theta} + \mathrm{H}_{n+1}^{(1)}\left(k_0\left|z^{\beta}\right|\right)\left[\frac{z^{\beta}}{\left|z^{\beta}\right|}\right]^{n+1}\cdot \bar{z}^{\beta-1}\mathrm{e}^{-\mathrm{i}\theta}\right\} \tag{31}$$

## 6 Numerical results

To demonstrate the theoretical formulation described in the previous section, a FORTRAN code was implemented to analyze the dynamic stress concentration factors (DSCFs) around the elliptical cavity in a radially inhomogeneous medium with power-law density variation disturbed by the elastic shear wave propagating horizontally with $\alpha = 0$. By truncating a finite system from Eq. (27), numerical calculations are carried out with the effects of inhomogeneous parameter, wave number and the aspect ratio of the elliptical cavity. The following dimensionless variables are defined: the reference wave number is $k_0 a$, the inhomogeneous parameter is $\beta a$, and the ratio of major and minor semi-axis is $b/a$.

To check the accuracy of the present solution, a degeneration of the distribution of DSCF around an elliptical cavity in an inhomogeneous medium by the present method is plotted in Fig. 2 with $\beta a = 1.0$ and $b/a = 1.0$, which is in agreement with available results of a homogeneous medium with a circular cavity (Pao and Mow, 1973).

For the cases of $b/a = 0.8$ and $1.5$, Figs. 3-4 show the distributions of dynamic stress concentration around an elliptical cavity with different inhomogeneous parameters when $k_0 a = 0.5$ and $1.0$. Due to the symmetry of the density profile and the horizontal propagation for the incident wave, the patterns of the DSCF are symmetrical about the $x$-axis. The forward scattering is dominant and the maximum values of DSCF for different inhomogeneous parameters always occur at $\theta = \pi$ in the illumination region. It is interesting to note that the effects of the $\beta a = 0.3$ and $0.6$ on the maximum DSCF are the opposite for the case of $k_0 a = 0.5$ and $1.0$.

Figure 5 displays the distribution of dynamic stress concentration around an elliptical cavity with $\beta a = 0.5$ for the given aspect ratio $b/a = 0.6$, $1.0$ and $1.5$, when $k_0 a = 0.1$, $0.5$ and $1.0$. It can be seen that the

