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# Robust $H_\infty$ control for aseismic structures with uncertainties in model parameters

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**Abstract:** This paper presents a robust $H_\infty$ output feedback control approach for structural systems with uncertainties in model parameters by using available acceleration measurements and proposes conditions for the existence of such a robust output feedback controller. The uncertainties of structural stiffness, damping and mass parameters are assumed to be norm-bounded. The proposed control approach is formulated within the framework of linear matrix inequalities, for which existing convex optimization techniques, such as the LMI toolbox in MATLAB, can be used effectively and conveniently. To illustrate the effectiveness of the proposed robust $H_\infty$ strategy, a six-story building was subjected both to the 1940 El Centro earthquake record and to a suddenly applied Kanai-Tajimi filtered white noise random excitation. The results show that the proposed robust $H_\infty$ controller provides satisfactory results with or without variation of the structural stiffness, damping and mass parameters.

**Keywords:** structural control; robust $H_\infty$ control; linear matrix inequality; dynamic output feedback; parameter uncertainty; seismic excitation

## 1 Introduction

Since the pioneering work of Yao (1972), a variety of control devices and methods have been studied for use in civil engineering structures. Much attention has been given to the control of tall slender structures, e.g. masts and long bridges, to reduce damage caused by earthquakes, wind or other environmental forces. Bakioglu and Aldemir (2001) proposed a numerical algorithm for the sub-optimal solution of optimal closed-open loop control based on the prediction of on-coming earthquake excitations, in which Taylor series and the Kalman filtering technique were both used; Akhiev *et al*. (2002) proposed a multipoint instantaneous optimal control of structures; Wu and Yang (2000) applied the LQG control strategy to reduce the acceleration responses of a 310m TV tower subjected to strong winds with an active mass driver installed on the upper observation deck; and Song *et al*. (2006) presented a precise integration strategy when using the LQG control strategy to avoid collision between adjacent tall buildings during earthquakes.

The above control methods are all based on the so-called “single nominal structural model.” In fact, uncertainties in the modeling of structures always exist and complete information about a structure is never available; in addition, various simplifying assumptions during modeling can lead to model errors. Neglecting these uncertainties may cause instability or degradation of controlled structural systems. Various control methods thus appeared to improve the robustness of the controlled systems. Wang (2003) proposed an active control approach for structural systems with uncertainties in model parameters by combining a robust state feedback control and a modified Kalman filter. In 2004, Wang (2004) further proposed a modified LQG control method, known as the LQG-$\alpha$ method, by introducing a relative stability vector and a gain parameter. Yang *et al*. (2004) proposed two $H_\infty$-based control strategies for civil engineering structures in which the peak values of the control resources were penalized or constrained. Yuen and Beck (2003) presented a reliability-based robust control methodology. Other robust control methods have also been applied to structural control, e.g. fuzzy logic control (Samali *et al*., 2004) and sliding mode control (Moon *et al*., 2003).

Considering the inevitability of uncertainties in

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the modeling of structures, a new robust $H_\infty$ control strategy is developed in this paper. It is assumed that the uncertainties of structural stiffness, damping and mass parameters are norm-bounded and that the available measurements in structural systems are their absolute accelerations. In Section 2, the structural behavior is modeled. In Section 3, a robust $H_\infty$ output feedback control strategy is proposed based on linear matrix inequality (LMI) technology, and a sufficient condition for the existence of such a robust $H_\infty$ controller is also derived. In Section 4, numerical results are given for a six-story building excited first by the 1940 El Centro NS ground acceleration record and then by a suddenly applied Kanai-Tajimi filtered white noise random excitation. The results show that the proposed robust $H_\infty$ controller behaves satisfactorily, even for different perturbation cases of the structural stiffness, damping and mass matrices.

## 2 Problem definitions

Consider a linear shear-type building structure installed with actuators with negligible dynamic characteristics and subjected to horizontal earthquake excitations. Its equation of motion is

$$\bar{\boldsymbol{M}}\ddot{\boldsymbol{q}}(t)+\bar{\boldsymbol{C}}\dot{\boldsymbol{q}}(t)+\bar{\boldsymbol{K}}\boldsymbol{q}(t)=-\bar{\boldsymbol{M}}\boldsymbol{L}\ddot{x}_{\mathrm{g}}(t)+\boldsymbol{H}\boldsymbol{u}(t) \tag{1}$$

where: $t$ is time; $\bar{\boldsymbol{M}}$, $\bar{\boldsymbol{C}}$ and $\bar{\boldsymbol{K}}$ are the lumped mass matrix, damping matrix and stiffness matrix of the building, respectively; $\boldsymbol{H}$ is the location matrix of the actuators; $\boldsymbol{q}$ is the structure displacement response vector relative to the ground; $\boldsymbol{L}$ is the unit index vector; $\ddot{x}_{\mathrm{g}}$ is the ground acceleration; and $\boldsymbol{u}$ is the active control force vector. In state space, Eq. (1) is expressed as

$$\dot{\boldsymbol{x}}(t)=\tilde{\boldsymbol{A}}\boldsymbol{x}(t)+\tilde{\boldsymbol{B}}_1\ddot{x}_{\mathrm{g}}(t)+\tilde{\boldsymbol{B}}_2\boldsymbol{u}(t) \tag{2}$$

where

$$\boldsymbol{x}=\begin{Bmatrix}\boldsymbol{q}\\ \dot{\boldsymbol{q}}\end{Bmatrix},\qquad \tilde{\boldsymbol{A}}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}\\ -\bar{\boldsymbol{M}}^{-1}\bar{\boldsymbol{K}} & -\bar{\boldsymbol{M}}^{-1}\bar{\boldsymbol{C}}\end{bmatrix}$$

$$\tilde{\boldsymbol{B}}_1=\begin{Bmatrix}\boldsymbol{0}\\ -\boldsymbol{L}\end{Bmatrix},\qquad \tilde{\boldsymbol{B}}_2=\begin{bmatrix}\boldsymbol{0}\\ \bar{\boldsymbol{M}}^{-1}\boldsymbol{H}\end{bmatrix}$$

