Earthq Eng & Eng Vib (2012) 11: 273-280 **DOI**: 10.1007/s11803-012-0117-1

Free vibrations of simply supported nonhomogeneous isotropic rectangular plates of bilinearly varying thickness and elastically restrained edges against rotation using Rayleigh-Ritz method

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Abstract: This paper addresses the free transverse vibrations of thin simply supported nonhomogeneous isotropic rectangular plates of bilinearly varying thickness with elastically restrained edges against rotation. The Gram-Schmidt process has been used to generate two-dimensional boundary characteristic orthogonal polynomials, which have been used in the Rayleigh-Ritz method to study the problem. The lowest three frequencies have been computed for various values of nonhomogeneous parameters, thickness parameters, aspect ratio and flexibility parameters. A comparison of the results with those available in the literature has been made. Three-dimensional mode shapes for the specified plate have been presented.

Keywords: nonhomogeneous; rectangular; bilinearly varying thickness; elastically restrained

1 Introduction

In various engineering fields, i.e., aeronautical, civil and naval engineering, the vibrations of many structures are analyzed by modelling them as rectangular plates with simply supported edges. But the condition of simply supportness can never be achieved as it experiences rotation along edges up to certain extent. Thus, these plates can be modelled as simply supported plates with elastically restrained edges against rotation. In this regard, free transverse vibrations of rectangular plates of uniform/non-uniform thickness with all possible classical boundary conditions up to 1985 are given by Leissa (1969, 1978, 1981, 1987a, 1987b). A survey of the work on the vibrations of rectangular plates of uniform/non-uniform thickness with elastically restrained edges against rotation and/or translation up to 1995 has been given by Grossi and Bhat (1995). Later, Zhou (1996) presented transverse vibration of rectangular plates with elastical restraints using the Ritz method. Eigenfrequencies of tapered rectangular plates with intermediate line supports have been studied by Cheung and Zhou (1999a,b) using the Rayleigh-Ritz approach. In another paper, Cheung and Zhou (2000) studied vibrations of rectangular plates with elastic intermediate line-supports and edge constraints using the Rayleigh-Ritz method. Free vibration of Mindlin rectangular plates with elastically restrained edges have been presented by Zhou (2001) using the Rayleigh-Ritz method. Ashour (2004) has investigated vibration of variable thickness plates that were elastically restrained against translation and rotation using the finite strip transition matrix technique. Vibrations of rectangular plates with general elastic boundary supports have been investigated by Li (2004) using the Rayleigh-Ritz method. Malekzadeh and Shahpari (2005) have analyzed free vibration of variable thickness thin and moderately thick plates with elastically restrained edges using the differential quadrature method. Li *et al.* (2009) presented an exact series solution for the transverse vibration of rectangular plates with general elastic boundary supports. Zhang and Li (2009) presented vibrations of rectangular plates with arbitrary non-uniform elastic edge restraints using the Fourier series method. The authors have found only a few papers dealing with vibration of isotropic rectangular plates in which the thickness of the plate varies in both directions; these are reported in Laura and Grossi (1979), Singh and Saxena (1996), Sakiyama and Huang (1998), Cheung and Zhou (1999a,b), Zhou (2002), Cheung and Zhou (2003), Malekzadeh and Shahpari (2005) and Huang *et al.* (2007).

Nonhomogeneous elastic plates find their applications in the design of space vehicles, modern missiles and aircraft wings. Various models for the nonhomogeneity of the plate material have been assumed by researchers and a brief review has been given in Lal *et al.* (2010). Very recently, in a series of papers, Kumar and Lal (2011a,b) and Lal and Kumar (2011c,d,e) studied free transverse vibrations of nonhomogeneous rectangular plates of uniform/varying thickness.

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Received November 7, 2011; **Accepted** March 27, 2012

The present paper deals with the free transverse vibrations of thin simply supported nonhomogeneous rectangular plates of varying thickness and elastically restrained against rotation using two-dimensional boundary characteristic orthogonal polynomials in the Rayleigh-Ritz method on the basis of classical plate theory. Polynomials, which are a product of boundary polynomials (that represent geometric boundary conditions and kinematics) with a complete two-dimensional simple polynomial, are obtained and used in the Gram-Schmidt process to generate orthogonal polynomials. These orthogonal polynomials can be generated once for various values of aspect ratio and the use of orthonormal polynomials in the Rayleigh-Ritz method leads to a standard eigenvalue problem instead of a generalized eigenvalue problem. Nonhomogeneity of the plate is assumed to arise due to linear variations in Young's modulus and the density of the plate material with the in-plane variables. The thickness of the plate varies bidirectionally and is the Cartesian product of linear variations along the two concurrent edges of the plate. The effect of nonhomogeneity parameters, flexibility parameters, aspect ratio and thickness on the lowest three natural frequencies of the rectangular plates has been studied. Mode shapes for different values of flexibility parameters have been plotted. Comparison of results with those available in the literature has been presented.