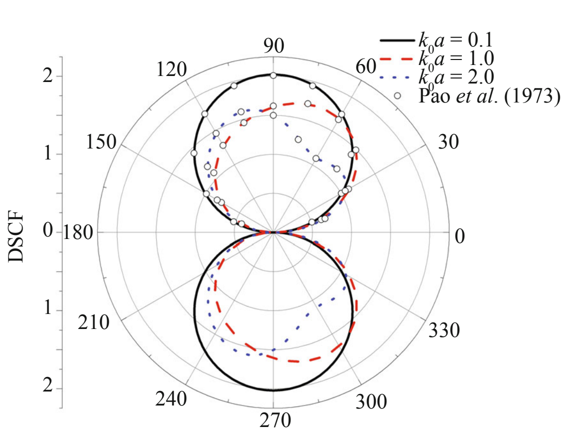
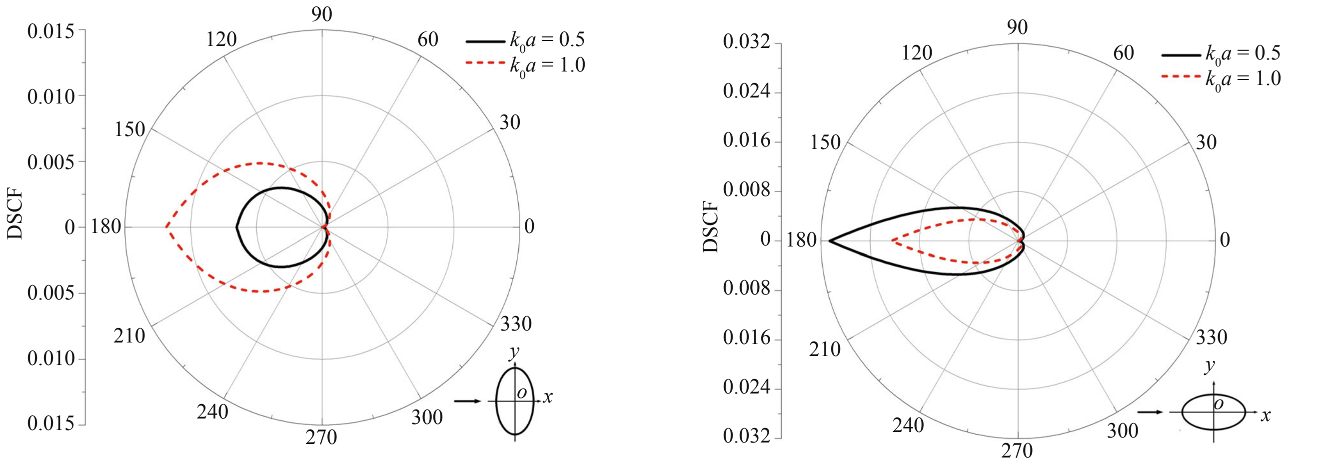
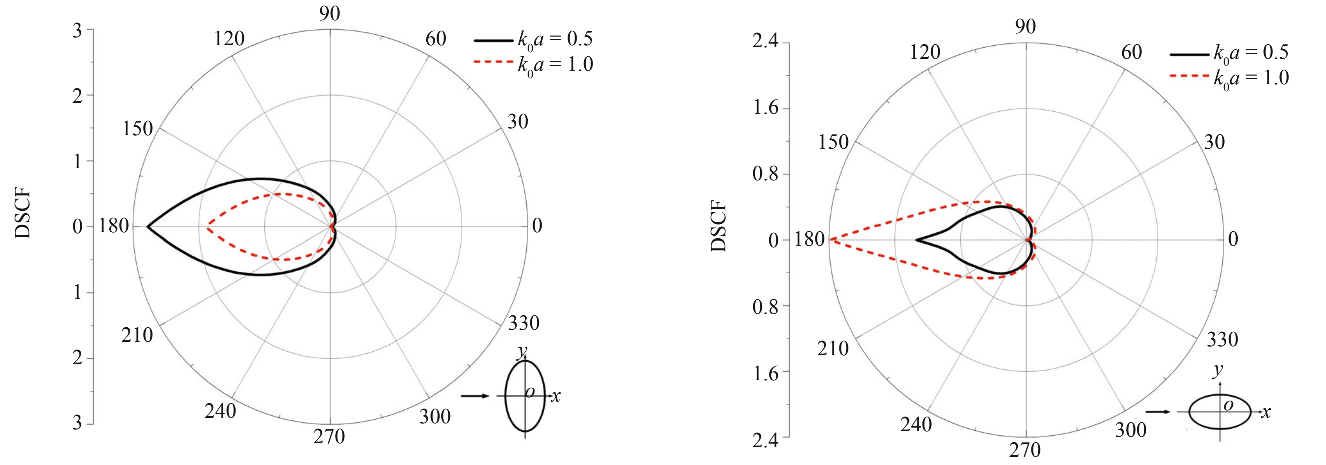



