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A whole-space transform formula of cylindrical wave functions for scattering problems

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Abstract: The theory of elastic wave scattering is a fundamental concept in the study of elastic dynamics and wave motion, and the wave function expansion technique has been widely used in many subjects. To supply the essential tools for solving wave scattering problems induced by an eccentric source or multi-sources as well as multi-scatters, a whole-space transform formula of cylindrical wave functions is presented and its applicability to some simple cases is demonstrated in this study. The transforms of wave functions in cylindrical coordinates can be classified into two basic types: interior transform and exterior transform, and the existing Graf's addition theorem is only suitable for the former. By performing a new replacement between the two coordinates, the exterior transform formula is first deduced. It is then combined with Graf's addition theorem to establish a whole-space transform formula. By using the whole-space transform formula, the scattering solutions by the sources outside and inside a cylindrical cavity are constructed as examples of its application. The effectiveness and advantages of the whole-space transform formula is illustrated by comparison with the approximate model based on a large cycle method. The whole-space transform formula presented herein can be used to perform the transform between two different cylindrical coordinates in the whole space. In addition, its concept and principle are universal and can be further extended to establish the coordinate transform formula of wave functions in other coordinate systems.

Keywords: scattering; transform of wave functions; whole-space transform; eccentric source; multi-sources; multi-scatters

1 Introduction

The elastic wave scattering theory is the foundation of research on elastic dynamics and wave motion. Although existing numerical methods are powerful for use in solving complicated problems, analytical solutions for simple configurations are still needed to provide insight into the physical aspects of the problem and to check the accuracy of approximate solutions as a benchmark. Therefore, the wave functions expansion technique has been widely used in the research on many subjects such as local site effects of seismic waves, wave scattering and diffraction, dynamic soil-structure interaction, dynamic stress concentration, etc. (Trifunac, 1973; Eringen and Suhubi, 1975; Cao and Lee, 1989; Yuan, 1994; Fang, 1995; Liang *et al.*, 2002; Hayir *et al.*, 2004; He *et al.*., 2004; Li and Zhao, 2004; Zhang, 2010; Gao *et al.*, 2012; Zhang *et al.*, 2012).

Although many scattering problems have been solved by the wave functions expansion technique, the target in the past has mostly been limited to cases of a single scatter which are concentric with the cylindrical coordinates. As research progresses, questions related to an eccentric source or multi-sources and multi-scatters have been investigated, and in these cases, the transforms of wave functions between two or more cylindrical coordinates have become essential tools in obtaining solutions for a variety of problems. However, the currently available methods are only suitable for interior transform problems and cannot meet the demands of many other types of problems.

2 Two basic types of transform problems

In terms of the relation between the boundary of the scattering problem to be solved and the region surrounded by the two cylindrical centers to be transformed, the transforms of wave functions between two cylindrical coordinates are defined as either interior or exterior in this study. A boundary located in the inner region of the circle formed by the two origins is called the interior transform problem, and when a boundary is located in the outer region of the circle, it is called an exterior transform problem.

The vibration of a cylindrical cavity due to excitation of the harmonic line source outside the cavity is shown

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in Fig. 1, representing the scattering of the harmonic waves by the circle cavity at origin O_1 with a distance of h to the line source outside the cavity. When the wave function expansion technique is used to solve the problem, the wave functions at origin O_1 in the polar coordinate system (r_1,θ_1) should be transformed first to that at origin O in the polar coordinate system (r,θ), and then the constants to be determined can be obtained according to the boundary condition of the cavity. In this case, the boundary of the cavity is within the circle region at origin O with radius h, which belongs to the interior transform problem, and the transform from O_1 to O can be called an interior transform.

By contrast, the vibration of a cylindrical cavity due to the harmonic line source inside a cavity is shown in Fig. 2, representing the scattering of the harmonic waves by the circle cavity at origin O_1 with a distance of h to the line source inside the cavity. Similarly, when the wave function expansion technique is used to solve the problem, the wave functions at origin O_1 in the coordinate system (r_1,θ_1) should be transformed first to the expressions at origin O in the coordinate system (r,θ), and then the constants to be determined can be obtained according to the boundary condition of the cavity. In this case, the boundary of the cavity is outside the circle region at origin O with radius h, which belongs to the exterior transform problem, and the transform from O_1 to O can be called an exterior transform.