2 Formulation

Consider a simply supported nonhomogeneous isotropic rectangular plate $\{0 \le x \le a, 0 \le y \le b\}$ of varying thickness $h(x, y)$ with elastically restrained edges against rotation, where a and b are the length and the breadth of the plate, respectively. The x– and y– axes are taken along the edges of the plate and z–axis is perpendicular to the xy– plane. The middle surface is z=0 and its origin is at one of the corners of the plate as shown in Fig. 1.

The expressions for strain energy and kinetic energy of the plate are given by

$$V = \frac{1}{2}\int_0^a \int_0^b D\,[w_{xx}{}^2 + 2\upsilon w_{xx} w_{yy} + w_{yy}{}^2 + 2(1-\upsilon) w_{xy}{}^2]\,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

and

$$T = \frac{1}{2}\int_0^a \int_0^b \rho h \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}y \tag{2}$$

where $D = \dfrac{Eh^3}{12(1-\upsilon^2)}$ is the flexural rigidity, $E\,(x,\ y)$ is the Young's modulus, $\rho\ (x,y)$ is the density, υ is the Poisson's ratio and t is the time.

For a harmonic solution, the deflection function $w(x,y,t)$ is assumed to be

$$w(x,y,t) = \bar{W}(x,y)\sin\omega t \tag{3}$$

where ω is the circular frequency.

Using Eq. (3) in Eqs. (1) and (2), the expressions for maximum strain energy and kinetic energy of the plate become

$$V_{\max} = \frac{1}{2}\int_0^a \int_0^b D\,[\bar{W}_{xx}{}^2 + 2\upsilon \bar{W}_{xx}\bar{W}_{yy} + \bar{W}_{yy}{}^2 + 2(1-\upsilon)\bar{W}_{xy}{}^2]\,\mathrm{d}x\,\mathrm{d}y \tag{4}$$

and

$$T_{\max} = \frac{\omega^2}{2}\int_0^a \int_0^b \rho h \bar{W}^2\,\mathrm{d}x\,\mathrm{d}y. \tag{5}$$

The maximum strain energy associated with the rotational restraints in the edges is given as follows:

$$U_{\max} = \frac{1}{2}[r_1 \int_0^a \bar{W}_y{}^2(x,0)\,\mathrm{d}x + r_2 \int_0^a \bar{W}_y{}^2(x,b)\,\mathrm{d}x + r_3 \int_0^b \bar{W}_x{}^2(0,y)\,\mathrm{d}y + r_4 \int_0^b \bar{W}_x{}^2(a,y)\,\mathrm{d}y] \tag{6}$$

here $\bar{W}(x,y)$ represents the maximum transverse displacement at the point (x, y), subscripts denote the partial derivative and $r_i (i = 1,...,4)$ are the rotational spring constants.

The Rayleigh quotient is given as

$$\omega^2 = \frac{V_{\max} + U_{\max}}{(T_{\max}/\omega^2)} \tag{7}$$

Introducing the non-dimensional variables $X = x/a$, $Y = y/b$, $W = \bar{W}/a$ together with the assumption that the variation of Young's modulus, density of the plate material and thickness of the plate is

$$E = E_0(1+\alpha_1 X + \alpha_2 Y)\,, \quad \rho = \rho_0(1+\beta_1 X + \beta_2 Y)$$
$$\text{and} \quad h(X,Y) = a h_0 (1+\gamma_1 X)(1+\gamma_2 Y) \tag{8}$$

Now, satisfying the essential boundary conditions, assume that

$$W(X,Y) = \sum_{k=1}^{N} d_k \hat{\phi}_k (X,Y) \tag{9}$$

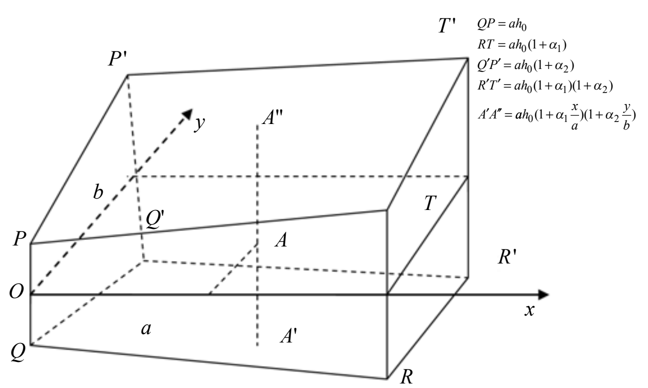

Fig. 1 Geometry of the plate

where N is the order of approximation to obtain the desired accuracy, d_k are unknowns and $\hat{\phi}_k$ are orthonormal polynomials which are generated using the Gram-Schmidt process (Singh and Chakraverty, 1994) as follows:

Orthogonal polynomials ϕ_k over the region $0 \le x \le 1$, $0 \le y \le 1$ have been generated with the help of linearly independent set of functions $L_k = l\, l_k, k = 1,2,3,\ldots\ldots$, with

$$l = X(1-X)Y(1-Y),$$
$$l_k = \{1, X, Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3, \ldots\ldots\ldots\ldots\},$$
$$\phi_1 = L_1, \quad \phi_k = L_k - \sum_{j=1}^{k-1} \alpha_{kj}\phi_j,$$
$$\alpha_{kj} = \frac{<L_k, \phi_j>}{<\phi_j, \phi_j>}, \quad j = 1,2,3,\ldots\ldots,(k-1), \quad k = 2,3,4,\ldots\ldots\ldots,N. \tag{10}$$

The inner product of the functions say, ϕ_1 and ϕ_2 is defined as

$$<\phi_1, \phi_2> = \int_0^1 \int_0^1 (1+\beta_1 X + \beta_2 Y)(1+\gamma_1 X)(1+\gamma_2 Y)\, \phi_1(X,Y)\, \phi_2(X,Y)\, \mathrm{d}X\, \mathrm{d}Y \tag{11}$$

where $(1+\beta_1 X + \beta_2 Y)(1+\gamma_1 X)(1+\gamma_2 Y)$ is the weight function and the norm of the function ϕ_1 is given by

$$\| \phi_1 \| = <\phi_1, \phi_1>^{1/2} = \left[\int_0^1 \int_0^1 (1+\beta_1 X + \beta_2 Y)(1+\gamma_1 X) (1+\gamma_2 Y)\phi_1^2(X,Y)\, \mathrm{d}X\, \mathrm{d}Y \right]^{1/2} \tag{12}$$

The normalization can be done by using

$$\hat{\phi}_k = \frac{\phi_k}{\| \phi_k \|} \tag{13}$$

Using Eqs. (8) and (9) into Eq. (7) and minimization of the resulting expression for ω^2 with respect to d_k leads to the standard eigenvalue problem

$$\sum_{k=1}^{N} (a_{jk} - \Omega^2 \delta_{jk})\, d_k = 0, \; j = 1,2,3,\ldots\ldots\ldots,N \tag{14}$$

where

$$a_{jk} = \int_0^1 \int_0^1 F[\hat{\phi}_j^{XX}\hat{\phi}_k^{XX} + \upsilon\mu^2(\hat{\phi}_j^{XX}\hat{\phi}_k^{YY} + \hat{\phi}_k^{XX}\hat{\phi}_j^{YY}) + 2(1-\upsilon)\mu^2\hat{\phi}_j^{XY}\hat{\phi}_k^{XY} + \mu^4\hat{\phi}_j^{YY}\hat{\phi}_k^{YY}]\mathrm{d}X\, \mathrm{d}Y + \mu R_1 \int_0^1 \left[\hat{\phi}_j^{Y}\hat{\phi}_k^{Y}\right]_{Y=0} \mathrm{d}X + \mu R_2 \int_0^1 \left[\hat{\phi}_j^{Y}\hat{\phi}_k^{Y}\right]_{Y=1} \mathrm{d}X + R_3 \int_0^1 \left[\hat{\phi}_j^{X}\hat{\phi}_k^{X}\right]_{X=0} \mathrm{d}Y + R_4 \int_0^1 \left[\hat{\phi}_j^{X}\hat{\phi}_k^{X}\right]_{X=1} \mathrm{d}Y \tag{15}$$

$$R_1 = \frac{12(1-\upsilon^2)r_1}{E_0 h_0^3 b^2}, R_2 = \frac{12(1-\upsilon^2)r_2}{E_0 h_0^3 b^2}, R_3 = \frac{12(1-\upsilon^2)r_3}{E_0 h_0^3 a^2}$$

and $$R_4 = \frac{12(1-\upsilon^2)r_4}{E_0 h_0^3 a^2}$$

$$F = (1+\alpha_1 X + \alpha_2 Y)(1+\gamma_1 X)^3(1+\gamma_2 Y)^3$$

$$\Omega^2 = \frac{12\rho_0 a^2(1-\upsilon^2)\omega^2}{E_0 h_0^2}, \mu = \frac{a}{b}$$

and $$\delta_{jk} = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{if } j \ne k \end{cases}$$