Practically, it is impossible to establish an accurate model for the real structure. Thus, the nominal mass, damping and stiffness matrices of the system are denoted as $\boldsymbol{M}_0$, $\boldsymbol{C}_0$ and $\boldsymbol{K}_0$. To account for the uncertainties in the structural parameters, Eq. (2) is reformulated as

$$\dot{\boldsymbol{x}}(t)=(\boldsymbol{A}+\Delta\boldsymbol{A})\boldsymbol{x}(t)+\tilde{\boldsymbol{B}}_1\ddot{x}_{\mathrm{g}}(t)+(\boldsymbol{B}_2+\Delta\boldsymbol{B}_2)\boldsymbol{u}(t) \tag{3}$$

where

$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}\\ -\boldsymbol{M}_0^{-1}\boldsymbol{K}_0 & -\boldsymbol{M}_0^{-1}\boldsymbol{C}_0\end{bmatrix},\qquad \boldsymbol{B}_2=\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{M}_0^{-1}\boldsymbol{H}\end{bmatrix}$$

and $\Delta\boldsymbol{A}$ and $\Delta\boldsymbol{B}_2$ are perturbation matrices that reflect the variations of the mass, damping and stiffness matrices of the system. Although these uncertainties may all exist, only structural stiffness perturbation is considered for controller design in this paper, i.e., $\bar{\boldsymbol{M}}=\boldsymbol{M}_0$, $\bar{\boldsymbol{C}}=\boldsymbol{C}_0$ and $\bar{\boldsymbol{K}}=\boldsymbol{K}_0(1+0.01\alpha)$, where parameter $\alpha$ is the percentage variation of stiffness. In reality, it is not wise to consider the worst case of all possible parameter perturbations in controller design because to do so would make the controller too conservative. As a result, $\Delta\boldsymbol{A}$ and $\Delta\boldsymbol{B}_2$ have the form:

$$[\Delta\boldsymbol{A}\quad \Delta\boldsymbol{B}_2]=\boldsymbol{D}_1\boldsymbol{F}[\boldsymbol{E}_1\quad \boldsymbol{E}_2] \tag{4}$$

in which

$$\boldsymbol{D}_1=\begin{bmatrix}\boldsymbol{0}\\ -\boldsymbol{M}_0^{-1}\boldsymbol{K}_0\end{bmatrix},\qquad \boldsymbol{E}_1=[0.01\bar{\alpha}\cdot\boldsymbol{I}\quad \boldsymbol{0}]$$
$$\boldsymbol{E}_2=\boldsymbol{0},\qquad \boldsymbol{F}=\bar{\delta}\boldsymbol{I} \tag{5}$$

The scalar $\bar{\alpha}$ ($\geq 0$) represents the maximum possible percentage change of the stiffness, with $\alpha\in[-\bar{\alpha},\ \bar{\alpha}]$, while scalar $\bar{\delta}$ represents an uncertain value bounded by -1 and 1.

Assume now that the absolute accelerations of all floors are adopted for measurement. The measurement vector $\boldsymbol{y}$ can then be expressed as

$$\boldsymbol{y}=\tilde{\boldsymbol{C}}_2\boldsymbol{x}+\tilde{\boldsymbol{D}}_{22}\boldsymbol{u}+\boldsymbol{v} \tag{6}$$

in which

$$\tilde{\boldsymbol{C}}_2=[-\bar{\boldsymbol{M}}^{-1}\bar{\boldsymbol{K}}\quad -\bar{\boldsymbol{M}}^{-1}\bar{\boldsymbol{C}}],\qquad \tilde{\boldsymbol{D}}_{22}=\bar{\boldsymbol{M}}^{-1}\boldsymbol{H} \tag{7}$$

and $\boldsymbol{v}$ is the measurement noise vector, which is assumed to be a white noise vector and to satisfy

$$\mathrm{E}[\boldsymbol{v}(t)]=\boldsymbol{0},\quad \mathrm{E}[\boldsymbol{v}(t_1)\boldsymbol{v}^{\mathrm{T}}(t_2)]=\tilde{\boldsymbol{V}}\delta(t_2-t_1)$$
$$\mathrm{E}[\ddot{x}_{\mathrm{g}}(t_1)\boldsymbol{v}^{\mathrm{T}}(t_2)]=\boldsymbol{0} \tag{8}$$

where E is the expectation operator, $t_1$ and $t_2$ are two arbitrary times, $\delta$ is the Dirac delta function and $\tilde{\boldsymbol{V}}$ is the intensity matrix of the measurement noise.

The aim of robust controller design is to restrict the responses of structures with possible parameter uncertainties, while not requiring excessive control forces. Now, denote the controlled output vector as

$$\boldsymbol{z}=\boldsymbol{C}_1\boldsymbol{x}+\boldsymbol{D}_{12}\boldsymbol{u} \tag{9}$$

in which $\boldsymbol{C}_1$ and $\boldsymbol{D}_{12}$ are constant matrices that are specified by the designer.