According to the above analysis, if the relation between the boundary in which the stress or displacement conditions need to be satisfied and the circle region surrounded by the cylindrical centers is taken as the division criterion, the transforms of wave functions between the two cylindrical coordinates can actually be divided into two basic types, i.e., the interior transform and the exterior transform.

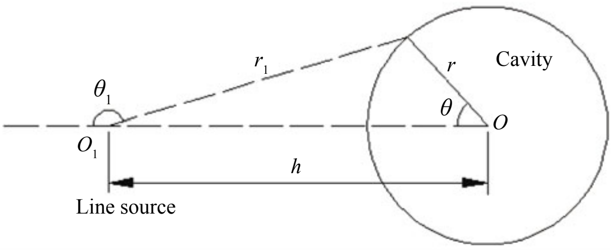

Fig. 1 Schematic of the interior transform problem

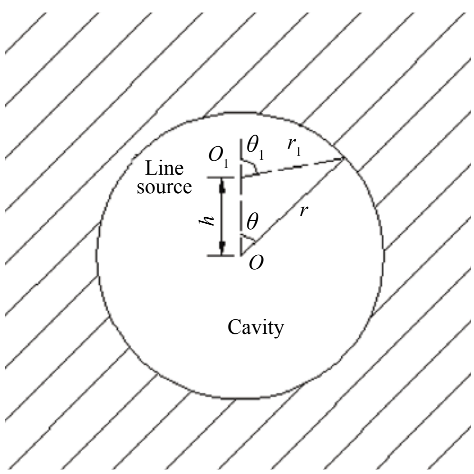

Fig. 2 Schematic of exterior transform problem

3 Graf's addition theorem

If Z, W, $\bar{\omega}$, χ and ψ are real values of variables, angle ψ can be defined as shown in Fig. 3 and expressed by

$$Z - W\cos\chi = \bar{\omega}\cos\psi \tag{1}$$

$$W\sin\chi = \bar{\omega}\sin\psi \tag{2}$$

then

$$C_n(\bar{\omega})\begin{Bmatrix}\cos n\psi \\ \sin n\psi\end{Bmatrix} = \sum_{n=-\infty}^{\infty} C_{n+m}(Z)\mathrm{J}_m(W)\begin{Bmatrix}\cos m\chi \\ \sin m\chi\end{Bmatrix} \tag{3}$$

Equation (3) identically holds when $C_n(x)$ is taken as the first kind Bessel functions $\mathrm{J}_n(x)$ and holds for $W<Z$ when $C_n(x)$ is taken as the first kind Hankel functions $\mathrm{H}_n^{(1)}(x)$.

Equation (3) is exactly the Graf's addition theorem (Yuan, 1994). Note that when $C_n(x)$ is taken as $\mathrm{H}_n^{(1)}(x)$, Eq. (3) holds within the circle region with radius Z for $W<Z$ as shown in Fig. 3. Therefore, Graf's addition theorem can be regarded as the interior transform formula for $\mathrm{H}_n^{(1)}(x)$ between the coordinates ($\bar{\omega}$, ψ) and (W, χ) with a distance of Z and as a result, to solve scattering models in the whole or half space, the theorem itself is only suitable for the interior transform.

Equation (3) can be directly applied to the interior transform between the coordinates (r_1, θ_1) and (r, θ) within the circular region Ω as shown in Fig. 4 by a simple alteration as follows:

$$\varpi = r_1, \psi = \pi - \theta_1, Z = h, W = r, \chi = \theta \tag{4}$$

Then, Eqs. (1) and (2) are satisfied and inserting Eq. (4) into Eq. (3) leads to

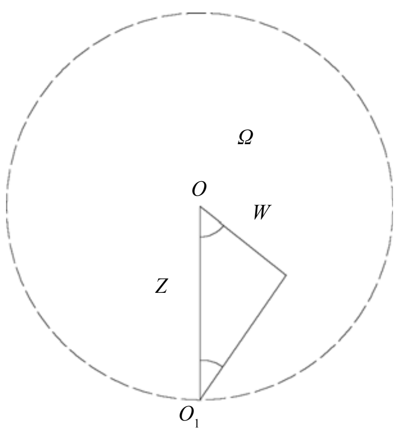

Fig. 3 Applicable region of Graf's addition theorem

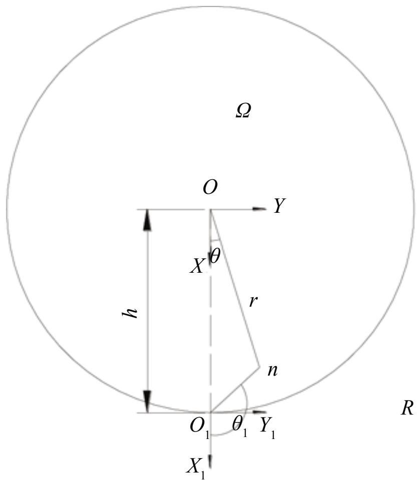

Fig. 4 Transform in the interior region *Ω*

$$C_n(r_1)\begin{Bmatrix}\cos n\theta_1\\ \sin n\theta_1\end{Bmatrix}=(-1)^n\sum_{n=-\infty}^{\infty}C_{n+m}(h)\mathrm{J}_m(r)\begin{Bmatrix}\cos m\theta\\ -\sin m\theta\end{Bmatrix}\quad(5)$$

Equation (5) identically holds for $C_n(x)=\mathrm{J}_n(x)$ and holds for $C_n(x)=\mathrm{H}_n^{(1)}(x)$ in *Ω*. Equation (5) can be further expressed by a general form as follows:

$$C_n(r_1)\begin{Bmatrix}\cos n\theta_1\\ \sin n\theta_1\end{Bmatrix}=(-1)^n\sum_{m=0}^{\infty}\mathrm{J}_m(r)\frac{\varepsilon_m}{2}\begin{Bmatrix}\left[\mathrm{H}_{m+n}^{(1)}(h)+(-1)^n\mathrm{H}_{m-n}^{(1)}(h)\right]\cos m\theta\\ -\left[\mathrm{H}_{m+n}^{(1)}(h)-(-1)^n\mathrm{H}_{m-n}^{(1)}(h)\right]\sin m\theta\end{Bmatrix}\quad(6)$$

Especially for $C_n(x)=\mathrm{H}_n^{(1)}(x)$, Eq. (5) yields

$$H_n^{(1)}(r_1)\begin{Bmatrix}\cos n\theta_1\\ \sin n\theta_1\end{Bmatrix}=(-1)^n\sum_{m=0}^{\infty}\mathrm{J}_m(r)\frac{\varepsilon_m}{2}\begin{Bmatrix}\left[\mathrm{H}_{m+n}^{(1)}(h)+(-1)^n\mathrm{H}_{m-n}^{(1)}(h)\right]\cos m\theta\\ -\left[\mathrm{H}_{m+n}^{(1)}(h)-(-1)^n\mathrm{H}_{m-n}^{(1)}(h)\right]\sin m\theta\end{Bmatrix}\quad(7)$$

where $\varepsilon_0=1$, $\varepsilon_n=2$, for $n=1, 2, 3,\ldots\ldots$

As the specific expression of Graf's addition theorem, Eq. (7) is currently the basic tool for transform of cylindrical wave functions for scattering problems in a cylindrical coordinate system. Equation (7) can be directly applied to the interior transform problem as shown in Fig. 1, but cannot be used for the exterior transform problem as shown in Fig. 2; thus, a new transform formula is needed.