The integrals involved in Eq. (15) have been evaluated using the formula (Singh and Chakraverty, 1994)

$$\int_0^1 \int_0^1 X^{p_1}(1-X)^{p_2} Y^{p_3}(1-Y)^{p_4}\, \mathrm{d}X\, \mathrm{d}Y = \frac{p_1!\, p_2!\, p_3!\, p_4!}{(p_1+p_2+1)!(p_3+p_4+1)!}$$

3 Results and discussion

In this study the edges of the plate are simply supported and elastically restrained against rotation. The numerical values of the frequency parameter Ω have been obtained by solving Eq. (14) employing the Jacobi method. The lowest three eigenvalues have been reported. The values of various plate parameters are taken as follows:

Nonhomogeneity parameters: $\alpha_1, \alpha_2 = -0.5, -0.3, -0.1, 0.1, 0.3, 0.5$; density parameters: $\beta_1, \beta_2 = -0.5, -0.3, -0.1, 0.1, 0.3, 0.5$; thickness parameters: $\gamma_1, \gamma_2 = -0.5, -0.3, -0.1, 0.1, 0.3, 0.5$; aspect ratio: $a/b = 0.5, 1.0, 1.5, 2.0$; flexibility parameters: $R_1 = R_2 = R_3 = R_4 = R = 10$, 100, 1000, 10000, 1000000 and $\upsilon = 0.3$.

To choose the appropriate value of the order of approximation N, a computer program developed in C++ to evaluate the frequency parameter Ω was run for different values of N. The accuracy of the results increases as the value of N increases. In all the above computations, N = 56 has been fixed. The use of polynomials in the Rayleigh-Ritz method leads to an instability condition but no such problem occurred up to N=56 as the calculations have been performed using double precision arithmetic. Table 1 shows the convergence of the frequency parameter Ω with N for a particular set of plate parameters, where a maximum value of N was required.

The results are presented in Figs. 2–7. Figure 2 shows the effect of the nonhomogeneity parameter α_1 on the frequency parameter Ω for α_2= 0.5, β_1=±0.5, β_2=0.5, γ_1=γ_1=0.5, R=10,1000, a/b=1 for first two modes of vibration. It is observed that it increases as the values of α_1 increase. Further, it decreases as the values of β_1

Table 1 Convergence of frequency parameter Ω of nonhomogeneous simply supported square plates with elastically restraint edges against rotation for R=1000000

	a/b=1 $\alpha_1=\alpha_2=\beta_1=\beta_2=\gamma_1=\gamma_2=-0.5$		
N	Mode I	Mode II	Mode III
10	3679.59	7627.05	36469.3
20	20.6893	47.4579	52.8973
30	19.6699	39.9854	43.1367
40	19.3877	39.4130	39.9418
50	19.3330	38.7145	39.3875
53	19.3234	38.4439	39.3860
54	19.3228	38.4430	39.3859
55	19.3228	38.4430	39.3859
56	19.3228	38.4430	39.3859

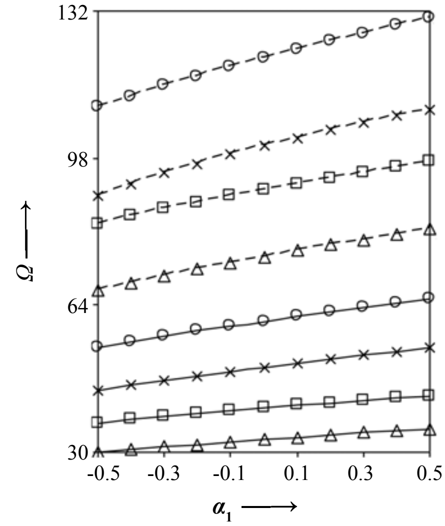

Fig. 2 Frequency parameter Ω —:Mode I; ----:Mode II; for $\alpha_2=\beta_2=\gamma_1=\gamma_2$=0.5, a/b=1: □, β_1=−0.5, R=10; o, β_1=−0.5, R=1000; Δ, β_1=0.5, R=10; ×, β_1=0.5, R=1000