**Fig. 2 Verification of DSCF by the degeneration procedure with $\beta a$ = 1.0 and $b/a$ = 1.0**




(a) $b/a$ = 0.8 (b) $b/a$ = 1.5

**Fig. 3 Distribution of dynamic stress concentration around an elliptical cavity with $\beta a$ = 0.3**




(a) $b/a$ = 0.8 (b) $b/a$ = 1.5

**Fig. 4 Distribution of dynamic stress concentration around an elliptical cavity with $\beta a$ = 0.6**

distributions of DSCF are symmetrical due to the $x$-axis corresponding to the horizontal propagation of the incident wave. It is evident that the maximum of the DSCF occurs at $\theta = \pi$ in the illumination region. Furthermore, the maximum of the DSCF occurs from the small aspect ratio to the larger one as the frequency increases. It is concluded that the aspect ratio has a significant effect on the distribution of the DSCF compared to the case of the circular cavity for $\beta a = 0.5$. As shown in Fig. 6, the distribution of dynamic stress concentration around an elliptical cavity with $\beta a = 0.3$ for $b/a = 0.6$, 0.8 and 1.5 is plotted, when $k_0 a = 0.1$, 0.5 and 1.0. It is noted that the distributions of the DSCF are symmetrical due to the $x$-axis corresponding to the horizontal propagation of the incident wave. The maximum of the DSCF occurs at $b/a = 1.5$ in the illumination region.

The effect of the dimensionless wave number on the dynamic stress concentration with $b/a = 0.8$, $\theta = \pi/2$ is plotted in Fig. 7 for the case of $\beta a = 0.1$, 0.5, 0.8 and 1.0, respectively. It can be seen that the frequency effects the DSCF slightly, when $\beta a$ is small. As the inhomogeneous parameter increases, the effect is significant at low frequency. At higher frequency, the DSCF decreases and tends to a smooth condition contrary to the results of the homogeneous medium.

Figure 8 shows the effect of the inhomogeneous parameter on the DSCF versus the inhomogeneous parameter with $b/a = 0.8$, $\theta = \pi/2$ for the case of $k_0 a = 0.1$, 0.5, 1.0 and 2.0, respectively. It is obvious that the DSCFs arrive at the same value and tend to zero for $0 < \beta a < 0.45$, which is consistent with the results shown in Fig. 8. The DSCF increases and shifts as $\beta a$ increases. Meanwhile, there are some decreases under the combined action of $\beta a$ and $k_0 a$.