4 Exterior transform formula

The exterior transform formula is presented first. As shown in Fig. 5, the region *R* represents the exterior space, i.e., the remaining part when a circular cylinder of radius *h* is removed from the whole space, and the scattering solution is concerned within the exterior region *R*. To perform a transform between the two coordinates (r_1, θ_1) and (r, θ) in *R*, the variables in Eqs. (1) and (2) can be replaced by

$$\varpi=r_1,\ \ \psi=\theta_1-\theta,\ \ Z=r,\ \ W=h,\ \ \chi=\theta\quad(8)$$

Then, from the coordinate relations in Fig. 5, Eqs. (1) and (2) are satisfied and substituting Eq. (8) into Eq. (3) results in

$$C_n(r_1)\begin{Bmatrix}\cos n\theta_1\\ \sin n\theta_1\end{Bmatrix}=\sum_{m=-\infty}^{\infty}C_{n+m}(r)\mathrm{J}_m(h)\begin{Bmatrix}\cos(n+m)\theta\\ \sin(n+m)\theta\end{Bmatrix}\quad(9)$$

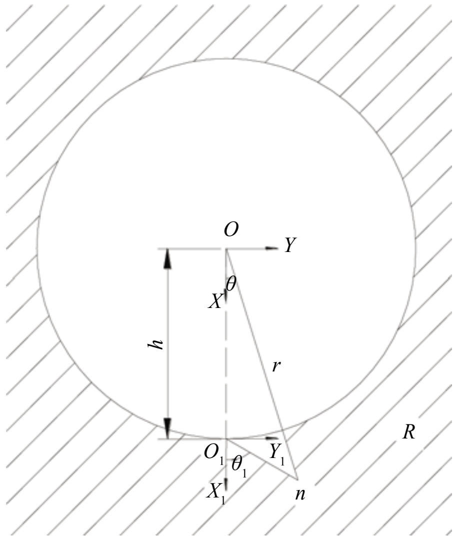

Fig. 5 Transform in the exterior region *R*

Equation (5) always holds for $C_n(x)=J_n(x)$ and holds for $C_n(x)=\mathrm{H}_n^{(1)}(x)$ in *R*. Equation (9) can be expressed by the following general form

$$C_n(r_1)\begin{Bmatrix}\cos n\theta_1\\ \sin n\theta_1\end{Bmatrix}=(-1)^n\sum_{m=0}^{\infty}C_m(r)\frac{\varepsilon_m}{2}\begin{Bmatrix}\left[\mathrm{J}_{m+n}(h)+(-1)^n\mathrm{J}_{m-n}(h)\right]\cos m\theta\\ -\left[\mathrm{J}_{m+n}(h)-(-1)^n\mathrm{J}_{m-n}(h)\right]\sin m\theta\end{Bmatrix}\quad(10)$$

Especially for the cases of $C_n(x)=\mathrm{H}_n^{(1)}(x)$, Eq. (10) can be written as

$$\mathrm{H}_n^{(1)}(r_1)\begin{Bmatrix}\cos n\theta_1\\ \sin n\theta_1\end{Bmatrix}=(-1)^n\sum_{m=0}^{\infty}\mathrm{H}_m^{(1)}(r)\frac{\varepsilon_m}{2}\begin{Bmatrix}\left[\mathrm{J}_{m+n}(h)+(-1)^n\mathrm{J}_{m-n}(h)\right]\cos m\theta\\ -\left[\mathrm{J}_{m+n}(h)-(-1)^n\mathrm{J}_{m-n}(h)\right]\sin m\theta\end{Bmatrix}\quad(11)$$

Note that Eq. (11) is obviously different from Eq. (7) in the region of interest and Eq. (11) can be used in outer region *R*. Thus, Eq. (11) can be called the exterior transform formula.

5 Whole-space transform formula

Combining Graf's addition theorem with the exterior transform formula presented by the author, a whole-space transform formula is developed in this study. For a scattering model in a whole or half space, the total region can be divided into two regions as shown in Fig. 6,

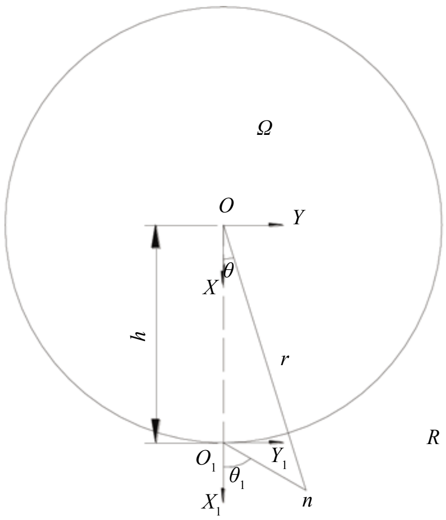

Fig. 6 Complete-region transform

i.e., the interior region Ω and the exterior region R, corresponding to inner and outer parts of the circle at origin O with radius h, respectively. By combining Eq. (7) with Eq. (11) a complete-region transform formula for $\mathrm{H}_n^{(1)}(x)$ can be established as follows:

$$\mathrm{H}_n^{(1)}(r_1)\begin{Bmatrix}\cos n\theta_1\\ \sin n\theta_1\end{Bmatrix}=$$
$$\begin{cases}(-1)^n\sum\limits_{m=0}^{\infty}\mathrm{J}_m(r)\dfrac{\varepsilon_m}{2}\begin{Bmatrix}\left[\mathrm{H}_{m+n}^{(1)}(h)+(-1)^n\mathrm{H}_{m-n}^{(1)}(h)\right]\cos m\theta\\ -\left[\mathrm{H}_{m+n}^{(1)}(h)-(-1)^n\mathrm{H}_{m-n}^{(1)}(h)\right]\sin m\theta\end{Bmatrix}\cdots\cdots\cdots r\in\Omega\\ (-1)^n\sum\limits_{m=0}^{\infty}\mathrm{H}_m^{(1)}(r)\dfrac{\varepsilon_m}{2}\begin{Bmatrix}\left[\mathrm{J}_{m+n}(h)+(-1)^n\mathrm{J}_{m-n}(h)\right]\cos m\theta\\ -\left[\mathrm{J}_{m+n}(h)-(-1)^n\mathrm{J}_{m-n}(h)\right]\sin m\theta\end{Bmatrix}\cdots\cdots\cdots r\in R\end{cases} \tag{12}$$

where $\varepsilon_0 = 1$, $\varepsilon_n = 2$, for $n = 1, 2, 3, \ldots\ldots$

Note that the parts, $\mathrm{J}_m(r)$ and $\mathrm{H}_m^{(1)}(r)$, for the interior region Ω and the exterior region R in Eq. (12), respectively, correspond to the expressions of wave functions for the interior and exterior scattering problems, respectively. Moreover, the regions Ω and R complement each other and both of them just cover the whole space. Therefore, Eq. (12) can be called as the whole-space transform formula and can be employed in performing any transform between two different cylindrical coordinates in the whole space.

Thus, the idea of dividing the transform of wave functions into the two types of interior and exterior regions is finally expressed by the whole-space transform formula, Eq. (12).

6 Example applications of the derived formulas

6.1 Scattering of the line source outside a cavity

The solution for the scattering of the line source outside a cavity as shown in Fig. 1 can be obtained based on Eq. (12).

The harmonic waves of the line source at origin O_1 can be written as

$$\sigma^{(i)}=\sigma_0\mathrm{H}_0^{(1)}\left(\beta r_1\right)\mathrm{e}^{-\mathrm{i}\omega t} \tag{13}$$

where σ_0 and β represent the stress amplitude and wave number of the incident waves, respectively.

To satisfy the traction-free boundary condition on the cavity, the key point is to transform Eq. (13) into the corresponding expression in coordinate (r, θ) by using the complete-region transform formula, Eq. (12). Substituting Eq. (12) into Eq. (13) results in

$$\sigma^{(i)}=\sigma_0\sum_{m=0}^{\infty}\varepsilon_m\mathrm{J}_m\left(\beta r\right)\mathrm{H}_m^{(1)}\left(\beta h\right)\cos m\theta\mathrm{e}^{-\mathrm{i}\omega t} \tag{14}$$

The scattering field to be determined can be expressed as

$$\sigma^{(d)}=\sigma_0\sum_{m=0}^{\infty}A_m\mathrm{H}_m^{(1)}\left(\beta r\right)\cos m\theta\mathrm{e}^{-\mathrm{i}\omega t} \tag{15}$$

where A_m are constants to be determined. The radius of the cavity is taken as a. The traction-free boundary condition on the cavity, i.e., $\sigma^{(i)}+\sigma^{(d)}=0$, leads to

$$A_m=\frac{\varepsilon_m\mathrm{J}_m(\beta a)}{\mathrm{H}_m^{(1)}(\beta a)} \tag{16}$$