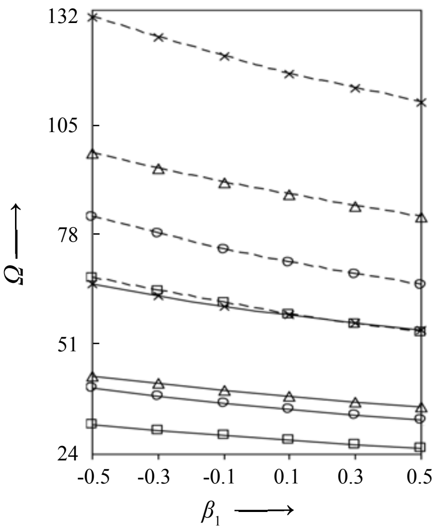

Fig. 3 Frequency parameter Ω—:Mode I; ----:Mode II; for $\alpha_1=\alpha_2=\beta_2=\gamma_2$=0.5, a/b=1: □, γ_1=−0.5, R=10; o, γ_1=−0.5, R=1000; Δ, γ_1=0.5, R=10; ×, γ_1=0.5, R=1000

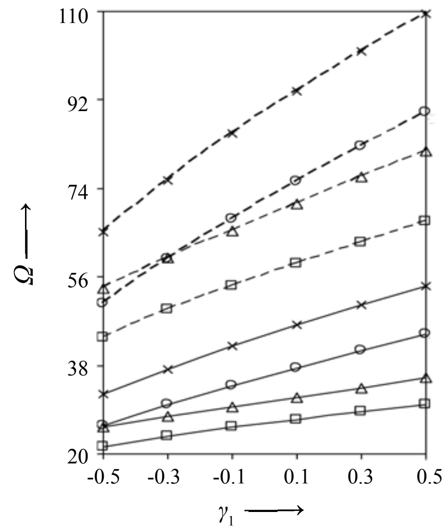

Fig. 4 Frequency parameter Ω —:Mode I; ----:Mode II; for $\alpha_2=\beta_1=\beta_2=\gamma_2$=0.5, a/b=1: □, α_1=−0.5, R=10; o, α_1=−0.5, R=1000; Δ, α_1=0.5, R=10; ×, α_1=0.5, R=1000

increase and increases as the values of R increase. Figure 3 depicts the behavior of the frequency parameter Ω with the density parameter β_1 for $\alpha_1=\alpha_2$=0.5, β_2=0.5, γ_1=±0.5, γ_2=0.5, R=10, 1000, a/b=1 for the first two modes of vibration. It is observed that it decreases as the values of β_1 increase and increases as the values of thickness parameter γ_1 and flexibility parameters represented by R increase. The effect of the thickness parameter γ_1 on frequency parameter Ω is shown in Fig. 4 for α_1=±0.5, α_2=0.5, $\beta_1=\beta_2$=0.5. γ_2=0.5, R=10, 1000, a/b=1. It is observed that it increases as the values of γ_1, α_1 and R increase. Figure 5 shows the behavior of the frequency parameter Ω with an aspect ratio a/b for $\alpha_1=\alpha_2$=0.5, β_1=±0.5, β_2=0.5, $\gamma_1=\gamma_1$=0.5, R=10,1000, for the first two modes of vibration. It is observed that the frequency parameter Ω increases as the values of the aspect ratio a/b and R increase, while it decreases as the values of β_1 increase. Further, it is noticed that the rate of increase of Ω with a/b is more pronounced for a/b>1 as compared to a/b<1 in the case of the first mode, while it is reversed in the case of the second mode, i.e., the rate of increase of Ω with a/b is more pronounced for a/b<1 as compared to a/b>1. Figure 6 demonstrates the effect of R on the frequency parameter Ω for α_1=±0.5, α_2=0.5, $\beta_1=\beta_2$=0.5, γ_1=±0.5, γ_2=0.5, a/b=1 for the first two modes of vibration. It is observed that the frequency parameter Ω

increases as the values of α_1, γ_1 and R increase. It is also observed that in the case of the first and second modes, the value of the frequency parameter Ω becomes almost constant in the neighborhood of R=100 for $\alpha_1=\gamma_1=-0.5$, while for $\alpha_1=-0.5$, $\gamma_1=0.5$, $\alpha_1=0.5$, $\gamma_1=-0.5$, $\alpha_1=0.5$, $\gamma_1=0.5$, it becomes constant in the neighborhood of R=1000 Further, as the value of R approaches infinity, the value of the frequency parameter Ω approaches that of the fully clamped plate. As the taper ratio γ_1/γ_2 (γ_1=0.25) increases from 0.5 to 2.0, the value of the frequency parameter Ω decreases for $\alpha_1=\alpha_2=\beta_1=\beta_2=0.5$, R=1000, a/b=1 for the first three modes of vibration as seen in Fig. 7. A comparison of the frequency parameter Ω for simply supported homogeneous isotropic square plates of uniform/non-uniform thickness with elastically restrained edges against rotation is shown in Table 2. Further, a comparison of the frequency parameter Ω for the limiting case of rotational stiffness tending to infinity for nonhomogeneous simply supported isotropic square plates of uniform/non-uniform thickness is shown in Table 3. In the computer program, the number 1,000,000 is considered to represent infinity. Three-dimensional mode shapes for different values of flexibility parameters have been plotted using MATLAB software and are shown in Figs. 8–12.

