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High Order Numerical Code for Hyperbolic Mild-slope Equations with Nonlinear Dispersion Relation

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Abstract Based on the hyperbolic mild-slope equations derived by Copeland (1985), a numerical model is established in unstaggered grids. A composite 4 th-order Adam-Bashforth-Moulton (ABM) scheme is used to solve the model in the time domain. Terms involving the first order spatial derivatives are differenced to $O(\Delta x)^4$ accuracy utilizing a five-point formula. The nonlinear dispersion relationship proposed by Kirby and Dalrymple (1986) is used to consider the nonlinear effect. A numerical test is performed upon wave propagating over a typical shoal. The agreement between the numerical and the experimental results validates the present model. Biodistribution and applications are also summarized.

Key words hyperbolic mild-slope equations; Adams-Bashforth-Moulton scheme; nonlinear dispersion property; wave

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1 Introduction

Wave prediction is quite important for coastal engineers who utilize and protect coastal regions by designing, constructing and maintaining breakwaters, harbors, beach resorts, *etc*. As a typical numerical model, the elliptic mild-slope equation firstly developed by Berkhoff (1972) is known as a linear model which can handle wave phenomena such as refraction, diffraction, shoaling, and reflection. However, when solving the elliptic equation, all the boundary conditions need to be specified, and the solution of a matrix is necessary, which usually requires a great deal of computational time. As an alternative to the elliptic mild-slope equation, its parabolic approximations have also been used to solve the water wave transformation efficiently. But, in the case where significant wave reflection occurs or wave directions are appreciably deviated from the presumed direction, the solution of the parabolic equation contains errors. As another alternative, hyperbolic equations have also been used with the same accuracy but with less computational time compared to the elliptic equation. Zheng *et al.* (2001) used the ADI scheme in staggered grids to solve the hyperbolic model of Madsen and Larsen (1987). The present paper presents a new approach to solving mild-slope equations in unstaggered grids. For time-marching, a composite 4th-order Adam-Bashforth-Moulton (ABM) scheme is used to solve Copeland's model (1985). And the terms involving the first order spatial derivatives are differenced to $O(\Delta x)^4$ accuracy utilizing a five-point formula.

A brief outline of the present paper is as follows. In section 2, the numerical model and numerical scheme are presented. In section 3, a numerical experiment is conducted upon an elliptic shoal. Numerical simulations with two dispersion relations are presented: the linear Stokes dispersion relation and the weakly nonlinear dispersion relation by Kirby and Dalrymaple (1986). Finally, the conclusions are given in section 4.

2 Numerical Model

2.1 Governing Equations

Copeland (1985) derived the hyperbolic mild-slope equations, and the expressions for the model in the present paper are given in the following forms:

$$\eta_t + \frac{C}{C_g}(P_x + Q_y) = 0, \tag{1}$$

$$P_t + CC_g\eta_x = 0, \tag{2}$$

$$Q_t + CC_g\eta_y = 0, \tag{3}$$

where η is the surface elevation, C and C_g are the wave phase speed and wave group velocity respectively, g is the gravitational acceleration, P and Q are the fluxes in

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the x and y directions respectively. At a constant water depth, $P = C_g \eta \cos\theta$ and $Q = C_g \eta \sin\theta$, where θ is the wave angle. In the computation, we first use the linear Stokes wave dispersion relation to calculate the wave number. At a relatively stable stage of computation, we use the weak nonlinearity dispersion relation proposed by Kirby and Dalrymaple, as given below:

$$\omega^2 = gk(1 + f_1 \varepsilon^2 D)\tanh(kh + f_2 \varepsilon), \qquad (4)$$

where $f_1 = [\tanh(kh)]^5$, $f_2 = [kh/\sinh(kh)]^4$,

$$D = [\cosh(4kh) + 8 - 2\tanh(2kh)]/8[\sinh(kh)]^4,$$

$\varepsilon = kA$, A is the wave amplitude, k is wave number and h is water depth.

As the amplitude in formula (4) is unknown, only when the wave reaches a relatively stable stage can we utilize formula (4) to calculate the new wave numbers using the wave amplitude calculated after every two wave periods and the wave numbers are recalculated three times in the whole computation.