Eqs. (3) and (6) are rewritten as

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} + \Delta\boldsymbol{A})\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{w} + (\boldsymbol{B}_2 + \Delta\boldsymbol{B}_2)\boldsymbol{u} \tag{10}$$

$$\boldsymbol{y} = (\boldsymbol{C}_2 + \Delta\boldsymbol{C}_2)\boldsymbol{x} + \boldsymbol{D}_{21}\boldsymbol{w} + (\boldsymbol{D}_{22} + \Delta\boldsymbol{D}_{22})\boldsymbol{u} \tag{11}$$

where

$$\boldsymbol{w} = [\ddot{x}_g \quad \boldsymbol{v}^T]^T \qquad \boldsymbol{B}_1 = [\tilde{\boldsymbol{B}}_1 \quad \boldsymbol{0}]$$

$$\boldsymbol{C}_2 = [-\boldsymbol{M}_0^{-1}\boldsymbol{K}_0 \quad -\boldsymbol{M}_0^{-1}\boldsymbol{C}_0] \qquad \boldsymbol{D}_{21} = [\boldsymbol{0} \quad \boldsymbol{I}]$$

$$\boldsymbol{D}_{22} = \boldsymbol{M}_0^{-1}\boldsymbol{H} \qquad [\Delta\boldsymbol{C}_2 \quad \Delta\boldsymbol{D}_{22}] = \boldsymbol{D}_2\boldsymbol{F}[\boldsymbol{E}_1 \quad \boldsymbol{E}_2]$$

$$\boldsymbol{D}_2 = -\boldsymbol{M}_0^{-1}\boldsymbol{K}_0$$

## 3 Robust $H_\infty$ output feedback control

Consider a general uncertain linear system as follows:

$$(\Sigma_1): \begin{cases} \dot{\boldsymbol{x}} = (\boldsymbol{A} + \Delta\boldsymbol{A})\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{w} + (\boldsymbol{B}_2 + \Delta\boldsymbol{B}_2)\boldsymbol{u} \\ \boldsymbol{z} = \boldsymbol{C}_1\boldsymbol{x} + \boldsymbol{D}_{12}\boldsymbol{u} \\ \boldsymbol{y} = (\boldsymbol{C}_2 + \Delta\boldsymbol{C}_2)\boldsymbol{x} + \boldsymbol{D}_{21}\boldsymbol{w} + (\boldsymbol{D}_{22} + \Delta\boldsymbol{D}_{22})\boldsymbol{u} \end{cases} \tag{12}$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$ is the state vector, $\boldsymbol{w} \in \boldsymbol{R}^q$ is the disturbance input, $\boldsymbol{u} \in \boldsymbol{R}^m$ is the control input, $\boldsymbol{z} \in \boldsymbol{R}^p$ is the controlled output, $\boldsymbol{y} \in \boldsymbol{R}^r$ is the measured output, $\boldsymbol{A}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, $\boldsymbol{D}_{12}$, $\boldsymbol{D}_{21}$ and $\boldsymbol{D}_{22}$ are real constant matrices of appropriate dimensions that describe the nominal system, and $\Delta\boldsymbol{A}$, $\Delta\boldsymbol{B}_2$, $\Delta\boldsymbol{C}_2$ and $\Delta\boldsymbol{D}_{22}$ are proper real-valued uncertain matrices. The parameter uncertainties considered here are norm-bounded and of the form

$$\begin{aligned} [\Delta\boldsymbol{A} \quad \Delta\boldsymbol{B}_2] &= \boldsymbol{D}_1\boldsymbol{F}[\boldsymbol{E}_1 \quad \boldsymbol{E}_2] \\ [\Delta\boldsymbol{C}_2 \quad \Delta\boldsymbol{D}_{22}] &= \boldsymbol{D}_2\boldsymbol{F}[\boldsymbol{E}_1 \quad \boldsymbol{E}_2] \end{aligned} \tag{13}$$

where $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are known constant matrices and $\boldsymbol{F}$ is an unknown matrix satisfying $\boldsymbol{F}^T\boldsymbol{F} \le \boldsymbol{I}$ and that has Lebesgue measurable elements.

Consider now a full-order dynamic output feedback controller in the form

$$\begin{cases} \dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}_k\hat{\boldsymbol{x}}(t) + \boldsymbol{B}_k\boldsymbol{y}(t) \\ \boldsymbol{u}(t) = \boldsymbol{C}_k\hat{\boldsymbol{x}}(t) \end{cases} \tag{14}$$

for the system $(\Sigma_1)$. The controller matrices $\boldsymbol{A}_k$, $\boldsymbol{B}_k$ and $\boldsymbol{C}_k$ will be designed based on the theorem described below.

First, a sufficient condition for the existence of an $H_\infty$ output feedback controller for a linear system without parameter uncertainties is given based on the LMI approach. The system without parameter uncertainties is defined as:

$$(\Sigma_2): \begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{w} + \boldsymbol{B}_2\boldsymbol{u} \\ \boldsymbol{z} = \boldsymbol{C}_1\boldsymbol{x} + \boldsymbol{D}_{12}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}_2\boldsymbol{x} + \boldsymbol{D}_{21}\boldsymbol{w} + \boldsymbol{D}_{22}\boldsymbol{u} \end{cases} \tag{15}$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$ is the state of the system, $\boldsymbol{w} \in \boldsymbol{R}^q$ is the disturbance input, $\boldsymbol{u} \in \boldsymbol{R}^m$ is the control input, $\boldsymbol{z} \in \boldsymbol{R}^p$ is the controlled output, $\boldsymbol{y} \in \boldsymbol{R}^r$ is the measured output, $\boldsymbol{A}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, $\boldsymbol{D}_{12}$, $\boldsymbol{D}_{21}$ and $\boldsymbol{D}_{22}$ are the same as in system $(\Sigma_1)$.