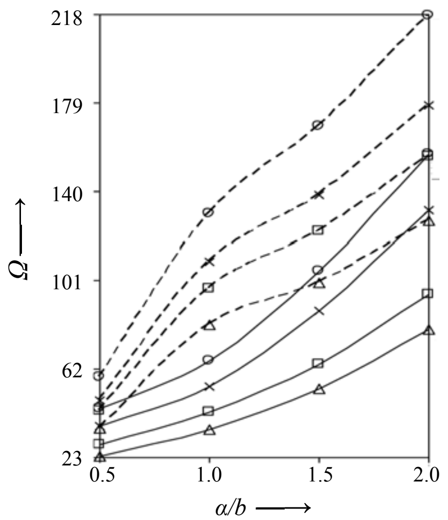

Fig. 5 Frequency parameter Ω —:Mode I; ----:Mode II; for $\alpha_1=\alpha_2=\beta_2=\gamma_1=\gamma_2=0.5$: □, $\beta_1=-0.5$, R=10; o, $\beta_1=-0.5$, R=1000; Δ, $\beta_1=0.5$, R=10; ×, $\beta_1=0.5$, R=1000

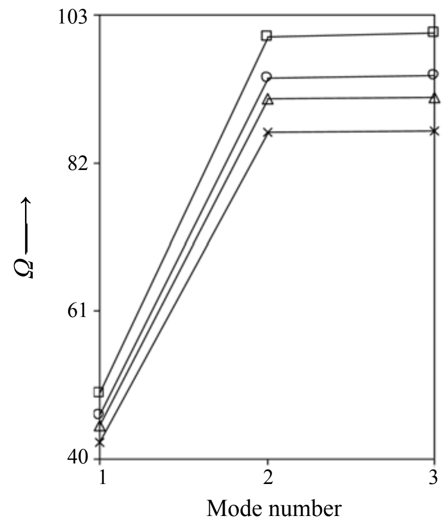

Fig. 7 Frequency parameter Ω for $\alpha_1=\alpha_2=\beta_1=\beta_2=0.5$, $\gamma_1=0.25$, R=1000, a/b=1: □, $\gamma_1/\gamma_2=0.5$; o, $\gamma_1/\gamma_2=0.75$; Δ, $\gamma_1/\gamma_2=1.0$; ×, $\gamma_1/\gamma_2=2.0$

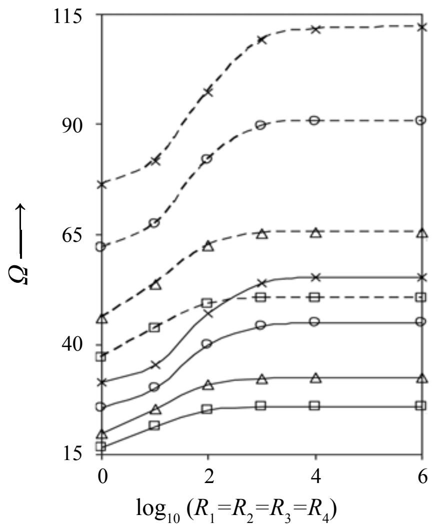

Fig. 6 Frequency parameter Ω —:Mode I; ----:Mode II; for $\alpha_2=\beta_1=\beta_2=\gamma_2=0.5$, a/b=1: □, $\alpha_1=-0.5$, $\gamma_1=-0.5$; o, $\alpha_1=-0.5$, $\gamma_1=0.5$; Δ, $\alpha_1=0.5$, $\gamma_1=-0.5$; ×, $\alpha_1=0.5$, $\gamma_1=0.5$

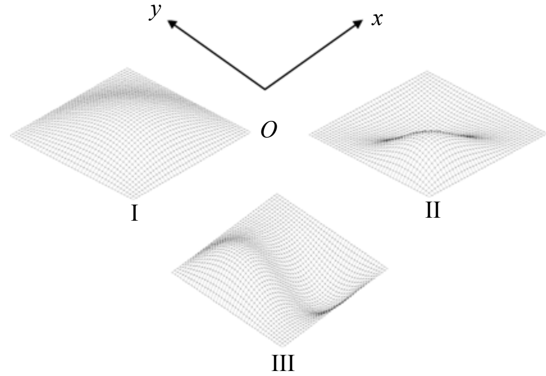

Fig. 8 Mode shapes of simply supported square (a/b=1) plate for $\alpha_1=\alpha_2=\beta_1=\beta_2=\gamma_1=\gamma_2=0.5$, R=0