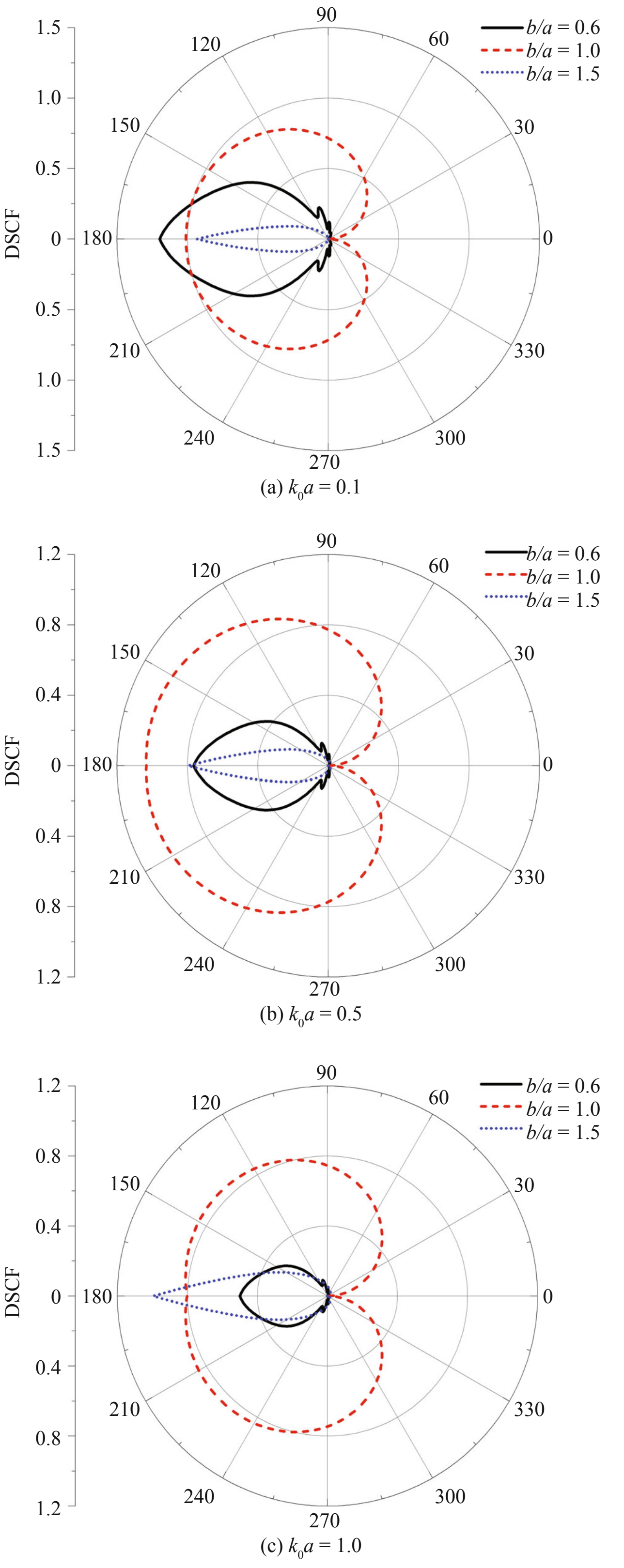



**Fig. 5 Distribution of dynamic stress concentration around an elliptical cavity with $\beta a = 0.5$**

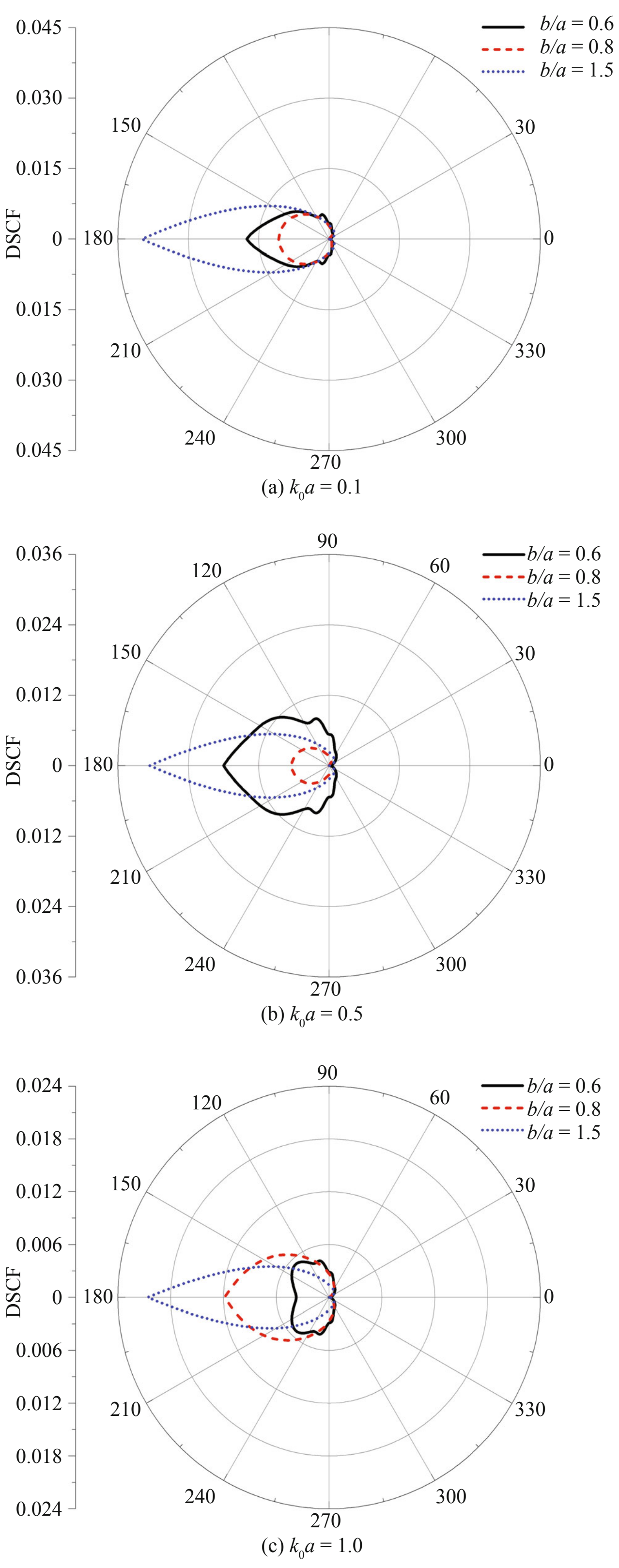



**Fig. 6 Distribution of dynamic stress concentration around an elliptical cavity with $\beta a = 0.3$**

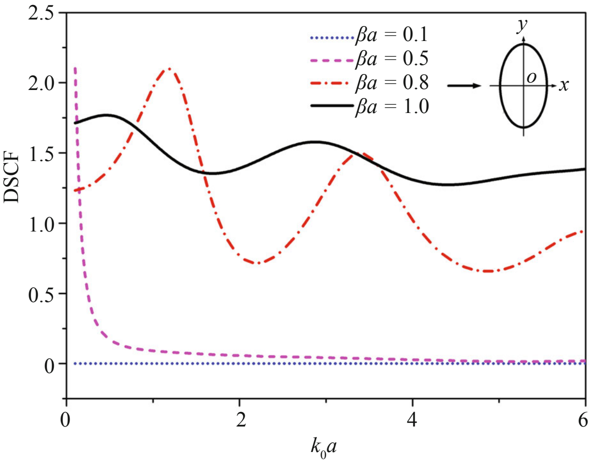



**Fig. 7 Dynamic stress concentration versus the dimensionless reference wave number with $b/a$ = 0.8, $\theta = \pi/2$**

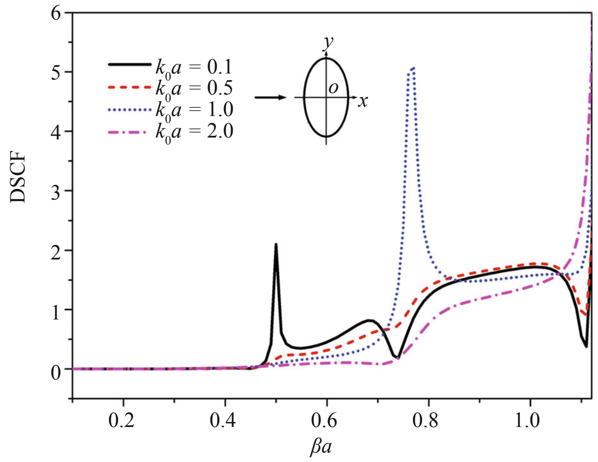



**Fig. 8 Dynamic stress concentration versus the dimensionless reference wave number with $b/a$ = 0.8, $\theta = \pi/2$**

## 7 Conclusions

Based on the complex function method, scattering of shear waves in an infinite inhomogeneous medium with an elliptical cylindrical cavity has been investigated in the study. For an inhomogeneous medium, it is assumed that the density varies as a power-law function with the distance from the origin and the elastic modulus is constant. Conformal mapping and variable transformation are introduced to the inhomogeneous wave motion equation with an elliptical cavity to the standard Helmholtz equation with a circular cavity. The displacement and stress fields are formulated. By truncating the set of infinite algebraic equations, the distributions of dynamic stress concentration in the neighborhood of an elliptical cavity are calculated numerically. It shows that the wave number, inhomogeneous parameters and different values of the aspect ratio have a significant influence on the dynamic stress concentration factors around the elliptical cavity. The dynamic stress analysis method for the radially inhomogeneous medium developed in this study is of practical importance in engineering design.

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