*Lemma 1* (Chilali and Gahinet, 1996; Scherer *et al.*, 1997): There exists an output feedback $H_\infty$ controller of the form (14) for the system $(\Sigma_2)$ and $H_\infty$ disturbance attenuation $\gamma>0$ if there exist matrices $\boldsymbol{R}=\boldsymbol{R}^T>\boldsymbol{0}$, $\boldsymbol{S}=\boldsymbol{S}^T>\boldsymbol{0}$ and matrices $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$, $\hat{\boldsymbol{C}}$ such that

$$\begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{12}^T & \boldsymbol{Q}_{22} \end{bmatrix} < \boldsymbol{0} \tag{16}$$

$$\begin{bmatrix} \boldsymbol{R} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{S} \end{bmatrix} > \boldsymbol{0} \tag{17}$$

in which

$$\boldsymbol{Q}_{11} = \begin{bmatrix} \boldsymbol{A}\boldsymbol{R} + \boldsymbol{R}\boldsymbol{A}^T + \boldsymbol{B}_2\hat{\boldsymbol{C}} + \hat{\boldsymbol{C}}^T\boldsymbol{B}_2^T & \boldsymbol{A} + \hat{\boldsymbol{A}}^T \\ \hat{\boldsymbol{A}} + \boldsymbol{A}^T & \boldsymbol{S}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{S} + \hat{\boldsymbol{B}}\boldsymbol{C}_2 + \boldsymbol{C}_2^T\hat{\boldsymbol{B}}^T \end{bmatrix}$$

$$\boldsymbol{Q}_{12} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{R}\boldsymbol{C}_1^T + \hat{\boldsymbol{C}}^T\boldsymbol{D}_{12}^T \\ \boldsymbol{S}\boldsymbol{B}_1 + \hat{\boldsymbol{B}}\boldsymbol{D}_{21} & \boldsymbol{C}_1^T \end{bmatrix}$$

$$\boldsymbol{Q}_{22} = \begin{bmatrix} -\gamma\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\gamma\boldsymbol{I} \end{bmatrix}$$

The inequalities above are linear in $\boldsymbol{R}$, $\boldsymbol{S}$, $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$, $\hat{\boldsymbol{C}}$ and $\gamma$. Thus, for any given solution of this LMI system, it is possible to find nonsingular, real-valued square matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ that satisfy $\boldsymbol{M}\boldsymbol{N}^T=\boldsymbol{I}-\boldsymbol{R}\boldsymbol{S}$ and the controller matrices $\boldsymbol{A}_k$, $\boldsymbol{B}_k$ and $\boldsymbol{C}_k$ then follow from

$$\boldsymbol{A}_k = \boldsymbol{N}^{-1}(\hat{\boldsymbol{A}} - \boldsymbol{S}\boldsymbol{A}\boldsymbol{R} - \boldsymbol{S}\boldsymbol{B}_2\hat{\boldsymbol{C}} - \hat{\boldsymbol{B}}\boldsymbol{C}_2\boldsymbol{R} - \hat{\boldsymbol{B}}\boldsymbol{D}_{22}\hat{\boldsymbol{C}})\boldsymbol{M}^{-T} \tag{18}$$

$$\boldsymbol{B}_k = \boldsymbol{N}^{-1}\hat{\boldsymbol{B}} \tag{19}$$

$$\boldsymbol{C}_k = \hat{\boldsymbol{C}}\boldsymbol{M}^{-T} \tag{20}$$

Next, another auxiliary system is introduced in order to establish the equivalence between robust $H_\infty$ and a scaled $H_\infty$ control problem:

$$(\Sigma_3): \quad \begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + [\varepsilon \boldsymbol{D}_1 \quad \gamma^{-1}\boldsymbol{B}_1]\tilde{\boldsymbol{w}} + \boldsymbol{B}_2\boldsymbol{u} \\ \tilde{\boldsymbol{z}} = \begin{bmatrix} \varepsilon^{-1}\boldsymbol{E}_1 \\ \boldsymbol{C}_1 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} \varepsilon^{-1}\boldsymbol{E}_2 \\ \boldsymbol{D}_{12} \end{bmatrix}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}_2\boldsymbol{x} + [\varepsilon \boldsymbol{D}_2 \quad \gamma^{-1}\boldsymbol{D}_{21}]\tilde{\boldsymbol{w}} + \boldsymbol{D}_{22}\boldsymbol{u} \end{cases} \tag{21}$$

where: $\boldsymbol{x} \in \boldsymbol{R}^n$ is the state; $\tilde{\boldsymbol{w}} \in \boldsymbol{R}^{q+i}$ is the disturbance input; $\boldsymbol{u} \in \boldsymbol{R}^m$ is the control input; $\tilde{\boldsymbol{z}} \in \boldsymbol{R}^{p+j}$ is the controlled output; $\boldsymbol{y} \in \boldsymbol{R}^r$ is the measured output; $\boldsymbol{A}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{D}_{12}$, $\boldsymbol{D}_{21}$ and $\boldsymbol{D}_{22}$ are the same as in the system ($\Sigma_1$); $\varepsilon > 0$ is a parameter to be chosen and; $\gamma > 0$ is the disturbance attenuation performance that it is hoped to achieve for the system ($\Sigma_1$).