Table 2 Comparison of frequency parameter Ω of homogeneous ($\alpha_1=\alpha_2=\beta_1=\beta_2=0$) simply supported square (a/b=1) plates with elastically restraint edges against rotation for γ_2=0 and υ=0.3

Reference	γ_1	R_1	R_2	R_3	R_4	Mode I	Mode II	Mode III
Grossi and Bhat (1985)	0.2	1	0	0	0	22.27	-	-
Present						22.1344	-	-
Grossi and Bhat (1985)	0.2	10	0	0	0	24.23	-	-
Present						23.9692	-	-
Grossi and Bhat (1985)	0.2	100	0	0	0	25.86	-	-
Present						25.5538	-	-
Grossi and Bhat (1985)	0.2	1000	0	0	0	26.14	-	-
Present						25.8389	-	-
Huang *et al*. (2007)	0.0	1	1	0	0	20.639	49.721	50.830
Kobayashi and Sonoda (1991)						20.639	49.721	50.830
Hung *et al*. (1993)						20.639	49.721	50.829
Present						20.6394	49.7207	50.8295
Huang *et al*. (2007)	0	100	100	0	0	28.165	54.109	67.133
Kobayashi and Sonoda (1991)						28.165	54.109	67.133
Hung *et al*. (1993)						28.165	54.109	67.133
Present						28.1650	54.1090	67.1331
Li *et al*. (2009)		1	1	1	1	21.500	51.187	51.187
Present						21.5019	51.1914	51.1914
Li (2004)	0	10	10	10	10	28.50	60.22	60.22
Li *et al*. (2009)						28.501	60.215	60.215
Present						28.5022	60.2166	60.2166
Li (2004)		20	20	20	20	31.08	64.31	64.31
Present						31.0812	64.3061	64.3071
Li (2004)		100	100	100	100	34.67	70.78	70.78
Present						34.6716	70.7881	70.7881
Li *et al*. (2009)		1000	1000	1000	1000	35.842	73.103	73.103
Present						35.8449	73.1235	73.1235
Hung *et al*. (1993)		0	0	5	5	22.965	50.811	55.066
Present						22.9650	50.811	55.0659
Kobayashi and Sonoda (1991)		0	0	100	100	28.165	54.109	67.133
Grossi and Bhat (1995)						28.168	54.122	67.140
Present						28.1650	54.1091	67.1331

Table 3 Comparison of frequency parameter Ω of nonhomogeneous simply supported square (a/b=1)plates with elastically restrained edges against rotation for R=1000000

Reference	$\alpha_1=\alpha_2$	$\beta_1=\beta_2$	$\gamma_1=\gamma_2$	Mode I	Mode II	Mode III
($R_1=R_2$=1000000, $R_3=R_4$=0)						
Lal *et al*. (2010)	-0.5	-0.5	0	28.3295	54.2284	68.4010
Present				28.3295	54.2281	68.4008
Lal *et al*. (2010)	0.5	0.5	0	28.8934	54.7060	69.2490
Present				28.8933	54.7055	69.2486
Lal and Kumar (2011b)	-0.5	-0.5	-0.5	15.8288	29.6139	36.5592
Present				15.8292	29.6153	36.5555
Lal and Kumar (2011b)	0.5	0.5	0.5	44.7632	84.3274	105.8450
Present				44.7624	84.3271	105.8420
($R_1=R_2=R_3=R_4$=1000000)						
Lal *et al*. (2010)	-0.5	-0.5	0	35.1734	72.0304	72.6703
Present				35.1753	72.0364	72.6747
Lal *et al*. (2010)	0.5	0.5	0	35.9133	73.2942	73.3265
Present				35.9131	73.2938	73.3260
Lal and Kumar (2011b)	-0.5	-0.5	-0.5	19.3194	38.3353	39.3757
Present				19.3228	38.4430	39.3859
Lal and Kumar (2011b)	0.5	0.5	0.5	55.3822	111.9460	112.8270
Present				55.3863	111.9530	112.8410

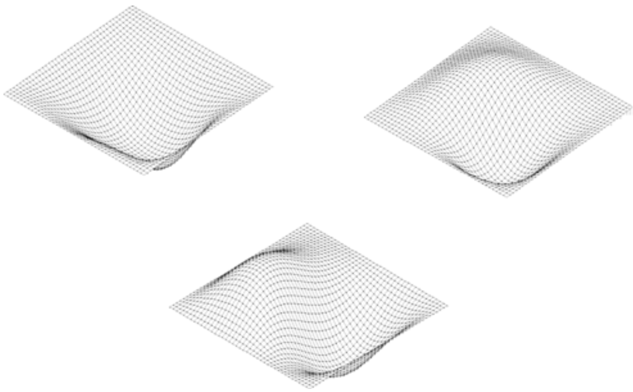

Fig. 9 Mode shapes of simply supported square (a/b=1) plate for $\alpha_1=\alpha_2=\beta_1=\beta_2=\gamma_1=\gamma_2=0.5$: R=100