The solution for scattering of the line source outside the cavity can then be obtained as

$$\sigma^{(d)}=\sigma_0\sum_{m=0}^{\infty}\frac{\varepsilon_m\mathrm{J}_m(\beta a)}{\mathrm{H}_m(\beta a)}\mathrm{H}_m^{(1)}\left(\beta r\right)\cos m\theta\mathrm{e}^{-\mathrm{i}\omega t} \tag{17}$$

6.2 Scattering of the line source inside a cavity

The solution for the scattering of the line source inside a cavity with radius a shown in Fig. 2 can also be obtained based on Eq. (12). The harmonic waves of the line source at origin O_1 can be written as

$$\sigma^{(i)}=\sigma_0\mathrm{H}_0^{(1)}\left(\beta r_1\right)\mathrm{e}^{-\mathrm{i}\omega t} \tag{18}$$

Similarly, transforming Eq. (13) in coordinate (r_1, θ_1) into that in coordinate (r, θ) by using Eq. (12), and then inserting Eq. (12) into Eq. (18) leads to

$$\sigma^{(i)}=\sigma_0\sum_{m=0}^{\infty}\varepsilon_m\mathrm{J}_m\left(\beta h\right)\mathrm{H}_m^{(1)}\left(\beta r\right)\cos m\theta\mathrm{e}^{-\mathrm{i}\omega t} \tag{19}$$

The scattering field to be determined can be given by

$$\sigma^{(d)}=\sigma_0\sum_{m=0}^{\infty}A_m\mathrm{H}_m^{(1)}\left(\beta r\right)\cos m\theta\mathrm{e}^{-\mathrm{i}\omega t} \tag{20}$$

Satisfying the traction-free boundary condition in the cavity gives

$$A_m=-\varepsilon_m\mathrm{J}_m(\beta h) \tag{21}$$

The radius of the cavity is again assumed to be a. The solution for scattering of the line source inside a cavity can be then given by

$$\sigma^{(d)} = \sigma_0 \sum_{m=0}^{\infty} \frac{\varepsilon_m \mathrm{J}_m(\beta a)}{\mathrm{H}_m(\beta a)} \mathrm{H}_m^{(1)}(\beta r) \cos m\theta \mathrm{e}^{-\mathrm{i}\omega t} \tag{22}$$

6.3 Solution for out-plane motion of a circular canyon in a half-space

The scattering model of the plane SH waves by an arbitrary circular canyon in a half-space is illustrated in Fig. 7. The key point to obtain the exact solution is that the origin O_1 of the scattering field that satisfies a traction-free boundary condition on the surface of the half-space is an eccentric source to the canyon center O. In order to let the scattering field satisfy the stress condition on the surface of the circular canyon, the wave field at origin O_1 should be transformed into that at origin O of the canyon cycle so that the transform is effective in the exterior region, i.e., the remaining space from which the circle shown in Fig. 7 is removed.

If Graf's addition theorem, Eq. (11), is employed to conduct the above transform, the transformed wave field from O_1 to O only holds within the cycle in Fig. 7 and the stress condition on the canyon surface cannot be satisfied. Because of this limitation, many researchers have employed an approximate model by a large cycle method as shown in Fig. 8, in which the flat surface is replaced by a large circular arc. In terms of the assumption of the large cycle, the approximate solution for out-of-plane motion of a circular canyon on the ground surface is obtained (Cao and Lee, 1989).

As shown in Fig. 8, by using the large circle model, the wave field at origin O_1 that satisfies the stress condition on the surface of the large circle can be transformed to that at origin O of the canyon center. In this case, the surface of the circular canyon is located within the circle at origin O with radius OO_1 and Graf's addition theorem can be directly employed.