*Lemma 2* (Xie *et al*., 1992): Let $\gamma > 0$ be a prescribed level of $H_\infty$ disturbance attenuation and $\boldsymbol{K}(s)$ denoting a given linear dynamic controller; then, the system ($\Sigma_1$) is quadratically stabilizable with disturbance attenuation $\gamma$ via the output feedback controller $\boldsymbol{K}(s)$ if and only if there exists a constant $\varepsilon > 0$ such that the closed-loop system corresponding to system ($\Sigma_3$) and $\boldsymbol{K}(s)$ is stable with unitary disturbance attenuation.

Combining Lemma 1 with Lemma 2, the following theorem can be derived:

*Theorem 1*: There exists a robust $H_\infty$ output feedback controller of the form (14) for the system ($\Sigma_1$) which makes $H_\infty$ disturbance attenuation $\gamma > 0$, if there exist $\varepsilon > 0$, matrices $\boldsymbol{R}=\boldsymbol{R}^{\mathrm{T}} > \boldsymbol{0}$, $\boldsymbol{S}=\boldsymbol{S}^{\mathrm{T}} > \boldsymbol{0}$ and matrices $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{C}}$ such that

$$\begin{bmatrix} \boldsymbol{J}_{11} & \boldsymbol{J}_{12} \\ \boldsymbol{J}_{12}^{\mathrm{T}} & \boldsymbol{J}_{22} \end{bmatrix} < \boldsymbol{0} \tag{22}$$

$$\begin{bmatrix} \boldsymbol{R} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{S} \end{bmatrix} > \boldsymbol{0} \tag{17}$$

where

$$\boldsymbol{J}_{11} = \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{12}^{\mathrm{T}} & \boldsymbol{Q}_{22} \end{bmatrix}$$

$$\boldsymbol{J}_{12} = \begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{R}\boldsymbol{E}_1^{\mathrm{T}} + \hat{\boldsymbol{C}}^{\mathrm{T}}\boldsymbol{E}_2^{\mathrm{T}} \\ \boldsymbol{S}\boldsymbol{D}_1 + \hat{\boldsymbol{B}}\boldsymbol{D}_2 & \boldsymbol{E}_1^{\mathrm{T}} \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{J}_{22} = \begin{bmatrix} -\varepsilon^{-1}\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\varepsilon\boldsymbol{I} \end{bmatrix}$$

Matrices $\boldsymbol{Q}_{11}$, $\boldsymbol{Q}_{12}$ and $\boldsymbol{Q}_{22}$ are the same as in Eq. (16).

If the above matrix inequalities are solvable, the controller matrices $\boldsymbol{A}_k$, $\boldsymbol{B}_k$ and $\boldsymbol{C}_k$ can be obtained as those in Lemma 1. The proof of Theorem 1 is straightforward by combining Lemma 1 with Lemma 2 and is thus omitted.

The procedures to design a robust $H_\infty$ output feedback controller of the form (14) for the system ($\Sigma_1$) can be summarized as follows:

(1) Specify $\varepsilon > 0$ and solve the following convex optimization problem and save the minimum $\gamma$ as $\gamma_{\mathrm{opt}}$

$$\min_{\boldsymbol{R}, \boldsymbol{S}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{\boldsymbol{C}}} \gamma$$

$$\text{s.t. (a) Eq. (22)} \tag{23}$$

$$\text{(b) Eq. (17)}$$

(2) If problem (23) is not solvable, increase or decrease $\varepsilon$ and repeat step (1).

(3) Repeat steps (1) and (2). Compare different $\gamma_{\mathrm{opt}}$ obtained from steps (1) and (2) and save the minimum $\gamma_{\mathrm{opt}}$ and the corresponding $\varepsilon$ as $\gamma_{\mathrm{opt}}^*$ and $\varepsilon^*$.

(4) Specify $\varepsilon = \varepsilon^*$, $\gamma > \gamma_{\mathrm{opt}}^*$ and search the feasible solutions of $\boldsymbol{R}$, $\boldsymbol{S}$, $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{C}}$ with Eqs. (22) and (17) satisfied. Save $\boldsymbol{R}$, $\boldsymbol{S}$, $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{C}}$.

(5) Compute the invertible matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ by using the singular value decomposition of $\boldsymbol{I}$-$\boldsymbol{RS}$.

(6) Solve Eqs. (18)-(20) for the controller matrices $\boldsymbol{A}_k$, $\boldsymbol{B}_k$ and $\boldsymbol{C}_k$.

(7) Repeat steps (4)-(6) until a desired robust $H_\infty$ controller is obtained.

*Remark*: Steps (1) and (4) can be implemented using the functions *mincx* and *feasp* in the LMI toolbox of MATLAB.