2.2 Numerical Scheme

In the unstaggered grids, the equations of (1)-(3) are solved by a composite 4th ABM scheme proposed by Wei and Kirby (1995) in the time domain. In the numerical simulation, a predict-correct scheme is applied.

1) At the predicting stage:

$$\frac{\eta^n - \eta^{n-1}}{\Delta t} = -\frac{1}{12} C/C_g (23P_x^{n-1} - 16P_x^{n-2} + 5P_x^{n-3} + 23Q_y^{n-1} - 16Q_y^{n-2} + 5Q_y^{n-3}), \qquad (5)$$

$$\frac{P^n - P^{n-1}}{\Delta t} = -\frac{1}{12} CC_g (23\eta_x^{n-1} - 16\eta_x^{n-2} + 5\eta_x^{n-3}), \qquad (6)$$

$$\frac{Q^n - Q^{n-1}}{\Delta t} = -\frac{1}{12} CC_g (23\eta_y^{n-1} - 16\eta_y^{n-2} + 5\eta_y^{n-3}), \qquad (7)$$

2) At the correcting stage:

$$\frac{\eta^n - \eta^{n-1}}{\Delta t} = -\frac{1}{24} C/C_g (9P_x^n + 19P_x^{n-1} - 5P_x^{n-2} + P_x^{n-3} + 9Q_y^n + 19Q_y^{n-1} - 5Q_y^{n-2} + Q_y^{n-3}), \qquad (8)$$

$$\frac{P^n - P^{n-1}}{\Delta t} = -\frac{1}{24} CC_g (9\eta_x^n + 19\eta_x^{n-1} - 5_x^{n-2} + \eta_x^{n-3}), \qquad (9)$$

$$\frac{Q^n - Q^{n-1}}{\Delta t} = -\frac{1}{24} CC_g (9\eta_y^n + 19\eta_y^{n-1} - 5\eta_y^{n-2} + \eta_y^{n-3}). \qquad (10)$$

As for the terms involving the first spatial derivatives, we use the following expression:

$$f_{(i,j)x} = \frac{1}{12\Delta x}(-f_{i+2,j} + 8f_{i+1,j} - 8f_{i-1,j} + f_{i-2,j}),$$

$$\text{for} \quad 2 \le i \le MI - 2. \qquad (11)$$

For other points, we use the off-center five-point scheme by Wei and Kirby (1995).

2.3 Boundaries and Initial Conditions

The initial boundary is given as

$$\eta(x_0, y, t) = A\sin(k\cos\theta_0 x_0 + k\sin\theta_0 y - \omega t), \qquad (12)$$

$$P(x_0, y, t) = C_g \eta \cos\theta_0, \qquad (13)$$

$$Q(x_0, y, t) = C_g \eta \sin\theta_0, \qquad (14)$$

where x_0 is the detailed place of the initial wave, θ_0 the initial wave angle, ω the wave frequency, k the wave number, and t is the time. As for the two lateral boundaries, we use the reflective boundary. A sponge layer is used at the open boundary.

2.4 Main Process

After all the conditions are given, in each time step, we first use the formula for the predicting stage to calculate f^* (η, P and Q), and then use those for the correcting stage to calculate f. When the following accuracy is reached, no further calculation is required

$$abs\,[(f - f^*)/f] \le 0.001. \qquad (15)$$

3 Validation of the Model

To validate the present model, numerical computation is performed upon a complex topography. Berkhoff *et al.* (1982) conducted experiments of a wave propagating over an elliptic shoal. The water depth and experimental set-up are given in Fig.1. In Berkhoff's experiment, the wave period was 1.0 s and the wave amplitude was 0.0232 m. In the computation, the space steps are $\Delta x = \Delta y = 0.1$ m and the time step is $\Delta t = 0.02$ s, and the total time is 40 s. Numerical results are obtained by computing the surface elevation data from 36 s to 40 s, and are presented in Fig.2, where symbol H stands for wave height.