However, the ratio of the radius of the large circle to the radius of the canyon shown in Fig. 8 depends on the study purpose. If the ratio is taken as a small value, the flat surface in the exact model in Fig. 7 cannot be simulated well, and the multiple reflections of the

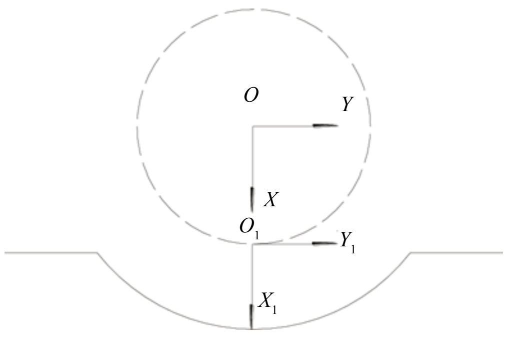

Fig. 7 Exact model for scattering of plane SH waves by surface canyon with arbitrary circular-cross-section in a half space

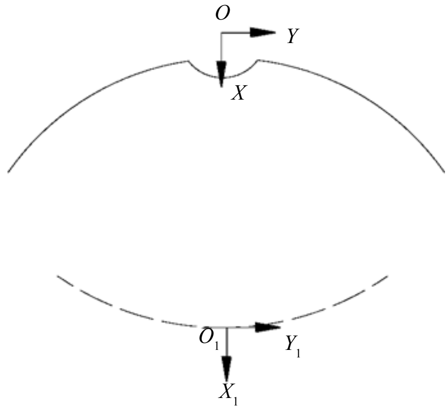

Fig. 8 Approximate model by a large cycle

waves inside the large circle as well as between the canyon and the large circle would result in a significant energy concentration near the canyon. As a result, larger displacements appear in some observation points on the ground surface. Contrarily, if a large ratio is selected, the conversion of the solution is hard to guarantee because of the obvious difficulties in the calculation of the special functions with large arguments and high orders. Meanwhile, from a physical point of view, the scattering field resulting from a local site should mainly reflect in the near-field with a finite region near the site. The performance of transforming the scattering source far away from the local site into the near-field is always in contradiction to the physical nature of the problem in a certain sense. Therefore, the reason for using the approximate model of a large cycle is mainly due to the limitation of the coordinate transform methods that are currently available and is not consistent with the mechanical characteristics of the problem.

If the whole-space transform formula is employed in the model as shown in Fig. 7, the exact solution will be easily obtained. Based on Eq. (12), the assumption of a large circle can be removed and the key problem for out-plane motion of a circular canyon on the ground surface is then solved (Yuan and Liao, 1994). Moreover, one of the key problems (Yuan, 1994) in obtaining the solution for the dynamic response of an arc-layered alluvial valley under the incidence of plane SH waves can also be solved by Eq. (12).

6.4 Other issues

Although the research target here is to investigate the transforms of the Bessel functions in cylindrical coordinates, the division of the transform types, the concept of complete region and the principle of forming the new formula herein are universal and may be easily extended to establish the coordinate transform formula of wave functions in other coordinates such as ellipsoidal coordinates. This means that in addition to the application to different kinds of circular scatters, the principle of the whole-space formula presented herein may be useful in the study of corresponding scattering and diffraction problems by the scatters with more complex shapes.

7 Conclusions

With regard to wave function expansion techniques in elastic wave scattering problems, two basic types of wave function transforms in cylindrical coordinates are distinguished and the whole-space transform formula is developed herein. Applications of the new formulas are demonstrated and prospects for their further use are briefly discussed. The main results are as follows:

(1) According to the relation between the boundary in which the stress or displacement conditions need to be satisfied and the circular region surrounded by the cylindrical centers, the transforms of wave functions between the two cylindrical coordinates can usually be divided into two types: the interior transform and the exterior transform, while the existing Graf's addition theorem is only suitable for the interior transform.

(2) The idea about division of wave function transforms into the above two types can be expressed by the whole-space transform formula presented herein, which eliminates the limitations in existing formulas and can meet the need to perform the transform between two different cylindrical coordinates in the whole space.

(3) By using the whole-space transform formula, the diffraction solutions by sources outside and inside a cylindrical cavity are obtained and a comparison is made for scattering of plane SH waves by a circular canyon in a half-space by using an approximate model with a large circular arc. Based on these, the effectiveness and advantage of the new formulas are verified

(4) The division of the transform types, the concept of complete region and the principle of forming the new formulas here are universal and may be further extended to construct a coordinate transform formula of wave functions in other coordinate systems.

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