## 4 Simulations

To demonstrate the effectiveness of the proposed robust $H_\infty$ control strategy, a six-story shear building subjected to seismic forcing was investigated (Arfiadi and Hadi, 2006), see Fig. 1. It was assumed that all the masses of the building are concentrated at its floor levels, such that the nominal mass of every story is $m_0 = 3.456 \times 10^5\,\mathrm{kg}$. The shear stiffness and damping coefficient for each story are $k_0 = 3.404 \times 10^8\,\mathrm{N/m}$ and $c_0 = 2.937 \times 10^6\,\mathrm{N \cdot s/m}$, respectively. Actuators were installed at the first and fourth stories, as shown. At the controller design stage, the possible structural stiffness perturbation was considered and its maximum possible perturbation was taken to be 20%, i.e. $\bar{\alpha}=20$ in Eq. (5). The horizontal absolute accelerations of each story were chosen as the measurement feedback quantities and the measurement noises were assumed to be white and uncorrelated with each other and with the seismic excitation. The intensity matrix of the measurement noises is selected as $\tilde{\boldsymbol{V}} = 1.17 \times 10^{-4}\boldsymbol{I}\ \mathrm{m^2/s^4}$

and the controlled output parameter matrices in Eq. (9) are selected as

$$\boldsymbol{C}_1 = 1.5\times\begin{bmatrix}100\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & 10\boldsymbol{I}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}, \qquad \boldsymbol{D}_{12} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\\ 1\times10^{-5}\boldsymbol{I}\end{bmatrix}$$

The $H_\infty$ disturbance attenuation levels of the closed-loop system for different selected values of $\varepsilon$ are shown in Fig. 2, where $\gamma_{\mathrm{opt}}$ and $\varepsilon$ are those of the optimization problem (23). Clearly, the minimum $\gamma_{\mathrm{opt}}$ is obtained for $\varepsilon \cong 1\times10^{-5}$. With other performances, such as $H_2$ performance, of the closed-loop system also taken into account, $\varepsilon = 1\times10^{-5}$ and $\gamma = 200$ are used in the practical controller.

Different types of seismic excitations were used as disturbance inputs to the building to verify the robustness of the control strategy. Thus a deterministic earthquake acceleration record and a non-stationary random ground acceleration excitation with specified PSD (power spectral density) and modulation function are applied to the building. Moreover, the simulations covered perturbations for different structural stiffness, damping and mass matrices.

### 4.1 Simulation results for the El Centro deterministic earthquake acceleration record

The 1940 El Centro NS earthquake ground acceleration record, with its amplitude scaled to 0.2g, was used as the excitation source and all calculations were performed for its entire duration. Table 1 gives peak responses of the building for different parameter perturbations, where $q_1$ is the horizontal displacement of the base floor relative to the ground, $\ddot{q}_{6a}$ is the horizontal absolute acceleration of the top floor, $u_1$ and $u_2$ are the horizontal components of the control forces and $\Delta\boldsymbol{K}$, $\Delta\boldsymbol{C}$ and $\Delta\boldsymbol{M}$ denote the variations of the stiffness, damping and mass matrices, respectively. The displacement and acceleration responses show that the proposed robust $H_\infty$ controller behaves satisfactorily for several alternative combinations of stiffness, damping and mass perturbations, e.g., for the perturbation case $\Delta\boldsymbol{K}$= +20% and $\Delta\boldsymbol{C}$= +20%, the top floor acceleration was 4.91m/s$^2$ without control, but the active bracings reduced it to 3.09m/s$^2$, i.e. there was a 37% reduction. Time histories for $q_1$, $\ddot{q}_{6a}$, $u_1$ and $u_2$ with perturbations of $\Delta\boldsymbol{K}$= +20% and $\Delta\boldsymbol{C}$= +20% are shown in Fig. 3 for the first 15s,

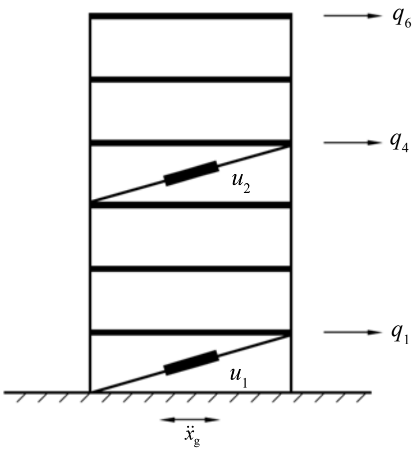


**Fig. 1 Model of building**

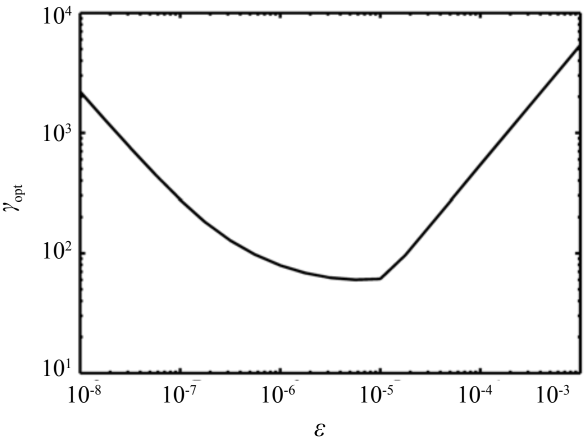


**Fig. 2 $H_\infty$ disturbance attenuation levels of the closed-loop system with different $\varepsilon$ selected**

**Table 1 Peak responses of the building to El Centro earthquake**

| Perturbation cases | | | Peak values | | | |
|---|---|---|---|---|---|---|
| $\Delta\boldsymbol{K}$ | $\Delta\boldsymbol{C}$ | $\Delta\boldsymbol{M}$ | $q_1$ (cm) | $\ddot{q}_{6a}$ (m/s$^2$) | $u_1$ (kN) | $u_2$ (kN) |
| 0 | 0 | 0 | 0.97 (2.16) | 2.77 (5.08) | 3052 | 3062 |
| +20% | 0 | 0 | 0.86 (1.56) | 3.15 (5.23) | 3656 | 3222 |
| -20% | 0 | 0 | 1.16 (2.27) | 2.25 (5.22) | 2470 | 2728 |
| +20% | +20% | 0 | 0.85 (1.52) | 3.09 (4.91) | 3575 | 3174 |
| -20% | -20% | 0 | 1.18 (2.44) | 2.30 (5.66) | 2512 | 2779 |
| +20% | 0 | +10% | 0.92 (1.66) | 3.17 (4.86) | 3567 | 3323 |
| -20% | 0 | -10% | 1.09 (2.09) | 2.31 (4.73) | 2615 | 2738 |