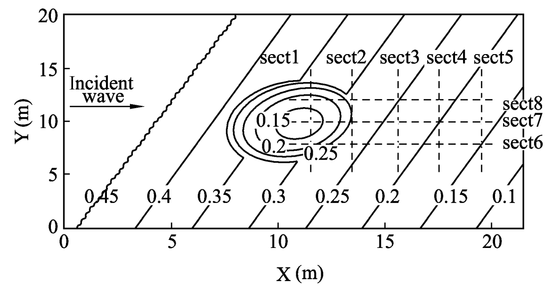

Fig.1 Computational topography of an elliptical shoal.

From Fig.2, it can be seen clearly that the numerical results obtained by the present method agree well with the experimental results. And the numerical results with linear dispersion relation give relatively poor agreement for sections 5-8, which shows that nonlinear dispersion, for instance, the relation in formula (4), plays a very important role in the wave simulation. Besides, from the comparison in sections 6-8, we can see that there is strong reflection in the results from the present model, which needs to be studied in future work.

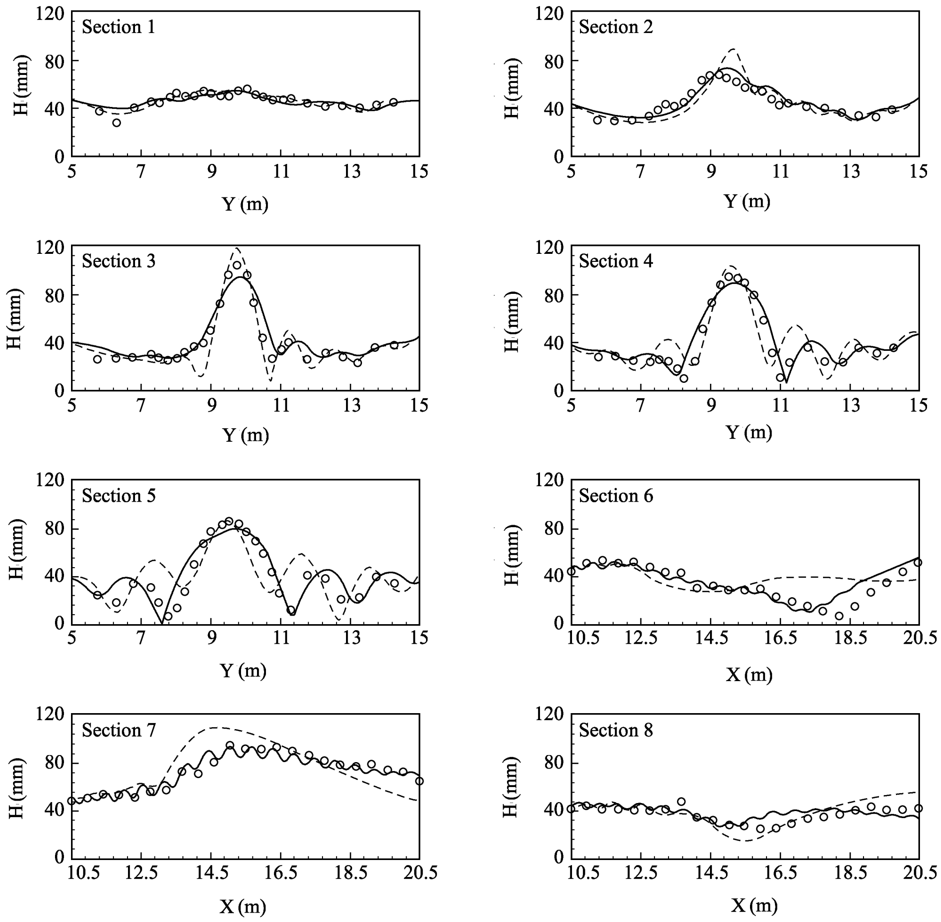

Fig.2 Comparison of numerical results with the experimental data. o: Experimental result; Solid line: present model; Dashed line: model with linear dispersion.

4 Conclusions

Based on the hyperbolic model of Copeland (1985), a high order numerical code is developed with the method proposed by Wei and Kirby (1995). In the numerical computation, weakly nonlinear dispersion relation is used. From the comparison between the computation results from the present model and the experiment results, we can conclude that the present method is capable of solving the hyperbolic mild-slope equations, and is expected to be of use in engineering.

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