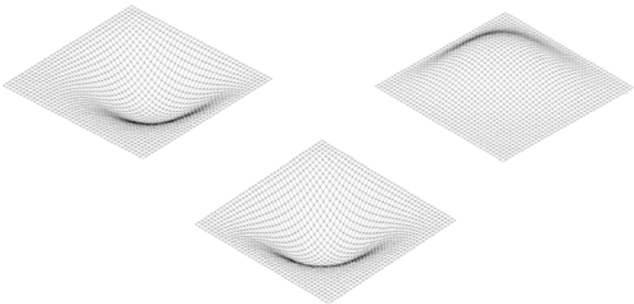

Fig. 10 Mode shapes of simply supported square (a/b=1) plate for $\alpha_1=\alpha_2=\beta_1=\beta_2=\gamma_1=\gamma_2=0.5$: R=1000000

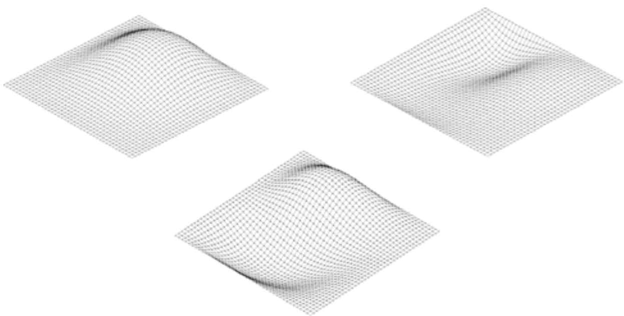

Fig. 11 Mode shapes of simply supported square (a/b=1) plate for $\alpha_1=\alpha_2=\beta_1=\beta_2=0.5$, $\gamma_1=\gamma_2=0$, $R_1=R_2$=1000000, $R_3=R_4$=0

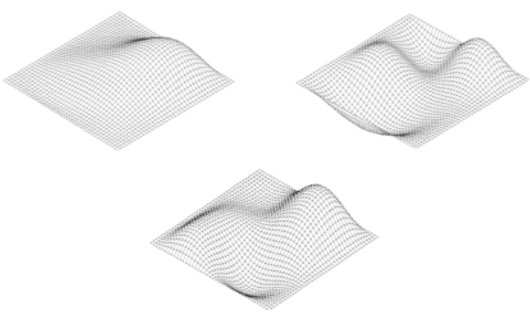

Fig. 12 Mode shapes of simply supported square (a/b=1) plate for $\alpha_1=\alpha_2=\beta_1=\beta_2=\gamma_1=\gamma_2=0.5$: $R_1=R_2$=1000000, $R_3=R_4$=0

4 Conclusions

The effect of nonhomogeneity caused by the dependence of Young's modulus and the density of the plate material on both the variables x and y on the natural frequencies of simply supported isotropic rectangular plates of thickness varying bidirectionally and elastically restrained edges against rotation has been studied using boundary characteristic orthogonal polynomials in the Rayleigh-Ritz method on the basis of classical plate theory. It is observed that the frequency parameter Ω increases as the plate becomes more and more stiff towards the edge $x=a$ and $y=b$ due to the increasing values of the parameters α_1,α_2, and it is the reverse with the increasing values of density parameters β_1 and β_2 . This also increases as the plate becomes thicker and thicker towards the edges $x=a$ and $y=b$. Further, an increase in the value of a/b and the flexibility parameters R increases the frequency. The percentage variations in the value of the frequency parameter Ω for the first mode of vibration are -11.6 to 8.8, -13.9 to 11.0 and -13.9 to 11.0, respectively, for R=100,10000,1000000 and $\gamma_1=\gamma_2$=0.5 when the nonhomogeneity arises due to the change in only α_1 from -0.5 to 0.5. These variations are -14.6 to 10.3, -14.2 to 10.1 and -14.2 to 10.1, respectively, for R=100,10000,1000000 and $\gamma_1=\gamma_2$=0.5 when the nonhomogeneity arises due to the change in only β_1 from -0.5 to 0.5. The present analysis will be very useful to design engineers in obtaining the desired frequency by varying one or more of the plate parameters considered here.

Acknowledgement

Constructive and valuable comments from the learned reviewers are gratefully acknowledged.

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