*Note*: Results are for robust $H_\infty$ control, except those in brackets are not controlled which includes the peak values. The beneficial effects of the proposed robust $H_\infty$ controller are again seen.

### 4.2 Simulation results for a random seismic excitation

A suddenly applied 40s duration Kanai-Tajimi filtered white noise was used to represent the ground acceleration excitation induced by an earthquake. Its two-sided power spectral density (PSD) and uniform modulation function have the following forms (Xu and Zhang, 2002; Gao, 2007):

$$S_{\ddot{x}_g\ddot{x}_g}(\omega) = \frac{\omega_g^4 + 4\xi_g^2\omega_g^2\omega^2}{(\omega_g^2 - \omega^2)^2 + 4\xi_g^2\omega_g^2\omega^2} S_0$$

$$g(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

where: $\omega$ denotes frequency in rad/s; $\xi_g$=0.65; $\omega_g$=15.0 rad/s; $S_0$=4.65×10$^{-4}$ m$^2$/(rad s$^3$); $|\omega| \in [\omega_a, \omega_b]$ and; $\omega_a$ and $\omega_b$ are the lower and upper cut-off frequencies in rad/s, with selected values of $\omega_a$= 0.04rad/s and $\omega_b$= 200rad/s. To evaluate the responses of buildings under random seismic excitation, the pseudo-excitation method (PEM) was used (Lin *et al.*, 2001; Lin and Zhang, 2005; Song *et al.*, 2006), which has been proven to be a rigorous, exact and highly efficient method for random vibration analysis.

Figure 4 depicts the time-varying standard deviations of $q_1$ and $\ddot{q}_{6a}$, for perturbations of $\Delta\boldsymbol{K}$= +20% and $\Delta\boldsymbol{C}$= +20%. Note that with actuators, the vibrations of the controlled building become steady in well under 1s, whereas an order of 10s is required for the uncontrolled building. Also, the standard deviations of the stationary responses of $q_1$ and $\ddot{q}_{6a}$ are reduced for the controlled building, particularly so for $q_1$, the base floor displacement. Standard deviations of the stationary responses are given in Table 2 for different perturbation cases and similar conclusions can be drawn to those in Section 4.1.

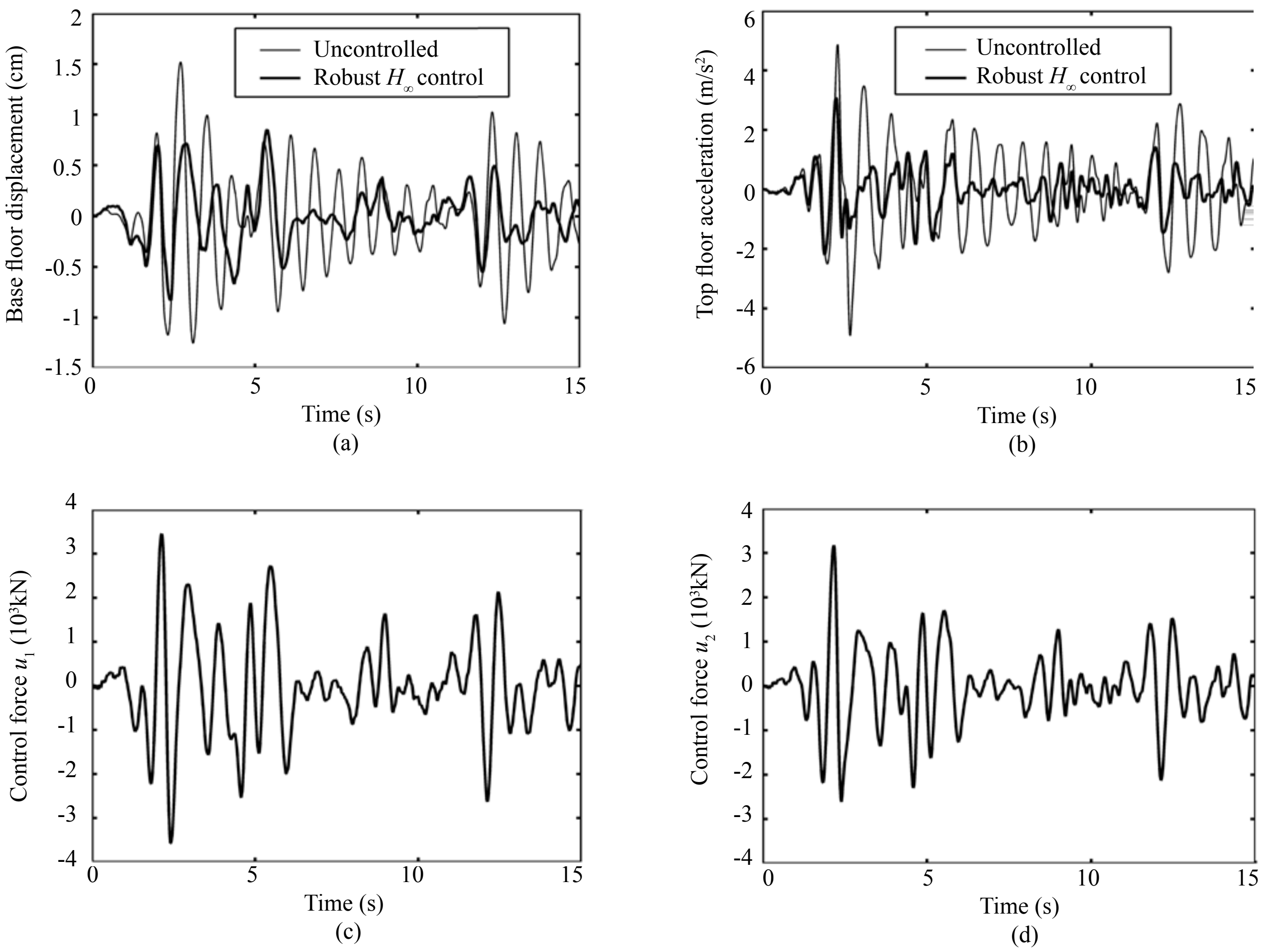


**Fig. 3 Time history responses of the building to El Centro earthquake when ΔK= +20% and ΔC= +20%**

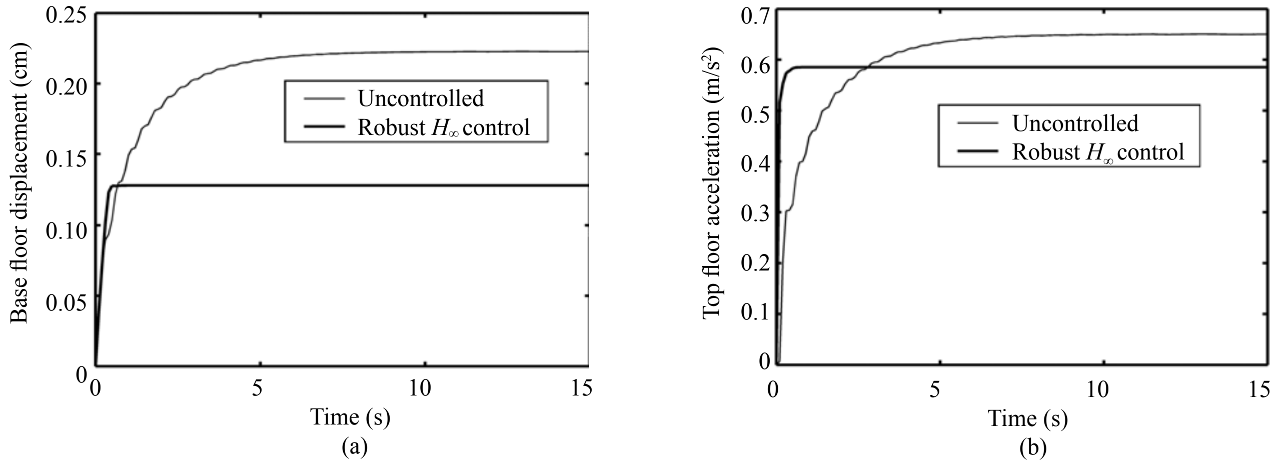


**Fig. 4 Time-varying standard deviation responses of the building to suddenly applied Kanai-Tajimi filtered white noise excitation, when ΔK= +20% and ΔC= +20%**

**Table 2 Standard deviation of stationary responses of the controlled building to Kanai-Tajimi filtered white noise excitations**

| Perturbation cases | | | Standard deviation | | | |
|---|---|---|---|---|---|---|
| $\Delta \boldsymbol{K}$ | $\Delta \boldsymbol{C}$ | $\Delta \boldsymbol{M}$ | $q_1$ (cm) | $\ddot{q}_{6a}$ (m/s$^2$) | $u_1$ (kN) | $u_2$ (kN) |
| 0 | 0 | 0 | 0.15 (0.26) | 0.576 (0.635) | 430 | 316 |
| +20% | 0 | 0 | 0.13 (0.24) | 0.589 (0.707) | 457 | 332 |
| -20% | 0 | 0 | 0.19 (0.29) | 0.562 (0.564) | 402 | 300 |
| +20% | +20% | 0 | 0.13 (0.22) | 0.586 (0.648) | 453 | 327 |
| -20% | -20% | 0 | 0.19 (0.32) | 0.566 (0.635) | 406 | 305 |
| +20% | 0 | +10% | 0.14 (0.26) | 0.588 (0.697) | 458 | 337 |
| -20% | 0 | -10% | 0.18 (0.26) | 0.564 (0.568) | 401 | 294 |

*Note*: Figures given in brackets are for the uncontrolled building

## 5 Conclusions

Based on LMIs, this paper proposes a new robust $H_\infty$ output feedback control strategy for structures with uncertainties in model parameters, and, in addition, derives a sufficient condition for the existence of such an output feedback controller. The proposed control strategy has been verified by aseismic control of a six-story building structure subjected first to the 1940 El Centro earthquake record and then to a suddenly applied Kanai-Tajimi filtered white noise random excitation. Simulation results show that the proposed robust control strategy achieves satisfactory benefits even for large model parameter perturbations.

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